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GROUND STATE SOLUTION FOR A CLASS FRACTIONAL HAMILTONIAN SYSTEMS*

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Abstract In this paper, we consider a class of Hamiltonian systems of the form ${}_tD^\alpha_\infty({}_{-\infty}D^\alpha_t u(t)) + L(t)u(t) - \nabla W(t, u(t)) = 0$ where $\alpha \in (\frac{1}{2}, 1)$, ${}_{-\infty}D^\alpha_t$ and ${}_tD^\alpha_\infty$ are left and right Liouville-Weyl fractional derivatives of order α on the whole axis R respectively. Under weaker superquadratic conditions on the nonlinearity and asymptotically periodic assumptions, ground state solution is obtained by mainly using Local Mountain Pass Theorem, Concentration-Compactness Principle and a new form of Lions Lemma respect to fractional differential equations.

Keywords Fractional Hamiltonian systems, ground state, local mountain pass theorem, concentration-compactness principle.

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1. Introduction

Consider the following fractional Hamiltonian systems

$$ {}_tD^\alpha_\infty({}_{-\infty}D^\alpha_t u(t)) + L(t)u(t) - \nabla W(t, u(t)) = 0 \tag{1.1} $$

where $\alpha \in \left(\frac{1}{2}, 1\right)$, $L : R \to R^{N^2}$ is a symmetric matrix valued function, $W \in C^1(R \times R^N, R)$ and $\nabla W(t, x) = (\partial W/\partial x)(t, x)$.

Fractional differential equations both ordinary and partial ones are applied in mathematical modeling of processes in physics, mechanics, control theory, biochemistry, bioengineering and economics. Fractional differential operators have got attention from many researchers that is mainly due to its application as a model for physical phenomena exhibiting anomalous diffusion. Therefore the theory of fractional differential equations is an area intensively developed during last decades [1, 7, 15, 20, 23]. The monographs [9, 14, 17] enclose a review of methods of solving which is an extension of procedures from differential equations theory.

In [8], for the first time, Jiao and Zhou showed that the critical point theory is an effective approach to tackle the existence of solutions for the following fractional

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boundary value problem

$$\begin{cases} {}_tD_T^\alpha({}_0D_t^\alpha u(t)) = \nabla W(t,u(t)), \\ u(0) = u(T). \end{cases} \tag{1.2}$$

The authors study the existence of problem (1.2) by establishing corresponding variational structure in some suitable fractional space and applying the least action principle and Mountain Pass theorem. Motivated by the above work, more and more authors began considering fractional Hamiltonian systems, see [4, 16, 21, 24–26]. In [21], the author shows system (1.1) possesses a nontrivial solution via the mountain pass theorem, by assuming that L and W satisfy the following hypotheses:

(L') $L \in C(R, R^{N^2})$ is a symmetric and positively definite matrix for all $t \in R$ and there exists a continuous function $l : R \to R$ such that $l(t) > 0$ for all $t \in R$ and

$$(L(t)x, x) \geq l(t)|x|^2, \ l(t) \to \infty \text{ as } |t| \to \infty.$$

(AR) there exists a constant $\mu > 2$ such that,

$$0 < \mu W(t,x) \leq (\nabla W(t,x), x)$$

for all $t \in R$ and $x \in R^N \backslash \{0\}$.

(H_1) there exits $\bar{W} \in C(R^N, R)$ such that

$$|W(t,x)| + |\nabla W(t,x)| \leq |\bar{W}(x)|$$

for all $x \in R^N$ and $x \in R$.

After then, some authors are interested in the existence of solutions for (1.1) under some new super-quadratic conditions instead of (AR), see [4, 16, 24]. In [24] and [25, 26], the authors consider the subquadratic case by assuming $W(t,x) = a(t)V(x)$, where $a \in C(R, R^+)$, $a(t) \to 0$ as $|t| \to \infty$ and V satisfies

(H_2) $V(x) \geq b_1(t)|x|^s$ and $|\nabla V(x)| \leq b_2(t)|x|^s$ for all $(t,x) \in R \times R^N$, where $1 < s < 2$ is a constant, $b_1 : R \to R^+$ is a bounded continuous function, and $b_2 : R \to R^+$ is a continuous function with proper integrability on R.

To our best knowledge, so far no study has conducted on the existence of ground state solutions (i.e., nontrivial solutions with least possible energy) for the fractional Hamiltonian systems. Our interests mainly concentrate on the existence of ground state solutions of system (1.1) under general superquadratic potentials.

The following conditions are assumed.

(L) $(L(t)x, x) := (L^\infty(t)x, x) - (L^0(t)x, x)$, where $L^\infty(t)$ and $L^0(t)$ are symmetric measurable matrix functions and L^∞ is T-periodic in t, there exist $0 < l_0 < l^\infty$

$$0 \leq (L(t)x, x) \leq (L^\infty(t)x, x) \leq l^\infty|x|^2, \ \ l_0|x|^2 \leq (L^\infty(t)x, x) \tag{1.3}$$

for all $(t,x) \in R \times R^N$, where $L^0 : R \to R^{N^2}$ such that for every $\varepsilon > 0$, the set $\left\{t \in R : \sup_{x \neq 0} \frac{|L^0(t)x|}{|x|} \geq \varepsilon\right\}$ has finite Lebesgue measure.

(W_0) $|\nabla W(t,x)| = o(|x|)$ as $|x| \to 0$ uniformly in $t \in R$, $W(t,0) \equiv 0$ and $W(t,x) \geq 0$ for all $(t,x) \in R \times R^N$.

(W_1) $F(t,x) \geq 0$, there exist $\eta \geq 1$ and $b \in L^1(R, R \setminus R^-)$ such that

$$F(t, \varsigma x) \leq \eta F(t, \zeta x) + b(t)$$

for all $(t,x)\in R\times R^N$ and $0\le\varsigma\le\zeta$, where $F(t,x)=\frac{1}{2}(\nabla W(t,x),x)-W(t,x)$.

(W_2) There exists $s_0>0$ such that

$$\frac{1-s^2}{2}(\nabla W(t,x),x)\ge\int_s^1(\nabla W(t,x),\theta x)d\theta=W(t,x)-W(t,sx)$$

for all $(t,x)\in R\times R^N$ and $s\in[0,s_0]$.

(W_3) There exists $W^\infty\in C(R\times R^N,R)$ such that $(\nabla W(t,x),x)\ge(\nabla W^\infty(t,x),x)$ and $|\nabla W^0(t,x)|\le h(t)|x|^{p-1}$ for all $(t,x)\in R\times R^N$, where $\nabla W^0(t,x)=\nabla W(t,x)-\nabla W^\infty(t,x)$, $2<p<+\infty$, $h\in L^\infty(R,R)$ such that for every $\varepsilon>0$, the set $\{t\in R:|h(t)|\ge\varepsilon\}$ has finite Lebesgue measure.

(W_4) $W^\infty(t,x)$ is T-periodic in t.

(W_5) $\lim_{|x|\to+\infty}\frac{|\nabla W^\infty(t,x)|}{|x|}=+\infty$ uniformly in $t\in R$.

(W_6) The mapping $\tau\to\left(\frac{\nabla W^\infty(t,\tau x)}{\tau},x\right)$ is strictly increasing in $\tau\in(0,1]$ for all $x\ne0$ and $t\in R$.

Theorem 1.1. *Assume (L), (W_0), one of (W_1) or (W_2), and (W_3)-(W_6). Then problem (1.1) possesses a nontrivial ground state solution.*

If $L^0=0,W^0=0$, systems (1.1) reduces to the periodic case. As a corollary of Theorem 1.1, Theorem 1.2 is still a new result.

Theorem 1.2. *Assume (W_0) and*

(L'') *$L(t)\in C(R,R^{N^2})$ is T-periodic in t and there are constants $0<\lambda_1<\lambda_2$ such that*

$$\lambda_1|x|^2\le(L(t)x,x)\le\lambda_2|x|^2 \text{ for all } (t,x)\in R\times R^N.$$

(W_7) *$W\in C^1(R\times R^N,R)$ is T-periodic in t.*

(W_8) *$\frac{W(t,x)}{|x|^2}\to\infty$ uniformly in t as $|x|\to\infty$.*

(W_9) *$\tau\to\frac{(\nabla W(t,\tau x),x)}{\tau}$ is strictly increasing of $\tau>0$ for all $x\ne0$ and $t\in R$.*

Then problem (1.1) possesses a nontrivial ground state solution.

Remark 1.1. It seems Theorem 1.1 is the first result on the existence of ground state solution for the fractional Hamiltonian system. Linking theorem and the Nehari manifold methods are two most commonly methods to obtain ground state solutions. Since we remove the strictly monotonic condition on W and the technical space decomposable condition, so the Linking theorem and the Nehari manifold methods are invalid here. Our methods are different from the ones in previous papers on ground state solutions.

2. Preliminary Results

2.1. Liouville-Weyl Fractional Calculus

Definition 2.1. The left and right Liouville-Weyl fractional integrals of order $0<\alpha<1$ on the hole axis R are defined by

$$_{-\infty}I_t^\alpha u(t):=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^t(t-h)^{\alpha-1}u(h)dh$$

$$ {}_tI^{\alpha}_{\infty}u(t) := \frac{1}{\Gamma(\alpha)}\int_t^{\infty}(h-t)^{\alpha-1}u(h)dh $$

respectively, where $t \in R$.

Definition 2.2. The left and right Liouville-Weyl fractional derivatives of order $0 < \alpha < 1$ on the hole axis R are defined by

$$ {}_{-\infty}D^{\alpha}_{t}u(t) := \frac{d}{dt}{}_{-\infty}I^{1-\alpha}_{t}u(t), $$

$$ {}_tD^{\alpha}_{\infty}u(t) := \frac{d}{dt}{}_tI^{1-\alpha}_{\infty}u(t), $$

respectively, where $t \in R$.

The Definition 2.1 and 2.2 may be written in an alternative form:

$$ {}_{-\infty}D^{\alpha}_{t}u(t) := \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{u(t)-u(t-h)}{h^{\alpha+1}}dh, $$

$$ {}_tD^{\alpha}_{\infty}u(t) := \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{u(t)-u(t+h)}{h^{\alpha+1}}dh. $$

Recalling that the Fourier transform $\mathcal{F}(u)(\xi)$ of $u(t)$ is defined by

$$ \mathcal{F}(u)(\xi) := \int_{-\infty}^{\infty}e^{-it\xi}u(t)dt. $$

We establish the Fourier transform properties of the fractional integral and fractional differential operators as follows

$$ \mathcal{F}({}_{-\infty}I^{\alpha}_{t}u)(\xi) := (i\xi)^{-\alpha}\mathcal{F}(\phi)(\xi), $$

$$ \mathcal{F}({}_tI^{\alpha}_{\infty}u)(\xi) := (-i\xi)^{-\alpha}\mathcal{F}(\phi)(\xi), $$

$$ \mathcal{F}({}_{-\infty}D^{\alpha}_{t}u)(\xi) := (i\xi)^{\alpha}\mathcal{F}(\phi)(\xi), $$

$$ \mathcal{F}({}_tD^{\alpha}_{\infty}u)(\xi) := (-i\xi)^{\alpha}\mathcal{F}(\phi)(\xi). $$

2.2. Fractional Derivative Spaces

In this section we introduce some fractional spaces for more detail see [5,6]. Let us recall that for any $\alpha > 0$, the semi-norm

$$ |u|_{I^{\alpha}_{-\infty}} := \|{}_{-\infty}D^{\alpha}_{t}u\|_{L^2} $$

and norm

$$ \|u\|_{I^{\alpha}_{-\infty}} := \left(\|u\|^2_{L^2} + |u|^2_{I^{\alpha}_{-\infty}}\right)^{\frac{1}{2}} $$

and let the space $I^{\alpha}_{-\infty}(R, R^N)$ denote the completion of $C^{\infty}_0(R, R^N)$ with respect to the norm $\|\cdot\|_{I^{\alpha}_{-\infty}}$, i.e.,

$$ I^{\alpha}_{-\infty}(R, R^N) = \overline{C^{\infty}_0(R, R^N)}^{\|\cdot\|_{I^{\alpha}_{-\infty}}}. $$

Next we define the fractional Sobolev space $H^\alpha(R,R^N)$ in terms of the Fourier transform. For $0<\alpha<1$, define the semi-norm

$$|u|_\alpha := \||\xi|^\alpha \mathcal{F}(u)\|_{L^2}$$

and the norm

$$\|u\|_{H^\alpha} := \left(\|u\|_{L^2}^2 + |u|_\alpha^2\right)^{\frac{1}{2}}$$

and let

$$H^\alpha(R,R^N) := \overline{C_0^\infty(R,R^N)}^{\|\cdot\|_\alpha}.$$

We note that a function $u\in L^2(R,R^N)$ belongs to $I_{-\infty}^\alpha(R,R^N)$ if and only if

$$|\xi|^\alpha \mathcal{F}(u) \in L^2(R,R^N).$$

In particular, it follows from the integral property of Fourier transform that

$$|u|_{I_{-\infty}^\alpha} = \|_{-\infty}D_t^\alpha u\|_{L^2} = \||\xi|^\alpha \mathcal{F}(u)\|_{L^2} = |u|_\alpha.$$

Therefore $I_{-\infty}^\alpha(R,R^N)$ and $H^\alpha(R,R^N)$ are equal and have equivalent semi-norm and norm.

