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INFINITELY MANY SOLUTIONS FOR A CLASS OF SUBLINEAR SCHRÖDINGER EQUATIONS*

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Abstract In this paper, we investigate the Schrödinger equation, which satisfies that the potential is asymptotical 0 at infinity in some measure-theoretic and the nonlinearity is sublinear growth. By using variant symmetric mountain lemma, we obtain infinitely many solutions for the problem. Moreover, if the nonlinearity is locally sublinear defined for $|u|$ small, we can also get the same result. In which, we show that these solutions tend to zero in $L^\infty(\mathbb{R}^N)$ by the Brézis-Kato estimate.

Keywords Sublinear Schrödinger equation, local sublinear nonlinearities, symmetric mountain lemma, variational methods.

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1. Introduction and main results

In recent years, many authors considered the following Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geqslant 3$, V is a potential and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. Problem (1.1) appeared in the context of various physical models. Knowledge of the solution of problem (1.1) has a great importance for studying wave solution for the following nonlinear Schrödinger equation

$$ih\frac{\partial \Psi}{\partial t} = -h^2 \triangle \Psi + W(x)\Psi - f(x,\Psi) \quad \text{for all } x \in \Omega, \tag{1.2}$$

where $h > 0$ and $\Omega \subseteq \mathbb{R}^N$ is a bounded domain. Problem (1.2) is one of the main objects of the quantum physics, because it appears in problem involving nonlinear optics, plasma, physics and condensed matter physics (also see [2, 13, 29]).

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There exist a lot of papers in which the authors considered the existence and multiplicity of nontrivial solutions for problem (1.1) over the past several decades. Generally speaking, authors discussed about potential V and nonlinearities f under different conditions in problem (1.1), and obtained the existence of results by some analysis methods. In the present paper, we observe that interesting conditions on nonlinearities f have been studied. Next, we will recall some of them.

To begin with, for the case that f is critical at infinity, that is

$$\lim_{|t|\to\infty}\frac{f(x,t)}{t^{2^*-1}}=C>0,$$

where $2^*=\frac{2N}{N-2}$ is called the critical Sobolev embedding exponent, investigated very intensely by many authors. For example, Benci and Cerami [5] obtained the existence of positive solution for problem (1.1). Deng [16] got multiple positive solutions for problem (1.1).

The second case that f is superlinear and subcritical at infinity, videlicet,

$$\lim_{|t|\to\infty}\frac{f(x,t)}{t}=+\infty \quad\text{and}\quad \lim_{|t|\to\infty}\frac{f(x,t)}{t^{2^*-1}}=0,$$

which was considered by Berestycki and Lions who used the constrained minimization method to obtain a ground state solution in [9], and obtained multiplicity solutions in [10] for problem (1.1). Liu etc [22] proved the existence of positive ground state solutions for problem (1.1). [14, 24] also considered problem (1.1) under the same nonlinear conditions.

For the third case that f is asymptotically linear at infinity, viz.,

$$\lim_{|t|\to\infty}\frac{f(x,t)}{t}=C>0,$$

which was researched by many researchers. In [18], Ding and Lee supposed that f is asymptotically linear and symmetric, and got infinitely many geometrically distinct solutions. Liu etc [23] obtained multiple solutions for problem (1.1) when the energy functional has a mountain-pass geometry. Liu etc [25] proved that problem (1.1) has a bound state solution.

At the last case, the sublinear Schrödinger equations have been extensively studied by many authors. We refer, for instance, [4,6–8,11–13, 15, 21, 28, 31] and the references therein. However, to author's knowledge, a fraction of these considered the existence of infinitely many solutions for problem (1.1) in $\mathbb{R}^N$. For a coercive potential V, Ding [17] treated the existence and multiplicity of nontrivial solutions for a class of sublinear elliptic systems corresponding to problem (1.1). Motivated by the work of [17], Zhang and Wang [31] assumed the coercive condition on V and the following condition of f holds,

(f_0) $f(x,t) = \mu h(x)|t|^{\mu-2}t$, where $\mu \in (1,2)$ is a constant and $h : \mathbb{R}^N \to \mathbb{R}$ is a positive continuous function such that $h \in L^{\frac{2}{2-\mu}}(\mathbb{R}^N)$,

and obtained infinitely many nontrivial solutions for problem (1.1).

Bao and Han [7] supposed that V is asymptotical constant and f satisfies

(f_1) $f(x,t) = h(x)|t|^{\mu-2}t$, where $h \in L^{\infty}(\mathbb{R}^N)$ and $h(x) > 0$, a.e. in $\mathbb{R}^N$,

and applied the dual fountain theorem to get infinitely many solutions with negative critical values converging to 0.

Bahrouni [3] assumed that $f(x,t) = a(x)|u|^{p-1}u + b(x)|u|^{q-1}u$ in problem (1.1), where $0 < q < p < 1$ and a, b satisfy the following conditions

(a_1) $a \in L^{\infty}(\mathbb{R}^N) \cap L^{\frac{2}{1-p}}(\mathbb{R}^N)$ and there exists $\beta > 0$ such that $a(x) < -\beta$ for any $x \in \mathbb{R}^N$,

(b_1) $b \in L^{\frac{p+1}{p-q}}(\mathbb{R}^N)$ and $b > 0$,

then obtained infinitely many nontrivial solutions by using the dual fountain theorem.

