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BIFURCATION OF LIMIT CYCLES FROM THE GLOBAL CENTER OF A CLASS OF INTEGRABLE NON-HAMILTON SYSTEMS*

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Abstract In this paper, we consider the bifurcation of limit cycles for system $\dot{x} = -y(x^2 + a^2)^m,\ \dot{y} = x(x^2 + a^2)^m$ under perturbations of polynomials with degree n, where $a \neq 0$, $m \in \mathbb{N}$. By using the averaging method of first order, we bound the number of limit cycles that can bifurcate from periodic orbits of the center of the unperturbed system. Particularly, if $m = 2, n = 5$, the sharp bound is 5.

Keywords Limit cycle, averaging function, bifurcation.

MSC(2010) 37G15, 34C05.

1. Introduction and statement of the main results

In the qualitative theory of real planar differential systems, one of the main open problems is the determination of limit cycles. A classical way to produce limit cycles is to perturb a system, which has a center, so that limit cycles bifurcate in the perturbed system from some of the periodic orbits surrounding the center of the unperturbed system ([3,5]). Recently, many people have considered the number of limit cycles bifurcating from the period annulus of a system

$$\begin{cases} \dot{x} = -yG(x, y) + \varepsilon P_n(x, y), \\ \dot{y} = xG(x, y) + \varepsilon Q_n(x, y), \end{cases} \tag{1.1}$$

where $P_n(x, y), Q_n(x, y)$ are polynomial of degree n, $G(0, 0) \neq 0$ and ε is a small parameter.

J. Llibre *et al.* [13] studied the case of one line ($G(x, y) = 1 + x$). A. Buica and J. Llibre [2] studied the case of two orthogonal lines ($G(x, y) = (x + a)(y + b)$). B. Coll *et al.* [7] studied the case of three lines, two of them parallel and the other perpendicular ($G(x, y) = (x + a)(y + b)(x + c)$). A. Atabaigi *et al.* [1] studied the case of four lines, two of them parallel and the other perpendicular ($G(x, y) = (x^2 - a^2)(y^2 - b^2)$). A. Gasull *et al.* [8] investigated the case of $G(x, y) = \prod_{j=1}^{k_1}(x - a_j)\prod_{l=1}^{k_2}(y - b_l)$ with $a_i \neq a_j$ and $b_i \neq b_j$ for $i \neq j$. J. Giné and J. Llibre [10] studied

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the case of $G(x,y)=0$ being a conic in $\mathbb{R}^2$ for $n=3$. A. Gasull *et al.* [9] considered the case of $G(x,y)=(1-y)^m$ with $m\in\mathbb{N}$. G. Xiang and M. Han [16] studied the case of $G(x,y)=ax^2+bx+1$ with $a\neq 0$. G. Chang and M. Han [4] considered the case of $G(x,y)=\prod\limits_{i=1}^{m}[(x-a_i)^2+(y-b_i)^2]^{k_i}$. Y. Xiong [17] investigated the case of $G(x,y)=\prod\limits_{i=1}^{m}\left[(x-a_i)^2+y^2-b_i^2\right]^{k_i}$. S. Li *et al.* [12] considered the case of $G(x,y)=y^2+ax^2+bx+c$ with $c\neq 0$.

Most cases of the above, system $(1.1)_{\varepsilon=0}$ has invariant set that is formed by parallel and/or orthogonal invariant lines. When the center is global, there are few results ([16] when $b^2-4a<0$). Motivated by [9], in this paper we will consider the bifurcation of limit cycles from the center of system

$$\begin{cases}\dot{x}=-y(x^2+a^2)^m,\\ \dot{y}=x(x^2+a^2)^m,\end{cases}\tag{1.2}$$

under perturbations of polynomials with degree n, where $a\neq 0$, $m\in\mathbb{N}$. Let $x_1=\frac{1}{a}x,\ y_1=\frac{1}{a}y,\ t_1=a^{2m}t$. Then system (1.2) is transformed into

