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# A WEAK GALERKIN FINITE ELEMENT METHOD FOR THE SECOND ORDER ELLIPTIC PROBLEMS WITH MIXED BOUNDARY CONDITIONS*

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**Abstract** In this paper, a weak Galerkin finite element method is proposed and analyzed for the second-order elliptic equation with mixed boundary conditions. Optimal order error estimates are established in both discrete $H^1$ norm and the standard $L^2$ norm for the corresponding WG approximations. The numerical experiments are presented to verify the efficiency of the method.

**Keywords** Galerkin finite element methods, discrete gradient, second-order elliptic problems, mixed boundary conditions.

**MSC(2010)** 65N15, 65N30, 35J50.

## 1. Introduction

For the past few years, researchers have been investigating Galerkin methods based on fully discontinuous approximating spaces. Weak Galerkin (WG) finite element method is one of the Galerkin methods that use the discontinuous approximations. Wang and Ye were the first two authors to introduce and analyze the weak Galerkin method for the second-order elliptic problems in [18]. From then on, weak Galerkin method is being widely used and developed for other problems including the Stokes equations [19], Helmholtz equations [15], Maxwell equations [9], and biharmonic equations [11, 13, 14, 20], etc. Weak Galerkin method attributes to finite element technique to study partial differential equations such that the differential operators are approximated by weak forms as distributions. The basic idea of weak Galerkin finite element methods is to use the weak functions and their weak derivatives in algorithm design. The continuity is recouped by the stabilizer through a suitable boundary integral defined on the boundary of elements. The general elliptic equation has been studied using standard Galerkin methods [4, 6, 7], various discontinuous Galerkin methods [1, 2, 5, 16, 17], and the weak Galerkin method [8, 10, 18]. However, the weak Galerkin study was limited to the Dirichlet boundary conditions.

In this paper, we consider the following second-order elliptic equation with mixed

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boundary conditions which seeks an unknown function $u = u(x)$ satisfying,

$$-\nabla \cdot (a\nabla u) = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = g_1 \quad \text{on } \Gamma_D, \tag{1.2}$$

$$a\nabla u \cdot \mathbf{n} = g_2 \quad \text{on } \Gamma_N, \tag{1.3}$$

where $\Omega$ is a polygonal or polyhedral domain in $\mathbb{R}^d$ ($d = 2, 3$), $a = (a_{ij}(x))_{d\times d} \in [L^\infty(\Omega)]^{d^2}$ is a symmetric matrix-valued function and $\mathbf{n}$ is the unit outwards normal vector on $\partial\Omega$. Let $\Gamma_D$ and $\Gamma_N$ be partitions of the boundary of $\Omega$ such that $\Gamma_D \neq \phi$, $\Gamma_D \cup \Gamma_N = \partial\Omega$, and $\Gamma_D \cap \Gamma_N = \phi$. Assume that the matrix $a$ satisfies the following property: there exists a constant $\alpha > 0$ such that

$$\alpha\xi^T\xi \leq \xi^T a\xi, \quad \forall \xi \in \mathbb{R}^d,$$

where $\xi$ is a column vector and $\xi^t$ is the transpose of $\xi$.

Second-order elliptic equation (1.1) has been studied in [12] using weak Galerkin method with Dirichlet boundary conditions and achieved the optimal order of convergence in both $H^1$ and $L^2$ norms. The purpose of this paper is to extend the results for the second-order elliptic equations in [12] to mixed boundary conditions. We concentrate on two-dimensional problems only (i.e., $d = 2$). Using the results given in this paper, one can easily extend to higher-dimensions and it will be a straightforward generalization of our work. We use weak functions of the form $v = \{v_0, v_b\}$, where the function $v$ takes the value $v_0$ inside each element and takes the value $v_b$ on the boundary of each element. Both $v_0$ and $v_b$ are approximated by polynomials in $P_k(T)$ and $P_{k-1}(e)$ respectively, where $T$ represents an element and $e$ represents an edge of $T$, $k$ is non-negative integer. The corresponding weak Galerkin solution converge with rate of $O(h^k)$ and $O(h^{k+1})$ to the exact solution of (1.1)-(1.3) in discrete $H^1$ norm and in standard $L^2$ norm respectively, provided that the exact solution of the original problem is sufficiently smooth. In this paper, the secondary objective is to study flexibility, reliability, and the accuracy of the proposed WG method by presenting various numerical tests strengthened by examples of different cases of Dirichlet and Neumann boundary conditions. Our numerical results show an optimal order of convergence for $k = 1$ on triangular meshes in two-dimensions.

