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PSEUDO ALMOST PERIODIC IN DISTRIBUTION SOLUTIONS AND OPTIMAL SOLUTIONS TO IMPULSIVE PARTIAL STOCHASTIC DIFFERENTIAL EQUATIONS WITH INFINITE DELAY*

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Abstract In this paper, we study a general class of impulsive partial stochastic differential equations with infinite delay and pseudo almost periodic coefficients in Hilbert spaces. Firstly, a more appropriate concept of pseudo almost periodic in distribution for stochastic processes of infinite class is introduced. Secondly, the existence of pseudo almost periodic in distribution mild solutions is investigated by utilizing the interpolation theory, the stochastic analysis techniques and fixed point theorem. The existence of optimal mild solutions of the systems is also proved. Finally, an example is provided to show the effectiveness of the theoretical results.

Keywords Impulsive partial stochastic differential equations, pseudo almost periodic in distribution solutions, optimal solutions, infinite delay, fixed point.

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1. Introduction

The concept of the pseudo almost periodicity, which is a natural generalization of almost periodicity and finds its application in various fields. For instance, the existence of pseudo almost periodic solutions are among the most attractive topics in qualitative theory of differential equations with finite and infinite delay (see [7, 9, 11, 29]). On the other hand, it should be pointed out that there has been an intense interest in studying several extensions of this concept such as pseudo almost periodic stochastic processes. The study of the existence of pseudo almost periodic solutions is one of the most attractive topics in the qualitative theory of stochastic differential equations in Hilbert spaces due to both its mathematical interest and the applications in physics; see [2, 3, 8, 23, 28] and the references therein.

The theory of impulsive differential equations has been an object of increasing interest because of its wide applicability in biology, medicine, and in many other fields [22]. Therefore, it seems interesting to study the existence and stability of pseudo almost periodic solutions to abstract impulsive differential equations (see [5, 16, 25]). However, besides impulse effects and delays, stochastic effects like-

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wise exist in real systems. In recent years, several interesting results on impulsive partial stochastic systems have been reported in [18, 20, 21, 27] and the references therein. Further, the authors in [26] established the existence and exponential stability of p-mean pseudo almost periodic solutions for impulsive nonautonomous partial stochastic evolution equations. However, as indicated in [?, 14, 19], it appears that almost periodicity in distribution sense is a more appropriate concept relatively to solutions of stochastic differential equations. Recently, the authors in [1, 24] studied pseudo almost automorphic and pseudo almost periodic in distribution solutions of stochastic differential equations in a Hilbert space.

In this paper, we study the existence of pseudo almost periodic in distribution mild solutions and optimal mild solutions to the following impulsive partial stochastic differential equations with infinite delay:

$$d[x(t) - h(t, x_t)] = [Ax(t) + g(t, x_t)]dt + f(t, x_t)dW(t),\ t \in \mathbb{R}, t \neq t_i, i \in \mathbb{Z}, \quad (1.1)$$

$$\Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i(x(t_i)), \quad i \in \mathbb{Z}, \quad (1.2)$$

where A is the infinitesimal generator of an uniformly exponentially stable semigroup of linear operators on a Hilbert space $L^p(\mathbb{P}, \mathbb{H})$; the coefficients $h, g : \mathbb{R} \times \mathcal{B} \to L^p(\mathbb{P}, \mathbb{H})$ and $f : \mathbb{R} \times \mathcal{B} \to L^p(\mathbb{P}, L_2^0)$ are appropriate functions, $\mathcal{B}$ is a abstract phase space defined in the next section. Also, the history $x_t : (-\infty, 0] \to L^p(\mathbb{P}, \mathbb{H})$, defined by $x_t(\theta) = x(t+\theta)$ for each $\theta \in (-\infty, 0]$. $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$, where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \leq t\}$. The functions $I_i, i \in \mathbb{Z}$, satisfy suitable conditions which will be established later. The notations $x(t_i^+), x(t_i^-)$ represent the right-hand side and the left-hand side limits of $x(\cdot)$ at t_i, respectively.

To the best of our knowledge, the existence of pseudo almost periodic in distribution mild solutions and optimal mild solutions to impulsive partial stochastic functional differential equations, especially, impulsive partial neutral differential equations with infinite delay, is an untreated topic and this is the main motivation of the present paper. In this work, we will introduce the concept of piecewise pseudo almost periodic in distribution for stochastic processes, which, in turn generalizes all the above-mentioned concepts. Then, we study and obtain the existence of pseudo almost periodic in distribution mild solutions to for nonlinear impulsive stochastic system by using the interpolation theory, the stochastic analysis techniques and the Krasnoselskii-Schaefer type fixed point theorem. Moreover, we investigate the existence of optimal mild solutions of the system with infinite delay and pseudo almost periodic coefficients. The known results appeared in [?, 14, 19] are generalized to the impulsive stochastic evolution equations settings and the case of infinite delay conditions.

The paper is organized as follows. In Section 2, we introduce some notations and necessary preliminaries. In Section 3, we give the existence of pseudo almost periodic in distribution mild solutions. In Section 4, we the existence of optimal mild solutions is proved. Finally, an example is given to illustrate our results in Section 5.

2. Preliminaries

Throughout the paper, $\mathbb{N}, \mathbb{Z}$ and $\mathbb{R}$ stand for the set of natural numbers, integers, real numbers, respectively. We assume that $(\mathbb{H}, \|\cdot\|), (\mathbb{K}, \|\cdot\|)$ are real separable

Hilbert spaces and $(\Omega, \mathcal{F}, \mathbb{P})$ is supposed to be a filtered complete probability space. The notation $L^p(\mathbb{P}, \mathbb{H})$, for $p \geq 1$ stands for the space of all $\mathbb{H}$-valued random variables x such that $E\|x\|^p = \int_\Omega \|x\|^p d\mathbb{P} < \infty$. Then $L^p(\mathbb{P}, \mathbb{H})$ is a Hilbert space when it is equipped with its natural norm $\|\cdot\|_p$ defined by $\|x\|_p = (\int_\Omega E\|x\|^p d\mathbb{P})^{1/p} < \infty$ for each $x \in L^p(\mathbb{P}, \mathbb{H})$. Let $C(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})), BC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ stand for the collection of all continuous functions from $\mathbb{R}$ into $L^p(\mathbb{P}, \mathbb{H})$, the Banach space of all bounded continuous functions from $\mathbb{R}$ into $L^p(\mathbb{P}, \mathbb{H})$, equipped with the sup norm, respectively. We let $L(\mathbb{K}, \mathbb{H})$ be the space of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$, equipped with the usual operator norm $\|\cdot\|_{L(\mathbb{K},\mathbb{H})}$; in particular, this is simply denoted by $L(\mathbb{H})$ when $\mathbb{K} = \mathbb{H}$. $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ with covariance operator Q, that is $E\langle W(t), x\rangle_\mathbb{K}\langle W(s), y\rangle_\mathbb{K} = (t \wedge s)\langle Qx, y\rangle_\mathbb{K}$, for all $x, y \in \mathbb{K}$, where Q is a positive, self-adjoint, trace class operator on $\mathbb{K}$. Furthermore, $L_2^0(\mathbb{K}, \mathbb{H})$ denotes the space of all Q-Hilbert-Schmidt operators from $\mathbb{K}$ to $\mathbb{H}$ with the norm $\|\psi\|^2_{L_2^0} = \mathrm{Tr}(\psi Q \psi^*) < \infty$ for any $\psi \in L(\mathbb{K}, \mathbb{H})$.

Definition 2.1 ([2]). A stochastic process $x : \mathbb{R} \to L^p(\mathbb{P}, \mathbb{H})$ is said to be bounded if there exists a constant $M_0 > 0$ such that

$$E\|x(t)\|^p \leq M_0, \quad t \in \mathbb{R}.$$

Definition 2.2 ([2]). A stochastic process $x : \mathbb{R} \to L^p(\mathbb{P}, \mathbb{H})$ is said to be continuous provided that for any $s \in \mathbb{R}$,

$$\lim_{t\to s} E \parallel x(t) - x(s) \parallel^p = 0.$$

Let $\mathbb{T}$ be the set consisting of all real sequences $\{t_i\}_{i\in\mathbb{Z}}$ such that $\varsigma = \inf_{i\in\mathbb{Z}}(t_{i+1} - t_i) > 0$, $\lim_{i\to\infty} t_i = \infty$ and $\lim_{i\to-\infty} t_i = -\infty$. For $\{t_i\}_{i\in\mathbb{Z}} \in \mathbb{T}$, let $PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ be the space consisting of all bounded piecewise continuous processes $f : \mathbb{R} \to L^p(\mathbb{P}, \mathbb{H})$ such that $f(\cdot)$ is continuous at t for any $t \notin \{t_i\}_{i\in\mathbb{Z}}$ and $f(t_i) = f(t_i^-)$ for all $i \in \mathbb{Z}$; let $PC(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}), L^p(\mathbb{P}, \mathbb{H}))$ be the space formed by all piecewise continuous processes $f : \mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}) \to L^p(\mathbb{P}, \mathbb{H})$ such that for any $x \in L^p(\mathbb{P}, \mathbb{K})$, $f(\cdot, x) \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ and for any $t \in \mathbb{R}$, $f(t, \cdot)$ is continuous at $x \in L^p(\mathbb{P}, \mathbb{K})$.

Definition 2.3 ([2]). A stochastic process $f \in C(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ is said to be p-mean almost periodic if for every sequence of real numbers $\{s'_n\}$, there exist a subsequence $\{s_n\}$ and a process $\tilde{f} \in C(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ such that

$$\lim_{n\to\infty} E \parallel f(t + s_n) - \tilde{f}(t) \parallel^p = 0$$

for all $t \in R$. Denote by $AP(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ the set of such processes.

Definition 2.4 ([22]). A sequence $\{x_k\}$ is called p-mean almost periodic if for every sequence of integer numbers $\{\alpha'_n\}$, there exist a subsequence $\{\alpha_n\}$ and a sequence $\{\tilde{x}_k\}$ such that

$$\lim_{n\to\infty} E \parallel x_{k+\alpha_n} - \tilde{x}_k \parallel^p = 0$$

holds for all $k \in \mathbb{N}$. Denote by $AP(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}))$ the set of such sequences.

Define $l^\infty(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H})) = \{x : \mathbb{Z} \to L^p(\mathbb{P}, \mathbb{H}) : \|x\| = \sup_{k\in\mathbb{Z}}(E\|x(k)\|^p)^{1/p} < \infty\}$, and

$$PAP_0(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H})) = \left\{x \in l^\infty(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H})) : \lim_{k\to\infty} \frac{1}{2k} \sum_{j=-k}^{k} E \parallel x(j) \parallel^p = 0\right\}.$$

Definition 2.5. A sequence $\{x_k\}_{k\in\mathbb{Z}} \in l^\infty(\mathbb{Z}, L^p(\mathbb{P},\mathbb{H}))$ is called p-mean pseudo almost periodic if $x_k = x_k^1 + x_k^2$, where $x_k^1 \in AP(\mathbb{Z}, L^p(\mathbb{P},\mathbb{H})), x_k^2 \in PAP_0(\mathbb{Z}, L^p(\mathbb{P},\mathbb{H}))$. Denote by $PAP(\mathbb{Z}, L^p(\mathbb{P},\mathbb{H}))$ the set of such sequences.