Analogous to $I_{-\infty}^\alpha(R,R^N)$, we introduce $I_\infty^\alpha(R,R^N)$. Let the semi-norm

$$|u|_{I_\infty^\alpha} := \|_tD_\infty^\alpha u\|_{L^2}$$

and norm

$$\|u\|_{I_\infty^\alpha} := \left(\|u\|_{L^2}^2 + |u|_{I_\infty^\alpha}^2\right)^{\frac{1}{2}},$$

and let

$$I_\infty^\alpha = \overline{C_0^\infty(R,R^N)}^{\|\cdot\|_{I_\infty^\alpha}}.$$

Moreover, $I_{-\infty}^\alpha(R,R^N)$ and $I_\infty^\alpha(R,R^N)$ are equivalent, with equivalent semi-norm and norm.

Let $\alpha\in(0,1)$ and $r\in(1,+\infty)$. We define the fractional Sobolev space $W^{\alpha,r}(R,R^N)$ as follows

$$W^{\alpha,r}(R,R^N) = \left\{u\in L^r(R,R^N) : \int_R\int_R \frac{|u(t)-u(t-h)|^r}{|h|^{1+\alpha r}}dhdt < \infty\right\}.$$

The space $W^{\alpha,r}$ is endowed with the norm

$$\|u\|_{\alpha,r} = \left(\|u\|_{L^r}^r + \int_R\int_R \frac{|u(t)-u(t-h)|^r}{|h|^{1+\alpha r}}dhdt\right)^{1/r}.$$

It follows from the Proposition 4.24 of [5] that the space $W^{\alpha,r}(R,R^N)$ is a Banach space.

The space $H^\alpha(R,R^N)$ coincide with the space $W^{\alpha,2}(R,R^N)$, which follows from the following proposition.

Lemma 2.1. *For $0<\alpha<1$ and $r>1$, $\int_R ||\xi|^\alpha\mathcal{F}(u)(\xi)|^r d\xi<\infty$ if and only if $\int_R\int_R \frac{|u(t)-u(t-h)|^r}{|h|^{1+\alpha r}}dhdt<\infty$. Especially, for $r=2$ we can get that $u\in H^\alpha(R,R^N)$ if and only if $u\in W^{\alpha,2}(R,R^N)$.*

Proof. Using $|1-e^{i\omega}| = 2\sin\left(\frac{\omega}{2}\right)$, we have

$$\begin{aligned}\int_R\int_R \frac{|u(t)-u(t-h)|^r}{h^{1+r\alpha}}dhdt &= \int_R \frac{1}{h^{1+r\alpha}}\int_R |e^{2i\pi h\xi}-1|^r|\mathcal{F}(u)(\xi)|^r d\xi dh\\ &= \int_R |\mathcal{F}(u)(\xi)|^r d\xi \int_R \frac{2^r\sin^r(\pi h\xi)}{h^{1+r\alpha}}dhd\xi\\ &= \int_R (\pi\xi)^{r\alpha}|\mathcal{F}(u)(\xi)|^r d\xi \int_R \frac{2^r\sin^r(l)}{l^{1+r\alpha}}dl \qquad (2.1)\\ &< \infty\end{aligned}$$

because the integral $\int_R \frac{\sin^r(l)}{l^{1+r\alpha}}dl$ convergences for $\alpha\in(0,1)$ and $r>1$. Conversely, these computations show that

$$\int_R\int_R \frac{|u(t)-u(t-h)|^r}{|h|^{1+\alpha r}}dhdt < \infty \Rightarrow \int_R ||\xi|^\alpha \mathcal{F}(u)(\xi)|^r d\xi < \infty.$$

□

Lemma 2.2 (Theorem 4.47, [5]). *Let $\alpha\in(0,1)$ and $r\in(1,+\infty)$. We have*

(i) If $\alpha r<1$, then $W^{\alpha,r}(R,R^N)\hookrightarrow L^s(R,R^N)$ for every $r<s<\frac{r}{1-\alpha r}$;

(ii) If $\alpha r=1$, then $W^{\alpha,r}(R,R^N)\hookrightarrow L^s(R,R^N)$ for every $r<s<\infty$;

(iii) If $\alpha r>1$, then $W^{\alpha,r}(R,R^N)\hookrightarrow L^\infty(R,R^N)$.

If $\alpha>\frac{1}{2}$, it follows from Lemma 2.1 and Lemma 2.2 that $H^\alpha(R,R^N)\hookrightarrow L^\infty(R,R^N)$. Since

$$\int_R |u(t)|^s dt \le \|u\|_{L^\infty}^{s-2}\|u\|_{L^2}^2$$

for all $s\in[2,+\infty)$, which together with Lemma 2.2 implies that $H^\alpha(R,R^N)\hookrightarrow L^s(R,R^N)$ for all $s\in[2,\infty]$. In particular, for all $s\in[2,+\infty)$ and $s=+\infty$, there exist constants C_s and C_∞ such that

$$\|u\|_{L^s}\le C_s\|u\|_{H^\alpha}, \qquad (2.2)$$

$$\|u\|_{L^\infty}\le C_\infty\|u\|_{H^\alpha} \qquad (2.3)$$

for all $u\in H^1(R,R^N)$. Here $L^s(R,R^N)(2\le s<+\infty)$ denote the Banach spaces of function on R with values in R^N under the norms

$$\|u\|_{L^s} = \left(\int_R |u|^s dt\right)^{1/s}.$$

$L^\infty(R,R^N)$ is the Banach space of essentially bounded functions from R into R^N equipped with the norm

$$\|u\|_\infty = \operatorname{ess\,sup}\{|u| : t\in R\}.$$

In order to establish our result via critical point theory, we firstly introduce a new fractional space

$$E^\alpha := \left\{u\in H^\alpha(R,R^N) : \int_R \left(|{}_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t),u(t))\right)dt<\infty\right\}.$$

The space E^α is a Hilbert space with the inner product

$$\langle u,v\rangle_{E^\alpha}=\int_R[({}_{-\infty}D_t^\alpha u(t),{}_{-\infty}D_t^\alpha u(t))+(L(t)u(t),u(t))]\,dt$$

and the corresponding norm

$$\|u\|^2_{E^\alpha}=\langle u,u\rangle_{E^\alpha}.$$

Lemma 2.1 in [21] shows that E^α is continuously embedded in $H^\alpha(R,R^N)$ if L is positively bounded from below. Since in our Theorem 1.1 L is not continuous and does not have positive lower bounds, it is not obvious that $\|\cdot\|_{E^\alpha}$ and $\|\cdot\|_{H^\alpha}$ are equivalent, which will be proved in following Lemma 2.3.

Lemma 2.3. *Suppose L satisfies (L). Then there exist two positive constants d_1 and d_2 such that $d_1\|u\|^2_{H^\alpha}\le\|u\|^2_{E^\alpha}\le d_2\|u\|^2_{H^\alpha}$ for all $u\in E^\alpha$.*

Proof. Since $0\le(L(t)x,x)\le(L^\infty(t)x,x)\le l^\infty|x|^2$ for all $(t,x)\in R\times R^N$, one has $\|u\|^2_{E^\alpha}\le\max\{1,l^\infty\}\|u\|^2_{H^\alpha}$. Thus we can choose $d_2=\max\{1,l^\infty\}$. Set $\Omega_\varepsilon=\left\{t\in R:\sup_{x\neq0}\frac{|L^0(t)x|}{|x|}\ge\varepsilon\right\}$ and $\Omega_\varepsilon(T)=\left\{R\setminus B_T:\sup_{x\neq0}\frac{|L^0(t)x|}{|x|}\ge\varepsilon\right\}$. It follows from (L) that $\mathrm{meas}(\Omega_\varepsilon)<\infty$ for any $\varepsilon>0$. We claim that

$$\mathrm{meas}(\Omega_\varepsilon(T))\to0\text{ as }T\to\infty. \tag{2.4}$$

In order to prove (2.4), it suffices to prove

$$\lim_{n\to\infty}\mathrm{meas}(\Omega_\varepsilon\cap(R\setminus B_{T_n}))=0$$

for each sequence $\{T_n\}\subset R$ such that $T_n\to\infty$. Consider the real function $f:R\to R$ given by $f(t)=\chi_{\Omega_\varepsilon}(t)$, that is

$$f(t)=\begin{cases}1\ \text{for }t\in\Omega_\varepsilon\\0\text{ for }t\notin\Omega_\varepsilon.\end{cases}$$

Then $f\in L^1(R,R)$ and $\|f\|_{L^1}=\int_R|f|dt=\mathrm{meas}(\Omega_\varepsilon)$. Moreover, defining the sequence of functions $f_n:R\to R$ by $f_n(t)=\chi_{\Omega_\varepsilon\cap(R\setminus B_{T_n})}(t)$, it follows from that $|f_n|\le|f|$. Since $f_n\to0$ almost everywhere in R as $n\to\infty$, our claim follows from Lebesgue's Dominated Convergence Theorem.

It follows from (2.4) that we can find $T_\varepsilon>0$ such that $\mathrm{meas}(\Omega_\varepsilon(T_\varepsilon))<\varepsilon$. Consequently,

$$\begin{aligned}\int_R(L^0(t)u,u)dt&=\int_{B_{T_\varepsilon}}(L^0(t)u,u)dt+\int_{R\setminus B_{T_\varepsilon}}(L^0(t)u,u)dt\\&=\int_{B_{T_\varepsilon}}(L^0(t)u,u)dt+\int_{\left\{t\in R\setminus B_{T_\varepsilon}:\sup_{x\neq0}\frac{|L^0(t)x|}{|x|}<\varepsilon\right\}}(L^0(t)u,u)dt\\&\quad+\int_{\left\{t\in R\setminus B_{T_\varepsilon}:\sup_{x\neq0}\frac{|L^0(t)x|}{|x|}\ge\epsilon\right\}}(L^0(t)u,u)dt\\&\le\int_{B_{T_\varepsilon}}(L^0(t)u,u)dt+\varepsilon\int_{R\setminus B_{T_\varepsilon}}|u|^2dt+l^\infty\int_{\Omega_\varepsilon(T_\varepsilon)}|u|^2dt\end{aligned}$$

$$\leq \int_{B_{T_\varepsilon}} (L^0(t)u,u)dt + \varepsilon \int_{R\setminus B_{T_\varepsilon}} |u|^2 dt$$
$$+ l^\infty \text{meas}(\Omega_\varepsilon(T_\varepsilon))^{1/3} \left(\int_{\Omega_\varepsilon(T_\varepsilon)} |u|^3 dt \right)^{2/3}$$
$$\leq \int_{B_{T_\varepsilon}} (L^0(t)u,u)dt + \varepsilon \int_{R\setminus B_{T_\varepsilon}} |u|^2 dt$$
$$+ l^\infty C_3^2 \varepsilon^{\frac{1}{3}} \int_{R\setminus B_{T_\varepsilon}} (|_{-\infty}D_t^\alpha u|^2 + |u|^2)dt. \tag{2.5}$$