This paper is organized as follows. In section 2, we present the definition of the weak gradient operator and develop the weak Galerkin finite element scheme. Some technical estimates are presented in section 3 which will be used later. Section 4 is dedicated to deriving the error equation and the optimal order error estimates of $H^1$ and $L^2$ for the WG finite element approximations. Finally, some numerical results are presented in section 5 that confirm the theory developed in earlier sections.

## 2. Weak Galerkin Finite Element Schemes

Let $\mathcal{T}_h$ be a partition of $\Omega$ with elements $T$ and their edges $e$. For every element $T \in \mathcal{T}_h$, let $h_T$ be the diameter of $T$ and the mesh size $h = \max_{T\in\mathcal{T}_h} h_T$. We define the weak gradient as follows:

**Definition 2.1.** The discrete weak gradient operator, denoted by $\nabla_w v$, is defined as the unique polynomial $\nabla_w v|_T \in [P_{k-1}(T)]^2$ satisfying the following equation

$$(\nabla_w v, \mathbf{q})_T = -(v_0, \nabla \cdot \mathbf{q})_T + \langle v_b, \mathbf{q} \cdot \mathbf{n}\rangle_{\partial T}, \quad \forall \mathbf{q} \in [P_{k-1}(T)]^2, \tag{2.1}$$

where $\mathbf{n}$ is the unit outward normal vector of $\partial T$.

Our weak formulation will use the following vector spaces of functions on $\Omega$. For a given integer $k \geq 1$, we define

$$\begin{aligned} V_h &= \{v = \{v_0, v_b\} : v_0|_T \in P_k(T), v_b|_e \in P_{k-1}(e), T \in \mathcal{T}_h, e \in \partial T\}, \\ V_h^0 &= \{v \in V_h : v_b|_e = 0, e \in \Gamma_D\}. \end{aligned}$$

The notation $e \in \partial T$ means that $e$ is an edge of element $T$. Also note that any function $v \in V_h$ has a single value $v_b$ on each edge $e$.

Next, we introduce two projection operators by using local $L^2$-projections. For each element $T \in \mathcal{T}_h$, we denote the $L^2$-projection by $Q_0$ from $L^2(T)$ onto $P_k(T)$. Similarly, for each edge face $e$, let $Q_b$ be $L^2$-projection from $L^2(e)$ onto $P_{k-1}(e)$. We denote $R_h$ be the $L^2$-projection onto $[P_{k-1}(T)]^2$. Note that $R_h$ is a composition of locally defined $L^2$-projections into the polynomial space $P_{k-1}(T)$ for each element $T \in \mathcal{T}_h$.

Now we introduce two bilinear forms on $V_h$. For all $v, w \in V_h$,

$$\begin{aligned} a(v, w) &= \sum_{T \in \mathcal{T}_h} (a\nabla_w v, \nabla_w w)_T, \\ s(v, w) &= \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle Q_b v_0 - v_b, Q_b w_0 - w_b \rangle_{\partial T}. \end{aligned}$$

Next, we denote $a_s(.,.)$ be the stabilization of $a(.,.)$ given by

$$a_s(v, w) = a(v, w) + s(v, w). \tag{2.2}$$

The weak formulation for boundary value problem (1.1)-(1.3) is given as:

**Weak Galerkin Algorithm 1.** The numerical approximation for (1.1)-(1.3) can be obtained by seeking $u_h = \{u_0, u_b\} \in V_h$ such that $u_b = Q_b g_1$ on $\Gamma_D$ and

$$a_s(u_h, v) = (f, v_0) + \langle g_2, v_b \rangle_{\Gamma_N}, \quad \forall v \in V_h^0. \tag{2.3}$$

Next, for any $v \in V_h$, we define $|\!|\!| v |\!|\!|$ as

$$|\!|\!| v |\!|\!|^2 := \sum_{T \in \mathcal{T}_h} (a\nabla_w v, \nabla_w v)_T + \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle Q_b v_0 - v_b, Q_b v_0 - v_b \rangle_{\partial T}. \tag{2.4}$$

The fact that $|\!|\!| . |\!|\!|$ defines a norm in finite element space $V_h^0$ can easily be verified.

The following lemma is about the uniqueness of the solution of weak Galerkin formulation.