With the help of the genus properties in critical point theory, Chen and Tang [15] obtained infinitely many solutions for problem (1.1) when V is a positive potential and the following hypothesises on f hold.

(f_2) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exist two constants $1 < \mu_1 < \mu_2 < 2$ and two functions $h_1 \in L^{\frac{2}{2-\mu_1}}(\mathbb{R}^N, \mathbb{R}^+)$ and $h_2 \in L^{\frac{2}{2-\mu_1}}(\mathbb{R}^N, \mathbb{R}^+)$ such that

$$|f(x,t)| \le \mu_1 h_1(x)|t|^{\mu_1-1} + \mu_2 h_2(x)|t|^{\mu_2-1} \quad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

(f_3) There exist an open set $\Omega \subset \mathbb{R}^N$ and three constants $\delta, \eta > 0$ and $\mu_3 \in (1,2)$ such that

$$F(x,t) \ge \eta |t|^{\mu_3} \quad \text{for all } (x,t) \in \Omega \times [-\delta, \delta],$$

where $F(x,t) := \int_0^t f(x,s)ds$.

In [13], Bahrouni etc assumed that $f(x,t) = h(x)g(t)$, in which h and g satisfy the following conditions

(V_1) $V \in L^{\frac{N}{2}}(\mathbb{R}^N)$ and $|V^-|_{\frac{N}{2}} < S$, where S is the best Sobolev constant,

$$S = \inf\left\{\|u\|^2 \;\Big|\; u \in \mathcal{D}^{1,2}(\mathbb{R}^N),\ \int |u|^{2^*}dx = 1\right\}.$$

(g_0) $g \in C(\mathbb{R}, \mathbb{R})$ and there exist $c > 0$ and $\mu \in (1, 2)$ such that $|g(t)| \leq c|t|^{\mu-1}$ for all $t \in \mathbb{R}$.

(g_1) $\lim\limits_{t\to 0} \frac{G(t)}{|t|^2} = +\infty$, where $G(t) = \int_0^t g(s)ds$ for all $t \in \mathbb{R}$.

(g_2) G is positive on $\mathbb{R} \setminus \{0\}$.

(h_0) $h \in L^{\frac{2^*}{2^*-\mu}}(\mathbb{R}^N)$ and there exist $y \in \mathbb{R}^N$ and $R_0 > 0$ such that $h(x) > 0$ for all $x \in \overline{B_{R_0}(y)}$.

The authors got the following result

Theorem. A. *Suppose that* (V_1), (g_0)–(g_2) *and* (h_0) *hold. Then problem* (1.1) *possesses infinitely many nontrivial solutions.*

Motivated by the above works, we only care about infinitely many solutions of problem (1.1) involving sublinear nonlinearities. Then we try to search for weaker conditions about f, precisely, assume that (V_1) holds and f satisfies a new set of hypothesises in problem (1.1).

(f_4) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exist a constant $\mu \in (1, 2)$ and a function $h \in L^{\frac{2^*}{2^*-\mu}}(\mathbb{R}^N)$ such that

$$|f(x,t)| \leq \mu|h(x)||t|^{\mu-1}.$$

(f_4') $f \in C(\mathbb{R}^N \times [-\delta, \delta], \mathbb{R})$ with $\delta > 0$ and there exist constants $\mu \in (1, 2)$, $p \in (\frac{N}{2}, +\infty)$ and a function $h \in L^{\frac{2^*}{2^*-\mu}}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ such that

$$|f(x,t)| \leq \mu|h(x)||t|^{\mu-1}.$$

(f_5) There exist $x_0 \in \mathbb{R}^N$ and a constant $r_0 > 0$ such that

$$\limsup_{t\to 0}\left(\inf_{|x-x_0|\leq r_0} t^{-2}F(x,t)\right) = +\infty,$$
$$\liminf_{t\to 0}\left(\inf_{|x-x_0|\leq r_0} t^{-2}F(x,t)\right) > -\infty,$$

where $F(x,t) = \int_0^t f(x,s)ds$.

(f_6) f is odd with respect to t.

Formally, solutions of problem (1.1) should arise as the critical points of functional

$$\Phi(u) := \frac{1}{2}\int |\nabla u|^2 + V(x)u^2 dx - \int F(x,u)dx. \tag{1.3}$$

It is not convenient to think of functional Φ defined in $H^1(\mathbb{R}^N)$, since in this space the first integral in Φ is not a norm for (V_1), and this poses serious difficulties. We will work in the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$. The equation we are dealing with then becomes

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^N). \end{cases} \tag{1.4}$$

Our main results are as follows

Theorem 1.1. *Suppose that* (V_1) *and* (f_4)–(f_6) *hold. Then problem* (1.4) *possesses infinitely many nontrivial solutions.*

Theorem 1.2. *Suppose that* (V_1) *and* (f_4')–(f_6) *hold. Then problem* (1.4) *possesses infinitely many nontrivial solutions.*

Remark 1.1. In the present paper, Theorem 1.1 generalizes Theorem A in [13], because the conditions of Theorem 1.1 contain the conditions of Theorem A and there are many functions f satisfying (f_4)–(f_6) which do not satisfy the conditions in [13]. For instance, let a cut-off function $\chi \in C_0(\mathbb{R})$ such that $0 \le \chi(t) \le 1$, $\chi(t) \equiv 1$ for $|t| \le \frac{1}{2}$ and $\chi(t) \equiv 0$ for $|t| > 1$, and define

$$f(x,t) = \frac{\sin^2 x_1}{1+|x|^N}\chi(t)|t|^{-\frac{1}{2}}t + \frac{\cos^2 x_2}{1+|x|^N}\chi(t)|t|^{-\frac{1}{3}}t.$$

Evidently, f satisfies the conditions of Theorem 1.1 just letting $h(x) = \frac{1}{1+|x|^N}$ and $\mu = \frac{3}{2}$.