$$\begin{cases}\dot{x}=-y(x^2+1)^m,\\ \dot{y}=x(x^2+1)^m,\end{cases}$$

here we omit the subscript 1. Hence, we consider the following system

$$\begin{cases}\dot{x}=-y(x^2+1)^m+\varepsilon P_n(x,y),\\ \dot{y}=x(x^2+1)^m+\varepsilon Q_n(x,y),\end{cases}\tag{1.3}$$

where $P_n(x,y)=\sum\limits_{i+j=0}^{n}p_{i,j}x^iy^j$, $Q_n(x,y)=\sum\limits_{i+j=0}^{n}q_{i,j}x^iy^j$, $0<|\varepsilon|\ll 1$. And the system $(1.3)_{\varepsilon=0}$ has a global center.

The objective of this paper is to estimate the number of limit cycles that bifurcate from the periodic orbits of the periodic annulus $\mathcal{D}$ of the origin of system (1.3), where

$$\mathcal{D}=\left\{(x,y):0<\sqrt{x^2+y^2}<+\infty\right\}.$$

Our results indicate that the upper bound is independent with m when $[\frac{n+1}{2}]\geq m$. Our method is the first order averaging method. Denote by H(n) the maximum number of limit cycles bifurcating from the period annulus $\mathcal{D}$ of system $(1.3)_{\varepsilon=0}$ up to the first order averaging method, then we have the following result:

Theorem 1.1. *Consider system (1.3) with* $0<|\varepsilon|\ll 1$.
(1) If $[\frac{n+1}{2}]<m$, *then* $H(n)\leq[\frac{n+1}{2}]+m-2$.
(2) If $[\frac{n+1}{2}]\geq m$, *then* $H(n)\leq 2[\frac{n+1}{2}]-1$.

Corollary 1.1. *If in system (1.3)* $m=2,\ n=5$, *and* $0<|\varepsilon|\ll 1$, *then* $H(5)=5$.

This paper is organized as follows. In Section 2, we introduce the first averaging method and obtain the expression of averaging function. In Section 3, the main results is proved. In the last section, we give a conjecture.

2. The expression of averaging function

In the plane, the method of Abelian integrals and the averaging theory are essentially equivalent, but each has its own advantages. For example, when the associated Abelian integrals are complicated or we need to study the orbits of the non-autonomous differential systems, the averaging method displays more flexibility. Recently, M. Han [11] studied the number of periodic solutions of piecewise smooth periodic equations by average method.

The following lemma provides a first order approximation in ε for the periodic differential system, for a proof see Theorem 2.6.1 of Sanders and Verhulst [14] and Theorem 11.5 of Verhulst [15]. The original theorem is given for a system of differential equations, but since we will use it only for one differential equation, we state them in this case.

Lemma 2.1 (Theorem 2.6.1, [14],Theorem 11.5, [15])**.** *Consider the following two initial value problems*

$$\dot{x} = \varepsilon F(t,x) + \varepsilon^2 G(t,x,\varepsilon), \quad x(0) = x_0, \tag{2.1}$$

and

$$\dot{y} = \varepsilon f^0(y), \quad y(0) = x_0, \tag{2.2}$$

where $x,\ y,\ x_0 \in U$ an open interval of $\mathbb{R}$, $t \in [0,+\infty)$, F and G are T-periodic in the variable t, and $f^0(y)$ is the averaged function of $F(t,y)$ with respect to t, i.e.,

$$f^0(y) = \frac{1}{T}\int_0^T F(t,y)\mathrm{d}t. \tag{2.3}$$

Suppose: (i) $F, \partial F/\partial x, \partial^2 F/\partial x^2, G$ and $\partial G/\partial x$ are defined, continuous and bounded by a constant independent on ε in $[0,+\infty)\times U$ and $\varepsilon \in (0,\varepsilon_0]$; (ii) T is a constant independent of ε; (iii) $y(t)$ belongs to U on the timescale $1/\varepsilon$. Then the following statements hold.