Definition 2.6 (Compare with [22])**.** For $\{t_i\}_{i\in\mathbb{Z}} \in \mathbb{T}$, the stochastic process $f \in PC(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$ is said to be p-mean piecewise almost periodic if the following conditions are fulfilled:

(i) $\{t_i^j = t_{i+j} - t_i\}, j \in \mathbb{Z}$, is equipotentially almost periodic, that is, for every sequence of integer numbers $\{\alpha'_n\}$, there exist a subsequence $\{\alpha_n\}$ and a sequence $\{\tilde{t}_i\}$ such that $\lim_{n\to\infty} |t_{i+\alpha_n} - t_i - \tilde{t}_i| = 0$ for all $i \in \mathbb{Z}$.

(ii) For any $\varepsilon > 0$, there exists a positive number $\tilde{\delta} = \tilde{\delta}(\varepsilon)$ such that if the points t' and t'' belong to a same interval of continuity of f and $|t' - t''| < \tilde{\delta}$, then $E \parallel f(t') - f(t'') \parallel^p < \varepsilon$.

(iii) For every sequence of real numbers $\{s'_n\}$, there exist a subsequence $\{s_n\}$ and a process $\tilde{f} \in C(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$ such that

$$\lim_{n\to\infty} E \parallel f(t+s_n) - \tilde{f}(t) \parallel^p = 0$$

for all $t \in \mathbb{R}$ satisfying the condition $|t - t_i| > \varepsilon$ for any $\varepsilon > 0, i \in \mathbb{Z}$.

We denote by $AP_T(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$ the collection of all the p-mean piecewise almost periodic functions. Obviously, the space $AP_T(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$ endowed with the sup norm defined by $\parallel f \parallel_\infty = (\sup_{t\in\mathbb{R}} E \parallel f(t) \parallel^p)^{1/p}$ for any $f \in AP_T(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$ is a Banach space. Throughout the rest of this paper, we always assume that $\{t_i^j\}$ are equipotentially almost periodic. Let $UPC(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$ be the space of all stochastic functions $f \in PC(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}))$ such that f satisfies the condition (ii) in Definition 2.6

Definition 2.7 (Compare with [22])**.** The stochastic process $f \in PC(\mathbb{R}\times L^p(\mathbb{P},\mathbb{K}), L^p(\mathbb{P},\mathbb{H}))$ is said to be p-mean piecewise almost periodic in $t \in \mathbb{R}$ uniform in $x \in L^p(\mathbb{P},\mathbb{K})$ if for every sequence of real numbers $\{s'_n\}$, there exist a subsequence $\{s_n\}$ and a process $\tilde{f} \in C(\mathbb{R} \times L^p(\mathbb{P},\mathbb{K}), L^p(\mathbb{P},\mathbb{H}))$ such that

$$\lim_{n\to\infty} E \parallel f(t+s_n, x) - \tilde{f}(t,x) \parallel^p = 0$$

for every bounded or compact set $K \subset L^p(\mathbb{P},\mathbb{K}), x \in K$, and $t \in \mathbb{R}$ satisfying $|t - t_i| > \varepsilon$ for any $\varepsilon > 0$, $i \in \mathbb{Z}$. Denote by $AP_T(\mathbb{R} \times L^p(\mathbb{P},\mathbb{K}), L^p(\mathbb{P},\mathbb{H}))$ the set of all such processes.

We need to introduce the new space of functions defined for each $q > 0$ by

$$PC_T^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}), q) = \Big\{ f \in PC(\mathbb{R}, L^p(\mathbb{P},\mathbb{H})) : \lim_{t\to\infty} \Big(\sup_{\theta\in[t-q,t]} E\|f(\theta)\|^p \Big) = 0 \Big\},$$

$$PAP_T^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}), q) = \Big\{ f \in PC(\mathbb{R}, L^p(\mathbb{P},\mathbb{H})) : \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r}\Big(\sup_{\theta\in[t-q,t]} E\|f(\theta)\|^p \Big) dt = 0 \Big\},$$

$$\begin{aligned} &PAP_T^0(\mathbb{R} \times L^p(\mathbb{P},\mathbb{K}), L^p(\mathbb{P},\mathbb{H}), q) \\ =&\Big\{ f \in PC(\mathbb{R} \times L^p(\mathbb{P},\mathbb{K}), L^p(\mathbb{P},\mathbb{H})) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \Big(\sup_{\theta\in[t-q,t]} E \parallel f(\theta, x) \parallel^p \Big) dt = 0 \\ &\quad \text{uniformly with respect to } x \in K, \\ &\quad \text{where } K \text{ is an arbitrary compact subset of } L^p(\mathbb{P},\mathbb{K}) \Big\}, \end{aligned}$$

where in both cases the limit (as $r \to \infty$) is uniform in compact subset of $L^p(\mathbb{P}, \mathbb{K})$.

Now, we recall the definition of fading memory space (phase space) $\mathcal{B}$ axiomatically presented in [10]. Let $\mathcal{B}$ denote the vector space of functions $x_t : (-\infty, 0] \to L^p(\mathbb{P}, \mathbb{H})$ defined as $x_t(s) = x(t+s)$ for $s \in \mathbb{R}^-$, endowed with a seminorm denoted by $\| \cdot \|_{\mathcal{B}}$. A Banach space $(\mathcal{B}, \| \cdot \|_{\mathcal{B}})$ which consists of such functions $\varphi : (-\infty, 0] \to L^p(\mathbb{P}, \mathbb{H})$ is called a fading memory space, if it satisfies the following axioms due to Hale and Kato (see e.g., in [10]).

(A) If $x : (-\infty, \tau + b] \to L^p(\mathbb{P}, \mathbb{H})$ with $b > 0, \tau \in \mathbb{R}$ is continuous on $[\tau, \tau + b]$ and $x_\tau \in \mathcal{B}$, then for every $t \in [\tau, \tau + b]$ the following conditions hold:

   (i) x_t is in $\mathcal{B}$;

   (ii) $\| x(t) \| \leq \tilde{H} \| x_t \|_{\mathcal{B}}$;

   (iii) $\| x_t \|_{\mathcal{B}} \leq \tilde{K}(t-\tau) \sup\{\| x(s) \| : \tau \leq s \leq t\} + \tilde{M}(t-\tau) \| x_\tau \|_{\mathcal{B}}$, where $\tilde{H} \geq 0$ is a constant; $\tilde{K}, \tilde{M} : [0, \infty) \to [1, \infty)$, $\tilde{K}$ is continuous and $\tilde{M}$ is locally bounded, and $\tilde{H}, \tilde{K}, \tilde{M}$ are independent of $x(\cdot)$.

(B) For the function $x(\cdot)$ in (A), x_t is a $\mathcal{B}$-valued function on $\mathbb{R}$.

(C) The space $\mathcal{B}$ is complete.

(D) If $\{\xi^n\}_{n \in N}$ is a sequence of continuous functions with compact support defined from $(-\infty, 0]$ into $L^p(\mathbb{P}, \mathbb{H})$, which converges to ξ uniformly on compact subsets of $(-\infty, 0]$ and if $\{\xi^n\}_{n \in N}$ is a cauchy sequence in $\mathcal{B}$, then $\xi \in \mathcal{B}$ and $\xi^n \to \xi$ in $\mathcal{B}$.

Definition 2.8 ([12]). Let $S(t) : \mathcal{B} \to \mathcal{B}$ be a C_0-semigroup defined by $S(t)\xi(\theta) = \xi(0)$ on $[-t, 0]$ and $S(t)\xi(\theta) = \xi(t + \theta)$ on $[-\infty, -t]$. The phase space $\mathcal{B}$ is called a fading memory space if $\| S(t)\xi \|_{\mathcal{B}} \to 0$ as $t \to \infty$ for each $\xi \in \mathcal{B}$ with $\xi(0) = 0$.

Remark 2.1 ([12]). By axiom (D), there exists a constant $\mathcal{K} > 0$ such that $\| \xi \|_{\mathcal{B}} \leq \mathcal{K} \sup_{\theta \leq 0} \| \xi(\theta) \|$ for every $\xi \in \mathcal{B}$ bounded continuous. Moreover, if $\mathcal{B}$ is a fading memory, we assume that $\max\{\tilde{K}(t), \tilde{M}(t)\} \leq \mathcal{K}_0$ for $t \geq 0$. Further, it should be mentioned that the phase $\mathcal{B}$ is a uniform fading memory space if and only if axiom (D) holds, the function $\tilde{K}(t)$ is bounded and $\lim_{t\to\infty} \tilde{M}(t) = 0$.

The next result is a consequence of the phase space axioms.

Lemma 2.1. *Let $x : \mathbb{R} \to L^p(\mathbb{P}, \mathbb{H})$ be an $\mathcal{F}_t$-adapted measurable process such that for $t \geq \tau$ the $\mathcal{F}_\tau$-adapted process $x_\tau = \varphi \in L_2^0(\Omega, \mathcal{B})$, then*

$$\| x_s \|_{\mathcal{B}} \leq \mathcal{K}_0 [E \| \varphi \|_{\mathcal{B}} + \sup_{s \in \mathbb{R}} E \| x(s) \|].$$

Similar to [7], one has.

Lemma 2.2. *The spaces $PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), q)$ and $PAP_T^0(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}), L^p(\mathbb{P}, \mathbb{H}), q)$ endowed with the uniform convergence topology are Banach spaces.*

Definition 2.9. A function $f \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ is said to be p-mean piecewise pseudo almost periodic if it can be decomposed as $f = f_1 + f_2$, where $f_1 \in AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ and $f_2 \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), q)$. Denoted by $PAP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), q)$ the set of all such functions.

$PAP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), q)$ is a Banach space with the sup norm $\|\cdot\|_\infty$.

Similar to [29], one has

Remark 2.2. (i) $PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), q)$ is a translation invariant set of $PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$. (ii) $PC_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), q) \subset PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), q)$.

Lemma 2.3. *Let $\{f_n\}_{n\in\mathbb{N}} \subset PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), q)$ be a sequence of functions. If f_n converges uniformly to f, then $f \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), q)$.*

One can refer to Lemma 2.5 in [7] for the proof of Lemma 2.3.

Definition 2.10. A function $f \in PC(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}), L^p(\mathbb{P}, \mathbb{H}))$ is said to be p-mean piecewise pseudo almost periodic if it can be decomposed as $f = f_1 + f_2$, where $f_1 \in AP_T(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}), L^p(\mathbb{P}, \mathbb{H}))$ and $f_2 \in PAP_T^0(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}), L^p(\mathbb{P}, \mathbb{H}), q)$.

Denoted by $PAP_T(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{K}), L^p(\mathbb{P}, \mathbb{H}), q)$ the set of all such functions.

Let $\mathcal{P}(\mathbb{H})$ be the space of all Borel probability measures on $\mathbb{H}$ endowed with the metric:

$$d_{BL}(\mu, \nu) := \sup\left\{\left|\int f d\mu - \int f d\nu\right| : \| f \|_{BL} \leq 1\right\}, \quad \mu, \nu \in \mathcal{P}(\mathbb{H}),$$

where f is Lipschitz continuous real-valued function on $\mathbb{H}$ with the norm

$$\| f \|_{BL} = \max\{\| f \|_L, \| f \|_\infty\}, \| f \|_L = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\| x - y \|}, \| f \|_\infty = \sup_{x\in\mathbb{H}} |f(x)|.$$

We denote by $\mathrm{law}(x(t))$ the distribution of the random variable $x(t)$. We say that x has almost periodic in one-dimensional distribution if the mapping $t \to \mathrm{law}(x(t))$ from $\mathbb{R}$ to $(\mathcal{P}(\mathbb{H}), d_{BL})$ is almost periodic.

Definition 2.11 (Compare with [14])**.** For $\{t_i\}_{i\in\mathbb{Z}} \in \mathbb{T}$, the stochastic process $f \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ is said to be piecewise almost periodic in distribution if the following conditions are fulfilled:

(i) $\{t_i^j, j \in \mathbb{Z}\}$ is equipotentially almost periodic.