Since $L(t)$ is positive definite in B_{T_ε}, there exits $l_\varepsilon > 0$ such that $(L(t)x,x) \geq l_\varepsilon |x|^2$ for all $(t,x) \in B_{T_\varepsilon} \times R^N$, which together with (2.5) that

$$\|u\|_{E^\alpha}^2 = \int_R |_{-\infty}D_t^\alpha u|^2 dt + \int_R (L^\infty(t)u,u)dt - \int_R (L^0(t)u,u)dt$$
$$\geq \int_R |_{-\infty}D_t^\alpha u|^2 dt + \int_R (L^\infty(t)u,u)dt - \int_{B_{T_\varepsilon}} (L^0(t)u,u)dt$$
$$-\varepsilon \int_{R\setminus B_{T_\varepsilon}} |u|^2 dt - l^\infty C_3^2 \varepsilon^{\frac{1}{3}} \int_{R\setminus B_{T_\varepsilon}} (|_{-\infty}D_t^\alpha u|^2 + |u|^2)dt$$
$$= \int_{B_{T_\varepsilon}} |_{-\infty}D_t^\alpha u|^2 dt + \int_{B_{T_\varepsilon}} ((L^\infty(t) - L^0(t))u,u)dt + \int_{R\setminus B_{T_\varepsilon}} |_{-\infty}D_t^\alpha u|^2 dt$$
$$+ \int_{R\setminus B_{T_\varepsilon}} (L^\infty(t)u,u)dt - \varepsilon \int_{R\setminus B_{T_\varepsilon}} |u|^2 dt - l^\infty C_3^2 \varepsilon^{\frac{1}{3}} \int_{R\setminus B_{T_\varepsilon}} (|_{-\infty}D_t^\alpha u|^2 + |u|^2)dt$$
$$\geq \int_{B_{T_\varepsilon}} |_{-\infty}D_t^\alpha u|^2 dt + l_\varepsilon \int_{B_{T_\varepsilon}} |u|^2 dt$$
$$+(1 - l^\infty C_3^2 \varepsilon^{\frac{1}{3}}) \int_{R\setminus B_{T_\varepsilon}} |_{-\infty}D_t^\alpha u|^2 dt + (l_0 - l^\infty C_3^2 \varepsilon^{\frac{1}{3}} - \varepsilon) \int_{R\setminus B_{T_\varepsilon}} |u|^2 dt. \tag{2.6}$$

Choose an appropriate $\varepsilon_0 > 0$ such that $a := 1 - l^\infty C_3^2 \varepsilon_0^{\frac{1}{3}} > 0$ and $b := l_0 - l^\infty C_3^2 \varepsilon_0^{\frac{1}{3}} - \varepsilon_0 > 0$. Then it follows from (2.6) that

$$\|u\|_{E^\alpha}^2 \geq \min\{1, a, b, l_{\varepsilon_0}\} \int_R (|_{-\infty}D_t^\alpha u|^2 + |u|^2)dt.$$

Thus we can choose $d_1 = \min\{1, a, b, l_{\varepsilon_0}\}$ and the proof of Lemma 2.3 is completed. □

Remark 2.1. It follows from Lemma 2.2 and Lemma 2.3 that $E^\alpha \hookrightarrow L^s(R, R^N)$ for any $s \in [2, +\infty]$. In particular, there exist constants which still denoted by C_s and C_∞ such that

$$\|u\|_{L^s} \leq C_s \|u\|_{E^\alpha}, \|u\|_{L^\infty} \leq C_\infty \|u\|_{E^\alpha}, \forall u \in E^\alpha.$$

Lemma 2.4. *Suppose (L), (W_3) hold. Assume u_n is bounded in E^α and $u_n \to 0$ in $L_{loc}^s(R, R^N)$, for any $s \in [2, +\infty]$. Then up to a sequence, one has*

$$\int_R (W(t,u_n) - W^\infty(t,u_n))dt \to 0 \tag{2.7}$$

and

$$\int_R (\nabla W(t,u_n) - \nabla W^\infty(t,u_n), u_n)dt \to 0 \tag{2.8}$$

as $n \to \infty$.

Proof. We just prove (2.7), and the proof of (2.8) is similar with (2.7). By the mean value theorem, there exists $s_n \in [0,1]$ such that

$$W(t,u_n) - W^\infty(t,u_n) = (\nabla W^0(t, s_n u_n), u_n).$$

Set $\Omega_\varepsilon = \{t \in R : |h(t)| \geq \varepsilon\}$ and $\Omega_\varepsilon(T_\varepsilon) = \{t \in R \setminus B_{T_\varepsilon} : |h(t)| \geq \varepsilon\}$. Since for any $\varepsilon > 0$, $\mathrm{meas}(\Omega_\varepsilon) < \infty$, (2.4) can still be proved. It follows that there exists $T_\varepsilon > 0$ such that $\mathrm{meas}(\Omega_\varepsilon(T_\varepsilon)) < \varepsilon$. Therefore, one has

$$\begin{aligned}
&\int_R |(W(t,u_n) - W^\infty(t,u_n))|dt = \int_R |(\nabla W^0(t,s_n u_n), u_n)|dt \leq \int_R h(t)|u_n|^p \\
&= \int_{B_{T_\varepsilon}} h(t)|u_n|^p dt + \int_{\{t\in R\setminus B_{T_\varepsilon} : |h(t)|<\varepsilon\}} h(t)|u_n|^p dt \\
&\quad + \int_{\{t\in R\setminus B_{T_\varepsilon} : |h(t)|\geq\varepsilon\}} h(t)|u_n|^p dt \\
&\leq I_1 + I_2 + I_3.
\end{aligned} \tag{2.9}$$

It is clear that

$$I_1 \leq \|h\|_{L^\infty} \int_{B_{T_\varepsilon}} |u_n|^p dt = o_n(1),$$

which is deduced by $u_n \to 0$ in $L^p_{loc}(R, R^N)$ for all $p \in [2, +\infty]$. Moreover,

$$\begin{aligned}
I_2 &= \int_{\{t\in R\setminus B_{T_\varepsilon} : |h(t)|<\varepsilon\}} h(t)|u_n|^p dt \\
&\leq \varepsilon \|u_n\|^p_{L^p}, \\
I_3 &\leq \|h\|_{L^\infty} \int_{\Omega_\varepsilon(T_\varepsilon)} |u_n|^p dt \\
&\leq \|h\|_{L^\infty} \mathrm{meas}(\Omega_\varepsilon(T_\varepsilon))^{1/2} \left(\int_R |u_n|^{2p} dt\right)^{1/2} \\
&\leq \|h\|_{L^\infty} \varepsilon^{1/2} \|u_n\|^p_{L^{2p}}.
\end{aligned}$$

To summarize,

$$\begin{aligned}
&\int_R |(W(t,u_n) - W^\infty(t,u_n))|dt \\
&\leq o_n(1) + \varepsilon\|u_n\|^p_{L^p} + \|h\|_{L^\infty}\varepsilon^{1/2}\|u_n\|^p_{L^{2p}} \\
&\to 0
\end{aligned}$$

as $n \to \infty$, for the arbitrary of ε. □

Lemma 2.5. *Assume (L) and* (W_3)*. If* $\{u_n\}$ *is bounded in* E^α *and* $|y_n| \to +\infty$*, for any* $\varphi \in C_0^\infty(R, R^N)$*, one has*

$$\int_R ((L(t) - L^\infty(t))u_n, \varphi(t - y_n))dt = o_n(1), \tag{2.10}$$

$$\int_R (\nabla W(t,u_n) - \nabla W^\infty(t,u_n), \varphi(t-y_n))dt = o_n(1). \tag{2.11}$$

Proof. (i) Set

$$\Omega_\varepsilon = \left\{ t \in R : \sup_{x\neq 0} \frac{|L^0(t)x|}{|x|} \geq \varepsilon \right\} \text{ and } \Omega_\varepsilon(T) = \left\{ R \setminus B_T : \sup_{x\neq 0} \frac{|L^0(t)x|}{|x|} \geq \varepsilon \right\}.$$

It follows from (2.4) that we can find $T_\varepsilon > 0$ such that $\text{meas}(\Omega_\varepsilon(T_\varepsilon)) < \varepsilon$. Then, we have

$$\begin{aligned}
&\int_{\left\{t\in R: \frac{|L^0(t)x|}{|x|}\geq\varepsilon\right\}} |u|^2 dt \\
&= \int_{\left\{t\in B_{T_\varepsilon}: \frac{|L^0(t)x|}{|x|}\geq\varepsilon\right\}} |u|^2 dt + \int_{\left\{t\in R\setminus B_{T_\varepsilon}: \frac{|L^0(t)x|}{|x|}\geq\varepsilon\right\}} |u|^2 dt \\
&\leq \int_{B_{T_\varepsilon}} |u|^2 dt + \text{meas}(\Omega_\varepsilon(T_\varepsilon))^{1/3} \left(\int_{\Omega_\varepsilon(T_\varepsilon)} |u|^3 dt \right)^{2/3} \\
&\leq \int_{B_{T_\varepsilon}} |u|^2 dt + C_3^2 \varepsilon^{\frac{1}{3}} \|u\|_{E^\alpha}^2.
\end{aligned} \tag{2.12}$$

By using reduction to absurdity, we can conclude from (L) that

$$\sup_{|x|\neq 0} \frac{|L^0(t)x|}{|x|} \leq A$$

for some $A > 0$ and all $t \in R$, which together with (2.12) implies

$$\begin{aligned}
&\int_R \left|(L^0(t)u_n(t), \varphi(t-y_n))\right| dt \\
&\leq \int_{\left\{t\in R: \sup_{x\neq 0} \frac{|L^0(t)x|}{|x|}\geq\varepsilon\right\}} A|u_n(t)||\varphi(t-y_n)|dt \\
&\quad + \int_{\left\{t\in R: \sup_{x\neq 0} \frac{|L^0(t)x|}{|x|}<\varepsilon\right\}} \varepsilon|u_n(t)||\varphi(t-y_n)|dt \\
&\leq A\|u_n\|_{L^2} \left(\int_{\left\{t\in R: \sup_{x\neq 0} \frac{|L^0(t)x|}{|x|}\geq\varepsilon\right\}} |\varphi(t-y_n)|^2 dt \right)^{\frac{1}{2}} + \varepsilon\|u_n\|_{L^2}\|\varphi\|_{L^2} \\
&\leq A\|u_n\|_{L^2} \left(\int_{B_{T_\varepsilon}} |\varphi(t-y_n)|^2 dt + C_3^2 \varepsilon^{\frac{1}{3}} \|\varphi\|_{E^\alpha}^2 \right)^{\frac{1}{2}} + \varepsilon\|u_n\|_{L^2}\|\varphi\|_{L^2} \\
&\leq C\varepsilon^{\frac{1}{6}} + C\varepsilon + o_n(1)
\end{aligned}$$

for some constant $C > 0$, in which

$$\int_{B_{T_\varepsilon}} |\varphi(t-y_n)|^2 dt = o_n(1) \tag{2.13}$$

is obtained by using the Lebesgue's dominated convergence theorem. In view of the arbitrary of ε, we complete the proof of (2.10).