**Lemma 2.1.** *The weak Galerkin finite element scheme (2.3) has a unique solution.*

**Proof.** Let $u_h^{(1)}$ and $u_h^{(2)}$ be two solutions of (2.3). Then $e_h = u_h^{(1)} - u_h^{(2)}$ satisfies the equation

$$a_s(e_h, v) = 0, \quad \forall v \in V_h^0.$$

Note that $e_h \in V_h^0$. Letting $v = e_h$, we get

$$|\!|\!| e_h |\!|\!|^2 = a_s(e_h, e_h) = 0.$$

Which implies $e_h = 0$, hence $u_h^{(1)} = u_h^{(2)}$. This concludes the proof. □

## 3. Some Estimates

In this section, we are going to present some technical results that are used in later sections. In what follows, $C$ denotes a generic constant which is independent of the mesh size $h$ and the functions in the estimates. For simplicity of analysis, we assume that the coefficient $a$ in the boundary value problem (1.1)-(1.3) is a piecewise constant matrix on each element $T$ of $\mathcal{T}_h$. The result can be extended to variable matrices, provided that the matrix $a$ is piecewise sufficiently smooth.

Firstly, we are going to present the trace inequality established in [18] for functions on general shape regular partitions. Let $T$ be an element with $e$ as an edge. For any function $\varphi \in H^1(T)$, the following trace inequality holds true (see [18]):

$$\|\varphi\|_e^2 \le C(h_T^{-1}\|\varphi\|_T^2 + h_T\|\nabla\varphi\|_T^2). \tag{3.1}$$

The next lemma presents the commutative property of $L^2$ projections $Q_h$ and $R_h$.

**Lemma 3.1** (Lemma 5.1 [12]). *Let $Q_h$ and $R_h$ be the $L^2$ projection operators as defined earlier. Then, on each element $T \in \mathcal{T}_h$, we have the following commutative property*

$$\nabla_w(Q_h\phi) = R_h(\nabla\phi), \quad \forall\phi \in H^1(T). \tag{3.2}$$

The following lemma provides some estimates for the projection operators $Q_h$ and $R_h$. The proof of lemma can be found in [18].

**Lemma 3.2** (Lemma 4.1 [18]). *Let $\mathcal{T}_h$ be a finite element partition of $\Omega$ that is shape regular. For all $\phi \in H^{k+1}(\Omega)$, we have*

$$\sum_{T\in\mathcal{T}_h}\|Q_0\phi-\phi\|_T^2 + \sum_{T\in\mathcal{T}_h} h_T^2\|\nabla(Q_0\phi-\phi)\|_T^2 \le Ch^{2(k+1)}\|\phi\|_{k+1}^2, \tag{3.3}$$

$$\sum_{T\in\mathcal{T}_h}\|a(R_h\nabla\phi-\nabla\phi)\|_T^2 \le Ch^{2k}\|\phi\|_{k+1}^2. \tag{3.4}$$

**Lemma 3.3.** *For any $\phi \in H^1(T)$ and $v \in V_h$, we have*

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}(\nabla v_0, a\nabla\phi)_T = &\sum_{T\in\mathcal{T}_h}(a\nabla_w Q_h\phi, \nabla_w v)_T + \sum_{T\in\mathcal{T}_h}\langle v_0 - v_b, (aR_h\nabla\phi)\cdot\mathbf{n}\rangle_{\partial T\setminus\Gamma_N}\\ &+\sum_{T\in\mathcal{T}_h}\langle v_0-v_b,(aR_h\nabla\phi)\cdot\mathbf{n}\rangle_{\partial T\cap\Gamma_N}.\end{aligned} \tag{3.5}$$

**Proof.** Using the definition of discrete weak gradient (2.1), Lemma 3.1, and integration by parts, we get

$$\begin{aligned}&(\nabla_w v, a\nabla_w Q_h\phi)_T\\ &=(\nabla_w v, aR_h\nabla\phi)_T\\ &=-(v_0,\nabla\cdot(aR_h\nabla\phi))_T + \langle v_b,(aR_h\nabla\phi)\cdot\mathbf{n}\rangle_{\partial T\setminus\Gamma_N} + \langle v_b,(aR_h\nabla\phi)\cdot\mathbf{n}\rangle_{\partial T\cap\Gamma_N}\\ &=(\nabla v_0, aR_h\nabla\phi)_T - \langle v_0-v_b,(aR_h\nabla\phi)\cdot\mathbf{n}\rangle_{\partial T\setminus\Gamma_N} - \langle v_0-v_b,(aR_h\nabla\phi)\cdot\mathbf{n}\rangle_{\partial T\cap\Gamma_N}\\ &=(\nabla v_0, a\nabla\phi)_T - \langle v_0-v_b,(aR_h\nabla\phi)\cdot\mathbf{n}\rangle_{\partial T\setminus\Gamma_N} - \langle v_0-v_b,(aR_h\nabla\phi)\cdot\mathbf{n}\rangle_{\partial T\cap\Gamma_N}.\end{aligned}$$