(ii) $f \in UPC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$.

(iii) The law$\mu(t)$ of $f(t)$ is a $\mathcal{P}(\mathbb{H})$-valued almost periodic mapping, i.e. for every sequence of real numbers $\{s_n'\}$ there exist a subsequence $\{s_n\}$ and a $\mathcal{P}(\mathbb{H})$-valued continuous mapping $\tilde{\mu}(t)$ such that

$$\lim_{n\to\infty} d_{BL}(\mu(t + s_n) - \tilde{\mu}(t)) = 0$$

hold for all $t \in \mathbb{R}$ satisfying the condition $|t - t_i| > \varepsilon$ for any $\varepsilon > 0, i \in \mathbb{Z}$.

Definition 2.12. A stochastic process $f \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ is said to be piecewise pseudo almost periodic in distribution of class q if it can be decomposed as $f = f_1 + f_2$, where f_1 is piecewise almost periodic in distribution and $f_2 \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), q)$.

Now, we introduce some notions and properties about hyperbolic semigroups and intermediate spaces.

Let $\mathbb{H}$ and $\mathbb{Z}$ be Hilbert spaces, with norms $\|\cdot\|$, $\|\cdot\|_{\mathbb{Z}}$ respectively, and suppose that $\mathbb{Z}$ is continuously embedded in $\mathbb{H}$, that is, $\mathbb{Z} \hookrightarrow \mathbb{H}$.

Definition 2.13 ([17]). A semigroup $\{T(t)\}_{t\geq 0}$ is hyperbolic, that is, there exist a projection P and constants $M, \delta > 0$ such that each $T(t)$ commutes with P, Ker$P$ is invariant with respect to $T(t)$, $T(t) : \mathbb{R}(Q) \to \mathbb{R}(Q)$ is invertible and for every $x \in \mathbb{H}$

$$\| T(t)Px \| \leq Me^{-\delta t} \| x \|, \text{ for } t \geq 0;$$

$$\| T(t)Qx \| \leq Me^{\delta t} \| x \|, \text{ for } t \leq 0;$$

where $Q := I - P$ and, for $t < 0, T(t) = T(-t)^{-1}$.

Definition 2.14 ([17]). A linear operator $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ (not necessarily densely defined) is said to be sectorial if the following hold: There exist constants $\omega \in \mathbb{R}, \theta \in (\frac{\pi}{2}, \pi)$, and $M > 0$ such that $\rho(A) \subset S_{\theta,\omega} := \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}$,

$$\| R(\lambda, A) \| \leq \frac{M}{|\lambda - \omega|}, \lambda \in S_{\theta,\omega}.$$

Definition 2.15 ([17]). Let $0 \leq \alpha \leq 1$. A Banach space $\mathbb{Y}$ such that $\mathbb{Z} \hookrightarrow \mathbb{Y} \hookrightarrow \mathbb{H}$ is said to the class J_α between $\mathbb{H}$ and $\mathbb{Z}$ if there is a constant $c > 0$ such that

$$\| x \|_{\mathbb{Y}} \leq c \| x \|^{1-\alpha} \| x \|_{\mathbb{Z}}^{\alpha}, \ x \in \mathbb{Z}.$$

In this case we write $\mathbb{Y} \in J_\alpha((X), \mathbb{Z})$.

Definition 2.16 ([17]). Let $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ be a sectorial operator. A Banach space $(\mathbb{H}_\alpha, \| \cdot \|_\alpha)$,$\alpha \in (0, 1)$, is said to be an intermediate space between $\mathbb{H}$ and $D(A)$ if $\mathbb{H}_\alpha \in J_\alpha(\mathbb{H}, D(A))$.

Lemma 2.4 ([17]). *Let $(T(t))_{t\geq 0}$ be a hyperbolic analytic semigroup on $\mathbb{H}$ with generator A. For $\alpha \in (0, 1)$, let $(\mathbb{H}_\alpha, \| \cdot \|_\alpha)$ be intermediate spaces between $\mathbb{H}$ and $D(A)$. Then there are positive constants $C(\alpha), M(\alpha), \delta$ and γ such that*

$$\| T(t)Px \|_\alpha \leq M(\alpha)t^{-\alpha}e^{-\gamma t} \| x \|, \quad t > 0. \tag{2.1}$$

$$\| T(t)Qx \|_\alpha \leq C(\alpha)e^{\delta t} \| x \|, \quad t \leq 0. \tag{2.2}$$

Lemma 2.5 ([17]). *Let $0 < \alpha, \beta < 1$. Then*

$$\| AT(t)Px \|_\alpha \leq ct^{\beta-\alpha-1}e^{-\gamma t} \| x \|_\beta, \quad t > 0. \tag{2.3}$$

$$\| AT(t)Qx \|_\alpha \leq ce^{\delta t} \| x \|_\beta, \quad t \leq 0. \tag{2.4}$$

Next, we introduce a useful compactness criterion on $PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), q)$.

Let $\tilde{h} : \mathbb{R} \to \mathbb{R}^+$ be a continuous function such that $\tilde{h}(t) \geq 1$ for all $t \in \mathbb{R}$ and $\tilde{h}(t) \to \infty$ as $|t| \to \infty$. Define

$$PC^0_{\tilde{h}}(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), q) = \left\{ f \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})) : \lim_{|t|\to\infty} \left(\sup_{\theta\in[t-q,t]} \frac{E \| f(\theta) \|^p}{\tilde{h}(\theta)} \right) = 0 \right\}$$

endowed with the norm $\| f \|_{\tilde{h}} = \sup_{t\in\mathbb{R}}(\sup_{\theta\in[t-q,t]} \frac{E\|f(\theta)\|^p}{\tilde{h}(\theta)})$, it is a Banach space.

Lemma 2.6. *A set $B \subseteq PC^0_{\tilde{h}}(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), q)$ is relatively compact if and only if it verifies the following conditions:*

(i) $\lim_{|t|\to\infty} (\sup_{\theta\in[t-q,t]} \frac{E\|f(t)\|^p}{\tilde{h}(t)}) = 0$ *uniformly for $f \in B$.*

(ii) $B(t) = \{f(t) : f \in B\}$ *is relatively compact in* $L^p(\mathbb{P}, \mathbb{H})$ *for every* $t \in \mathbb{R}$.

(iii) *The set* B *is equicontinuous on each interval* $(t_i, t_{i+1})(i \in Z)$.

One can refer to Lemma 4.1 in [16] for the proof of Lemma 2.6.

Lemma 2.7 (Krasnoselskii-Schaefer type fixed point theorem [4])**.** *Let* Φ_1, Φ_2 *be two operators such that:*

(a) Φ_1 *is a contraction, and*

(b) Φ_2 *is completely continuous.*

Then either:

(i) *the operator equation* $x = \Phi_1 x + \Phi_2 x$ *has a solution, or*

(ii) *the set* $G = \{x \in \mathbb{H} : \lambda\Phi_1(\frac{x}{\lambda}) + \lambda\Phi_2 x = x\}$ *is unbounded for* $\lambda \in (0, 1)$.

3. Existence of pseudo almost periodic in distribution mild solution

In this section, we investigate the existence of pseudo almost periodic in distribution mild solution for system (1.1)-(1.2). We begin introducing the followings concepts of mild solutions.

Definition 3.1. An $\mathcal{F}_t$ -progressively measurable process $x : \mathbb{R} \to L^p(\mathbb{P}, \mathbb{H}), \sigma > 0$, is called a mild solution of system (1.1)-(1.2), if $x_\sigma = \varphi \in \mathcal{B}$, the function $s \to AT(t-s)h(s, x_s)$ is integrable on $\mathbb{R}$ and for every $t \geq \sigma, \sigma \in \mathbb{R}$ and $\sigma \neq t_i, i \in \mathbb{Z}$,

$$\begin{aligned} x(t) =& T(t-\sigma)[\varphi(\sigma) - h(\sigma, \varphi)] + h(t, x_t) \\ &+ \int_\sigma^t AT(t-s)h(s, x_s)ds + \int_\sigma^t T(t-s)g(s, x_s)ds \\ &+ \int_\sigma^t T(t-s)f(s, x_s)dW(s) + \sum_{\sigma < t_i < t} T(t-t_i)I_i(x(t_i)). \end{aligned} \tag{3.1}$$

Additionally, we make the following hypotheses:

(H1) The operator A is sectorial and generates a hyperbolic semigroup $(T(t))_{t\geq 0}$. Moreover, $T(t)$ is compact for $t > 0$.

(H2) If $0 < \alpha < \beta < 1$, then we let k_1, $k(\alpha)$ be the bound of the embedding $L^p(\mathbb{P}, \mathbb{H}_\beta) \hookrightarrow L^p(\mathbb{P}, \mathbb{H}_\alpha) \hookrightarrow L^p(\mathbb{P}, \mathbb{H})$, that is $E \parallel x \parallel^p \leq k_1 E \parallel x \parallel_\alpha^p$ for $x \in L^p(\mathbb{P}, \mathbb{H}_\alpha)$ and $E \parallel x \parallel_\alpha^p \leq k(\alpha) E \parallel x \parallel_\beta^p$ for $x \in L^p(\mathbb{P}, \mathbb{H}_\beta)$.

(H3) $h = h_1 + h_2 \in PAP_T(\mathbb{R} \times \mathcal{B}, L^p(\mathbb{P}, \mathbb{H}_\beta), q)$, where $h_1 \in AP_T(\mathbb{R} \times \mathcal{B}, L^p(\mathbb{P}, \mathbb{H}_\beta))$ and $h_2 \in PAP_T^0(\mathbb{R} \times \mathcal{B}, L^p(\mathbb{P}, \mathbb{H}_\beta), q)$. $g = g_1 + g_2 \in PAP_T(\mathbb{R} \times \mathcal{B}, L^p(\mathbb{P}, \mathbb{H}), q)$, where $g_1 \in AP_T(\mathbb{R} \times \mathcal{B}, L^p(\mathbb{P}, \mathbb{H}))$ and $g_2 \in PAP_T^0(\mathbb{R} \times \mathcal{B}, L^p(\mathbb{P}, \mathbb{H}), q)$. $f = f_1 + f_2 \in PAP_T(\mathbb{R} \times \mathcal{B}, L^p(\mathbb{P}, L_2^0), q)$, where $f_1 \in AP_T(\mathbb{R} \times \mathcal{B}, L^p(\mathbb{P}, L_2^0))$ and $f_2 \in PAP_T^0(\mathbb{R} \times \mathcal{B}, L^p(\mathbb{P}, L_2^0), q)$. $I_i = I_{i,1} + I_{i,2} \in PAP(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}))$, where $I_{i,1} \in AP(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}))$ and $I_{i,2} \in PAP_0(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}))$.