(ii) Similarly with (2.9), we obtain that

$$\begin{aligned}
&\int_R |(\nabla W(t,u_n) - \nabla W^{\infty}(t,u_n), \varphi(t-y_n))|dt \\
\le& \int_{B_{T_\varepsilon}} h(t)|u_n|^{p-1}|\varphi(t-y_n)|dt + \int_{\{t\in R\setminus B_\varepsilon(T_\varepsilon):|\nabla W^0(t,u_n)|<\varepsilon\}} h(t)|u_n|^{p-1}|\varphi(t-y_n)|dt \\
&+ \int_{\Omega_\varepsilon(T_\varepsilon)} h(t)|u_n|^{p-1}|\varphi(t-y_n)|dt \\
=& I_4 + I_5 + I_6.
\end{aligned}$$

It is clear that

$$I_4 \le \|h\|_{L^\infty}\|u_n\|_{L^\infty}^{p-2}\|u_n\|_{L^2}\left(\int_{B_{T_\varepsilon}} |\varphi(t-y_n)|^2 dt\right)^{1/2} = o_n(1),$$

which is deduced by (2.13). Moreover,

$$\begin{aligned}
I_5 &= \int_{\{t\in R\setminus B_{T_\varepsilon}:|h(t)|<\varepsilon\}} h(t)|u_n|^{p-1}|\varphi(t-y_n)|dt \\
&\le \varepsilon\|u_n\|_{L^\infty}^{p-2}\left(\int_R |u_n|^2\right)^{1/2}\left(\int_R |\varphi(t-y_n)|^2\right)^{1/2} \\
&\le \varepsilon\|u_n\|_{L^\infty}^{p-2}\|u_n\|_{L^2}\|\varphi\|_{L^2}, \\
I_6 &\le \|h\|_{L^\infty}\|u_n\|_{L^\infty}^{p-1}\int_{\Omega_\varepsilon(T_\varepsilon)} |\varphi(t-y_n)|dt \\
&\le \|h\|_{L^\infty}\|u_n\|_{L^\infty}^{p-1}\mathrm{meas}(\Omega_\varepsilon(T_\varepsilon))^{1/2}\left(\int_R |\varphi(t-y_n)|^2 dt\right)^{1/2} \\
&\le \varepsilon^{1/2}\|h\|_{L^\infty}\|u_n\|_{L^\infty}^{p-1}\|\varphi\|_{L^2}.
\end{aligned}$$

To summarize,

$$\begin{aligned}
&\int_R |(W(t,u_n) - W^{\infty}(t,u_n))|dt \\
\le&\ o_n(1) + \varepsilon\|u_n\|_{L^\infty}^{p-2}\|u_n\|_{L^2}\|\varphi\|_{L^2} + \varepsilon^{1/2}\|h\|_{L^\infty}\|u_n\|_{L^\infty}^{p-1}C_2\|\varphi\|_{E^\alpha} \\
\to&\ 0
\end{aligned}$$

as $n\to\infty$, for the arbitrary of ε. □

In the proof of our results, we shall use the following lemma by Lions ([11,12]) which is well known as the concentration-compactness principle.

Lemma 2.6 (Lemma 1.1, [11]). *Let ρ_n be a sequence in $L^1(R,R)$ satisfying $\rho_n \ge 0$ in R and $\int_R \rho_n dt \to \eta$ which is a fixed constant. Then there exists a subsequence which we still denote by ρ_n satisfying one of the three following possibilities*

(i) (Vanishing):

$$\lim_{n\to\infty}\sup_{y\in R}\int_{y-l}^{y+l} \rho_n dt = 0$$

for all $l>0$;

(ii) (Compactness): There exists $\{y_n\} \subset R$ *satisfying* $\forall \varepsilon > 0$, $\exists l > 0$ *such that*

$$\int_{y_n-l}^{y_n+l} \rho_n dt \geq \eta - \varepsilon$$

for all n;

(iii) (Dichotomy): There exist $\alpha \in (0, \eta), \rho_n^1 \geq 0, \rho_n^2 \geq 0$, *and* $\rho_n^1, \rho_n^2 \in L^1(R, R)$ *such that*

(a) $\|\rho_n - (\rho_n^1 + \rho_n^2)\|_{L^1} \to 0$ *as* $n \to \infty$,
(b) $\int_R \rho_n^1 dt \to \alpha$ *as* $n \to \infty$,
(c) $\int_R \rho_n^2 dt \to \eta - \alpha$ *as* $n \to \infty$,
(d) *dist*(*supp* ρ_n^1, *supp* ρ_n^2) $\to \infty$ *as* $n \to \infty$.

If $\alpha = 1$, the following lemma corresponds to Lemma 1.1 in [12], which is well known as Lions Lemma.

Lemma 2.7. *Let* u_n *be a bounded sequence in* $L^q(R, R^N) \cap L^\infty(R, R^N)$, $1 \leq q < \infty$ *such that* $\mathcal{F}(-\infty D_t^\alpha u_n)$ $(0 < \alpha < 1)$ *is bounded in* $L^p(R, R^N)$, $\frac{1}{\alpha} < p < \infty$. *If, in addition, there exists* $l > 0$ *such that*

$$\sup_{y \in R} \int_{y-l}^{y+l} |u_n|^q dt \to 0$$

as $n \to \infty$, *then* $u_n \to 0$ *in* $L^s(R, R^N)$, *for all* $s \in (q, \infty)$.

Proof. Since $\{u_n\}$ is bounded in $L^\infty(R, R^N)$, then clearly we have for all $\beta \geq q$

$$\sup_{y \in R} \int_{y-l}^{y+l} |u_n|^\beta dt \to 0 \tag{2.14}$$

as $n \to \infty$. For $0 < \beta < q$, by Hölder inequality we also have

$$\sup_{y \in R} \int_{y-l}^{y+l} |u_n|^\beta dt \leq (2l)^{\frac{q-\beta}{q}} \left(\sup_{y \in R} \int_{y-l}^{y+l} |u_n|^q \right)^{\frac{\beta}{q}} \to 0 \tag{2.15}$$

as $n \to \infty$.

By Lemma 2.1 we have $\int_R \int_R \frac{|u_n(t) - u_n(t-h)|^p}{|h|^{1+p\alpha}} dh dt$ is bounded. Cover R by intervals $(y_i - l, y_i + l)$, $i \in N$, in such a way that each point of R is contained in at most 2 intervals. It follows from Lemma 2.2, $W^{\alpha,p}(R, R^N) \hookrightarrow L^\infty(R, R^N)$, there exists C independent of i such that

$$\|u_n\|_{L^\infty_{[y_i - l, y_i + l]}} \leq C \int_{y_i - l}^{y_i + l} \left(|u_n|^p + \int_R \frac{|u_n(t) - u_n(t-h)|^p}{|h|^{1+\alpha p}} dh \right) dt. \tag{2.16}$$

If $p \geq q$, it is clear that u_n is bounded in $L^p(R, R^N)$. If $p < q$, it follows from Hölder inequality that

$$\|u_n\|_{L^\infty_{[y_i - l, y_i + l]}} \leq C \int_{y_i - l}^{y_i + l} \left(|u_n|^q + \int_R \frac{|u_n(t) - u_n(t-h)|^p}{|h|^{1+\alpha p}} dh \right) dt. \tag{2.17}$$

Set $\theta = p$ if $p \geq q$, $\theta = q$ if $p < q$. In view of (2.16) and (2.17), for $s \in (q, \infty)$ one has

$$\begin{aligned}
\int_R |u_n|^s dt &\leq \sum_{i=1}^{\infty} \int_{y_i-l}^{y_i+l} |u_n|^s dt \\
&\leq \sum_{i=1}^{\infty} \|u_n\|_{L^\infty_{[y_i-l, y_i+l]}} \int_{y_i-l}^{y_i+l} |u_n|^{s-1} dt \\
&\leq C \sum_{i=1}^{\infty} \int_{y_i-l}^{y_i+l} |u_n|^\theta dt \int_{y_i-l}^{y_i+l} |u_n|^{s-1} dt \\
&\quad + C \sum_{i=1}^{\infty} \left(\int_{y_i-l}^{y_i+l} \left(\int_R \frac{|u_n(t) - u_n(t-h)|^p}{|h|^{1+\alpha p}} dh \right) dt \right) \int_{y_i-l}^{y_i+l} |u_n|^{s-1} dt \\
&\leq 2C \sup_{i \in N} \int_{y_i-l}^{y_i+l} |u_n|^{s-1} dt \int_R |u_n|^\theta dt \\
&\quad + 2C \sup_{i \in N} \int_{y_i-l}^{y_i+l} |u_n|^{s-1} dt \int_R \left(\int_R \frac{|u_n(t) - u_n(t-h)|^p}{|h|^{1+\alpha p}} dh \right) dt \\
&\to 0 \qquad (2.18)
\end{aligned}$$

as $n \to \infty$, which follows from (2.14), (2.15) and boundedness of $\int_R |u_n|^\theta dt$, $\int_R \left(\int_R \frac{|u_n(t) - u_n(t-h)|^p}{|h|^{1+\alpha p}} dh \right) dt$. □

Now we introduce some notations and some necessary definitions which will be used later. Let B be a real Banach space, $I \in C^1(B, R)$, which means that I is continuously Frechet-differentiable functional defined on B. Recall that $I \in C^1(B, R)$ is said to satisfy the (PS) condition if any sequence $\{q_n\}_{n \in N} \subset B$, for which $\{I(q_n)\}$ is bounded and $I'(q_n) \to 0$ as $n \to +\infty$ possesses a convergent subsequence in B.

Moreover, let B_r be the open ball in B with the radius r and centered at 0 and ∂B_r denotes its boundary, the following lemma is well known as Mountain Pass Theorem [18].

Lemma 2.8 ([18]). *Let B be a real Banach space and $I \in C^1(B, R)$ satisfying the (PS) condition. Suppose that $I(0) = 0$ and*

(A1) there are constants ρ, $\alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$;

(A2) there is an $e \in B \backslash \bar{B}_\rho$ such that $I(e) < 0$.

Then I possesses a critical value $c \geq \alpha$. Moreover c can be characterized as

$$c = \inf_{f \in \Gamma} \max_{s \in [0,1]} I(f(s)),$$

where

$$\Gamma = \{f \in C([0,1], B) : f(0) = 0, f(1) = e\}. \qquad (2.19)$$

As shown in [3], a deformation lemma can be proved with the $(Ce)_c$ condition replacing the usual (PS) condition, and it turns out that the Mountain Pass Theorem in [18] hold true under the $(Ce)_c$ condition. So Lemma 2.8 is still true under the weaker $(Ce)_c$ condition.

In the proof of results, the following Local Mountain Pass Theorem is also needed.

Lemma 2.9 (Theorem 2.3, [13]). *Let E be a real Banach space and $I \in C^1(E, R)$ satisfies $I(0) = 0$, (A1) and (A2). If there exists $\gamma_0 \in \Gamma$, Γ defined by (2.19), such that*

$$c = \max_{s\in[0,1]} I(\gamma_0(s)) > 0,$$

then I possesses a nontrivial point u at level c.

3. Proof of Theorem 1.1

Define the functional $I : E^\alpha \to R$ by

$$\begin{aligned} I(u) &= \int_R \left[\frac{1}{2} |{}_{-\infty}D_t^\alpha u(t)|^2 + \frac{1}{2}(L(t)u(t), u(t)) - W(t, u(t)) \right] dt \\ &= \frac{1}{2}\|u\|^2_{E^\alpha} - \int_R W(t, u(t))dt. \end{aligned} \tag{3.1}$$

Lemma 3.1. *Assume (L), (W_0) and (W_3)-(W_4). Then $I \in C^1(E^\alpha, R)$ and for all $u, v \in E^\alpha$ we have*

$$\langle I'(u), v\rangle = \int_R [({}_{-\infty}D_t^\alpha u(t), {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t), v(t)) - (\nabla W(t, u(t)), v(t))]\, dt.$$

Proof. We firstly show that $I : E^\alpha \to R$. Let $u \in E^\alpha$, then there exists $k > 0$ such that $\|u\|_{L^\infty} \le k$. In view of (W_0) and (W_3), it follows from standard arguments that for any $\delta > 0$ there exists $C_\delta > 0$ and $p > 2$ such that

$$|\nabla W^0(t, x)| \le \delta|x| + C_\delta|x|^{p-1} \tag{3.2}$$

for all $t \in R$ and $x \in R^N$. Define

$$\max_{t\in[0,T], 0<|x|\le k} \frac{|\nabla W^\infty(t, x)|}{|x|^{p-1}} := \lambda.$$

Then one has

$$\begin{aligned} |\nabla W(t, x)| &\le |\nabla W^\infty(t, x)| + |\nabla W^0(t, x)| \\ &\le \delta|x| + (\lambda + C_\delta)|x|^{p-1} \\ &\le \delta|x| + \lambda_\delta|x|^{p-1} \end{aligned} \tag{3.3}$$

and then

$$|W(t, x)| \le \frac{\delta}{2}|x|^2 + \frac{\lambda_\delta}{2}|x|^p \tag{3.4}$$

for all $t \in R$ and $|x| \le k$, where $\lambda_\delta = \lambda + C_\delta$. Hence, one has $\int_R W(t, u(t))dt < \infty$ and $I : E \to R$.