(a) On the time-scale $1/\varepsilon$ we have that $x(t) - y(t) = O(\varepsilon)$ as $\varepsilon \to 0$. (b) If p is an equilibrium point of the averaged system (2.2) such that

$$\left.\frac{\partial f^0}{\partial y}\right|_{y=p} \neq 0, \tag{2.4}$$

then system (2.1) has a T-periodic solution $\phi(t,\varepsilon) \to p$ as $\varepsilon \to 0$. (c) If (2.4) is negative, then the corresponding periodic solution $\phi(t,\varepsilon)$ of Eq.(2.1) in the space (t,x) is asymptotically stable for ε sufficiently small. If (2.4) is positive, then it is unstable.

Since the averaging theory does not tell any information on upper bound of the maximum number of periodic solution, we need the following results which can be obtained from Theorem 1.1 of [11].

Lemma 2.2. *Under the assumption of Lemma 2.1, if the averaged function $f^0(y)$ has at most k zeros on U, taking into account the multiplicity, then there exist at most k periodic solutions bifurcating from the period annulus of $(2.1)_{\varepsilon=0}$.*

Let $x = r\cos\theta, y = r\sin\theta$ and taking θ as the new independent variable, we transform the differential system (1.3) into the equivalent differential equation

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \varepsilon f_1(r,\theta) + \varepsilon^2 G(r,\theta,\varepsilon), \quad r \in (0,+\infty), \tag{2.5}$$

where

$$f_1(r,\theta)=\sum_{i+j=0}^{n} r^{i+j}\frac{p_{i,j}\cos^{i+1}\theta\sin^j\theta+q_{i,j}\cos^i\theta\sin^{j+1}\theta}{(r^2\cos^2\theta+1)^m}. \tag{2.6}$$

The averaged function of (2.6) is

$$\begin{aligned}f^0(r)&=\frac{1}{2\pi}\int_0^{2\pi} f_1(r,\theta)\mathrm{d}\theta\\&=\sum_{i=0}^{n} r^i\sum_{j=0}^{i+1}\omega_{i+1-j,j}\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos^{i+1-j}\theta\sin^j\theta}{(r^2\cos^2\theta+1)^m}\mathrm{d}\theta,\end{aligned}$$

where $\omega_{i,j}=p_{i-1,j}+q_{i,j-1}$, and $1\le i+j\le n+1$. Note that all coefficients $\omega_{i,j}$ remain arbitrary According to Lemma 2.1, every simple zero of the averaged function $f^0(r)$ provides a limit cycle of system (1.3). Note that in $(0,+\infty)$ the zeros of the function $f^0(r)$ coincide with the zeros of the function $rf^0(r)$. Therefore, in order to simplify further computation we consider the function $F^0(r)=rf^0(r)$.

Denote by

$$I_{i,j}(r)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos^i\theta\sin^j\theta}{(r^2\cos^2\theta+1)^m}\mathrm{d}\theta, \tag{2.7}$$

then we have

$$I_{i,2j+1}(r)=I_{2i+1,2j}(r)=0. \tag{2.8}$$

By induction, we can obtain

$$\begin{aligned}F^0(r)&=\sum_{i=1}^{n+1} r^i\sum_{j=0}^{i}\omega_{i-j,j}I_{i-j,j}(r)=\sum_{i=1}^{n+1} r^i\sum_{j=0}^{[\frac{i}{2}]}\omega_{i-2j,2j}I_{i-2j,2j}(r)\\&=\sum_{i=1}^{n+1} r^i\sum_{j=0}^{[\frac{i}{2}]}\omega_{i-2j,2j}\left(\sum_{k=0}^{j}\binom{j}{k}(-1)^k I_{i+2k-2j,0}(r)\right)\\&=\sum_{i=0}^{n+1} r^i I_{i,0}(r)P_i^{[\frac{n+1-i}{2}]}(r^2)\\&=\sum_{i=0}^{[\frac{n+1}{2}]} r^{2i}I_{2i,0}(r)P_{2i}^{[\frac{n+1-2i}{2}]}(r^2)\end{aligned} \tag{2.9}$$