(H4) Let $0 \leq \alpha < \beta < 1$ and there exist constants $L_h > 0$ such that

$$E \parallel h(t,\psi_1) - h(t,\psi_2) \parallel_\beta^p \leq L_h[|t_1 - t_2| + \parallel \psi_1 - \psi_2 \parallel_{\mathcal{B}}^p], \quad t \in \mathbb{R}, \psi_1, \psi_2 \in \mathcal{B},$$
$$E \parallel h(t,\psi) \parallel_\beta^p \leq L_h(\parallel \psi \parallel_{\mathcal{B}}^p + 1), \quad t \in \mathbb{R}, \psi \in \mathcal{B},$$
$$E \parallel h_1(t,\psi_1) - h_1(t,\psi_2) \parallel_\beta^p \leq L_h \parallel \psi_1 - \psi_2 \parallel_{\mathcal{B}}^p, \quad t \in \mathbb{R}, \psi_1, \psi_2 \in \mathcal{B}.$$

(H5) The functions $g : \mathbb{R} \times \mathcal{B} \to L^p(\mathbb{P}, \mathbb{H})$, $f : \mathbb{R} \times \mathcal{B} \to L^p(\mathbb{P}, L_2^0)$ are continuous with respect to ψ, and there exists a constant $\tilde{\mu}$ such that

$$\limsup_{\|\psi\|^p \to \infty} \left(\sup_{t \in \mathbb{R}} \frac{E \parallel g(t,\psi) \parallel^p + E \parallel f(t,\psi) \parallel_{L_2^0}^p}{\parallel \psi \parallel^p} \right) = \tilde{\mu}, \quad \psi \in \mathcal{B}.$$

(H6) The functions $I_i : L^p(\mathbb{P}, \mathbb{K}) \to L^p(\mathbb{P}, \mathbb{H})$ are continuous with respect to x, and there exist constants c_i such that

$$\limsup_{\|x\|^p \to \infty} \frac{E \parallel I_i(x) \parallel^p}{\parallel x \parallel^p} = c_i$$

for every $x \in L^p(\mathbb{P}, \mathbb{K}), i \in \mathbb{Z}$.

(H7) The functions $g_1(t, \cdot), f_1(t, \cdot)$ are uniformly continuous in each bounded subset of $\mathcal{B}$ uniformly in $t \in \mathbb{R}$, and $I_{i,1}(\cdot)$ are uniformly continuous in $x \in L^p(\mathbb{P}, \mathbb{K})$ uniformly in $i \in \mathbb{Z}$.

Theorem 3.1. *Assume that (H1)-(H7) are satisfied. Then system* (1.1)-(1.2) *has at least one pseudo almost periodic in distribution mild solution on* $\mathbb{R}$*, provided that*

$$\begin{aligned}
&18^{p-1}\mathcal{K}_0^p k_1 \Bigg\{ \Bigg(k(\alpha) + c^p \Bigg(\Gamma(1 + \frac{p(\beta - \alpha - 1)}{p-1}) \Bigg)^{p-1} \gamma^{-p(\beta-\alpha)} + \frac{1}{\delta^p} \Bigg) L_h \\
&+ \Bigg(\Bigg[(M(\alpha))^p \Bigg(\Gamma(1 - \frac{p}{p-1}\alpha)\gamma^{\frac{p}{p-1}\alpha - 1} \Bigg)^{p-1} \frac{1}{\gamma} + (C(\alpha))^p \frac{1}{\delta^p} \Bigg] \\
&+ C_p \Bigg[(M(\alpha))^p \Bigg(\Gamma(1 - \frac{p}{p-2}\alpha)(\frac{p}{p-2}\gamma)^{\frac{p}{p-2}\alpha - 1} \Bigg)^{\frac{p-2}{p}} \frac{2}{p\gamma} \\
&+ (C(\alpha))^p \Bigg(\frac{p-2}{p\delta} \Bigg)^{\frac{p-2}{p}} \frac{2}{p\gamma} \Bigg] \Bigg) \tilde{\mu} \\
&+ \Bigg[(M(\alpha))^p \varsigma^{-p\alpha} \frac{1}{(1 - e^{-\gamma\varsigma})^p} + (C(\alpha))^p \frac{1}{(1 - e^{-\varsigma\delta})^p} \Bigg] \sup_{i \in \mathbb{Z}} c_i \Bigg\} < 1
\end{aligned}$$

for $p > 2$, *and*

$$\begin{aligned}
&18\mathcal{K}_0^2 k_1 \Bigg\{ k(\alpha) + (M(\alpha))^2 \Bigg[(\Gamma(1 + 2(\beta - \alpha - 1)\gamma^{-2(\beta-\alpha-1)-1}) \frac{1}{\gamma} + \frac{1}{\delta^2} \Bigg] L_h \\
&+ [(M(\alpha))^2 (\Gamma(1 - 2\alpha)\gamma^{2\alpha - 2}) + (C(\alpha))^2 \frac{1}{\delta}] \\
&+ [(M(\alpha))^2 (\Gamma(1 - 2\alpha)(2\gamma)^{2\alpha - 1}) + (C(\alpha))^2 \frac{1}{2\delta}] \tilde{\mu} \\
&+ \Bigg[(M(\alpha))^2 \varsigma^{-2\alpha} \frac{1}{(1 - e^{-\gamma\varsigma})^2} + (C(\alpha))^2 \frac{1}{(1 - e^{-\varsigma\delta})^2} \Bigg] \sup_{i \in \mathbb{Z}} c_i \Bigg\} < 1
\end{aligned}$$

for $p = 2$.

Proof. Let $\mathcal{Y} = UPC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$. Consider the operator $\Phi : \mathcal{Y} \to PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}_\alpha))$ defined by

$$\begin{aligned}(\Phi x)(t) = h(t, x_t) &+ \int_{-\infty}^{t} AT(t-s)Ph(s, x_s)ds - \int_{t}^{\infty} AT(t-s)Qh(s, x_s)ds \\ &+ \int_{-\infty}^{t} T(t-s)Pg(s, x_s)ds - \int_{t}^{\infty} T(t-s)Qg(s, x_s)ds \\ &+ \int_{-\infty}^{t} T(t-s)Pf(s, x_s)dW(s) - \int_{t}^{\infty} T(t-s)Qf(s, x_s)dW(s) \\ &+ \sum_{t_i<t} T(t-t_i)PI_i(x(t_i)) - \sum_{t<t_i} T(t-t_i)QI_i(x(t_i)), \quad t \in \mathbb{R}.\end{aligned}$$

It is clear that Φ is a well-defined operator on $\mathcal{Y}$. We show that Φ has a fixed point, which in turn is a mild solution of the system (1.1)-(1.2). To prove which we shall employ Lemma 2.7, we divide the proof into several steps.

Step 1. $\Psi x \in \mathcal{Y}$.

Let $t', t'' \in (t_i, t_{i+1}), i \in \mathbb{Z}, t'' < t'$. By (H1), for any $\varepsilon > 0$, there exists $0 < \xi < \min\{\frac{\varepsilon}{\kappa}, (\frac{\varepsilon}{\kappa})^{1/p(\beta+\alpha)}, (\frac{\varepsilon}{\kappa})^{1/p}, (\frac{\varepsilon}{\kappa})^{p/2(p-1)}, (\frac{\varepsilon}{\kappa})^{p/2(p-1)}\}$, $\kappa = 18\bar{h}_1 + 18\bar{h}_2(1 + \frac{p(\beta-\alpha-1)}{p-1})^{1-p} + 18\bar{h}_2 + 18\bar{f}(1 - \frac{p}{p-2}\alpha)^{\frac{p}{p(2-\alpha)-2}} + 18\bar{f}$, such that $0 < t' - t'' < \xi$ and $\sup_{-\infty \le \theta \le 0} E \parallel x(t' + \theta) - x(t'' + \theta) \parallel^p < \frac{\varepsilon}{18\bar{h}_1}$, we have for $p > 2$,

$$\parallel T(t'-t'') - I \parallel^p \le \min\left\{\frac{\varepsilon}{18\bar{h}_2\bar{\delta}_1}, \frac{\varepsilon}{18\bar{g}\bar{\delta}_2}, \frac{\varepsilon}{18\bar{f}\bar{\delta}_3}, \frac{\bar{\delta}_4\varepsilon}{18\bar{\gamma}_1}\right\},$$

where $\bar{h}_1 = 9^{p-1}k(\alpha)L_h$, $\bar{h}_2 = 18^{p-1}c^p \parallel h \parallel^p_{\beta,\infty}$, $\bar{\delta}_1 = [\Gamma(1 + \frac{p(\beta-\alpha-1)}{p-1})]^{p-1}\gamma^{-\frac{p(\beta-\alpha)}{p-1}} + \frac{1}{\delta^p}$, $\bar{g} = 18^{p-1}[(M(\alpha))^p + (C(\alpha))^p] \parallel g \parallel^p_\infty$, $\bar{\delta}_2 = (\Gamma(1 - \frac{p}{p-1}\alpha))^{p-1}\gamma^{p(\alpha-1)} + \frac{1}{\delta^p}$, $\bar{f} = 18^{p-1}[(M(\alpha))^p + (C(\alpha))^p]C_p \parallel f \parallel^p_\infty$, $\bar{\delta}_3 = (\Gamma(1 - \frac{p\alpha}{p-2})(\frac{p\gamma}{p-2})^{\frac{p\alpha}{p-2}-1}))^{\frac{p-2}{p}}\frac{2}{p\gamma} + (\frac{p-2}{p}\delta)^{\frac{p-2}{p}}\frac{2}{p\delta}$, $\bar{\gamma}_1 = 18^{p-1}[(M(\alpha))^p + (C(\alpha))^p]\sup_{i\in\mathbb{Z}} \parallel I_i \parallel^p_\infty$, $\bar{\delta}_4 = (1 - e^{-\gamma\varsigma})^p + (1 - e^{-\delta\varsigma})^p$. Then we have for $p > 2$,

$$E \parallel (\Phi x)(t') - (\Phi x)(t'') \parallel^p_\alpha \le \sum_{j=1}^{9} \Xi_j,$$

where

$$\Xi_1 = 9^{p-1}E \parallel h(t', x_{t'}) - h(t'', x_{t''}) \parallel^p_\alpha,$$
$$\Xi_2 = 9^{p-1}E\left\| \int_{-\infty}^{t'} AT(t'-s)Ph(s, x_s)ds - \int_{-\infty}^{t''} AT(t''-s)Ph(s, x_s)ds \right\|^p_\alpha,$$
$$\Xi_3 = 9^{p-1}E\left\| \int_{t'}^{\infty} AT(t'-s)Qh(s, x_s)ds - \int_{-\infty}^{t''} AT(t''-s)Qh(s, x_s)ds \right\|^p_\alpha,$$
$$\Xi_4 = 9^{p-1}E\left\| \int_{-\infty}^{t'} T(t'-s)Pg(s, x_s)ds - \int_{-\infty}^{t''} T(t'-s)P(s)g(s, x_s)ds \right\|^p_\alpha,$$
$$\Xi_5 = 9^{p-1}E\left\| \int_{t'}^{\infty} T(t'-s)Qg(s, x_s)ds - \int_{t''}^{\infty} T(t'-s)Q(s)g(s, x_s)ds \right\|^p_\alpha,$$

$$\Xi_6 = 9^{p-1}E\left\| \int_{-\infty}^{t'} T(t'-s)Pf(s,x_s)dW(s) - \int_{-\infty}^{t''} T(t'-s)Pf(s,x_s)dW(s)\right\|_\alpha^p,$$
$$\Xi_7 = 9^{p-1}E\left\| \int_{t'}^{\infty} T(t'-s)Qf(s,x_s)dW(s) - \int_{t''}^{\infty} T(t'-s)Qf(s,x_s)dW(s)\right\|_\alpha^p,$$
$$\Xi_8 = 9^{p-1}E\left\| \sum_{t_i<t'} T(t'-t_i)PI_i(x(t_i)) - \sum_{t_i<t''} T(t''-t_i)PI_i(x(t_i))\right\|_\alpha^p,$$
$$\Xi_9 = 9^{p-1}E\left\| \sum_{t'<t_i} T(t'-t_i)QI_i(x(t_i)) - \sum_{t''<t_i} T(t''-t_i)QI_i(x(t_i))\right\|_\alpha^p.$$