Remark 1.2. On the one hand, we study problem (1.4) with the assumptions of Theorem 1.2 which have never been investigated. What's more, it should be noted that the nonlinear term $f(x,t)$ is only locally defined for $|t|$ small without any growth condition on t at infinity. Comparing to the nonlinearities f in [7, 13, 15, 31], one can easily see that the f under the assumptions of Theorem 1.2 is more general and weaker.

On the other hand, we present a different way from [7, 13, 15, 31] to search for solutions. The central task is to modify and extend $f(x,t)$ for t outside a neighborhood of 0, and obtain the existence of infinitely solutions for the modified problem by using a variant of symmetric mountain lemma [20]. And then we may conclude that these solutions tend to zero in $L^\infty(\mathbb{R}^N)$ by the Brézis-Kato estimate and the Moser iteration, which implies that these solutions are also solutions of problem (1.4).

Remark 1.3. The idea of (f_4') was inspired by [30] and (f_5) was first explored in [20].

The present paper is organized as follows. In the next section we present some preliminary results. We give the proof of Theorem 1.1 in Section 3. Section 4 is devoted to proof of Theorem 1.2.

2. Preliminaries

From now on, we will use the following notations

- $H^1(\mathbb{R}^N)$ is the usual Sobolev space endowed with the usual norm
$$\|u\|_H = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 dx\right)^{\frac{1}{2}}.$$
- $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is completion of $C_0^\infty(\mathbb{R}^N)$ with the usual norm
$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{1}{2}}.$$
- $L^p(\mathbb{R}^N)$ is the usual Lebesgue space endowed with the norm
$$|u|_p = \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{\frac{1}{p}} \text{ and } |u|_\infty = \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} |u(x)|,$$
for all $p \in [1, +\infty)$.
- meas$\{\Omega\}$ denotes the Lebesgue measure of the set Ω.
- $\langle \cdot, \cdot \rangle$ denotes action of dual.
- $B_R(y) := \{x \in \mathbb{R}^N : |x - y| < R\}$ and $B_R := B_R(0)$ and $B_R^c = \mathbb{R}^N \setminus B_R$.
- C, C_i $(i = 0, 1, 2, \cdots)$ denote various positive constants.
- For convenience of writing, the domain of an integral is $\mathbb{R}^N$ unless otherwise indicated.

Now we define the following new real functional
$$\|u\|_* := \left(\int |\nabla u|^2 + V(x)u^2 dx\right)^{\frac{1}{2}},$$
which is a norm in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ for (V_1).

Lemma 2.1. *Suppose that* (V_1) *holds. Then the norm* $\|u\|_*$ *is equivalent to the usual norm* $\|u\|$ *in* $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

Proof. On the one hand, we have

$$\begin{aligned}\int |\nabla u|^2 + V(x)u^2 dx &\leq \int |\nabla u|^2 dx + \left(\int |V(x)|^{\frac{N}{2}} dx\right)^{\frac{2}{N}} \left(\int |u|^{2^*} dx\right)^{\frac{2}{2^*}}\\ &\leq C\|u\|^2. \end{aligned} \tag{2.1}$$

On the other hand, one obtains

$$\begin{aligned}\int |\nabla u|^2 + V(x)u^2 dx &= \int |\nabla u|^2 + V^+(x)u^2 - V^-(x)u^2 dx\\ &\geq \int |\nabla u|^2 dx - \left(\int |V^-(x)|^{\frac{N}{2}} dx\right)^{\frac{2}{N}} \left(\int |u|^{2^*} dx\right)^{\frac{2}{2^*}}\\ &\geq \delta\|u\|^2, \end{aligned} \tag{2.2}$$

where $V^+ = \max\{V, 0\}$, $V^- = -\min\{V, 0\}$, and the last inequality of (2.2) holds by the fact that $1 - \frac{|V^-|_{\frac{N}{2}}}{S} \geq \delta > 0$. Combining (2.1) with (2.2), we obtain that there exist constant $C, \delta > 0$ such that

$$\delta\|u\|^2 \leq \|u\|_*^2 \leq C\|u\|^2,$$

which completes the proof. □

Then the energy functional Φ defined in (1.3) becomes

$$\Phi(u) = \frac{1}{2}\|u\|_*^2 - \Psi(u), \tag{2.3}$$

where

$$\Psi(u) := \int F(x, u) dx. \tag{2.4}$$

Lemma 2.2. *Suppose that* (V_1) *and* (f_4) *hold. Then* $\Phi : \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ *is differentiable and* Ψ' *is weakly continuous.*