Next we prove that $I \in C^1(E^\alpha, R)$. Rewrite I as following

$$I = I_1 - I_2$$

where

$$I_1 = \int_R \left[\frac{1}{2}|{}_{-\infty}D_t^\alpha u(t)|^2 + \frac{1}{2}(L(t)u(t), u(t))\right] dt,$$

$$I_2 = \int_R W(t, u(t))dt.$$

It is easy to check that $I_1 \in C^1(E^\alpha, R)$ and

$$\langle I_1'(u), v\rangle = \int_R [({}_{-\infty}D_t^\alpha u(t), {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t), v(t))]dt.$$

It remains to show that $I_2 \in C^1(E^\alpha, R)$. By the Mean value Theorem, for any $u, v \in E^\alpha$ and $h \in [0, 1]$ we have

$$W(t, u(t) + hv(t)) - W(t, u(t)) = (\nabla W(t, u(t) + h\theta(t)v(t), v(t)),$$

where $\theta(t) \in (0, 1)$. Given $u, v \in E^\alpha$, there exists a positive constant which still denoted by $k > 0$, such that

$$|u(t)| + |v(t)| < k$$

for all $t \in R$, so that,

$$|u(t) + h\theta(t)v(t)| < k$$

for all $t \in R$, which together with (3.3) implies

$$\begin{aligned}
&\int_R \max_{h\in[0,1]} |(\nabla W(t, u(t) + h\theta(t)v(t)), v(t))|dt \\
&\leq \int_R \delta[|u(t)||v(t)| + |v(t)|^2]dt \\
&\quad + \int_R \lambda_\delta 2^{p-1}[|u(t)|^{p-1}|v(t)| + |v(t)|^p]dt \\
&\leq \delta(\|u\|_{L^2}\|v\|_{L^2} + \|v\|_{L^2}^2) \\
&\quad + \lambda_\delta 2^{p-1}(\|u\|_{L^\infty}^{p-2}\|u\|_{L^2}\|v\|_{L^2} + \|v\|_{L^p}^p) \\
&< \infty.
\end{aligned}$$

Then by Lebesgue's Convergence Theorem, we have

$$\begin{aligned}
\langle I_2'(u), v\rangle &= \lim_{h\to 0^+} \frac{I_2(u + hv) - I_2(u)}{h} \\
&= \lim_{h\to 0^+} \int_R \frac{W(t, u(t) + hv(t)) - W(t, u(t))}{h} dt \\
&= \lim_{h\to 0^+} \int_R (\nabla W(t, u(t) + h\theta(t)v(t)), v(t))dt \\
&= \int_R (\nabla W(t, u(t)), v(t))dt.
\end{aligned}$$

Now we show that I_2' is continuous. Suppose $u_n \to u$ in E^α, by an easy computation, one has

$$\begin{aligned}
\sup_{\|v\|=1} |\langle I_2'(u_n) - I_2'(u), v\rangle| &= \sup_{\|v\|=1} \left|\int_R (\nabla W(t, u_n(t)) - \nabla W(t, u(t)), v(t))dt\right| \\
&\leq \sup_{\|v\|=1} \|\nabla W(t, u_n(t)) - \nabla W(t, u(t))\|_{L^2}\|v\|_{L^2} \\
&\leq C_2\|\nabla W(t, u_n(t)) - \nabla W(t, u(t))\|_{L^2}.
\end{aligned}$$

Since $u_n \to u$ in E^α, there exists a positive constant which still denoted by $k > 0$, such that

$$\sup_{n\in N} \|u_n\|_{L^\infty} \leq k, \|u\|_{L^\infty} \leq k. \tag{3.5}$$

which together with (3.3) implies

$$\begin{aligned}
&\int_R |\nabla W(t, u_n(t)) - \nabla W(t, u(t))|^2 dt \\
\leq &\int_R (|\nabla W(t, u_n(t))| + |\nabla W(t, u(t))|)^2 dt \\
\leq &\int_R 2\delta^2(|u_n| + |u|)^2 + 2{\lambda_\delta}^2(|u_n|^{p-1} + |u|^{p-1})^2 dt \\
\leq &\int_R 4\delta^2(|u_n|^2 + |u|^2) + 4{\lambda_\delta}^2(|u_n|^{2(p-1)} + |u|^{2(p-1)}) dt \\
< &\infty
\end{aligned} \tag{3.6}$$

for all $n \in N$. By using Lebesgue's Convergence Theorem, one has

$$\langle I_2'(u_n) - I_2'(u), v\rangle \to 0$$

as $n \to \infty$ uniformly with respect to v, which implies the continuity of I_2'. Now we have proved $I \in C^1(E^\alpha, R)$. □

Define the Nehari manifold

$$\mathcal{N} := \{u \in E^\alpha \setminus \{0\} : \langle I'(u), u\rangle = 0\}$$

and set

$$m := \inf_{u\in\mathcal{N}} I(u).$$

In order to prove Theorem 1.1, we study firstly the following periodic problem, namely,

$${}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u(t)) + L^\infty(t)u(t) - \nabla W^\infty(t, u(t)) = 0. \tag{3.7}$$

For system (3.7), we define the Nehari manifold

$$\mathcal{N}^\infty = \{u \in E^\alpha\setminus\{0\} : \langle {I^\infty}'(u), u\rangle\} = 0$$

and set

$$m^\infty := \inf_{u\in\mathcal{N}^\infty} I^\infty(u),$$

where

$$I^\infty(u) = \frac{1}{2}\int_R (|{}_{-\infty}D_t^\alpha u|^2 + (L^\infty(t)u, u))dt - \int_R W^\infty(t, u)dt.$$

Lemma 3.2. *Assume (L), (W_0), (W_3)-(W_5). Then for each $u \in E^\alpha \setminus \{0\}$, there exists $s_u > 0$ such that $s_u u \in \mathcal{N}$. Moreover, the maximum of $I(su)$ for $s \geq 0$ is achieved at s_u.*

Proof. It follows from (3.9) that for any $\delta > 0$, there exists $l_\delta > 0$ such that

$$0 \leq W(t, x) \leq \frac{1}{2}\delta|x|^2$$

for all $t \in R$ and $|x| < l_\delta$. Fix $u \in E^\alpha \setminus \{0\}$, then $\|u\|_{L^\infty} \le k$ for some $k > 0$. Take $0 < s < \frac{l_\delta}{k}$, then $|su(t)| < l_\delta$ for all $t \in R$, hence

$$\begin{aligned} f(s) &= \frac{s^2}{2}\|u\|_{E^\alpha}^2 - \int_R W(t, su(t))dt \\ &\ge \frac{s^2}{2}\|u\|_{E^\alpha}^2 - \frac{\delta}{2}s^2\|u\|_{L^2}^2 \\ &= \frac{s^2}{2}(1 - \delta C_2^2)\|u\|_{E^\alpha}^2. \end{aligned}$$

Fix δ sufficiently small, then there exists $s_0 > 0$, such that $f(s_0) > 0$. Set $\Omega = \{t \in R : |u(t)| > 0\}$, combining with Fatou's Lemma and (W_5), we have

$$\liminf_{s\to+\infty} \int_\Omega \frac{W(t, su)}{|su|^2}dt \ge \liminf_{s\to+\infty} \int_\Omega \frac{W^\infty(t, su)}{|su|^2}dt = +\infty.$$

Hence

$$\begin{aligned} \limsup_{s\to+\infty} \frac{f(s)}{s^2} &= \frac{1}{2}\|u\|_{E^\alpha}^2 - \liminf_{s\to+\infty} \int_R \frac{W(t, su)}{|s|^2}dt \\ &= \frac{1}{2}\|u\|_{E^\alpha}^2 - \liminf_{s\to+\infty} \int_R \frac{W(t, su)}{|su|^2}|u|^2dt \\ &= -\infty \end{aligned}$$

which deduces $f(s) \to -\infty$ as $s \to +\infty$. So there exists $s_u > 0$ such that $f(s_u) = \max_{s>0} f(s)$ and hence $f'(s_u) = 0$, i.e., $I(s_u u) = \max_{s>0} I(su)$ and $s_u u \in \mathcal{N}$. □

In view of the proof of Lemma 3.2, the following remarks are obvious. The functional I verifies the geometric conditions of the Mountain Pass Theorem.

Remark 3.1. Assume (L), (W_0), (W_3)-(W_5) hold. Then I satisfies $I(0) = 0$ and

(A1) there exists $\rho, \alpha > 0$ such that $I(u) \ge \alpha$ for all $\|u\| = \rho$;

(A2) there exists $e \in E^\alpha$ with $\|e\| > \rho$ such that $I(e) \le 0$.

Remark 3.2. The existence of s_u respect to I^∞ is unique, i.e., for each $u \in E^\alpha$, there exists a unique $s_u > 0$ such that $s_u u \in \mathcal{N}^\infty$ and the maximum of $I^\infty(su)$ is achieved at s_u. Assume that there exist $s'_u > s_u > 0$ such that $s'_u u,\ s_u u \in \mathcal{N}^\infty$, then we have

$$s_u^2\|u\|_{E^\alpha}^2 - \int_R (\nabla W^\infty(t, s_u u), s_u u)dt = 0,$$

and

$${s'}_u^2\|u\|_{E^\alpha}^2 - \int_R (\nabla W^\infty(t, s'_u u), s'_u u)dt = 0.$$

It follows from that

$$\int_R \left(\frac{\nabla W^\infty(t, s'_u u)}{s'_u} - \frac{\nabla W^\infty(t, s_u u)}{s_u}, u\right) dt = 0,$$

without loss of generality we may assume $1 \ge s'_u > s_u > 0$, which contradicts with (W_6).

Lemma 3.3 (Proposition 3.11, [19])**.** *Assume (L), (W_0), (W_3)-(W_6) hold. Then*

$$m^{\infty} := \inf_{u\in\mathcal{N}^{\infty}} I^{\infty}(u) = \inf_{u\in E^{\alpha}\setminus\{0\}} \max_{s>0} I^{\infty}(su). \tag{3.8}$$

Lemma 3.4. *Assume (L), (W_0), (W_1), (W_3)-(W_5) and suppose that $\{u_n\}$ is a Cerami sequence at a level $c>0$ for the function I. Then $\|u_n\|_{E^\alpha}$ is bounded.*

Proof. Let $\{u_n\}$ be a Cerami sequence at some level $c>0$, that is,

$$I(u_n)\to c, \tag{3.9}$$

$$(1+\|u_n\|_{E^\alpha})\|I'(u_n)\|\to 0 \tag{3.10}$$

as $n\to\infty$. Arguing by contradiction, we assume $\|u_n\|_{E^\alpha}\to\infty$. Define $v_n = 2\sqrt{c}(u_n/\|u_n\|_{E^\alpha})$, then

$$\|v_n\|_{E^\alpha} = 2\sqrt{c} \tag{3.11}$$

and there exists $v\in E^\alpha$ such that $v_n\rightharpoonup v$ in E^α, $v_n\to v$ in $L^2_{loc}(R)$ and $v_n(t)\to v(t)$ a.e. in R. For any $n\in N$, there exists $k_n\in N$ such that $\|v_n(\cdot+k_nT)\|_{L^\infty} = \max_{t\in R}|v_n(t)|$ occurs in $[0,T]$. Let $\bar v_n := v_n(\cdot+k_nT)$. Since $\{\bar v_n\}$ is also bounded in E^α, passing to a subsequence, we may assume that $\bar v_n\rightharpoonup \bar v$ in E^α, $\bar v_n\to\bar v$ in $L^s_{loc}(R)$, $s\in[2,+\infty]$ and $\bar v_n(t)\to\bar v(t)$ a.e. in R.

Case 1: $\bar v\not\equiv 0$.