By (H4) and Hölder's inequality, we have

$$\begin{aligned}\Xi_1 &\le 9^{p-1}k(\alpha)E \parallel h(t',x_{t'}) - h(t'',x_{t''}) \parallel_\beta^p \\ &\le 9^{p-1}k(\alpha)L_h[|t'-t''| + \parallel x_{t'} - x_{t''} \parallel_{\mathcal{B}}^p] \\ &\le 9^{p-1}k(\alpha)L_h[|t'-t''| + \sup_{s\in(-\infty,0]} E \parallel x(t'+s) - x(t''+s) \parallel^p] < \frac{\varepsilon}{9},\end{aligned}$$

$$\begin{aligned}\Xi_2 &\le 18^{p-1}E\left\| \int_{-\infty}^{t''} A[T(t'-t'')-I]T(t''-s)Ph(s,x_s)ds\right\|_\alpha^p \\ &\quad + 18^{p-1}E\left\| \int_{t''}^{t'} AT(t'-s)Ph(s,x_s)ds\right\|_\alpha^p \\ &\le 18^{p-1}c^p \parallel T(t'-t'') - I \parallel^p \left(\int_{-\infty}^{t''} (t''-s)^{\frac{p}{p-1}(\beta-\alpha-1)} e^{-\gamma(t''-s)}ds\right)^{p-1} \\ &\quad \times \left(\int_{-\infty}^{t''} e^{-\gamma(t''-s)} E \parallel h(s,x_s) \parallel_\beta^p ds\right) \\ &\quad + 18^{p-1}c^p\left(\int_{t''}^{t'} (t'-s)^{\frac{p}{p-1}(\beta-\alpha-1)} e^{-\gamma(t'-s)}ds\right)^{p-1} \\ &\quad \times \left(\int_{t''}^{t'} e^{-\gamma(t'-s)} E \parallel h(s,x_s) \parallel_\beta^p ds\right) \\ &\le 18^{p-1}c^p \parallel T(t'-t'') - I \parallel^p \left[\Gamma\left(1+\frac{p(\beta-\alpha-1)}{p-1}\right)\right]^{p-1} \gamma^{-\frac{p(\beta-\alpha)}{p-1}} \parallel h \parallel_{\beta,\infty}^p \\ &\quad + 18^{p-1}c^p\left(1+\frac{p(\beta-\alpha-1)}{p-1}\right)^{1-p}(t'-t'')^{p(\beta+\alpha)} \parallel h \parallel_{\beta,\infty}^p < \frac{\varepsilon}{9}.\end{aligned}$$

Similarly, we have

$$\Xi_3 \le 18^{p-1}c^p \parallel T(t'-t'') - I \parallel^p \frac{1}{\delta^p} \parallel h \parallel_{\beta,\infty}^p + 18^{p-1}c^p(t'-t'')^p \parallel h \parallel_{\beta,\infty}^p < \frac{\varepsilon}{9}.$$

By (H5) and Hölder's inequality, we have

$$\begin{aligned}\Xi_4 &\le 18^{p-1}E\left\| \int_{-\infty}^{t''} [T(t'-t'')-I]T(t''-s)Pg(s,x_s)ds\right\|_\alpha^p \\ &\quad + 18^{p-1}E\left\| \int_{t''}^{t'} T(t'-s)Pg(s,x_s)ds\right\|_\alpha^p\end{aligned}$$

$$\leq 18^{p-1}(M(\alpha))^p \parallel T(t'-t'')-I \parallel^p \left(\int_{-\infty}^{t''}(t''-s)^{-\frac{p}{p-1}\alpha}e^{-\gamma(t''-s)}ds\right)^{p-1}$$
$$\times\left(\int_{-\infty}^{t''}e^{-\gamma(t''-s)}E \parallel g(s,x_s)\parallel^p ds\right)$$
$$+18^{p-1}(M(\alpha))^p\left(\int_{t''}^{t'}(t'-s)^{-\frac{p}{p-1}\alpha}e^{-\gamma(t'-s)}ds\right)^{p-1}$$
$$\times\left(\int_{t''}^{t'}e^{-\gamma(t'-s)}E \parallel g(s,x_s)\parallel^p ds\right)$$
$$\leq 18^{p-1}(M(\alpha))^p \parallel T(t'-t'')-I \parallel^p \left(\Gamma(1-\frac{p}{p-1}\alpha)\right)^{p-1}\gamma^{p(\alpha-1)} \parallel g \parallel_\infty^p$$
$$+18^{p-1}(M(\alpha))^p\left(1-\frac{p}{p-1}\alpha\right)^{1-p}(t'-t'')^p \parallel g \parallel_\infty^p < \frac{\varepsilon}{9}.$$

Similarly, we have

$$\Xi_5 \leq 18^{p-1}(C(\alpha))^p\|T(t'-t'')-I\|^p(\frac{1}{\delta})^p\|g\|_\infty^p+18^{p-1}(C(\alpha))^p(t'-t'')^p\|g\|_\infty^p<\frac{\varepsilon}{9}.$$

By (H5) and the Ito integral [13], we have

$$\Xi_6 \leq 18^{p-1}E\left\| \int_{-\infty}^{t''}[T(t'-t'')-I]T(t''-s)Pf(s,x_s)dW(s)\right\|_\alpha^p$$
$$+18^{p-1}E\left\| \int_{t''}^{t'}T(t'-s)Pf(s,x_s)dW(s)\right\|_\alpha^p$$
$$\leq 18^{p-1}(M(\alpha))^pC_pE\left[\int_{-\infty}^{t''}(t''-s)^{-2\alpha}e^{-2\gamma(t''-s)}\|T(t'-t'')-I\|^2\|lf(s,x_s)\|_{L_2^0}^2ds\right]^{p/2}$$
$$+18^{p-1}(M(\alpha))^pC_pE\left[\int_{t''}^{t'}e^{-2\gamma(t'-s)} \parallel f(s,x_s)\parallel_{L_2^0}^2 ds\right]^{p/2}$$
$$\leq 18^{p-1}(M(\alpha))^pC_p \parallel T(t'-t'')-I \parallel^p$$
$$\times\left(\int_{-\infty}^{t''}(t''-s)^{-\frac{p}{p-2}\alpha}e^{-\frac{p}{p-2}\gamma(t''-s)}ds\right)^{\frac{p-2}{p}}\left(\int_{-\infty}^{t''}e^{-\frac{p}{2}\delta(t''-s)}ds\right) \parallel f \parallel_\infty^p$$
$$+18^{p-1}(M(\alpha))^pC_p\left(\int_{t''}^{t'}(t'-s)^{-\frac{p}{p-2}\alpha}e^{-\frac{p}{p-2}\gamma(t'-s)}ds\right)^{\frac{p-2}{p}}$$
$$\times\left(\int_{t''}^{t'}e^{-\frac{p}{2}\gamma(t'-s)}ds\right) \parallel f \parallel_\infty^p$$
$$\leq 18^{p-1}(M(\alpha))^pC_p\|T(t'-t'')-I\|^p\left(\Gamma(1-\frac{p\alpha}{p-2})(\frac{p\gamma}{p-2})^{\frac{p\alpha}{p-2}-1}\right)^{\frac{p-2}{p}}\frac{2}{p\gamma}\|f\|_\infty^p$$
$$+18^{p-1}(M(\alpha))^pC_p(1-\frac{p}{p-2}\alpha)^{\frac{p}{p-2}}(t'-t'')^{\frac{p(2-\alpha)-2}{p}} \parallel f \parallel_\infty^p < \frac{\varepsilon}{9}.$$

Similarly, we have

$$\Xi_7 \leq 18^{p-1}(C(\alpha))^pC_p \parallel T(t'-t'')-I \parallel^p \left(\frac{p-2}{p}\delta\right)^{\frac{p-2}{p}}\frac{2}{p\delta} \parallel f \parallel_\infty^p$$

$$+ 18^{p-1}(C(\alpha))^p C_p(t'-t'')^{\frac{2(p-1)}{p}} \parallel f \parallel_\infty^p < \frac{\varepsilon}{9}.$$

For $p = 2$, let $\varepsilon > 0$, there exists $0 < \xi < \min\{\frac{\varepsilon}{\kappa}, (\frac{\varepsilon}{\kappa})^{1/2(\beta+\alpha)}, (\frac{\varepsilon}{\kappa})^{1/2}\}$, $\kappa = 18\bar{h}_1 + 18\bar{h}_2(1+2(\beta-\alpha-1))^{-1} + 18\bar{h}_2 + 18\bar{f}$, such that $0 < t'-t'' < \xi$ and $\sup_{-\infty\leq\theta\leq 0} E \parallel x(t'+\theta) - x(t''+\theta) \parallel^2 < \frac{\varepsilon}{18\bar{h}_1}$, we have

$$\parallel T(t'-t'') - I \parallel^p \leq \min\left\{\frac{\varepsilon}{18\bar{h}\bar{\delta}_1}, \frac{\varepsilon}{18\bar{g}\bar{\delta}_2}, \frac{\varepsilon}{18\bar{f}\bar{\delta}_3}, \frac{\bar{\delta}_4\varepsilon}{18\bar{\gamma}_1}\right\},$$

where $\bar{h}_1 = 9k(\alpha)L_h$, $\bar{h}_2 = 18c^2 \parallel h_1 \parallel_{\beta,\infty}^2$, $\bar{\delta}_1 = \Gamma(1+2(\beta-\alpha-1)\gamma^{-2(\beta-\alpha)} + \frac{1}{\delta^2}$, $\bar{g} = 18[(M(\alpha))^2 + (C(\alpha))^2] \parallel g \parallel_\infty^2$, $\bar{\delta}_2 = \Gamma(1-2\alpha)\gamma^{2(\alpha-1)} + (\frac{1}{\delta})^2$, $\bar{f} = 18[(M(\alpha))^2 + (C(\alpha))^2] \parallel f \parallel_\infty^2$, $\bar{\delta}_3 = \Gamma(1-2\alpha)(2\gamma)^{2\alpha-1} + \frac{1}{2\delta}$, $\bar{\gamma}_1 = 18[(M(\alpha))^2 + (C(\alpha))^2]\sup_{i\in\mathbb{Z}} \|I_i\|_\infty^2$, $\bar{\delta}_4 = (1-e^{-\gamma\varsigma})^2 + (1-e^{-\delta\varsigma})^2$. Then we have

$$\begin{aligned}
\Xi_6 &\leq 18(M(\alpha))^2 \parallel T(t'-t'') - I \parallel^2 \left(\int_{-\infty}^{t''}(t''-s)^{-2\alpha}e^{-2\gamma(t''-s)}ds\right) \parallel f \parallel_\infty^2 \\
&\quad + 18(M(\alpha))^2\left(\int_{t''}^{t'}(t'-s)^{-2\alpha}ds\right) \parallel f \parallel_\infty^2 \\
&\leq 18(M(\alpha))^2 \parallel T(t'-t'') - I \parallel^2 \Gamma(1-2\alpha)(2\gamma)^{2\alpha-1} \parallel f \parallel_\infty^2 \\
&\quad + 18(M(\alpha))^2 \parallel f \parallel_\infty^2 (t'-t'') < \frac{\varepsilon}{9}.
\end{aligned}$$

Similarly, we have

$$\Xi_7 \leq 18(C(\alpha))^2 \parallel T(t'-t'') - I \parallel^2 \parallel f \parallel_\infty^2 \frac{1}{2\delta} + 18(C(\alpha))^2 \parallel f \parallel_\infty^2 (t'-t'') < \frac{\varepsilon}{9}.$$

By (H7) and Hölder's inequality again, we have

$$\begin{aligned}
\Xi_8 &= 9^{p-1}E\left\| \sum_{t_i<t''}[T(t'-t'') - I]T(t''-t_i)PI_i(x(t_i))\right\|_\alpha^p \\
&\leq 9^{p-1}(M(\alpha))^p \parallel T(t'-t'') - I \parallel^p \left(\sum_{t_i<t''}(t''-t_i)^{-\frac{p}{p-1}\alpha}e^{-\gamma(t''-t_i)}\right)^{p-1} \\
&\quad \times \left(\sum_{t_i<t''}e^{-\gamma(t''-t_i)}E \parallel I_i(x(t_i)) \parallel^p\right) \\
&\leq 9^{p-1}(M(\alpha))^p \parallel T(t'-t'') - I \parallel^p \varsigma^{-p\alpha}\left(\sum_{t_i<t''}e^{-\gamma(t''-t_i)}\right)^p \sup_{i\in\mathbb{Z}} \parallel I_i \parallel_\infty^p \\
&< \frac{\varepsilon}{9}.
\end{aligned}$$

Similarly, we have

$$\Xi_9 \leq 9^{p-1}(C(\alpha))^p \parallel T(t'-t'') - I \parallel^p \left(\sum_{t''<t_i}e^{\delta(t''-t_i)}\right)^p \sup_{i\in\mathbb{Z}} \parallel I_i \parallel_\infty^p < \frac{\varepsilon}{9}.$$

By the above discussion, one has

$$E \parallel (\Phi x)(t') - (\Phi x)(t'') \parallel_\alpha^p < \varepsilon.$$

Consequently, $\Phi x \in \mathcal{Y}$.