where $P_i^{[\frac{n+1-i}{2}]}(r^2)=\sum\limits_{j=0}^{[\frac{n+1-i}{2}]}c_{i,j}r^{2j}$ with $c_{i,j}=\sum\limits_{k=0}^{[\frac{i}{2}]}(-1)^k\binom{j+k}{k}\omega_{i-2k,2j+2k}$. Note that $c_{i,j}$ $((i,j)\neq(0,0))$ are independent coefficients with $c_{0,0}=\omega_{0,0}=0$.

Let

$$J_k^1(r^2)=\frac{1}{2\pi}\int_0^{2\pi}(r^2\cos^2\theta+1)^k\mathrm{d}\theta,\quad k\ge 0 \tag{2.10}$$

$$J_k^2(r^2)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{(r^2\cos^2\theta+1)^k}\mathrm{d}\theta,\quad k\ge 1 \tag{2.11}$$

then we have the following results.

Lemma 2.3. *For $i \geq 0$, let $\rho(i) = \frac{(2i-1)!!}{(2i)!!}$ and $\rho(0) \triangleq 1$, then*

$$J_k^1(r^2) = \sum_{i=0}^{k} \binom{k}{i} \rho(i) r^{2i}, \quad k \geq 0, \tag{2.12}$$

$$J_k^2(r^2) = \frac{\sum\limits_{i=0}^{k-1} \binom{k-1}{i} \rho(i) r^{2i}}{(r^2+1)^{\frac{2k-1}{2}}}, \quad k \geq 1. \tag{2.13}$$

Proof. The statement (2.12) follows from the direct calculation. We will prove the conclusion (2.13) by induction. For $k = 1$,

$$J_1^2(r^2) = \frac{1}{\sqrt{r^2+1}}.$$

By induction, the formula (2.13) holds for $k = l$, that is

$$J_l^2(r^2) = \frac{\sum\limits_{i=0}^{l-1} \binom{l-1}{i} \rho(i) r^{2i}}{(r^2+1)^{\frac{2l-1}{2}}}. \tag{2.14}$$

Denote by $R = r^2$, differentiating (2.14) both sides with respect to R, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}R} J_l^2(R) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{-l\cos^2\theta}{(R\cos^2\theta + 1)^{l+1}} \mathrm{d}\theta \\
&= \frac{1}{(R+1)^{\frac{2l+1}{2}}} \left[\sum_{i=0}^{l-1} \binom{l-1}{i} i\rho(i) R^i + \sum_{i=0}^{l-1} \binom{l-1}{i} i\rho(i) R^{l-1} \right. \\
&\quad \left. - \frac{2l-1}{2} \sum_{i=0}^{l-1} \binom{l-1}{i} \rho(i) R^i \right].
\end{aligned}$$

Multiply $-\frac{R}{l}$ on both sides of the above equation, we get

$$\begin{aligned}
J_l^2(R) - J_{l+1}^2(R) =& \frac{1}{(R+1)^{\frac{2l+1}{2}}} \left[-\frac{1}{l} \sum_{i=0}^{l-1} \binom{l-1}{i} i\rho(i) R^{i+1} - \frac{1}{l} \sum_{i=0}^{l-1} \binom{l-1}{i} i\rho(i) R^i \right. \\
& \left. + \frac{2l-1}{2l} \sum_{i=0}^{l-1} \binom{l-1}{i} \rho(i) R^{i+1} \right].
\end{aligned}$$