Applying summation and solving for $\sum_{T\in\mathcal{T}_h}(\nabla v_0, a\nabla\phi)_T$, we obtain

$$\sum_{T\in\mathcal{T}_h}(\nabla v_0, a\nabla\phi)_T = \sum_{T\in\mathcal{T}_h}(a\nabla_w Q_h\phi,\nabla_w v)_T + \sum_{T\in\mathcal{T}_h}\langle v_0-v_b,(aR_h\nabla\phi)\cdot\mathbf{n}\rangle_{\partial T\setminus\Gamma_N}$$

$$+\sum_{T\in\mathcal{T}_h}\langle v_0-v_b,(aR_h\nabla\phi)\cdot\mathbf{n}\rangle_{\partial T\cap\Gamma_N}.$$

Which concludes the proof. □

Next, We introduce a discrete $H^1$ semi-norm in the finite element space $V_h$ as follows:

$$\|v\|_{1,h}=\left(\sum_{T\in\mathcal{T}_h}(\|\nabla v_0\|_T^2+h_T^{-1}\|Q_bv_0-v_b\|_{\partial T}^2)\right)^{\frac{1}{2}}. \tag{3.6}$$

In the following lemma, we are going to present the equivalence of $\|\cdot\|_{1,h}$ to $|||\cdot|||$. The proof of the lemma can be found in [12].

**Lemma 3.4** (Lemma 5.3 [12]). *There exists two positive constants $C_1$ and $C_2$ such that for any $v=\{v_0,v_b\}\in V_h$, we have*

$$C_1\|v\|_{1,h}\le |||v|||\le C_2\|v\|_{1,h}.$$

**Lemma 3.5** (Lemma 5.4 [12]). *Assume that $\mathcal{T}_h$ is shape regular. Then for any $w\in H^{k+1}(\Omega)$ and $v=\{v_0,v_b\}\in V_h$, we have*

$$|s(Q_hw,v)|\le Ch^k\|w\|_{k+1}|||v|||, \tag{3.7}$$

$$\left|\sum_{T\in\mathcal{T}_h}\langle a(\nabla w-R_h\nabla w)\cdot\mathbf{n},v_0-v_b\rangle_{\partial T}\right|\le Ch^k\|w\|_{k+1}|||v|||. \tag{3.8}$$

# 4. Error Analysis

In this sections, some error estimates for the weak Galerkin finite element method solution $u_h$ will be established. The errors will be measured in two natural norms: the triple-bar norm as defined in (2.4) and the standard $L^2$ norm. First, we will present the error equation.

## 4.1. Error Equation

Let $u_h=\{u_0,u_b\}\in V_h$ be the weak Galerkin finite element solution arising from (2.3) and $u$ be the exact solution of (1.1)-(1.3). The $L^2$ projection of $u$ on to the finite element space $V_h$ is given as

$$Q_hu=\{Q_0u,Q_bu\}.$$

Let $e_h$ be the error between $L^2$ projection of the exact solution and the weak Galerkin finite element solution defined as:

$$e_h=\{e_0,e_b\}=\{Q_0u-u_0,Q_bu-u_b\}.$$

In the next theorem, we are going to present the error equation.

**Theorem 4.1.** *Let $e_h$ be the error defined as above. Then for any $v\in V_h^0$, we have*

$$a_s(e_h,v)=\sum_{T\in\mathcal{T}_h}\langle a(\nabla u-R_h\nabla u)\cdot\mathbf{n},v_0-v_b\rangle_{\partial T}+s(Q_hu,v). \tag{4.1}$$

**Proof.** Testing (1.1) by $v_0$ where $v = \{v_0, v_b\} \in V_h^0$ and using integration by parts, we get

$$\sum_{T\in\mathcal{T}_h} (a\nabla u, \nabla v_0)_T - \sum_{T\in\mathcal{T}_h} \langle a\nabla u\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial T\setminus\Gamma_N} - \sum_{T\in\mathcal{T}_h} \langle a\nabla u\cdot\mathbf{n}, v_0\rangle_{\partial T\cap\Gamma_N} = (f, v_0)$$

where we have used the fact that $\sum_{T\in\mathcal{T}_h} (\nabla u\cdot\mathbf{n}, v_b)_{\partial T\setminus\Gamma_N} = 0$.