Step 2. Φ has a fixed point in $\mathcal{Y}$.

To do this, we decompose Φ as $\Phi_1 + \Phi_2$ where

$$\begin{aligned}(\Phi_1 x)(t) &= \left[h(t,x_t) + \int_{-\infty}^{t} AT(t-s)Ph(s,x_s)ds\right] - \left[\int_{t}^{\infty} AT(t-s)Qh(s,x_s)ds\right]\\ &:= (\Phi_{11}x)(t) + (\Phi_{12}x)(t), \quad t \in \mathbb{R},\end{aligned}$$

$$\begin{aligned}(\Phi_2 x)(t) = &\left[\int_{-\infty}^{t} T(t-s)Pg(s,x_s)ds + \int_{-\infty}^{t} T(t-s)Pf(s,x_s)dW(s)\right.\\ &\left.+ \sum_{t_i<t} T(t-t_i)PI_i(x(t_i))\right] - \left[\int_{t}^{\infty} T(t-s)Qg(s,x_s)ds\right.\\ &\left.+ \int_{t}^{\infty} T(t-s)Qf(s,x_s)dW(s) + \sum_{t<t_i} T(t-t_i)QI_i(x(t_i))\right]\\ := &(\Phi_{21}x)(t) + (\Phi_{22}x)(t), \quad t \in \mathbb{R}.\end{aligned}$$

We will verify that Φ_1 is a contraction while Φ_2 is a completely continuous operator.

(1) Φ_1 is a contraction.

For $t \in \mathbb{R}$, and $x^*, x^{**} \in B_{r^*}$. From (H2) and (H4) and Lemma 2.1, we have

$$\begin{aligned}&E \parallel (\Phi_{11}x^*)(t) - (\Phi_{11}x^{**})(t) \parallel_\alpha^p\\ \le &2^{p-1}E\|h(t,x_t^*)-h(t,x_t^{**})\|_\alpha^p + 2^{p-1}E\left\|\int_{-\infty}^{t} AT(t-s)P[h(s,x_s^*)-h(s,x_s^{**})]ds\right\|_\alpha^p\\ \le &2^{p-1}(k(\alpha))^p L_h \parallel x_t^* - x_t^{**} \parallel_{\mathcal{B}}^p + 2^{p-1}\left(\int_{-\infty}^{t} (t-s)^{\frac{p}{p-1}(\beta-\alpha-1)} e^{-\gamma(t-s)}ds\right)^{p-1}\\ &\times\left(\int_{-\infty}^{t} e^{-\gamma(t-s)} E \parallel h(s,x_s^*) - h(s,x_s^{**}) \parallel_\beta^p ds\right)\\ \le &2^{p-1}k(\alpha)L_h \parallel x_t^* - x_t^{**} \parallel_{\mathcal{B}}^p + 2^{p-1}\left(\Gamma(1+\frac{p}{p-1}(\beta-\alpha-1))\gamma^{-\frac{p}{p-1}(\beta-\alpha-1)-1}\right)^{p-1}\\ &\times\left(\int_{-\infty}^{t} e^{-\gamma(t-s)} L_h \parallel x_s^* - x_s^{**} \parallel_{\mathcal{B}}^p ds\right)\\ \le &2^{p-1}L_h k_1 \mathcal{K}_0^p\left[k(\alpha) + \left(\Gamma(1+\frac{p}{p-1}(\beta-\alpha-1))\right)^{p-1}\gamma^{-p(\beta-\alpha)}\right] \parallel x^* - x^{**} \parallel_{\alpha,\infty}^p .\end{aligned}$$

Similarly, we have

$$E \parallel (\Phi_{12}x^*)(t) - (\Phi_{12}x^{**})(t) \parallel_\alpha^p \le L_h k_1 \mathcal{K}_0^p \frac{1}{\delta^p} \parallel x^* - x^{**} \parallel_{\alpha,\infty}^p .$$

Then, we have

$$E \parallel (\Phi_1 x^*)(t) - (\Phi_1 x^{**})(t) \parallel_\alpha^p \le L_0 \parallel x^* - x^{**} \parallel_{\alpha,\infty}^p .$$

Taking supremum over t,

$$\parallel \Phi_1 x^* - \Phi_1 x^{**} \parallel_{\alpha,\infty}^p \le L_0 \parallel x^* - x^{**} \parallel_{\alpha,\infty}^p .$$

where $L_0 = 3^{p-1}L_h k_1 \mathcal{K}_0^p[k(\alpha) + (\Gamma(1+\frac{p}{p-1}(\beta-\alpha-1)))^{p-1}\gamma^{-p(\beta-\alpha)} + \frac{1}{\delta^p}] < 1$. Hence, Φ_1 is a contractive operator with constant L_0.

(2) Φ_2 maps bounded sets into bounded sets in $\mathcal{Y}$.

Indeed, it is enough to show that there exists a positive constant $\mathcal{L}$ such that for each $x \in B_{r^*} = \{x \in \mathcal{Y} : \parallel x \parallel_{\alpha,\infty}^p \leq r^*\}$, $r^* > 0$, one has $\parallel \Phi_2 x \parallel_{\alpha,\infty} \leq \mathcal{L}$. By (H5) and (H6) it follows that there exist positive constants $\epsilon, \epsilon_i (i \in \mathbb{Z})$ and $\tilde{r}$ such that, for all $t \in \mathbb{R}$ and $\psi \in \mathcal{B}, x \in L^p(\mathbb{P}, \mathbb{K})$ with $\parallel \psi \parallel_{\mathcal{B}}^p > \tilde{r}, E \parallel x \parallel^p > \tilde{r}$,

$$E \parallel g(t,\psi) \parallel^p + E \parallel f(t,\psi) \parallel_{L_2^0}^p \leq (\tilde{\mu}+\epsilon) \parallel \psi \parallel_{\mathcal{B}}^p,$$
$$E \parallel I_i(x) \parallel^p \leq (c_i+\epsilon_i) E \parallel x \parallel^p, i \in \mathbb{Z}.$$

For $p > 2$, we have

$$\begin{aligned}\tilde{L} = 18^{p-1}\mathcal{K}_0^p k_1 \Bigg\{ & \Bigg(\bigg(k(\alpha) + c^p\Big(\Gamma(1+\frac{p(\beta-\alpha-1)}{p-1})\Big)^{p-1}\gamma^{-p(\beta-\alpha)} + \frac{1}{\delta^p}\bigg) L_h \\ & + \bigg(\Big[(M(\alpha))^p\Big(\Gamma(1-\frac{p}{p-1}\alpha)\gamma^{\frac{p}{p-1}\alpha-1}\Big)^{p-1}\frac{1}{\gamma} + (C(\alpha))^p\frac{1}{\delta^p}\Big] \\ & + C_p\Big[(M(\alpha))^p\Big(\Gamma(1-\frac{p}{p-2}\alpha)(\frac{p}{p-2}\gamma)^{\frac{p}{p-2}\alpha-1}\Big)^{\frac{p-2}{p}}\frac{2}{p\gamma} \\ & + (C(\alpha))^p\Big(\frac{p-2}{p\delta}\Big)^{\frac{p-2}{p}}\frac{2}{p\gamma}\Big]\bigg)(\tilde{\mu}+\epsilon) \\ & + \bigg[(M(\alpha))^p\varsigma^{-p\alpha}\frac{1}{(1-e^{-\gamma\varsigma})^p} + (C(\alpha))^p\frac{1}{(1-e^{-\varsigma\delta})^p}\bigg]\sup_{i\in\mathbb{Z}}(c_i+\epsilon_i)\Bigg\} < 1,\end{aligned}$$

and for $p = 2$, we have

$$\begin{aligned}\tilde{L} = 18\mathcal{K}_0^2 k_1 \Bigg\{ & k(\alpha) + (M(\alpha))^2\bigg[(\Gamma(1+2(\beta-\alpha-1)\gamma^{-2(\beta-\alpha-1)-1})\frac{1}{\gamma} + \frac{1}{\delta^2}\bigg] L_h \\ & + [(M(\alpha))^2(\Gamma(1-2\alpha)\gamma^{2\alpha-2}) + (C(\alpha))^2\frac{1}{\delta}] \\ & + [(M(\alpha))^2(\Gamma(1-2\alpha)(2\gamma)^{2\alpha-1}) + (C(\alpha))^2\frac{1}{2\delta}](\tilde{\mu}+\epsilon) \\ & + \bigg[(M(\alpha))^2\varsigma^{-2\alpha}\frac{1}{(1-e^{-\gamma\varsigma})^2} + (C(\alpha))^2\frac{1}{(1-e^{-\varsigma\delta})^2}\bigg]\sup_{i\in\mathbb{Z}}(c_i+\epsilon_i)\Bigg\} < 1.\end{aligned}$$

Let

$$\tilde{\nu} = \sup_{t\in\mathbb{R}}\{E \parallel g(t,\psi) \parallel^p + E \parallel f(t,\psi) \parallel_{L_2^0}^p : E \parallel \psi \parallel_{\mathcal{B}}^p \leq \tilde{r}\},$$
$$\tilde{\nu}_1 = \sup_{t\in\mathbb{R}, i\in\mathbb{Z}}\{E \parallel I_i(x) \parallel^p : E \parallel x \parallel^p \leq \tilde{r}\}.$$

Thus, we have for all $t \in \mathbb{R}$ and $\psi \in \mathcal{B}, x \in L^p(\mathbb{P}, \mathbb{K})$,

$$E \parallel g(t,\psi) \parallel^p + E \parallel f(t,\psi) \parallel_{L_2^0}^p \leq (\tilde{\mu}+\epsilon) \parallel \psi \parallel_{\mathcal{B}}^p + \tilde{\nu}, \tag{3.2}$$

$$E \parallel I_i(x) \parallel^p \leq (c_i+\epsilon_i) E \parallel x \parallel^p + \tilde{\nu}_1, i \in \mathbb{Z}. \tag{3.3}$$

On the other hand, for $x \in B_{r^*}$, from (H2) and Lemma 2.1, it follows that

$$\parallel x_s \parallel_{\mathcal{B}}^p \leq 2^{p-1}\mathcal{K}_0^p(\parallel \varphi \parallel_{\mathcal{B}}^p + k_1 r^*) := r'. \tag{3.4}$$