Hence, from (2.14), we can obtain

$$\begin{aligned}
J_{l+1}^2(R) =& \frac{1}{(R+1)^{\frac{2l+1}{2}}} \left[\sum_{i=0}^{l-1} \binom{l-1}{i} \rho(i) R^{i+1} + \sum_{i=0}^{l-1} \binom{l-1}{i} \rho(i) R^i + \frac{1}{l} \sum_{i=0}^{l-1} \binom{l-1}{i} i\rho(i) R^{i+1} \right. \\
& \left. + \frac{1}{l} \sum_{i=0}^{l-1} \binom{l-1}{i} i\rho(i) R^i - \frac{2l-1}{2l} \sum_{i=0}^{l-1} \binom{l-1}{i} \rho(i) R^{i+1} \right] \\
=& \frac{1}{(R+1)^{\frac{2l+1}{2}}} \left[\sum_{i=1}^{l} \binom{l-1}{i-1} \rho(i-1) R^i + \sum_{i=0}^{l-1} \binom{l-1}{i} \rho(i) R^i \right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{l}\sum_{i=1}^{l}\binom{l-1}{i-1}(i-1)\rho(i-1)R^i+\frac{1}{l}\sum_{i=1}^{l-1}\binom{l-1}{i}i\rho(i)R^i\\
&-\frac{2l-1}{2l}\sum_{i=1}^{l}\binom{l-1}{i-1}\rho(i-1)R^i\Bigg]\\
=&\frac{1}{(R+1)^{\frac{2l+1}{2}}}\left(R^0+\sum_{i=1}^{l-1}\sigma(i)R^i+\sigma(l)R^l\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma(i)=&\binom{l-1}{i-1}\rho(i-1)+\binom{l-1}{i}\rho(i)+\frac{1}{l}\binom{l-1}{i-1}(i-1)\rho(i-1)\\
&+\frac{1}{l}\binom{l-1}{i}i\rho(i)-\frac{2l-1}{2l}\binom{l-1}{i-1}\rho(i-1)\\
=&\binom{l}{i}\rho(i),\\
\sigma(l)=&\binom{l-1}{l-1}\rho(l-1)+\frac{1}{l}\binom{l-1}{l-1}(l-1)\rho(l-1)-\frac{2l-1}{2l}\binom{l-1}{l}\rho(l-1)=\rho(l),
\end{aligned}
$$

which implys (2.13). □

Then using Lemma 2.2, we divide the computation of the expression of $F^0(r)$ into two cases.

Case 1. $[\frac{n+1}{2}]<m$

For $i<m$, we have

$$
r^{2i}I_{2i,0}(r)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(r^2\cos^2\theta+1-1)^i}{(r^2\cos^2\theta+1)^m}\mathrm{d}\theta=\sum_{k=0}^{i}\binom{i}{k}(-1)^{i-k}J^2_{m-k}(r^2). \quad (2.15)
$$

Then, by (2.9) and (2.13), we can obtain that

$$
\begin{aligned}
F^0(r)=&\sum_{k=0}^{[\frac{n+1}{2}]}J^2_{m-k}(r^2)\sum_{i=k}^{[\frac{n+1}{2}]}P_{2i}^{[\frac{n+1-2i}{2}]}(r^2)\binom{i}{k}(-1)^{i-k}\\
=&\sum_{k=0}^{[\frac{n+1}{2}]}\frac{1}{(r^2+1)^{\frac{2m-2k-1}{2}}}\sum_{i=k}^{[\frac{n+1}{2}]}\left[\left(\binom{i}{k}(-1)^{i-k}\sum_{j=0}^{m-k-1}\binom{m-k-1}{j}\rho(j)r^{2j}\right)\right.\\
&\left.\times\left(\sum_{l=0}^{[\frac{n+1-2i}{2}]}c_{2i,l}r^{2l}\right)\right]\\
=&\sum_{k=0}^{[\frac{n+1}{2}]}\frac{1}{(r^2+1)^{\frac{2m-2k-1}{2}}}\sum_{i=k}^{[\frac{n+1}{2}]}\binom{i}{k}(-1)^{i-k}\sum_{s=0}^{m-k-1+[\frac{n+1-2i}{2}]}d_{i,s}r^{2s}\\
=&\sum_{k=0}^{[\frac{n+1}{2}]}\frac{1}{(r^2+1)^{\frac{2m-2k-1}{2}}}\sum_{j=0}^{m-k-1+[\frac{n+1-2k}{2}]}e_{k,j}r^{2j}\\
=&\frac{1}{(r^2+1)^{\frac{2m-1}{2}}}\sum_{k=0}^{[\frac{n+1}{2}]}(r^2+1)^k\sum_{j=0}^{m-k-1+[\frac{n+1-2k}{2}]}e_{k,j}r^{2j}
\end{aligned}
$$