Proof. For any $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, it follows from (f_4) and the Hölder inequality that

$$\begin{aligned}\int F(x, u) dx &\leq \int |h(x)||u|^\mu dx\\ &\leq \left(\int |h(x)|^{\frac{2^*}{2^*-\mu}} dx\right)^{\frac{2^*-\mu}{2^*}} \left(\int |u|^{2^*} dx\right)^{\frac{\mu}{2^*}}\\ &< +\infty. \end{aligned} \tag{2.5}$$

Then Ψ and Φ are well defined. Define an associated linear operator $G(u) : \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ such that

$$\langle G(u), v\rangle = \int f(x,u)v dx \quad \text{for all } v \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

From (f_4), one obtains

$$\begin{aligned} |\langle G(u), v\rangle| &\le \mu \int |h(x)||u|^{\mu-1}|v| dx \\ &\le \mu \left(\int |h(x)|^{\frac{2^*}{2^*-\mu}} dx\right)^{\frac{2^*-\mu}{2^*}} \left(\int |u|^{\frac{(\mu-1)2^*}{\mu}} |v|^{\frac{2^*}{\mu}} dx\right)^{\frac{\mu}{2^*}} \\ &\le C \left(\int |u|^{2^*} dx\right)^{\frac{\mu-1}{2^*}} \left(\int |v|^{2^*} dx\right)^{\frac{1}{2^*}} \\ &< +\infty, \end{aligned} \tag{2.6}$$

which implies that $G(u)$ is well defined and bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. For any $u, v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, by the Lagrange Theorem there exists a real number η such that $|\eta| \le |t|$ and

$$\begin{aligned} \left|\frac{F(x, u+tv) - F(x,u)}{t}\right| &= |f(x, u+\eta v)v| \\ &\le \mu 2^{\mu-1}|h|(|u|^{\mu-1}|v| + |v|^{\mu}). \end{aligned}$$

Combining (2.5) with (2.6), we obtain $|h||u|^{\mu-1}|v|$ and $|h||v|^{\mu} \in L^1(\mathbb{R}^N)$. By the dominated convergence theorem, one has

$$\begin{aligned} \lim_{t\to 0} \frac{\Psi(u+tv) - \Psi(u)}{t} &= \lim_{t\to 0} \int \frac{F(x,u+tv) - F(x,u)}{t} dx \\ &= \langle G(u), v\rangle. \end{aligned}$$

This means that Ψ is Gâteaux differentiable in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and the Gâteaux derivative of Ψ at u is $G(u)$. Next, we will prove that Ψ' is weakly continuous, that is to say, for all $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$

$$\int \big(f(x,u_n) - f(x,u)\big) v dx \to 0 \quad \text{as } u_n \rightharpoonup u \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^N).$$

Now let $u_n \rightharpoonup u$ as $n \to \infty$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, it follows from (f_4) that for any $\varepsilon > 0$ there exists $R > 0$ such that

$$\int_{B_R^c} |h(x)|^{\frac{2^*}{2^*-\mu}} dx \le \varepsilon.$$

In fact, one can easily see that

$$\int_{B_R^c} |h(x)||u_n|^\mu dx \leq \left(\int_{B_R^c} |h(x)|^{\frac{2^*}{2^*-\mu}} dx\right)^{\frac{2^*-\mu}{2^*}} \left(\int |u_n|^{2^*} dx\right)^{\frac{\mu}{2^*}}$$
$$\leq C\varepsilon^{\frac{2^*-\mu}{2^*}}. \tag{2.7}$$

From the Hölder inequality and (2.7), we obtain

$$\begin{aligned}
&\int_{B_R^c} |f(x,u_n) - f(x,u)||v|dx \\
\leq&\mu \int_{B_R^c} |h(x)|(|u_n|^{\mu-1} + |u|^{\mu-1})|v|dx \\
\leq&\mu \int_{B_R^c} |h(x)||u_n|^{\mu-1}|v|dx + \mu \int_{B_R^c} |h(x)||u|^{\mu-1}|v|dx \\
\leq&C\varepsilon^{\frac{2^*-\mu}{2^*}}.
\end{aligned} \tag{2.8}$$

On the other hand, since $h \in L^{\frac{2^*}{2^*-\mu}}(\mathbb{R}^N)$, for all $\varepsilon > 0$ there exists $\eta > 0$ such that

$$\int_I |h(x)|^{\frac{2^*}{2^*-\mu}} dx \leq \varepsilon,$$

for all $I \subset B_R$ with meas$\{I\} < \eta$ (see [19]). This deduces that

$$\begin{aligned}
\int_I |h(x)||u_n|^{\mu-1}|v|dx &\leq \left(\int_I |h(x)|^{\frac{2^*}{2^*-\mu}} dx\right)^{\frac{2^*-\mu}{2^*}} \left(\int_I |u_n|^{\frac{(\mu-1)2^*}{\mu}} |v|^{\frac{2^*}{\mu}} dx\right)^{\frac{\mu}{2^*}} \\
&\leq \varepsilon^{\frac{2^*-\mu}{2^*}} \left(\int_I |u_n|^{2^*} dx\right)^{\frac{\mu-1}{2^*}} \left(\int_I |v|^{2^*} dx\right)^{\frac{1}{2^*}} \\
&\leq C\varepsilon^{\frac{2^*-\mu}{2^*}}.
\end{aligned}$$