In this case meas$\{\Omega\}>0$, where $\Omega=\{t\in R: |\bar v(t)|>0\}$. Therefore

$$|u_n(t+k_nT)| = \frac{|\bar v_n(t)|}{2\sqrt{c}}\|u_n\|_{E^\alpha}\to+\infty$$

as $n\to\infty$, then by Fatou's Lemma, we have

$$\begin{aligned}\liminf_{n\to\infty}\int_R \frac{W(t,u_n)}{|u_n|^2}|v_n|^2dt &= \liminf_{n\to\infty}\int_R \frac{W(t+k_nT,u_n(t+k_nT))}{|u_n(t+k_nT)|^2}|v_n(t+k_nT)|^2dt\\ &\ge \liminf_{n\to\infty}\int_\Omega \frac{W(t+k_nT,u_n(t+k_nT))}{|u_n(t+k_nT)|^2}|v_n(t+k_nT)|^2dt\\ &\ge \liminf_{n\to\infty}\int_\Omega \frac{W^\infty(t,u_n(t+k_nT))}{|u_n(t+k_nT)|^2}|\bar v_n|^2dt\\ &= +\infty.\end{aligned}$$

Then

$$0=\limsup_{n\to\infty}\frac{I(u_n)}{\|u_n\|^2_{E^\alpha}} = \frac{1}{2}-\frac{1}{4c}\liminf_{n\to\infty}\int_R\frac{W(t,u_n)}{|u_n|^2}|v_n|^2dt\to-\infty,$$

which is a contradiction.

Case 2. $\bar v\equiv 0$.

Since $\bar v_n\to 0$ in L^∞_{loc}, in view of the definition of $\bar v_n$, we have

$$\|v_n\|_{L^\infty} = \|\bar v_n\|_{L^\infty} = \|\bar v_n\|_{L^\infty_{[0,T]}}\to 0$$

as $n\to\infty$. Fixing $\theta>0$, for any given $\delta>0$, when n large enough

$$|\theta v_n(t)|\le l_\delta$$

for all $t \in R$. By (3.9), we have

$$\int_R W(t, \theta v_n)\, dt \le \frac{\delta}{2}\theta^2 \|v_n\|_{L^2}^2$$

when n large enough. Since δ is arbitrary and v_n is bounded in E^α, we have

$$\int_R W(t, \theta v_n)\, dt \to 0 \tag{3.12}$$

as $n \to \infty$. Therefore,

$$I\left(\frac{2\sqrt{c}\theta}{\|u_n\|_{E^\alpha}} u_n\right) = I(\theta v_n) = \frac{1}{2}\|\theta^2 v_n\|_{E^\alpha}^2 - \int_R W(t, \theta v_n)\, dt \ge 2\theta^2 c + o_n(1). \tag{3.13}$$

Since $\|u_n\|_{E^\alpha} \to \infty$, then $\frac{2\sqrt{c}\theta}{\|u_n\|_{E^\alpha}} \in (0,1)$ for n sufficiently large, so

$$\max_{s\in[0,1]} I(su_n) \ge I\left(\frac{2\sqrt{c}\theta}{\|u_n\|_{E^\alpha}} u_n\right) \ge 2\theta^2 c + o_n(1). \tag{3.14}$$

By the continuity of I, there exists $s_n \in [0,1]$ such that $I(s_n u_n) = \max_{s\in[0,1]} I(su_n)$. Since $\frac{2\sqrt{c}}{\|u_n\|_{E^\alpha}} \in [0,1]$ when n large enough, we have

$$I(s_n u_n) \ge I(v_n) = I\left(\frac{2\sqrt{c}}{\|u_n\|_{E^\alpha}} u_n\right) = \|v_n\|_{E^\alpha} - \int_R W(t, v_n)dt = 2c + o_n(1).$$

Note that $I(u_n) \to c$, so $0 < s_n < 1$ and $\langle I'(s_n u_n), s_n u_n\rangle = o_n(1)$. Hence by (W_1), one has

$$\begin{aligned} I(s_n u_n) &= I(s_n u_n) - \frac{1}{2}\langle I'(s_n u_n), s_n u_n\rangle + o_n(1) \\ &= \int_R \left(\frac{1}{2}(\nabla W(t, s_n u_n), s_n u_n) - W(t, s_n u_n)\right) + o_n(1) \\ &= \int_R F(t, s_n u_n) + o_n(1) \\ &\le \eta \int_R F(t, u_n) + \int_R b(t)dt + o_n(1) \\ &\le \eta\left(I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle\right) + M_1 + o_n(1) \\ &\le \eta c + M_1 + o_n(1) \le M_2. \end{aligned} \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$2\theta^2 c + o_n(1) \le \max_{s\in[0,1]} I(su_n) = I(s_n u_n) \le M_2$$

which is a contradiction when θ is sufficiently large. Summarize the two cases, we have proved $\{u_n\} \in E^\alpha$ is bounded. □

Lemma 3.5. *Assume (L), (W_0) and (W_2)-(W_5) and suppose that $\{u_n\}$ is a Cerami sequence at a level $c > 0$ for the function I. Then $\|u_n\|_{E_\alpha}$ is bounded.*

Proof. The proof of this lemma follows the same steps of Lemma 3.4, with a change in the Case 2 where $\bar{v} \equiv 0$. Recalling (3.13) and keeping the same notations as in the previous lemma, we have

$$I\left(\frac{2\sqrt{c}\theta}{\|u_n\|_{E^\alpha}} u_n\right) \geq 2\theta^2 c + o_n(1).$$

Indeed, taking $0 \leq s \leq s_0$ and using (W_2) we obtain

$$\begin{aligned}
I(u)-I(su)-\frac{1-s^2}{2}\langle I'(u), u\rangle &= \frac{1}{2}\|u\|^2_{E^\alpha} - \int_R W(t,u)dt - \frac{s^2}{2}\|u\|^2_{E^\alpha} + \int_R W(t,su)dt \\
&\quad - \frac{1-s^2}{2}\|u\|^2_{E^\alpha} + \frac{1-s^2}{2}\int_R (\nabla W(t,u), u)dt \\
&= \int \left(-W(t,u) + W(t,su) + \frac{1-s^2}{2}(\nabla W(t,u), u)\right) dt \\
&\geq 0. \qquad (3.16)
\end{aligned}$$

Since $\frac{2\sqrt{c}\theta}{\|u_n\|_{E^\alpha}} \in [0, s_0]$ when n large enough, as a consequence of the (3.16) we then get

$$c + o_n(1) = I(u_n) \geq I\left(\frac{2\sqrt{c}\theta}{\|u_n\|_{E^\alpha}} u_n\right) \geq 2\theta^2 c + o_n(1).$$

Since θ can be chosen large enough, we produce a contradiction and the proof is finished. □

Lemma 3.6. *Assume (L), (W_0), one of (W_1) or (W_2), (W_3)-(W_5). Then I satisfies the $(Ce)_c$ condition for all $0 < c < m^\infty$.*

Proof. Let $\{u_n\}$ be a Cerami sequence at the level $0 < c < m^\infty$,

$$I(u_n) = \frac{1}{2}\|u_n\|^2_{E^\alpha} - \int_R W(t, u_n)dt \to c, \qquad (3.17)$$

$$\begin{aligned}
\langle I'(u_n), \phi\rangle &= \int_R ({}_{-\infty}D^\alpha_t u_n, {}_{-\infty}D^\alpha_t \phi) + (L(t)u_n, \phi)dt - \int_R (\nabla W(t, u_n), \phi)dt \\
&= o_n(1)\|\phi\|, \ \forall \phi \in E^\alpha. \qquad (3.18)
\end{aligned}$$

Then $\|u_n\|_{E^\alpha}$ are bounded by Lemma 3.4 and Lemma 3.5. Without loss of generality, we may assume that $\|u_n\|_{E^\alpha} \to a$.

Claim 1: $a > 0$.

If not, assuming by contradiction that $\|u_n\|_{E^\alpha} \to 0$, now we will deduce a contradiction. It follows from $\|u_n\|_{E^\alpha} \to 0$ that $\|u_n\|_{L^\infty} \to 0$. For any given $\delta > 0$, when n large enough, $|u_n(t)| \leq l_\delta$ for all $t \in R$. Recalling (3.4), one has

$$\int_R |W(t,u_n)|dt \leq \frac{1}{2}\int_R \delta|u_n|^2 dt + \frac{1}{2}\int_R \lambda_\delta |u_n|^p dt \leq \frac{\delta}{2}C_2^2\|u_n\|^2_{E^\alpha} + \frac{\lambda_\delta}{2}C_p^p\|u_n\|^p_{E^\alpha} \to 0$$

as $n \to \infty$, which is a contradiction. Now we finish the proof of Claim 1.

Next, we will check each one of the possible alternatives of Lemma 2.6 for $\rho_n = |{}_{-\infty}D^\alpha_t u_n|^2 + l^\infty|u_n|^2$.

Step 1. Vanishing:

$$\lim_{n\to\infty} \sup_{y \in R} \int_{y-l}^{y+l} |{}_{-\infty}D^\alpha_t u_n|^2 + l^\infty|u_n|^2 dt = 0$$

for all $l > 0$. Since u_n is bounded in E^α, there exists a constant $k > 0$ such that

$$\sup_{n\in N} \|u_n\|_{L^\infty} \leq k. \tag{3.19}$$

Recalling $\|_{-\infty}D_t^\alpha u_n\|_{L^2} = \|\mathcal{F}(_{-\infty}D_t^\alpha u_n)\|_{L^2}$, by Lemma 2.7, we have $u_n \to 0$ in $L^s(R, R^N)$ for all $s > 2$, which together with (3.3) and (3.19) implies

$$0 \leq \int_R (\nabla W(t, u_n), u_n)dt \leq \delta\|u_n\|_{L^2}^2 + \lambda_\delta\|u_n\|_{L^p}^p \to 0$$

as $n \to \infty$, for the arbitrary of δ. Taking $\phi = u_n$ in (3.18), it follows that

$$o_n(1) = \langle I'(u_n), u_n\rangle = \|u_n\|_{E^\alpha}^2 - \int_R (\nabla W(t, u_n), u_n)dt = \|u_n\|_{E^\alpha}^2 + o_n(1)$$

which is a contradiction. Now we can exclude this alternative.

Step 2. Dichotomy: There exists $\alpha_0 (0 < \alpha_0 < \alpha)$ such that for any given $\varepsilon > 0$, there is $R_0 > 0$ and sequences $\{y_n\} \subset R$, $\{R_n\} \subset R^+$, with $R_0 < R_1 < \cdots < R_n < R_{n+1} \to \infty$, such that

$$\alpha_0 - \varepsilon < \int_{|t-y_n|\leq \frac{R_0}{2}} (|_{-\infty}D_t^\alpha u_n|^2 + l^\infty |u_n|^2)dt < \alpha_0 + \varepsilon,$$

$$\int_{|t-y_n|\geq 3R_n} (|_{-\infty}D_t^\alpha u_n|^2 + l^\infty |u_n|^2)dt > \alpha - \alpha_0 - \varepsilon, \tag{3.20}$$

and in particular

$$\int_{\frac{R_0}{2}<|t-y_n|<3R_n} (|_{-\infty}D_t^\alpha u_n|^2 + l^\infty |u_n|^2)dt < 2\varepsilon. \tag{3.21}$$

Picking $\xi \in C_0^\infty(R)$, $\xi(t) = 1$ for $|t| \leq 1$, $\xi(t) = 0$ for $|t| \geq 2$, and $\varphi = 1 - \xi$, set

$$u_n^1 = \xi\left(\frac{\cdot - y_n}{R_0}\right)u_n,\ u_n^2 = \varphi\left(\frac{\cdot - y_n}{R_n}\right)u_n.$$