$$=\frac{1}{(r^2+1)^{\frac{2m-1}{2}}}\sum_{k=0}^{[\frac{n+1}{2}]}\sum_{l=0}^{[\frac{n+1-2k}{2}]+m-1} s_{k,l}r^{2l}$$
$$=\frac{1}{(r^2+1)^{\frac{2m-1}{2}}}\sum_{i=0}^{[\frac{n+1}{2}]+m-1} t_i r^{2i}$$

where

$$d_{i,s}=\sum_{j+l=s}\binom{m-k-1}{j}\rho(j)c_{2i,l},\quad j\le m-k-1, l\le[\frac{n+1-2i}{2}],$$
$$e_{k,j}=\sum_{i=k}^{[\frac{n+1}{2}]}\binom{i}{k}(-1)^{i-k}d_{i,j},\quad if\ j>m-k-1+[\frac{n+1-2i}{2}],\ d_{i,j}=0,$$
$$s_{k,l}=\sum_{i+j=l}\binom{k}{i}e_{k,j},\quad i\le k, j\le m-k-1+[\frac{n+1-2k}{2}],$$
$$t_i=\sum_{k=0}^{[\frac{n+1}{2}]}s_{k,i},\quad if\ i>[\frac{n+1-2k}{2}]+m-1,\ s_{k,i}=0.$$

Case 2. $[\frac{n+1}{2}]\ge m$

If $i>m$, similar to (2.15), we have

$$r^{2i}I_{2i,0}(r)=\sum_{k=0}^{m-1}\binom{i}{k}(-1)^{i-k}J^2_{m-k}(r^2)+\sum_{k=m}^{i}\binom{i}{k}(-1)^{i-k}J^1_{k-m}(r^2). \qquad (2.16)$$

Hence, by (2.9) and Lemma 2.2, we can obtain that

$$\begin{aligned}M^0(r)=&\sum_{k=0}^{m-1}J^2_{m-k}(r^2)\sum_{i=k}^{[\frac{n+1}{2}]}P_{2i}^{[\frac{n+1-2i}{2}]}(r^2)\binom{i}{k}(-1)^{i-k}\\&+\sum_{k=m}^{[\frac{n+1}{2}]}J^1_{k-m}(r^2)\sum_{i=k}^{[\frac{n+1}{2}]}P_{2i}^{[\frac{n+1-2i}{2}]}(r^2)\binom{i}{k}(-1)^{i-k}\\&\triangleq M^{01}(r)+M^{02}(r).\end{aligned}$$

From Case 1, we have

$$M^{01}(r)=\frac{1}{(r^2+1)^{\frac{2m-1}{2}}}\sum_{i=0}^{[\frac{n+1}{2}]+m-1}\tilde{t}_i r^{2i},$$

where $\tilde{t}_i$ are real constants which can be obtain similar to Case 1.

By (2.12), we can get that

$$M^{02}(r)=\sum_{k=m}^{[\frac{n+1}{2}]}\sum_{i=k}^{[\frac{n+1}{2}]}\left(\sum_{j=0}^{k-m}\binom{k-m}{j}\rho(j)r^{2j}\right)\left(\sum_{l=0}^{[\frac{n+1-2i}{2}]}c_{2i,l}r^{2l}\right)\binom{i}{k}(-1)^{i-k}$$