Hence

$$\int_I |f(x,u_n) - f(x,u)||v|dx \leq \mu \int_I |h(x)|(|u_n|^{\mu-1} + |u|^{\mu-1})|v|dx \leq C\varepsilon^{\frac{2^*-\mu}{2^*}}.$$

From the fact that $u_n \to u$ a.e. in B_R and the Vitali convergence theorem, we obtain that

$$\int_{B_R} |f(x,u_n) - f(x,u)||v|dx \to 0 \quad as\ n \to \infty. \tag{2.9}$$

Then for any $\varepsilon > 0$, combining (2.8) with (2.9), we get

$$\begin{aligned}\|\Psi'(u_n) - \Psi'(u)\| &= \sup_{\|v\|=1} \left|\langle G(u_n) - G(u), v\rangle\right| \\ &\le \sup_{\|v\|=1} \int_{B_R} |f(x,u_n) - f(x,u)||v|dx \\ &\quad + \sup_{\|v\|=1} \int_{B_R^c} |f(x,u_n) - f(x,u)||v|dx \\ &\le \varepsilon,\end{aligned}$$

which means that Ψ' is weakly continuous, thus Ψ and $\Phi : \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ are both differentiable. The proof of Lemma 2.2 is completed. □

Lemma 2.3. *Suppose that* (f_5) *holds. There exist two sequences* $\{\delta_m\}$, $\{M_m\}$ *and constants* δ, $C > 0$ *such that* δ_m, $M_m > 0$ *and*

$$\delta_m \to 0,\ M_m \to \infty \text{ as } m \to \infty,$$

$$F(x,u) \ge -Cu^2 \qquad \text{for all } x \in B_{r_0}(x_0) \text{ and } |u| \le \delta, \tag{2.10}$$

$$F(x,\delta_m)/\delta_m^2 \ge M_m \quad \text{for all } x \in B_{r_0}(x_0) \text{ and } m \in \mathbb{N}. \tag{2.11}$$

One can see that the genus plays an important role in proof of geometrical structure. Now, the definition and properties of the genus are listed here for the reader's convenience (see [1, 20, 27]).

Definition 2.1 (see [1,27])**.** Let X be a Banach space and A is a subset of X. A is said to be symmetric if $u \in A$ implies $-u \in A$. The family of all closed symmetric subsets of A which does not contain 0 is denoted by Γ. We define a genus $\gamma(A)$ of A by the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$. If the k does not exist, we say $\gamma(A) = +\infty$, we set $\gamma(\emptyset) = 0$. For each $k \in \mathbb{N}$, let $\Gamma_k = \{A \in \Gamma \mid \gamma(A) \ge k\}$.

Proposition 2.1 (see [20])**.** *Let* A *and* B *be closed symmetric subsets of* X*, which do not contain the origin. Then* (1)*–*(5) *below hold.*

(1) *If there is an odd continuous mapping from* A *to* B*, then* $\gamma(A) \le \gamma(B)$.
(2) *If there is an odd homeomorphism from* A *onto* B*, then* $\gamma(A) = \gamma(B)$.
(3) *If* $\gamma(B) < \infty$*, then* $\gamma(\overline{A \setminus B}) \ge \gamma(A) - \gamma(B)$.
(4) *If* A *is compact, then* $\gamma(A) < +\infty$ *and there exists a uniform neighborhood* $N_\delta(A)$ *of* A *such that* $\gamma(N_\delta(A)) = \gamma(A)$ *for* $\delta > 0$ *small enough.*

(5) *The n-dimensional sphere S^n has a genus of $n+1$ by the Borsuk-Ulam theorem.*

In order to prove the existence of infinitely solutions for problem (1.4), we use the variant symmetric mountain pass lemma given in [20], [26] and [27]. Until then, we give the definition of the (PS) sequence.

Definition 2.2. Let X be a Banach space and $J : X \to \mathbb{R}$ be a differentiable functional. A sequence $\{u_k\} \subseteq X$ such that

$$\{J(u_k)\} \text{ is bounded (in } \mathbb{R}) \text{ and}$$

$$J'(u_k) \to 0 \quad \text{as } k \to \infty,$$

is called a Palais-Smale sequence for J (shortly: $\{u_k\}$ is a (PS) sequence). If every Palais-Smale sequence for J has a converging subsequence, we say that J satisfies the Palais-Smale condition (shortly: J satisfies the (PS) condition).

Lemma 2.4 (see [20]). *Let X be an infinitely dimensional Banach space and $J \in C^1(X, \mathbb{R})$ such that*

(A_1) *$J(u)$ is even, bounded from below, $J(0) = 0$ and J satisfies the (PS) condition,*

(A_2) *for each $k \in N$, there exists an $W_k \in \Gamma_k$ such that $\sup_{u \in W_k} J(u) < 0$.*

Then either (B_1) or (B_2) holds

(B_1) *There exists a critical point sequence $\{u_k\}$ such that $J'(u_k) = 0$, $J(u_k) < 0$ and $u_k \to 0$.*

(B_2) *There exist two critical point sequences $\{u_k\}$ and $\{v_k\}$ such that*

$$J'(u_k) = 0,\ J(u_k) = 0,\ u_k \neq 0,\ u_k \to 0,$$

$$J'(v_k) = 0,\ J(v_k) < 0,\ J(v_k) \to 0$$

and $\{v_k\}$ converges to a non-zero limit.