Then, by (3.2)-(3.4), Hölder's inequality and the Ito integral, we have for $p > 2$,

$$
\begin{aligned}
& E \parallel (\Phi_{21}x)(t) \parallel_\alpha^p \\
\leq & 3^{p-1} E \Big\| \int_{-\infty}^t T(t-s)Pg(s,x_s)ds \Big\|_\alpha^p + 3^{p-1} E \Big\| \int_{-\infty}^t T(t-s)Pf(s,x_s)dW(s) \Big\|_\alpha^p \\
& + 3^{p-1} E \Big\| \sum_{t_i<t} T(t-t_i)PI_i(x(t_i)) \Big\|_\alpha^p \\
\leq & 3^{p-1}(M(\alpha))^p \Big(\int_{-\infty}^t (t-s)^{-\frac{p}{p-1}\alpha} e^{-\gamma(t-s)} ds \Big)^{p-1} \Big(\int_{-\infty}^t e^{-\gamma(t-s)} E \parallel g(s,x_s) \parallel^p ds \Big) \\
& + 3^{p-1}(M(\alpha))^p C_p E \Big[\int_{-\infty}^t (t-s)^{-2\alpha} e^{-2\gamma(t-s)} \parallel f(s,x_s) \parallel_{L_2^0}^2 ds \Big]^{p/2} \\
& + 3^{p-1}(M(\alpha))^p E \Big[\Big(\sum_{t_i<t} (t-t_i)^{-\frac{p}{p-1}\alpha} e^{-\gamma(t-t_i)} \Big)^{p-1} \Big(\sum_{t_i<t} e^{-\gamma(t-t_i)} \|I_i(x(t_i))\|^p \Big) \Big] \\
\leq & 3^{p-1}(M(\alpha))^p \Big(\Gamma(1-\frac{p}{p-1}\alpha)\gamma^{\frac{p}{p-1}\alpha-1} \Big)^{p-1} \Big(\int_{-\infty}^t e^{-\gamma(t-s)} [(\tilde{\mu}+\epsilon) \parallel x_s \parallel_{\mathcal{B}}^p + \tilde{\nu}] ds \Big) \\
& + 5^{p-1}(M(\alpha))^p C_p \Big(\Gamma(1-\frac{p}{p-2}\alpha)(\frac{p}{p-2}\gamma)^{\frac{p}{p-2}\alpha-1} \Big)^{\frac{p-2}{p}} \\
& \times \Big(\int_{-\infty}^t e^{-\frac{p}{2}\gamma(t-s)} [(\tilde{\mu}+\epsilon) \parallel x_s \parallel_{\mathcal{B}}^p + \tilde{\nu}] ds \Big) + 5^{p-1}(M(\alpha))^p \varsigma^{-p\alpha} \frac{1}{(1-e^{-\gamma\sigma})^{p-1}} \\
& \times \Big(\sum_{t_i<t} e^{-\gamma(t-t_i)} [(c_i+\epsilon_i) \parallel x(t_i) \parallel_{\mathcal{B}}^p + \tilde{\nu}_1] \Big) \\
\leq & 3^{p-1}(M(\alpha))^p \Big(\Gamma(1-\frac{p}{p-1}\alpha)\gamma^{\frac{p}{p-1}\alpha-1} \Big)^{p-1} \frac{1}{\gamma} [(\tilde{\mu}+\epsilon)r' + \tilde{\nu}] \\
& + 3^{p-1}(M(\alpha))^p C_p \Big(\Gamma(1-\frac{p}{p-2}\alpha)(\frac{p}{p-2}\gamma)^{\frac{p}{p-2}\alpha-1} \Big)^{\frac{p-2}{p}} \frac{2}{p\gamma} [(\tilde{\mu}+\epsilon)r' + \tilde{\nu}] \\
& + 3^{p-1}(M(\alpha))^p \varsigma^{-p\alpha} \frac{1}{(1-e^{-\gamma\varsigma})^p} \sup_{i\in\mathbb{Z}} [(c_i+\epsilon_i)k_1 r^* + \tilde{\nu}_1] := \mathcal{L}_1.
\end{aligned}
$$

For $p = 2$, we have

$$
\begin{aligned}
E \parallel (\Phi_{21}x)(t) \parallel_\alpha^p \leq & 3(M(\alpha))^2 (\Gamma(1-2\alpha)\gamma^{2\alpha-1}) [(\tilde{\mu}+\epsilon)r' + \tilde{\nu}] \\
& + 3(M(\alpha))^2 \varsigma^{-p\alpha} \frac{1}{(1-e^{-\gamma\varsigma})^2} \sup_{i\in\mathbb{Z}} [(c_i+\epsilon_i)k_1 r^* + \tilde{\nu}_1] \\
:= & \tilde{\mathcal{L}}_1.
\end{aligned}
$$

Similarly, we have for $p > 2$,

$$
\begin{aligned}
E\|(\Psi_{22}x)(t)\|_\alpha^p \leq & 3^{p-1}(C(\alpha))^p \frac{1}{\delta^p} [(\tilde{\mu}+\epsilon)r' + \tilde{\nu}] + 3^{p-1}(C(\alpha))^p C_p \Big(\frac{p-2}{p}\delta \Big)^{\frac{p-2}{p}} \frac{2}{p\delta} \\
& \times [(\tilde{\mu}+\epsilon)r' + \tilde{\nu}] + 3^{p-1}(C(\alpha))^p \Big(\frac{1}{1-e^{-\varsigma\delta}} \Big)^p \sup_{i\in\mathbb{Z}} [(c_i+\epsilon_i)k_1 r^* + \tilde{\nu}_1] \\
:= & \mathcal{L}_2.
\end{aligned}
$$

For $p = 2$, we have

$$E \parallel (\Phi_{22}x)(t) \parallel_\alpha^p \le 3(C(\alpha))^2 \frac{1}{\delta^2}[(\tilde{\mu}+\epsilon)r' + \tilde{\nu}] + 3(C(\alpha))^2 \frac{1}{2\delta}[(\tilde{\mu}+\epsilon)r' + \tilde{\nu}] + 3(C(\alpha))^2 \frac{1}{(1-e^{-\varsigma\delta})^2} \sup_{i\in\mathbb{Z}}[(c_i+\epsilon_i)k_1 r^* + \tilde{\nu}_1] := \tilde{\mathcal{L}}_2.$$

Take $\mathcal{L} = \max_{j=1,2}\{\mathcal{L}_j, \tilde{\mathcal{L}}_j\}$. Then for each $x \in B_{r^*}$, we have $\parallel \Phi_2 x \parallel_{\alpha,\infty} \le \mathcal{L}$.

(3) Φ_2 maps bounded sets into equicontinuous sets of $\mathcal{Y}$.

Let $\tau_1, \tau_2 \in (t_i, t_{i+1}), i \in \mathbb{Z}, \tau_1 < \tau_2$, and $x \in B_{r^*}$. Then, by (H1)-(H6), Hölder's inequality and the Ito integral, we have for $p > 2$,

$$\begin{aligned}
&E \parallel (\Phi_{21}x)(\tau_2) - (\Phi_{21}x)(\tau_1) \parallel_\alpha^p \\
&\le 6^{p-1}E\left\| \int_{-\infty}^{\tau_1} T(\tau_1 - s)[T(\tau_2-\tau_1) - I]Pg(s,x_s)ds \right\|_\alpha^p \\
&\quad + 6^{p-1}E\left\| \int_{\tau_1}^{\tau_2} T(\tau_2 - s)Pg(s,x_s)ds \right\|_\alpha^p \\
&\quad + 6^{p-1}E\left\| \int_{-\infty}^{\tau_1} T(\tau_1 - s)[T(\tau_2-\tau_1) - I]Pf(s,x_s)dW(s) \right\|_\alpha^p \\
&\quad + 6^{p-1}E\left\| \int_{\tau_1}^{\tau_2} T(\tau_2 - s)Pf(s,x_s)dW(s) \right\|_\alpha^p \\
&\quad + 3^{p-1}E\left\| \sum_{t_i<\tau_1} T(\tau_1 - t_i)[T(\tau_2-\tau_1) - I]PI_i(x(t_i)) \right\|_\alpha^p \\
&\le 6^{p-1}(M(\alpha))^p \parallel T(\tau_2-\tau_1) - I \parallel^p \left(\int_{-\infty}^{\tau_1} (\tau_1 - s)^{-\frac{p}{p-1}\alpha} e^{-\gamma(\tau_1 - s)} ds \right)^{p-1} \\
&\quad \times \left(\int_{-\infty}^{\tau_1} e^{-\gamma(\tau_1 - s)} E \parallel g(s,x_s) \parallel^p ds \right) \\
&\quad + 6^{p-1}(M(\alpha))^p \left(\int_{\tau_1}^{\tau_2} (\tau_2 - s)^{-\frac{p}{p-1}\alpha} e^{-\gamma(\tau_2 - s)} ds \right)^{p-1} \\
&\quad \times \left(\int_{\tau_1}^{\tau_2} e^{-\gamma(\tau_2 - s)} E \parallel g(s,x_s) \parallel^p ds \right) \\
&\quad + 6^{p-1}(M(\alpha))^p C_p E\bigg[\int_{-\infty}^{\tau_1} (\tau_1 - s)^{-2\alpha} e^{-2\gamma(\tau_1 - s)} \parallel T(\tau_2 - \tau_1) - I \parallel^2 \\
&\quad \times \parallel f(s,x_s) \parallel_{L_2^0}^2 ds \bigg]^{p/2} \\
&\quad + 6^{p-1}(M(\alpha))^p C_p E\left[\int_{\tau_1}^{\tau_2} (\tau_2 - s)^{-2\alpha} e^{-2\gamma(\tau_2 - s)} \parallel f(s,x_s) \parallel_{L_2^0}^2 ds \right]^{p/2} \\
&\quad + 3^{p-1}(M(\alpha))^p \parallel T(\tau_2-\tau_1) - I \parallel^p \left(\sum_{t_i<\tau_1} (\tau_1 - t_i)^{-\frac{p}{p-1}\alpha} e^{-\gamma(\tau_1 - t_i)} \right)^{p-1} \\
&\quad \times \left(\sum_{t_i<\tau_1} e^{-\gamma(\tau_1 - t_i)} E \parallel I_i(x(t_i)) \parallel^p \right) \\
&\le 6^{p-1}(M(\alpha))^p \parallel T(\tau_2-\tau_1) - I \parallel^p \left(\Gamma(1 - \frac{p\alpha}{p-1})\gamma^{\frac{p\alpha}{p-1}-1} \right)^{p-1}
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\int_{-\infty}^{\tau_1} e^{-\gamma(\tau_1 - s)}[(\tilde{\mu}+\epsilon) \parallel x_s \parallel_{\mathcal{B}}^p + \tilde{\nu}]ds \right) \\
&+ 6^{p-1}(M(\alpha))^p \left(\int_{\tau_1}^{\tau_2} (\tau_2 - s)^{-\frac{p}{p-1}\alpha} ds \right)^{p-1} \\
&\times \left(\int_{\tau_1}^{\tau_2} e^{-\gamma(\tau_2 - s)}[(\tilde{\mu}+\epsilon) \parallel x_s \parallel_{\mathcal{B}}^p + \tilde{\nu}]ds \right) \\
&+ 6^{p-1}(M(\alpha))^p C_p \parallel T(\tau_2-\tau_1) - I \parallel^p \left(\Gamma(1 - \frac{p\alpha}{p-2})(\frac{p\alpha}{p-2}\gamma)^{\frac{p\alpha}{p-2}-1} \right)^{\frac{p-2}{p}} \\
&\times \left(\int_{-\infty}^{\tau_1} e^{-\frac{p}{2}\gamma(\tau_1 - s)} m(s)\Theta(\parallel x_s \parallel_{\mathcal{C}}^p) ds \right) \\
&+ 6^{p-1}(M(\alpha))^p C_p \left(\int_{\tau_1}^{\tau_2} (\tau_2 - s)^{-\frac{p\alpha}{p-2}} ds \right)^{\frac{p-2}{p}} \\
&\times \left(\int_{\tau_1}^{\tau_2} e^{-\frac{p}{2}\gamma(\tau_2 - s)}[(\tilde{\mu}+\epsilon) \parallel x_s \parallel_{\mathcal{B}}^p + \tilde{\nu}]ds \right) \\
&+ 3^{p-1}(M(\alpha))^p \parallel T(\tau_2-\tau_1) - I \parallel^p \varsigma^{-p\alpha} \left(\sum_{t_i<\tau_1} e^{-\gamma(\tau_1 - t_i)} \right)^{p-1} \\
&\times \left(\sum_{t_i<\tau_1} e^{-\gamma(\tau_1 - t_i)}[(c_i+\epsilon_i) \parallel x(t_i) \parallel^p + \tilde{\nu}_1] \right) \\
\le\; & 6^{p-1}(M(\alpha))^p \parallel T(\tau_2-\tau_1) - I \parallel^p \left(\Gamma(1 - \frac{p\alpha}{p-1})\gamma^{\frac{p\alpha}{p-1}-1} \right)^{p-1} [(\tilde{\mu}+\epsilon)r' + \tilde{\nu}] \\
&+ 6^{p-1}(M(\alpha))^p \left(\int_{\tau_1}^{\tau_2} (\tau_2 - s)^{-\frac{p}{p-1}\alpha} ds \right)^{p-1} \left(\int_{\tau_1}^{\tau_2} e^{-\gamma(\tau_2 - s)} ds \right) \\
&\times [(\tilde{\mu}+\epsilon)r' + \tilde{\nu}] \\
&+ 6^{p-1}(M(\alpha))^p C_p \parallel T(\tau_2-\tau_1) - I \parallel^p \\
&\times \left(\Gamma(1 - \frac{p\alpha}{p-2})(\frac{p\alpha}{p-2}\gamma)^{\frac{p\alpha}{p-2}-1} \right)^{\frac{p-2}{p}} [(\tilde{\mu}+\epsilon)r' + \tilde{\nu}] \\
&+ 6^{p-1}(M(\alpha))^p C_p \left(\int_{\tau_1}^{\tau_2} (\tau_2 - s)^{-\frac{p}{p-2}\alpha} ds \right)^{\frac{p-2}{p}} \\
&\times \left(\int_{\tau_1}^{\tau_2} e^{-\frac{p}{2}\gamma(\tau_2 - s)} ds \right) [(\tilde{\mu}+\epsilon)r' + \tilde{\nu}] \\
&+ 3^{p-1}(M(\alpha))^p \parallel T(\tau_2-\tau_1) - I \parallel^p \varsigma^{-p\alpha} \frac{1}{(1-e^{-\gamma\varsigma})^p} \sup_{i\in\mathbb{Z}}[(c_i+\epsilon_i)k_1 r^* + \tilde{\nu}].
\end{aligned}
$$