By the definition of ξ and φ, there exists a constant $M' > 0$ such that $|u_n^i(t)| \leq M'|u_n(t)|$, for all $t \in R$ and $i = 1, 2$. On the other hand,

$$\begin{aligned}
|_{-\infty}D_t^\alpha u_n^1(t)| &\leq \int_0^\infty \frac{|u_n^1(t) - u_n^1(t-h)|}{|h|^{1+\alpha}}dh \\
&\leq \int_0^\infty \frac{\left|\xi\left(\frac{t-y_n}{R_0}\right)u_n(t) - \xi\left(\frac{t-y_n-h}{R_0}\right)u_n(t-h)\right|}{|h|^{1+\alpha}}dh \\
&\leq \int_0^\infty \frac{\left|\xi\left(\frac{t-y_n}{R_0}\right) - \xi\left(\frac{t-y_n-h}{R_0}\right)\right||u_n(t)|}{|h|^{1+\alpha}}dh \\
&\quad + \int_0^\infty \frac{\left|\xi\left(\frac{t-y_n-h}{R_0}\right)\right||u_n(t) - u_n(t-h)|}{|h|^{1+\alpha}}dh \\
&\leq |u_n(t)|(2M')^{1-\alpha/2}\int_0^1 \frac{\left|\xi\left(\frac{t-y_n}{R_0}\right) - \xi\left(\frac{t-y_n-h}{R_0}\right)\right|^{\alpha/2}}{|h|^{1+\alpha}}dh
\end{aligned}$$

$$
\begin{aligned}
&+|u_n(t)|\int_1^{\infty}\frac{\left|\xi\left(\frac{t-y_n}{R_0}\right)-\xi\left(\frac{t-y_n-h}{R_0}\right)\right|}{|h|^{1+\alpha}}dh\\
&+M'|_{-\infty}D_t^{\alpha}u_n(t)|\\
\le\ & \frac{1}{R_0^{\alpha/2}}|u_n(t)|(2M')^{1-\alpha/2}\|\dot{\xi}\|_{L^\infty}^{\alpha/2}\int_0^1\frac{1}{|h|^{1+\alpha/2}}dh\\
&+\frac{1}{R_0}|u_n(t)|\|\dot{\xi}\|_{L^\infty}\int_1^{\infty}\frac{1}{|h|^{\alpha}}dh+M'|_{-\infty}D_t^{\alpha}u_n(t)|\\
\le\ & M'|_{-\infty}D_t^{\alpha}u_n(t)|+M''|u_n(t)|
\end{aligned}
$$

for some $M''>0$, because of $\int_0^1\frac{1}{|h|^{1+\alpha/2}}dh<\infty$ and $\int_1^{\infty}\frac{1}{|h|^{\alpha}}dh<\infty$. Set $M=\max\{M',M''\}$, then we have

$$
|u_n^1(t)|\le M|u_n(t)|,\quad |_{-\infty}D_t^{\alpha}u_n^1(t)|\le M(|u_n(t)|+|_{-\infty}D_t^{\alpha}u_n(t)|) \tag{3.22}
$$

for all $t\in R$. Similarly, we can also get

$$
|u_n^2(t)|\le M|u_n(t)|,\quad |_{-\infty}D_t^{\alpha}u_n^2(t)|\le M(|u_n(t)|+|_{-\infty}D_t^{\alpha}u_n(t)|). \tag{3.23}
$$

It follows from (3.3) that $|W(t,u_n)|\le\frac{\delta}{2}|u_n|^2+\frac{\lambda_\delta}{2}|u_n|^p$, which together with (3.21), (3.22) and (3.23) implies

$$
\begin{aligned}
&|I(u_n)-I(u_n^1)-I(u_n^2)|\\
\le\ &\int_{R_0\le|t-y_n|\le 2R_n}(|_{-\infty}D_t^{\alpha}u_n|^2+(L(t)u_n,u_n)+|_{-\infty}D_t^{\alpha}u_n^1|^2+(L(t)u_n^1,u_n^1))dt\\
&+\int_{R_0\le|t-y_n|\le 2R_n}(|_{-\infty}D_t^{\alpha}u_n^2|^2+(L(t)u_n^2,u_n^2))dt\\
&+\int_{R_0\le|t-y_n|\le 2R_n}(|W(t,u_n)|+|W(t,u_n^1)|+|W(t,u_n^2)|)dt\\
\le\ &(1+6M^2)\int_{R_0\le|t-y_n|\le 2R_n}(|_{-\infty}D_t^{\alpha}u_n|^2+l^{\infty}|u_n|^2)dt\\
&+\frac{\delta}{2}(1+2M^2)\int_{R_0\le|t-y_n|\le 2R_n}|u_n|^2dt+\frac{\lambda_\delta}{2}(1+2M^P)\int_{R_0\le|t-y_n|\le 2R_n}|u_n|^Pdt\\
\le\ &2(1+6M^2)\varepsilon+\frac{\delta}{l^{\infty}}(1+2M^2)\varepsilon+\frac{\lambda_\delta}{l^{\infty}}(1+2M^p)\|u_n\|_{L^\infty}^{p-2}\varepsilon
\end{aligned}
$$

that is

$$
I(u_n)-I(u_n^1)-I(u_n^2)=o_\varepsilon(1), \tag{3.24}
$$

where $o_\varepsilon(1)\to 0$ as $\varepsilon\to 0$. Similarly, we get

$$
\begin{aligned}
&\left|\langle I'(u_n),u_n^1\rangle-\|u_n^1\|_{E^\alpha}^2+\int_R(\nabla W(t,u_n^1),u_n^1)dt\right|\\
=\ &\left|\int_{R_0\le|t-y_n|\le 2R_n}(\nabla W(t,u_n),u_n^1)-(\nabla W(t,u_n^1),u_n^1)dt\right|\\
=\ &o_\varepsilon(1),
\end{aligned} \tag{3.25}
$$

which together with (3.18) implies

$$\|u_n^1\|_{E^\alpha}^2 - \int_R (\nabla W(t,u_n^1), u_n^1)dt = o_n(1) + o_\varepsilon(1). \tag{3.26}$$

Similarly, we obtain

$$\|u_n^2\|_{E^\alpha}^2 - \int_R (\nabla W(t,u_n^2), u_n^2)dt = o_n(1) + o_\varepsilon(1). \tag{3.27}$$

We now consider the following two cases:

Case 1: $\{y_n\} \subset R$ is bounded.

Let Ω be a bounded domain in R. Since $\{y_n\} \subset R$ is bounded, for any given $t \in \Omega$, $\frac{t-y_n}{R_n} \le 1$ when n large enough. In view of the definition of φ, we have $u_n^2 = \varphi\left(\frac{\cdot - y_n}{R_n}\right) u_n \to 0$ in L^s_{loc} for all $s \in [2, +\infty]$. It follows from (2.8)

$$\int_R (\nabla W(t,u_n^2) - \nabla W^\infty(u_n^2), u_n^2)dt = o_n(1),$$

which together with (3.27) implies

$$\langle I^{\infty\prime}(u_n^2), u_n^2\rangle = \|u_n^2\|_{E^\alpha}^2 - \int_R \left(\nabla W^\infty(u_n^2), u_n^2\right) dt = o_n(1) + o_\varepsilon(1). \tag{3.28}$$

Similarly, we have

$$\int_R W(t,u_n^2) - W^\infty(t,u_n^2)dt = o_n(1),$$

so that

$$I(u_n^2) = I^\infty(u_n^2) + o_n(1) + o_\varepsilon(1). \tag{3.29}$$

Define $w_n^2 := u_n^2(\sigma_n t)$, where $\sigma_n \in R$ is a undetermined parameter, then

$$\begin{aligned}\langle I^{\infty\prime}(w_n^2), w_n^2\rangle &= \sigma_n^{2\alpha-1} \int_R |{}_{-\infty}D_t^\alpha u_n^2|^2 dt + \sigma_n^{-1}\left(\langle I^{\infty\prime}(u_n^2), u_n^2\rangle - \int_R |{}_{-\infty}D_t^\alpha u_n^2|^2 dt\right) \\ &= \sigma_n^{-1}\left((\sigma_n^{2\alpha}-1)\int_R |{}_{-\infty}D_t^\alpha u_n^2|^2 dt + \langle I^{\infty\prime}(u_n^2), u_n^2\rangle\right).\end{aligned}$$

We claim that $\int_R |{}_{-\infty}D_t^\alpha u_n^2|^2 dt > K > 0$ for some $K > 0$, if not, we have

$$\int_R |{}_{-\infty}D_t^\alpha u_n^2|^2 dt = \int_R ||\xi|^\alpha \mathcal{F}(u)(\xi)|^2 d\xi = 0$$

which together with Lemma 2.1 implies

$$\int_R \int_R \frac{|u_n(t) - u_n(t-h)|^2}{|h|^{1+2\alpha}} dh dt = 0$$

which means that u_n is a constant almost everywhere. Recalling the fact that $u_n \in E^\alpha$, we have $u_n(t) = 0$ a.e. $t \in R$, which is in contradiction with (3.20). Therefore we can choose proper σ_n such that

$$(\sigma_n^{2\alpha}-1)\int_R |{}_{-\infty}D_t^\alpha u_n^2|^2 dt + \langle I^{\infty\prime}(u_n^2), u_n^2\rangle = 0,$$

which gives $w_n^2 = u_n^2(\sigma_n t) \in \mathcal{N}^\infty$. Using (3.28), we obtain

$$\sigma_n^{2\alpha} - 1 = o_n(1) + o_\varepsilon(1),$$

which together with the arbitrary of ε shows $\sigma_n \to 1$ as $n \to \infty$. Now noting that

$$I^\infty(w_n^2) = \sigma_n^{-1}(\sigma_n^{2\alpha} - 1)\int_R |_{-\infty}D_t^\alpha u_n^2|^2 dt + (\sigma_n^{-1} - 1)I^\infty(u_n^2) + I^\infty(u_n^2),$$

recalling (3.29) and the boundedness of $I^\infty(u_n^2)$ and $\int_R |_{-\infty}D_t^\alpha u_n^2|^2 dt$, we have

$$I(u_n^2) \geq I^\infty(w_n^2) + o_n(1) + o_\varepsilon(1) \geq m^\infty + o_n(1) + o_\varepsilon(1). \tag{3.30}$$

On the other hand, in view of (3.26) and the fact that $F(t,x) \geq 0$ we have

$$\begin{aligned} I(u_n^1) &= \frac{1}{2}\|u_n^1\|_{E^\alpha}^2 - \int_R W(t,u_n^1)dt \\ &\geq \frac{1}{2}\int_R (\nabla W(t,u_n^1), u_n^1)dt - \int_R W(t,u_n^1)dt + o_n(1) + o_\varepsilon(1) \\ &= \frac{1}{2}\int_R F(t,u_n^1)dt + o_n(1) + o_\varepsilon(1) \\ &\geq o_n(1) + o_\varepsilon(1). \end{aligned} \tag{3.31}$$

Finally, (3.24), (3.30) and (3.31) yield

$$I(u_n) = I(u_n^1) + I(u_n^2) + o_\varepsilon(1) \geq m^\infty + o_n(1) + o_\varepsilon(1)$$

which contradicts (3.17) for ε small and n large.

Case 2: $\{y_n\} \subset R^N$ is not bounded.

Then, passing to a subsequence if necessary, we can assume that $|y_n| \to \infty$. In this case the support of u_n^1 is going to infinity and arguing similarly as above with the roles of u_n^1 and u_n^2 reversed, we gain a contradiction.