3. Proof of Theorem 1.1.

Recall the energy functional Φ associated to problem (1.4) is well defined and differentiable in above section. Now, we will prove that the functional Φ satisfies conditions (A_1) and (A_2) of Lemma 2.4 and obtain Theorem 1.1 in this section.

Lemma 3.1. *Suppose that (V_1) and (f_4) hold, then Φ is bounded from below and the (PS) sequence is bounded.*

Proof. Firstly, we claim that Φ is bounded from below. Recall the fact, the Young inequality, for any $a, b \geq 0$, $p, q > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$ one has

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}. \tag{3.1}$$

From (V_1), (f_4) and the Hölder inequality, we have for any $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$,

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\int |\nabla u|^2 + V(x)u^2 dx - \Psi(u) \\
&\geq \frac{1}{2}\|u\|_*^2 - \int |h(x)||u|^\mu dx \\
&\geq \frac{1}{2}\|u\|_*^2 - C\|u\|_*^\mu \\
&\geq \frac{1}{2}\|u\|_*^2 - \frac{1}{\mu}\mu C\|u\|_*^\mu
\end{aligned} \tag{3.2}$$

Set $a = \|u\|_*^\mu$, $b = \mu C$, $p = \frac{2}{\mu}$, $q = \frac{2}{2-\mu}$ and in (3.1) we have

$$\mu C\|u\|_*^\mu \leq \frac{\mu}{2}\|u\|_*^2 + \frac{2-\mu}{2}(\mu C)^{\frac{2}{2-\mu}}.$$

Hence

$$\Phi(u) \geq \frac{\mu-2}{2}\mu^{\frac{\mu}{2-\mu}} C^{\frac{2}{2-\mu}},$$

which yields that Φ is bounded from below.

In fact, for any (PS) sequence $\{u_n\}$, if $\|u_n\|_* \to +\infty$, we obtain that $\Phi(u_n) \to +\infty$ from (3.2), this contradicts the definition of the (PS) sequence. Thus the (PS) sequence is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. This completes the proof. □

Lemma 3.2. *Suppose that (V_1) and (f_4) hold. Then Φ satisfies the (PS) condition.*

Proof. Let $\{u_n\}$ be a (PS) sequence, which is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ obviously. Then, up to subsequence, there exists $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ as $n \to \infty$. In fact, we have

$$\begin{aligned}
o(1) =& \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \\
=& \int (f(x,u_n) - f(x,u))(u_n - u)dx + \|u_n - u\|_*^2.
\end{aligned} \tag{3.3}$$

From Lemma 2.2, we know that Ψ' is weakly continuous. Hence

$$\int (f(x,u_n) - f(x,u))(u_n - u)dx \to 0 \quad \text{as } n \to \infty. \tag{3.4}$$

According to (3.3) and (3.4), one gets

$$\|u_n - u\|_*^2 \to 0 \quad \text{as } n \to \infty,$$

which means that $u_n \to u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ as $n \to \infty$. Thus Φ satisfies the (PS) condition. This completes the proof. □

Proof of Theorem 1.1. In fact, there exists an open bounded domain $B_{r_0}(x_0)$, where r_0 and x_0 are given in (f_5). For any $k \in \mathbb{N}$, we take k disjoint bounded sets $\Omega_i (i = 1, \cdots, k)$ such that

(1) for any Ω_i, there exists a closed bounded set $E_i \subset \Omega_i$ with $\text{meas}\{E_i\} \geq a$ and $\min\{|x - y| \mid x \in E_i, y \in \partial\Omega_i\} \geq b$, where a, b is a positive constant,

(2) $\bigcup_{i=1}^k \Omega_i \subset B_{r_0}(x_0)$.

Defining a function $\varphi_i \in C_0^\infty(\mathbb{R}^N, [0, 1])$ satisfies $\varphi_i(x) \equiv 1$ for all $x \in E_i$ and $\varphi_i(x) \equiv 0$ for all $x \in \mathbb{R}^N \setminus \Omega_i$. Let

$$u(x) = \sum_{i=1}^k \lambda_i \varphi_i(x) \tag{3.5}$$

and

$$V_k = \left\{ \sum_{i=1}^k \lambda_i \varphi_i \;\middle|\; \sum_{1 \leq i \leq k} \lambda_i^2 = \rho^2 \right\}. \tag{3.6}$$

Evidently, V_k is homeomorphic to the sphere S^{k-1} by an odd mapping for any $\rho > 0$. From Proposition 2.1, one can easily see that $\gamma(V_k) = \gamma(S^{k-1}) = k$.