By setting $\phi = u$ in (3.5) and substituting in above equation, we obtain

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} (a\nabla_w Q_h u, \nabla_w v)_T =&(f, v_0) + \sum_{T\in\mathcal{T}_h} \langle g_2, v_0\rangle_{\partial T\cap\Gamma_N} - \sum_{T\in\mathcal{T}_h} \langle aR_h\nabla u\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial T\cap\Gamma_N}\\ &+ \sum_{T\in\mathcal{T}_h} \langle a(\nabla u - R_h\nabla u)\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial T\setminus\Gamma_N}.\end{aligned}$$

Adding the term $s(Q_h u, v)$ to both sides of the above equation gives rise to

$$\begin{aligned}a_s(Q_h u, v) =&(f, v_0) + \sum_{T\in\mathcal{T}_h} \langle g_2, v_0\rangle_{\partial T\cap\Gamma_N} - \sum_{T\in\mathcal{T}_h} \langle aR_h\nabla u\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial T\cap\Gamma_N}\\ &+ s(Q_h u, v) + \sum_{T\in\mathcal{T}_h} \langle a(\nabla u - R_h\nabla u)\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial T\setminus\Gamma_N}.\end{aligned} \tag{4.2}$$

Subtracting (2.3) from (4.2) yields

$$\begin{aligned}a_s(e_h, v) =& \sum_{T\in\mathcal{T}_h} \langle a(\nabla u - R_h\nabla u)\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial T\setminus\Gamma_N} + s(Q_h u, v)\\ &+ \sum_{T\in\mathcal{T}_h} \langle g_2 - aR_h\nabla u\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial T\cap\Gamma_N}.\end{aligned}$$

By combining first and third terms gives the error equation (4.1)

$$a_s(e_h, v) = \sum_{T\in\mathcal{T}_h} \langle a(\nabla u - R_h\nabla u)\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial T} + s(Q_h u, v),$$

which completes the proof. □

## 4.2. Error Estimates

In this section, we are going to derive the error estimates for the weak Galerkin finite element solution.

**Theorem 4.2** ($H^1$ error)**.** *Let $u_h \in V_h$ be the weak Galerkin finite element solution arising from (2.3)and $u \in H^{k+1}(\Omega)$ be the exact solution of the problem (1.1)-(1.3). Then, there exists a constant $C$ such that*

$$|||u_h - Q_h u||| \le Ch^k \|u\|_{k+1}. \tag{4.3}$$

**Proof.** Substituting $v = e_h$ in (4.1) and using the equation (2.4), we get

$$|||e_h|||^2 = a_s(e_h, e_h) = \sum_{T\in\mathcal{T}_h} \langle a(\nabla u - R_h\nabla u)\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial T} + s(Q_h u, e_h).$$

Using (3.7) and (3.8) gives us

$$|\!|\!| e_h |\!|\!|^2 \leq Ch^k \|u\|_{k+1} |\!|\!| e_h |\!|\!|,$$

which gives (4.3). This concludes the proof. □

Next, we are going to derive $L^2$ error estimate for the weak Galerkin finite element scheme. Consider the dual problem that seek $w \in H_0^1(\Omega)$ satisfying:

$$\begin{aligned} -\nabla\cdot(a\nabla w) &= e_0 \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \Gamma_D, \\ a\nabla w\cdot\mathbf{n} &= 0 \quad \text{on } \Gamma_N, \end{aligned} \tag{4.4}$$

with the $H^{1+s}$-regularity assumption $\|w\|_{1+s} \leq C\|e_0\|$ where $0 < s \leq 1$. From Theorem 1.1 in [3], we know $w \in H^2(\mathcal{T}_h)$ in many situations, where $H^2(\mathcal{T}_h)$ is a broken Sobolev space defined as follows:

$$H^2(\mathcal{T}_h) = \{v : v|_T \in H^2(T), \forall T \in \mathcal{T}_h\}.$$

**Theorem 4.3** ($L^2$ error)**.** *Assume that the exact solution $w$ of the dual problem (4.4) satisfies $w \in H^{1+s}(\Omega) \cap H^2(\mathcal{T}_h)$ with $s \in (0,1]$. Let $u$ and $u_h \in V_h$ be the solutions of the problem (1.1)-(1.3) and (2.3) respectively. Then, there exists a constant $C$ such that*