Step 3. Compactness: There exists a sequence $\{y_n\} \subset R$ satisfying for any $\varepsilon > 0$ there exists $l > 0$ such that

$$\left(\int_{-\infty}^{y_n - l} + \int_{y_n + l}^{+\infty}\right)(|_{-\infty}D_t^\alpha u_n|^2 + l^\infty |u_n|^2)dt < \varepsilon \tag{3.32}$$

for all $n \in N$. As in the case of dichotomy, if $|y_n| \to \infty$ (for some subsequence), we can get a contradiction to $I(u_n) \to c < m^\infty$. Therefore $\{y_n\} \subset R$ is a bounded sequence, and for every $\varepsilon > 0$, we can find $l' > 0$ such that

$$\left(\int_{-\infty}^{-l'} + \int_{l'}^{+\infty}\right)(|_{-\infty}D_t^\alpha u_n|^2 + l^\infty |u_n|^2)dt < \varepsilon. \tag{3.33}$$

Since $\{u_n\}$ is bounded, then $u_n \rightharpoonup u$ for some $u \in E^\alpha$. Noting the fact that $E^\alpha \hookrightarrow L^\infty(R, R^N)$ is continuous, there exists $l'' > l'$ such that

$$\left(\int_{-\infty}^{-l''} + \int_{l''}^{+\infty}\right)|u_n|^s dt < \frac{\varepsilon}{2} \text{ and } \left(\int_{-\infty}^{-l''} + \int_{l''}^{+\infty}\right)|u|^s dt < \frac{\varepsilon}{2}$$

for all $s \geq 2$. On the other hand, it is clear that $u_n \to u$ in $L^s([-l'', l''], R^N)$. Hence

$$\begin{aligned}\int_R |u_n - u|^s dt &= \left(\int_{-\infty}^{-l''} + \int_{l''}^{+\infty}\right) |u_n - u|^s dt + \int_{-l''}^{l''} |u_n - u|^s dt \\ &\leq 2^{s-1}\left(\int_{-\infty}^{-l''} + \int_{l''}^{+\infty}\right) (|u_n|^s + |u|^s) dt + \int_{-l''}^{l''} |u_n - u|^s dt \\ &\leq 2^{s-1}\varepsilon + o_n(1), \end{aligned} \tag{3.34}$$

which together with the arbitrary of ε implies $u_n \to u$ in $L^s(R, R^N)$ for all $s \geq 2$. Taking $\phi = u_n - u$ in (3.18), we have

$$\begin{aligned} &o_n(1) = \langle I'(u_n), u_n - u\rangle \\ &= \|u_n - u\|_{E^\alpha}^2 + \int_R ({}_{-\infty}D_t^\alpha u, {}_{-\infty}D_t^\alpha (u_n - u)) \\ &+(L(t)u, u_n - u)dt - \int_R (\nabla W(t, u_n), u_n - u)dt. \end{aligned} \tag{3.35}$$

Since $\{u_n\}$ is bounded in E^α, it follows from (3.3) that

$$\begin{aligned}\left|\int_R (\nabla W(t, u_n), u_n - u)dt\right| &\leq \int_R \delta |u_n||u_n - u| + \lambda_\delta |u_n|^{p-1}|u_n - u| dt \\ &\leq (\delta + \lambda_\delta \|u_n\|_{L^\infty}^{p-2})\|u_n\|_{L^2}^2 \|u_n - u\|_{L^2}^2 \to 0 \end{aligned} \tag{3.36}$$

as $n \to \infty$. We easily conclude from (3.35) and (3.36) that $\|u_n - u\|_{E^\alpha} \to 0$, that is, $u_n \to u$ in E^α. The proof of Lemma 3.5 is completed. □

Proof of Theorem 1.1. We divide two steps to prove systems (1.1) possesses a nontrivial ground state solution.

(a) By Remark 3.1 there exists $\{u_n\} \in E^\alpha$ such that

$$I(u_n) \to c \geq a > 0 \text{ and } (1 + \|u_n\|_{E^\alpha})I'(u_n) \to 0, \text{ as } n \to \infty.$$

If $0 < c < m^\infty$, applying Lemma 3.5 and Lemma 2.8, we conclude that I possesses a critical point at level c. Otherwise $c \geq m^\infty$. Let $u^\infty \in \mathcal{N}^\infty$ satisfying $I^\infty(u^\infty) = m^\infty$. It follows from Remark 3.2 that the maximum of $I^\infty(su^\infty)$ for $s > 0$ is only reached at $s = 1$, that is, $\max_{s\geq 0} I^\infty(su^\infty) = I^\infty(u^\infty) = m^\infty$. In view the proof of Lemma 3.2, there exits $s_0 > 0$ such that $I(s_0u^\infty) < 0$. Define a path $\hat{\gamma} : [0, 1] \to E^\alpha$ by $\hat{\gamma}(s) = ss_0u^\infty$, it is clear that $\hat{\gamma} \in \Gamma$. Consequently,

$$c = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} I(\gamma(s)) \leq \max_{s\in[0,1]} I(\hat{\gamma}(s)) \leq \max_{s\geq 0} I(su^\infty) = I^\infty(u^\infty) = m^\infty \leq c. \tag{3.37}$$

Thus

$$c = \max_{s\in[0,1]} I(\hat{\gamma}(s)).$$

By using Lemma 2.9, we obtain that I possesses a critical point at level c.

(b) In view of the above existence result it is well defined

$$m := \inf_{u\in\mathcal{N}} I(u).$$

Let $\{u_n\} \in E^\alpha$ be a minimizing sequence for I, by Ekeland's variational principle we may assume

$$I(u_n) \to m, I'(u_n) \to 0 \tag{3.38}$$

as $n \to \infty$. In this step we prove that m is achieved. Since u_n is a Cerami sequence, it follows from Lemma 3.4 that $\{u_n\}$ is bounded in E^α, then there exists $u \in E^\alpha$ such that up to a subsequence $u_n \rightharpoonup u$ in E^α, $u_n \to u$ in $L^s_{loc}(R)$ for all $s \in (2, +\infty)$, $u_n(t) \to u(t)$ a.e. in R.

Case 1. $u \neq 0$.

It is clear that $I(u) \geq m$. On the other hand, by using Fatou's Lemma, we have

$$\begin{aligned}
m &= \liminf_{n\to\infty} \left(I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle \right) \\
&= \liminf_{n\to\infty} \int_R \left(\frac{1}{2}(\nabla W(t,u_n), u_n) - W(t,u_n) \right) dt \\
&\geq \int_R \left(\frac{1}{2}(\nabla W(t,u), u) - W(t,u) \right) dt \\
&= I(u) - \frac{1}{2}\langle I'(u), u\rangle \\
&= I(u).
\end{aligned}$$

Hence $I(u) = m$ and $I'(u) = 0$.

Case 2. $u = 0$.

Define

$$\beta := \lim_{n\to\infty} \sup_{y\in R} \int_{y-1}^{y+1} u_n^2 dt.$$

If $\beta = 0$, by Lemma 2.7, $u_n \to 0$ in $L^s(R, R^N)$ for all $s \in (2, +\infty)$. Since $\{u_n\}$ is bounded in E^α, it follows from (3.3) that

$$\begin{aligned}
\left| \int_R W(t,u_n)dt \right| &\leq \frac{\delta}{2}\|u_n\|^2_{L^2} + \frac{\lambda_\delta}{2}\|u_n\|^p_{L^p} \to 0, \\
\left| \int_R (\nabla W(t,u_n), u_n)dt \right| &\leq \delta\|u_n\|^2_{L^2} + \lambda_\delta\|u_n\|^p_{L^p} \to 0
\end{aligned}$$

as $n \to \infty$, for the arbitrary of δ. Hence

$$\begin{aligned}
c &= I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle + o_n(1) \\
&= \int_R \left(\frac{1}{2}(\nabla W(t,u_n), u_n) - W(t,u_n) \right) dt + o_n(1) = o_n(1)
\end{aligned}$$

which is a contradiction. Thus $\beta > 0$. In view of the definition of supermum, up to a subsequence there exists $\{y_n\}$ such that

$$\int_{-1}^{1} u_n(t+y_n)^2 dt = \int_{y_n-1}^{y_n+1} u_n^2 dt \geq \frac{\beta}{2}. \tag{3.39}$$

Define $v_n := u_n(\cdot + y_n)$. Thus there exists a nonnegative function $v \in E^\alpha$ such that up to a subsequence, $v_n \rightharpoonup v$ in E^α, $v_n \to v$ in L^s_{loc} for all $s \in [2, +\infty]$ and

$v_n(t) \to v(t)$ a.e. in R. Obviously, $v \neq 0$. If $\{y_n\}$ is bounded, there exists $\hat{R} > 0$ such that

$$\int_{-\hat{R}}^{\hat{R}} u_n^2 dt \geq \int_{y_n-1}^{y_n+1} |u_n|^2 dt > \frac{\beta}{2},$$

which contradicts to $u_n \to 0$ in $L^2_{loc}(R, R^N)$. Thus $\{y_n\}$ is unbounded, without loss of generality, we may assume $|y_n| \to \infty$. For any $\varphi \in C_0^\infty(R, R^N)$, it follows from (2.10) and (2.11) that

$$\begin{aligned} o_n(1) &= \langle I'(u_n), \varphi(\cdot - y_n)\rangle \\ &= \int_R ({}_{-\infty}D_t^\alpha u_n, {}_{-\infty}D_t^\alpha \varphi(\cdot - y_n))dt + \int_R (L(t)u_n, \varphi(\cdot - y_n))dt \\ &\quad - \int_R (\nabla W(t, u_n), \varphi(\cdot - y_n))dt \\ &= \int_R ({}_{-\infty}D_t^\alpha u_n, {}_{-\infty}D_t^\alpha \varphi(\cdot - y_n))dt + \int_R (L^\infty(t)u_n, \varphi(\cdot - y_n))dt \\ &\quad - \int_R (\nabla W^\infty(t, u_n), \varphi(\cdot - y_n))dt \\ &= \int_R ({}_{-\infty}D_t^\alpha v_n, {}_{-\infty}D_t^\alpha \varphi)dt + \int_R (L^\infty(t)v_n, \varphi)dt - \int_R (\nabla W^\infty(t, v_n), \varphi)dt \\ &= \int_R ({}_{-\infty}D_t^\alpha v, {}_{-\infty}D_t^\alpha \varphi)dt + \int_R (L^\infty(t)v, \varphi)dt - \int_R (\nabla W^\infty(t, v), \varphi)dt \end{aligned}$$

which means v is a solution of (3.7). It follows from (2.7), $(\nabla W(t,x), x) \geq (\nabla W^\infty(t,x), x)$ and Fatou's Lemma that

$$\begin{aligned} m &= I(u_n) - \langle I'(u_n), u_n\rangle + o_n(1) = \int_R \left(\frac{1}{2}(\nabla W(t, u_n), u_n) - W(t, u_n)\right) + o_n(1) \\ &\geq \int_R \left(\frac{1}{2}(\nabla W^\infty(t, u_n), u_n) - W^\infty(t, u_n)\right) + o_n(1) \\ &= \int_R \left(\frac{1}{2}(\nabla W^\infty(t, v_n), v_n) - W^\infty(t, v_n)\right) + o_n(1) \\ &\geq \int_R \left(\frac{1}{2}(\nabla W^\infty(t, v), v) - W^\infty(t, v)\right) + o_n(1) \\ &= I^\infty(v) - \langle I^{\infty\prime}(v), v\rangle + o_n(1) = I^\infty(v) \\ &\geq m^\infty \end{aligned} \tag{3.40}$$

For any $u \in E^\alpha \setminus \{0\}$, by Lemma 3.2, there exists $s_u > 0$ such that $s_u u \in \mathcal{N}$ and the maximum of $I(su)$ for $s > 0$ is achieved at s_u and then $I(s_u u) \geq m$. Combining with the fact that $(L(t)x, x) \leq (L^\infty(t)x, x)$ and $W(t,x) \geq W^\infty(t,x)$, one has

$$m \leq I(s_u u) \leq I^\infty(s_u u) \leq \max_{s>0} I^\infty(su).$$

In view of the arbitrary of u and (3.8), we obtain

$$m \leq \inf_{u \in E^\alpha \setminus \{0\}} \max_{s>0} I^\infty(su) = m^\infty. \tag{3.41}$$

Combining (3.40) and (3.41), we have $I^\infty(v) = m^\infty = m$. Since v is a solution of (3.7), by Remark 3.2 we have

$$\max_{s>0} I^\infty(sv) = I^\infty(v).$$

By Lemma 3.2, there exists $s_1 > s_2 > 0$ such that $I(s_1 v) < 0$ and $s_2 v \in \mathcal{N}$. Define a path $\tilde{\gamma} : [0,1] \to E^\alpha$ by $\tilde{\gamma}(s) = s s_1 v$, it is clear that $\tilde{\gamma} \in \Gamma$. Therefore one has

$$m \le I(s_2 v) \le \max_{s\in[0,1]} I(\tilde{\gamma}(s)) \le \max_{s\in[0,1]} I^\infty(\tilde{\gamma}(s)) \le \max_{s>0} I^\infty(sv) = I^\infty(v) = m$$

which means that

$$m = \max_{s\in[0,1]} I(\tilde{\gamma}(s)).$$

By using Lemma 2.9, we obtain that I possesses a critical point at level m. Summarize the above two cases, we obtain that (1.1) has a nontrivial ground state solution in E^α. □

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