$$
\begin{aligned}
&= \sum_{k=m}^{[\frac{n+1}{2}]} \sum_{i=k}^{[\frac{n+1}{2}]} \binom{i}{k} (-1)^{i-k} \sum_{s=0}^{k-m+[\frac{n+1-2i}{2}]} D_{i,s} r^{2s} \\
&= \sum_{k=m}^{[\frac{n+1}{2}]} \sum_{j=0}^{k-m+[\frac{n+1-2k}{2}]} E_{k,j} r^{2j} \\
&= \sum_{k=m}^{[\frac{n+1}{2}]} \sum_{j=0}^{[\frac{n+1}{2}]-m} E_{k,j} r^{2j} \\
&= \sum_{i=0}^{[\frac{n+1}{2}]-m} T_i r^{2i}
\end{aligned}
$$

where

$$
\begin{aligned}
D_{i,s} &= \sum_{j+l=s} \binom{k-m}{j} \rho(j) c_{2i,l}, \quad j \leq k-m, l \leq [\frac{n+1-2i}{2}], \\
E_{k,j} &= \sum_{i=k}^{[\frac{n+1}{2}]} \binom{i}{k} (-1)^{i-k} D_{i,j}, \quad if\ j > k-m+[\frac{n+1-2i}{2}],\ D_{i,j} = 0, \\
T_i &= \sum_{k=m}^{[\frac{n+1}{2}]-i} E_{k,i}.
\end{aligned}
$$

Therefore,

$$
M^0(r) = \sum_{i=0}^{[\frac{n+1}{2}]-m} T_i r^{2i} + \frac{1}{(r^2+1)^{\frac{2m-1}{2}}} \sum_{i=0}^{[\frac{n+1}{2}]+m-1} \tilde{t}_i r^{2i}.
$$

From the above analysis, we have the following results.

Lemma 2.4. *If $[\frac{n+1}{2}] < m$, then*

$$
M^0(r) = \frac{P^{[\frac{n+1}{2}]+m-1}(r^2)}{(r^2+1)^{\frac{2m-1}{2}}}, \tag{2.17}
$$

if $[\frac{n+1}{2}] \geq m$, then

$$
M^0(r) = P^{[\frac{n+1}{2}]-m}(r^2) + \frac{P^{[\frac{n+1}{2}]+m-1}(r^2)}{(r^2+1)^{\frac{2m-1}{2}}}, \tag{2.18}
$$

where $P^k(r^2)$ are polynomials in variable r^2 with degree k.

3. Proof of main results

Based on the expression of averaged function obtained in Section 2, we give the proof of our main results. In the following discussion, we denote by $\#\{\varphi(r) = 0, r \in (a,b)\}$ the number of isolated zeros of $\varphi(r)$ on (a,b) (taking into account the multiplicity).

Lemma 3.1 ([8]). *Consider a function of the form*

$$F(x) = P^{n_0}(x) + \sum_{j=1}^{K} P^{n_j}(x)\frac{1}{\sqrt{x+c_j}},$$

where $P^{n_j}(x)$ are polynomials in variable x with degree n_j and c_j, $j = 1, \ldots, K$, are real constants. Then its number of real zeros, taking into account their multiplicities, $\#\{F(x) = 0, x \in (\max\{-c_j\}, +\infty)\}$, satisfies

$$\#\{F(x) = 0, x \in (\max\{-c_j\}, +\infty)\} \leq K\left(\max_{j=1,\ldots,K}(n_j) + 1\right) + n_0.$$

Here, if $P^{n_0}(x) = 0$, then $n_0 = -1$.