$$\|u - u_0\| \leq Ch^{k+s}\|u\|_{k+1}.$$

**Proof.** Testing the first equation of (4.4) with $e_0$, we get

$$\|e_0\|^2 = (-\nabla\cdot(a\nabla w), e_0).$$

From integration by parts, we get

$$\|e_0\|^2 = \sum_{T\in\mathcal{T}_h}(a\nabla w, \nabla e_0)_T - \sum_{T\in\mathcal{T}_h}\langle a\nabla w\cdot\mathbf{n}, e_0\rangle_{\partial T\setminus\Gamma_N},$$

since $\sum_{T\in\mathcal{T}_h}\langle a\nabla w\cdot\mathbf{n}, e_b\rangle_{\partial T\setminus\Gamma_N} = 0$, we can rewrite the above expression as

$$\|e_0\|^2 = \sum_{T\in\mathcal{T}_h}(a\nabla w, \nabla e_0)_T - \sum_{T\in\mathcal{T}_h}\langle a\nabla w\cdot\mathbf{n}\ ,\ e_0 - e_b\rangle_{\partial T\setminus\Gamma_N}. \tag{4.5}$$

Setting $\phi = w$ and $v = e_h$ in (3.5) gives

$$\sum_{T\in\mathcal{T}_h}(a\nabla w, \nabla e_0)_T = \sum_{T\in\mathcal{T}_h}(a\nabla_w Q_h w, \nabla_w e_h)_T + \sum_{T\in\mathcal{T}_h}\langle e_0 - e_b, (aR_h\nabla w)\cdot\mathbf{n}\rangle_{\partial T\setminus\Gamma_N}. \tag{4.6}$$

Substituting (4.6) in (4.5), we get

$$\|e_0\|^2 = a(Q_h w, e_h) + \sum_{T\in\mathcal{T}_h}\langle a(R_h\nabla w - \nabla w)\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial T\setminus\Gamma_N},$$

adding and subtracting the term $s(Q_h w, e_h)$, we obtain

$$\|e_0\|^2 = a_s(Q_h w, e_h) - s(Q_h w, e_h) + \sum_{T\in\mathcal{T}_h}\langle a(R_h\nabla w - \nabla w)\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial T\setminus\Gamma_N}. \tag{4.7}$$

It follows from the error equation (4.1) that

$$a_s(Q_hw, e_h) = \sum_{T\in\mathcal{T}_h} \langle a(\nabla u - R_h\nabla u)\cdot\mathbf{n}, Q_0w - Q_bw\rangle_{\partial T} + s(Q_hu, Q_hw). \tag{4.8}$$

By combining (4.7) with (4.8), we get

$$\begin{aligned}\|e_0\|^2 = &\sum_{T\in\mathcal{T}_h} \langle a(\nabla u - R_h\nabla u)\cdot\mathbf{n}, Q_0w - Q_bw\rangle_{\partial T} + s(Q_hu, Q_hw)\\ &- s(Q_hw, e_h) + \sum_{T\in\mathcal{T}_h} \langle a(R_h\nabla w - \nabla w)\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial T\setminus\Gamma_N}.\end{aligned} \tag{4.9}$$

Now we are going to bound the term on the right hand of equation (4.9). Using the Cauchy-Schwarz inequality and the definition of $Q_b$ we get

$$\begin{aligned}&\left|\sum_{T\in\mathcal{T}_h} \langle a(\nabla u - R_h\nabla u)\cdot\mathbf{n}, Q_0w - Q_bw\rangle_{\partial T}\right|\\ \le&\left(\sum_{T\in\mathcal{T}_h} \|a(\nabla u - R_h\nabla u)\|_{\partial T}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h} \|Q_0w - Q_bw\|_{\partial T}^2\right)^{1/2}\\ \le& C\left(\sum_{T\in\mathcal{T}_h} \|a(\nabla u - R_h\nabla u)\|_{\partial T}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h} \|Q_0w - w\|_{\partial T}^2\right)^{1/2}.\end{aligned} \tag{4.10}$$

From the trace inequality (3.1) and the estimate (3.3), we have

$$\left(\sum_{T\in\mathcal{T}_h} \|a(\nabla u - R_h\nabla u)\|_{\partial T}^2\right)^{1/2} \le Ch^{k-\frac{1}{2}}\|u\|_{k+1}, \tag{4.11}$$

$$\left(\sum_{T\in\mathcal{T}_h} \|Q_0w - w\|_{\partial T}^2\right)^{1/2} \le Ch^{s+\frac{1}{2}}\|w\|_{1+s}. \tag{4.12}$$