Since V_k is compact, there exists a constant $C_k > 0$ such that

$$\|u\|_* \leq C_k \quad \text{for all } u \in V_k. \tag{3.7}$$

From (2.3), (3.5) and the definition of Ω_i, we have for any $s \in (0, \frac{\delta}{2})$,

$$\Phi(su) = \frac{1}{2}\|su\|_*^2 - \sum_{i=1}^k \int_{\Omega_i} F(x, s\lambda_i \varphi_i(x))dx. \tag{3.8}$$

Obviously, there exist some integers $1 \leq i_u \leq k$ such that $\lambda_{i_u} \neq 0$, where λ_i is defined in (3.6). Rewrite the integral in (3.8) as follows,

$$\sum_{i=1}^k \int_{\Omega_i} F(x, s\lambda_i\varphi_i)dx = \int_{E_{i_u}} F(x, s\lambda_{i_u}\varphi_{i_u})dx + \int_{\Omega_{i_u} \setminus E_{i_u}} F(x, s\lambda_{i_u}\varphi_{i_u})dx$$

$$+\sum_{i\neq i_u}\int_{\Omega_i} F(x, s\lambda_i\varphi_i)dx. \tag{3.9}$$

From the fact that $E_{i_u} \subseteq \Omega_i \subseteq B_{r_0}(x_0)$ and combining with Lemma 2.3 we know

$$\int_{\Omega_{i_u}\setminus E_{i_u}} F(x, s\lambda_{i_u}\varphi_{i_u})dx + \sum_{i\neq i_u}\int_{\Omega_i} F(x, s\lambda_i\varphi_i)dx \geq -Cr_0^N\rho^2 s^2. \tag{3.10}$$

For any $\delta_m \in (0, \frac{\delta}{2})$, by (2.11) and (3.7)-(3.10), one sees

$$\begin{aligned}\Phi(\delta_m u) &\leq \frac{1}{2}\delta_m^2 C_k^2 + Cr_0^N\rho^2\delta_m^2 - \int_{E_{i_u}} F(x, \delta_m\lambda_{i_u}\varphi_{i_u})dx \\ &\leq \delta_m^2\left(\frac{1}{2}C_k^2 + Cr_0^N\rho^2 - \rho^2 a^N M_m\right),\end{aligned} \tag{3.11}$$

for the fact that $|\delta_m\lambda_{i_u}\varphi_{i_u}(x)| \equiv \delta_m\rho$ for all $x \in E_{i_u}$ and $|E_{i_u}| \geq a^N$. Since $\delta_m \to 0$ and $M_m \to \infty$ as $m \to \infty$, there exists $m_0 \in \mathbb{N}$ such that the right-hand side of (3.11) is negative for any $m \geq m_0$. Define

$$W_k := \{\delta_{m_0}u \mid u \in V_k\}.$$

Then we have

$$\gamma(W_k) = \gamma(V_k) = k \quad \text{and} \quad \sup_{u\in W_k}\Phi(u) < 0,$$

which deduces (A_2) of Lemma 2.4 holds. The definition of $\Phi(u)$ in (2.3) and (f_6) imply that $\Phi(u)$ is even functional and $\Phi(0) = 0$. Lemmas 2.2 and 3.1 deduce that $\Phi : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ is differentiable and satisfies (A_1) of Lemma 2.4. Thus, there exists a nontrivial critical point sequence $\{u_n\}$ for Φ satisfying $\Phi(u_n) \leq 0$, $\Phi'(u_n) \to 0$ for all $n \in \mathbb{N}$ and $u_n \to 0$ in $\mathbb{R}^N$ as $n \to \infty$. The $\{u_n\}$ is a solution sequence of problem (1.4). This completes the proof for theorem 1.1. □

4. Proof of Theorem 1.2.

In order to prove Theorem 1.2, via variational methods, we need to modify and extend f for u outside a neighborhood of 0 to get $\widetilde{f}$ as follows. Define a cut-off function $\varphi \in C_0(\mathbb{R})$ such that $0 \leq \varphi(t) \leq 1$, $\varphi(t) \equiv 1$ for $|t| \leq \frac{\delta}{2}$ and $\varphi(t) \equiv 0$ for $|t| \geq \delta$. Let

$$\widetilde{f}(x, u) := \varphi(u)f(x, u) \quad \text{for all } (x, u) \in \mathbb{R}^N \times \mathbb{R}$$

and

$$\widetilde{F}(x,t) := \int_0^t \widetilde{f}(x,s)ds.$$

The problem (1.4) may be modified as the following nonlinear Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = \widetilde{f}(x,u), \\ u \in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{4.1}$$

The energy functional $\widetilde{\Phi} : \mathbb{R}^N \to \mathbb{R}$ associated to problem (4.1) can be defined as

$$\widetilde{\Phi}(u) := \frac{1}{2}\int |\nabla u|^2 + V(x)u^2 dx - \int \widetilde{F}(x,u)dx.$$

It is easy to see that $\widetilde{f}$ satisfies the conditions of Theorem 1.1, then the existence of a nontrivial critical point sequence $\{u_n\}$ for $\widetilde{\Phi}$ could be proved. Next, we will prove that the $u_n \to 0$ in $L^\infty(\mathbb{R}^N)$ as $n \to \infty$.

Lemma 4.1. *Assume that* (V_1) *and* (f_4') *hold and* $\{u_n\}$ *is a nontrivial critical point sequence of* $\widetilde{\Phi}$ *satisfying* $u_n \to 0$ *in* $\mathcal{D}^{1,2}(\mathbb{R}^N)$ *as* $n \to \infty$, *then* $u_n \to 0$ *in* $L^\infty(\mathbb{R}^N)$ *as* $n \to \infty$.