Lemma 3.2 ([12]). *Consider $p+1$ linearly independent analytical functions $f_i : U \to \mathbb{R}$, $i = 0, 1, \ldots, p$, where $U \in \mathbb{R}$ is an interval. Suppose that there exists $j \in \{0, 1, \ldots, p\}$ such that f_j has constant sign. Then there exist $p+1$ constants C_i, $i = 0, \ldots, p$ such that $f(x) = \sum_{i=0}^{p} C_i f_i(x)$ has at least p simple zeros in U.*

Proof of Theorem 1.1. From Lemma 2.4 and Lemma 3.1, if $[\frac{n+1}{2}] < m$, we have

$$\#\{F^0(r) = 0, r \in [0, +\infty)\} \leq \left[\frac{n+1}{2}\right] + m - 1,$$

if $[\frac{n+1}{2}] \geq m$, we have

$$\#\{F^0(r) = 0, r \in [0, +\infty)\} \leq 2\left[\frac{n+1}{2}\right].$$

Since $F^0(0) = 0$, by Lemma 2.2, we finish the proof. □

Proof of Corollary 1.1. When $m = 2,\ n = 5$, from the Case 2 of Section 2, we can get that

$$F^0(r) = \lambda_1\left(1 - \frac{1}{(r^2+1)^{\frac{3}{2}}}\right) + \lambda_2 r^2 + \frac{1}{(r^2+1)^{\frac{3}{2}}}\left(\lambda_3 r^2 + \lambda_4 r^4 + \lambda_5 r^6 + \lambda_6 r^8\right),$$

where

$$\begin{cases} \lambda_1 = c_{4,0} - 2c_{6,0}, \\ \lambda_2 = c_{4,1} + \frac{1}{2}c_{6,0}, \\ \lambda_3 = c_{0,1} + \frac{1}{2}c_{2,0} - \frac{3}{2}c_{4,0} + 3c_{6,0}, \\ \lambda_4 = \frac{1}{2}c_{0,1} + \frac{1}{2}c_{2,1} - \frac{3}{2}c_{4,1} + c_{0,2}, \\ \lambda_5 = \frac{1}{2}c_{0,2} + \frac{1}{2}c_{2,2} + c_{0,3}, \\ \lambda_6 = \frac{1}{2}c_{0,3}. \end{cases}$$

Since $c_{i,j}$ are arbitrary real constants and

$$\det\frac{\partial(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)}{\partial(c_{4,0}, c_{6,0}, c_{0,1}, c_{4,1}, c_{2,2}, c_{0,3})} = -\frac{3}{16} \neq 0,$$

we have that λ_i $(i = 1, \ldots, 6)$ are arbitrary real constants. By Lemma 3.2, noting that r^2 does't change sign in $r \in (0, +\infty)$, we get that $M^0(r)$ can have at least 5 zeros in $(0, +\infty)$. In addition, from Theorem 1.1, $M^0(r)$ has at most 5 zeros in $(0, +\infty)$. By Lemma 2.1 and Lemma2.2, the conclusion is proved. □

4. Conjecture

From Section 2, noting that $c_{i,j}$ $((i,j) \neq 0)$ are arbitrary constants, we can obtain that the coefficients of $F^0(r)$ have $\frac{\left([\frac{n+1}{2}]+1\right)\left([\frac{n+1}{2}]+2\right)}{2} - 1 = \frac{1}{2}[\frac{n+1}{2}]\left([\frac{n+1}{2}]+3\right)$ arbitrary constants. From our conclusion, it seems to have the following results.

Conjecture. (i) If $[\frac{n+1}{2}] < m$ and $\frac{1}{2}[\frac{n+1}{2}]([\frac{n+1}{2}]+1) \geq m-1$, then $H(n) = [\frac{n+1}{2}] + m - 2$.
(ii) If $[\frac{n+1}{2}] < m$ and $\frac{1}{2}[\frac{n+1}{2}]([\frac{n+1}{2}]+1) < m-1$, then

$$\frac{1}{2}\left[\frac{n+1}{2}\right]\left(\left[\frac{n+1}{2}\right]+3\right) - 1 \leq H(n) \leq \left[\frac{n+1}{2}\right] + m - 2.$$

(iii) If $[\frac{n+1}{2}] \geq m$, then $H(n) = 2[\frac{n+1}{2}] - 1$.

Unfortunately, due to the complexity of the coefficients of $F^0(r)$, we can't prove that they are independent. So we can't give the lower bounds for general n degree perturbation.

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