Substituting (4.11) and (4.12) into (4.10), we get

$$\left|\sum_{T\in\mathcal{T}_h} \langle a(\nabla u - R_h\nabla u)\cdot\mathbf{n}, Q_0w - Q_bw\rangle_{\partial T}\right| \le Ch^{k+s}\|u\|_{k+1}\|w\|_{1+s}. \tag{4.13}$$

Similarly, it follows from the definition of $Q_b$, the trace inequality (3.1), and the estimate (3.3) that

$$\begin{aligned}|s(Q_hu, Q_hw)| &\le \sum_{T\in\mathcal{T}_h} h_T^{-1}\,|Q_0u - Q_bu, Q_0w - Q_bw|\\ &\le \left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_0u - u\|_{\partial T}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_0w - w\|_{\partial T}^2\right)^{1/2}\\ &\le Ch^{k+s}\|u\|_{k+1}\|w\|_{1+s}.\end{aligned} \tag{4.14}$$

The estimate (3.7) and (4.3) implies that

$$|s(Q_hw, e_h)| \le Ch^s\|w\|_{1+s}|\!|\!|e_h|\!|\!| \le Ch^{k+s}\|u\|_{k+1}\|w\|_{1+s}. \tag{4.15}$$

For the fourth term, the estimate (3.8) and (4.3) gives

$$\begin{aligned}\left|\sum_{T\in\mathcal{T}_h}\langle a(R_h\nabla w-\nabla w)\cdot\mathbf{n}, e_0-e_b\rangle_{\partial T\setminus\Gamma_N}\right| &\le \left|\sum_{T\in\mathcal{T}_h}\langle a(R_h\nabla w-\nabla w)\cdot\mathbf{n}, e_0-e_b\rangle_{\partial T}\right|\\ &\le Ch^{k+s}\|u\|_{k+1}\|w\|_{1+s}. \qquad (4.16)\end{aligned}$$

Substituting (4.13)-(4.16) into (4.9) yields

$$\|e_0\|^2 \le Ch^{k+s}\|u\|_{k+1}\|w\|_{1+s}.$$

By using the regularity assumption $\|w\|_2 \le C\|e_0\|$, we arrive at

$$\|e_0\| \le Ch^{k+s}\|u\|_{k+1},$$

which concludes the proof. □

## 5. Numerical Experiments

In this section, we are going to validate the proposed WG method by presenting some numerical experiments. Let us consider the second-order elliptic problem (1.1)-(1.3), with $a$ to be a unit matrix on the unit square $\Omega = [0,1]\times[0,1]$. We define the Neumann boundary as $\Gamma_N = \{(x,1)\in\mathbb{R}^2 : 0\le x\le 1\}$ and the Dirichlet boundary is defined as $\Gamma_D = \partial\Omega\setminus\Gamma_N$. Let $h=\frac{1}{n}(n=2,4,8,16,32,64,128)$ be the mesh sizes for different triangular meshes. The construction of the triangular mesh: First to obtain the square mesh, uniformly partition the square domain $\Omega$ into $n\times n$ sub-squares. Then divide each square element into two triangles by the diagonal with a positive slope. This completes the construction of the triangular mesh.

All the examples given below use these triangulations of $\Omega$. The lowest order ($k=1$) weak Galerkin element is used to find weak Galerkin solution $u_h=\{u_0,u_b\}$ where $u_0|_T\in P_1(T)$, and $u_b|_e\in P_0(e)$. Consider the following four exact solutions of (1.1)-(1.3) defined on $\Omega=[0,1]\times[0,1]$, which are

$$\begin{aligned} &u_1 = x^2(1-x)^2y^2(1-y)^2 &&\text{and}&& u_2=\sin(2\pi x)\sin(2\pi y),\\ &u_3=\cos(2\pi x)\cos(2\pi y) &&\text{and}&& u_4 = x^2(1-x)^2y^2(1-y)^2+x^2,\end{aligned}$$

with following types of boundary conditions,

$$\begin{aligned} u_1|_{\Gamma_D}&=0 &&\text{and}&& \left.\frac{\partial u_1}{\partial\mathbf{n}}\right|_{\Gamma_N}=0,\\ u_2|_{\Gamma_D}&=0 &&\text{and}&& \left.\frac{\partial u_2}{\partial\mathbf{n}}\right|_{\Gamma_N}\ne 0,\\ u_3|_{\Gamma_D}&\ne 0 &&\text{and}&& \left.\frac{\partial u_3}{\partial\mathbf{n}}\right|_{\Gamma_N}=0,\\ u_4|_{\Gamma_D}&\ne 0 &&\text{and}&& \left.\frac{\partial u_4}{\partial\mathbf{n}}\right|_{\Gamma_N}\ne 0.\end{aligned}$$