Proof. We shall use the Brézis-Kato estimate and the Moser iteration technique to show $u_n \to 0$ in $L^\infty(\mathbb{R}^N)$. For every critical point u of $\widetilde{\Phi}$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and any $K > 0$, define

$$u^K(x) := \begin{cases} u(x), & \text{if } |u(x)| \le K, \\ K, & \text{if } u(x) > K, \\ -K, & \text{if } u(x) < -K. \end{cases} \tag{4.2}$$

Note that, we have for all $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$,

$$\int \nabla u \cdot \nabla v + V(x)uvdx = \int \widetilde{f}(x,u)vdx. \tag{4.3}$$

For $\beta \ge 0$, take $v := |u^K|^{2\beta}u^K \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ in (4.3), thus we get

$$\int \widetilde{f}(x,u)|u^K|^{2\beta}u^K dx = \int (1+2\beta)|u^K|^{2\beta}\nabla u \cdot \nabla u^K + V(x)u|u^K|^{2\beta}u^K dx.$$

According to (V_1) and (4.2), we have

$$\frac{1}{(\beta+1)^2}\| \, |u^K|^{\beta+1}\|_*^2 \le \mu \int |h(x)||u|^{2\beta+\mu}dx.$$

Then it follows from (f_4') and the Hölder inequality that

$$\frac{S}{(\beta+1)^2}|u^K|_{2^*(\beta+1)}^{2(\beta+1)} \leq \mu|h|_p|u|_{(2\beta+\mu)p/(p-1)}^{2\beta+\mu}.$$

Letting $K \to +\infty$ in (4.2), we obtain

$$|u|_{2^*(\beta+1)} \leq [C(\beta+1)]^{\frac{1}{\beta+1}}|u|_{(2\beta+\mu)p/(p-1)}^{(2\beta+\mu)/(2\beta+2)}, \tag{4.4}$$

where $C = \left(\frac{\mu|h|_p}{S}\right)^{\frac{1}{2}}$.

In order to use the Moser iteration, for all $n \in \mathbb{N}$, we set

$$\begin{cases} 2^*(\beta_{n-1}+1) = \frac{(2\beta_n+\mu)p}{p-1}, & n > 0, \\ \beta_n = 0, \quad n = 0, \end{cases}$$

and obtain that

$$\beta_n = \frac{a}{b-1}(b^n - 1), \quad n > 0,$$

where $b = \frac{2^*(p-1)}{2p}$ and $a = \frac{2^*(p-1)-\mu p}{2p}$. Since $p > \frac{N}{2}$, one gets

$$b = \frac{2^*(p-1)}{2p} > 1 \quad \text{and} \quad a = \frac{2^*(p-1)-\mu p}{2p} > 0.$$

Then $\beta_n \to +\infty$ as $n \to \infty$ and $\frac{(2\beta_1+\mu)p}{p-1} = 2^*$. Combining with (4.4), we obtain

$$|u|_{2^*(\beta_n+1)} \leqslant \xi_n \xi_{n-1}^{\alpha_n} \cdots \xi_1^{\alpha_n\alpha_{n-1}\cdots\alpha_2}|u|_{2^*}^{\rho_n},$$

where $\xi_i = [C(\beta_i+1)]^{\frac{1}{\beta_i+1}}$, $\alpha_i = (2\beta_i+\mu)/(2\beta_i+2)$ and $\rho_n := \prod_{i=1}^n \alpha_i$.

It is quite clear that $\lim_{n\to\infty}\rho_n = \rho \in (0,1)$. At the same time, one sees that

$$\xi_n \xi_{n-1}^{\alpha_n} \cdots \xi_1^{\alpha_n\alpha_{n-1}\cdots\alpha_2} \leq \xi_1\xi_2\cdots\xi_n.$$

Note that,

$$\delta_n := \ln \prod_{i=1}^n \xi_i = \sum_{i=1}^n \frac{\ln[C(\beta_i+1)]}{\beta_i+1} \to \delta \quad \text{as } n \to \infty.$$

Hence we have

$$|u|_{2^*(\beta_n+1)} \leq e^{\delta_n}|u|_{2^*}^{\rho_n} \quad \text{for all } n \in \mathbb{N}. \tag{4.5}$$

Letting $n \to \infty$ in (4.5), we may conclude that

$$|u|_\infty \leq e^\delta \|u\|_{2^*}^\rho < +\infty. \tag{4.6}$$

Combining with the fact that $u_n \to 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and (4.6), we obtain that $u_n \to 0$ in $L^\infty(\mathbb{R}^N)$ as $n \to \infty$. The proof is completed. □

Proof of Theorem 1.2. By lemma 4.1, $\{u_n\}$ is a sequence of solutions for problem (4.1) with $u_n \to 0$ in $L^\infty(\mathbb{R}^N)$ and then $|u_n| < \delta$ as $n \to \infty$. Therefore, there exists $N \in \mathbb{N}$ such that $\{u_n\}$ is solutions of problem (1.4) for each $n \geq N$. This completes the proof of Theorem 1.2. □

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