The different cases of the boundary conditions of these exact solutions make them best choice to test for our problem. This enables us to test the effect of different boundary data on convergence rates. The source term of equation (1.1), Dirichlet,

**Table 1.** $H^1$ and $L^2$ norm errors and their convergence rates for $u_1$.

| $h$ | $\|\|\|u - u_h\|\|\|$ | order | $\|u - u_h\|$ | order |
|---|---|---|---|---|
| 1/2 | 1.14E-02 | | 2.44E-03 | |
| 1/4 | 8.41E-03 | 0.44 | 9.51E-04 | 1.36 |
| 1/8 | 4.53E-03 | 0.89 | 2.61E-04 | 1.87 |
| 1/16 | 2.31E-03 | 0.97 | 6.68E-05 | 1.97 |
| 1/32 | 1.16E-03 | 0.99 | 1.68E-05 | 1.99 |
| 1/64 | 5.81E-04 | 1.00 | 4.21E-06 | 2.00 |
| 1/128 | 2.91E-04 | 1.00 | 1.05E-06 | 2.00 |

**Table 2.** $H^1$ and $L^2$ norm errors and their convergence rates for $u_2$.

| $h$ | $\|\|\|u - u_h\|\|\|$ | order | $\|u - u_h\|$ | order |
|---|---|---|---|---|
| 1/2 | 1.22E+01 | | 2.35E+00 | |
| 1/4 | 6.03E+00 | 1.02 | 6.45E-01 | 1.87 |
| 1/8 | 3.13E+00 | 0.94 | 1.67E-01 | 1.95 |
| 1/16 | 1.58E+00 | 0.98 | 4.22E-02 | 1.99 |
| 1/32 | 7.94E-01 | 0.97 | 1.06E-02 | 2.00 |
| 1/64 | 3.97E-01 | 1.00 | 2.64E-03 | 2.00 |
| 1/128 | 1.99E-01 | 1.00 | 6.61E-04 | 2.00 |

**Table 3.** $H^1$ and $L^2$ norm errors and their convergence rates for $u_3$.

| $h$ | $\|\|\|u - u_h\|\|\|$ | order | $\|u - u_h\|$ | order |
|---|---|---|---|---|
| 1/2 | 2.83E+00 | | 5.77E-01 | |
| 1/4 | 6.02E+00 | -1.09 | 6.267E-01 | -0.12 |
| 1/8 | 3.13E+00 | 0.94 | 1.62E-01 | 1.95 |
| 1/16 | 1.58E+00 | 0.98 | 4.06E-02 | 1.99 |
| 1/32 | 7.94E-01 | 0.97 | 1.03E-02 | 2.00 |
| 1/64 | 3.97E-01 | 1.00 | 2.57E-03 | 2.00 |
| 1/128 | 1.99E-01 | 1.00 | 6.42E-05 | 2.00 |

**Table 4.** $H^1$ and $L^2$ norm errors and their convergence rates for $u_4$.

| $h$ | $\|\|\|u - u_h\|\|\|$ | order | $\|u - u_h\|$ | order |
|---|---|---|---|---|
| 1/2 | 6.48E-01 | | 1.43E-01 | |
| 1/4 | 3.27E-01 | 0.99 | 3.63E-02 | 1.97 |
| 1/8 | 1.64E-01 | 1.00 | 9.13E-03 | 1.99 |
| 1/16 | 8.20E-02 | 1.00 | 2.29E-03 | 2.00 |
| 1/32 | 4.10E-02 | 1.00 | 5.72E-04 | 2.00 |
| 1/64 | 2.05E-02 | 1.00 | 1.43E-04 | 2.00 |
| 1/128 | 1.03E-02 | 1.00 | 3.58E-05 | 2.00 |

and Neumann boundary conditions are computed to match the exact solutions. The results for test problems with exact solutions $u_1, u_2, u_3$ and $u_4$, are reported in Tables 1, 2, 3 and 4 respectively.

It can be seen from the above results that $u$ always achieve an optimal order. The rate of convergence for both $H^1$ and $L^2$ errors are of $O(h)$ and $O(h^2)$ respectively. In Table 3, we can notice that the convergence rate is negative for $h = 1/2$ but it improves as the mesh gets refiner. Numerical experiment results confirm the theory established in earlier sections of this article.

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