For $p = 2$, we have

$$
\begin{aligned}
& E \parallel (\Phi_{21}x)(\tau_2) - (\Phi_{21}x)(\tau_1)) \parallel_\alpha^2 \\
\le\; & 6(M(\alpha))^2 \parallel T(\tau_2-\tau_1) - I \parallel^2 (\Gamma(1-2\alpha)\gamma^{2\alpha-1})^{p-1}[(\tilde{\mu}+\epsilon)r' + \tilde{\nu}] \\
&+ 6(M(\alpha))^2 \left(\int_{\tau_1}^{\tau_2} (\tau_2 - s)^{-2\alpha} ds \right) \left(\int_{\tau_1}^{\tau_2} e^{-\gamma(\tau_2 - s)} ds \right) [(\tilde{\mu}+\epsilon)r' + \tilde{\nu}] \\
&+ 6(M(\alpha))^2 \parallel T(\tau_2-\tau_1) - I \parallel^2 \Gamma(1-2\alpha)\gamma^{2\alpha-1}[(\tilde{\mu}+\epsilon)r' + \tilde{\nu}]
\end{aligned}
$$

$$+ 6(M(\alpha))^2\left(\int_{\tau_1}^{\tau_2}(\tau_2-s)^{-2\alpha}ds\right)[(\tilde{\mu}+\epsilon)r'+\tilde{\nu}]$$
$$+ 3(M(\alpha))^2 \parallel T(\tau_2-\tau_1)-I \parallel^2 \varsigma^{-p\alpha}\frac{1}{(1-e^{-\gamma\varsigma})^2}\sup_{i\in\mathbb{Z}}[(c_i+\epsilon_i)k_1r^*+\tilde{\nu}_1].$$

The right-hand side of the above inequality is independent of $x \in B_{r^*}$ and tends to zero as $\tau_2 \to \tau_1$, since the compactness of $T(t)$ for $t > 0$ implies imply the continuity in the uniform operator topology. Thus, Φ_{21} maps B_{r^*} into an equicontinuous family of functions. Similarly, we can show that Φ_{22} maps B_{r^*} into an equicontinuous family of functions and hence Φ_2 maps B_{r^*} into an equicontinuous family of functions.

(4)The set $V(t) = \{(\Phi_2 x)(t) : x \in B_{r^*}\}$ is relatively compact in $\mathcal{Y}$.

For each $t \in \mathbb{R}$, and let ε be a real number satisfying $0 < \varepsilon < 1$. For $x \in B_{r^*}$, we define

$$\begin{aligned}(\Phi_{21,\varepsilon}x)(t) &= T(\varepsilon)\Big[\int_{-\infty}^{t-\varepsilon}T(t-\varepsilon-s)Pg(s,x_s)ds+\int_{-\infty}^{t-\varepsilon}T(t-\varepsilon-s)Pf(s,x_s)dW(s)\\ &\quad+\sum_{t_i<t-\varepsilon}T(t-\varepsilon-t_i)PI_i(x(t_i))\Big]\\ &= T(\varepsilon)[(\Phi_{21}x)(t-\varepsilon)].\end{aligned}$$

Since $T(t)(t > 0)$ is compact, then the set $V_\varepsilon(t) = \{(\Phi_{21,\varepsilon}x)(t) : x \in B_{r^*}\}$ is relatively compact in $L^p(\mathbb{P},\mathbb{H})$ for each $t \in \mathbb{R}$. Moreover, for every $x \in B_{r^*}$, we have for $p > 2$,

$$\begin{aligned}&E \parallel (\Phi_{21}x)(t)-(\Phi_{21,\varepsilon}x)(t) \parallel_\alpha^p\\ \le&3^{p-1}E\left\|\int_{t-\varepsilon}^{t}T(t-s)Pg(s,x_s)ds\right\|_\alpha^p+3^{p-1}E\left\|\int_{t-\varepsilon}^{t}T(t-s)Pf(s,x_s)dW(s)\right\|_\alpha^p\\ &+3^{p-1}E\left\|\sum_{t-\varepsilon<t_i<t}T(t-t_i)PI_i(x(t_i))\right\|_\alpha^p\\ \le&3^{p-1}(M(\alpha))^p\left(\int_{t-\varepsilon}^{t}(t-s)^{-\frac{p}{p-1}\alpha}e^{-\gamma(t-s)}ds\right)^{p-1}\left(\int_{t-\varepsilon}^{t}e^{-\gamma(t-s)}E \parallel g(s,x_s) \parallel^p ds\right)\\ &+3^{p-1}(M(\alpha))^pC_pE\left(\int_{t-\varepsilon}^{t}(t-s)^{-2\alpha}e^{-2\gamma(t-s)} \parallel f(s,x_s) \parallel_{L_2^0}^2 ds\right)^{p/2}\\ &+3^{p-1}(M(\alpha))^pE\left[\left(\sum_{t-\varepsilon<t_i<t}(t-t_i)^{-\frac{p}{p-1}\alpha}e^{-\gamma(t-t_i)}\right)^{p-1}\right.\\ &\left.\times\left(\sum_{t-\epsilon<t_i<t}e^{-\gamma(t-t_i)} \parallel I_i(x(t_i)) \parallel^p\right)\right]\\ \le&3^{p-1}(M(\alpha))^p\left(\int_{t-\varepsilon}^{t}(t-s)^{-\frac{p}{p-1}\alpha}ds\right)^{p-1}\left(\int_{t-\varepsilon}^{t}e^{-\gamma(t-s)}[(\tilde{\mu}+\epsilon) \parallel x_s \parallel_{\mathcal{B}}^p+\tilde{\nu}]ds\right)\\ &+3^{p-1}(M(\alpha))^pC_p\left(\int_{t-\varepsilon}^{t}(t-s)^{-\frac{p}{p-2}\alpha}ds\right)^{\frac{p-2}{p}}\left(\int_{t-\varepsilon}^{t}e^{-\frac{p}{2}\gamma(t-s)}[(\tilde{\mu}+\epsilon)\|x_s\|_{\mathcal{B}}^p+\tilde{\nu}]ds\right)\\ &+3^{p-1}(M(\alpha))^p\varsigma^{-p\alpha}\left(\sum_{t-\varepsilon<t_i<t}e^{-\gamma(t-t_i)}\right)^{p-1}\end{aligned}$$

$$\times \left(\sum_{t-\varepsilon<t_i<t} e^{-\gamma(t-t_i)}[(c_i+\epsilon_i)E \parallel x(t_i) \parallel^p + \tilde{\nu}_1] \right)$$

$$\leq 3^{p-1}(M(\alpha))^p \left(\int_{t-\varepsilon}^t (t-s)^{-\frac{p-1}{p}\alpha} ds \right)^{p-1} \left(\int_{t-\varepsilon}^t e^{-\gamma(t-s)} ds \right) [(\tilde{\mu}+\epsilon)r' + \tilde{\nu}]$$

$$+ 3^{p-1}(M(\alpha))^p C_p \left(\int_{t-\varepsilon}^t (t-s)^{-\frac{p}{p-2}\alpha} ds \right)^{\frac{p-2}{p}} \left(\int_{t-\varepsilon}^t e^{-\frac{p}{2}\delta(t-s)} ds \right) [(\tilde{\mu}+\epsilon)r' + \tilde{\nu}]$$

$$+ 3^{p-1}(M(\alpha))^p \varsigma^{-p\alpha} \left(\sum_{t-\varepsilon<t_i<t} e^{-\gamma(t-t_i)} \right)^p \sup_{i\in\mathbb{Z}} [(c_i+\epsilon_i)k_1 r^* + \tilde{\nu}_1].$$

For $p = 2$, we have

$$E \parallel (\Phi_{21}x)(t) - (\Phi_{21,\varepsilon}x)(t) \parallel_\alpha^2$$

$$\leq 3(M(\alpha))^2 \left(\int_{t-\varepsilon}^t (t-s)^{-2\alpha} ds \right) \left(\int_{t-\varepsilon}^t e^{-\gamma(t-s)} ds \right) [(\tilde{\mu}+\epsilon)r' + \tilde{\nu}]$$

$$+ 3(M(\alpha))^2 \left(\int_{t-\varepsilon}^t (t-s)^{-2\alpha} ds \right) [(\tilde{\mu}+\epsilon)r' + \tilde{\nu}]$$

$$+ 3(M(\alpha))^2 \varsigma^{-2\alpha} \left(\sum_{t-\varepsilon<t_i<t} e^{-\gamma(t-t_i)} \right)^2 \sup_{i\in\mathbb{Z}} [(c_i+\epsilon_i)k_1 r^* + \tilde{\nu}_1].$$

Therefore, letting $\varepsilon \to 0$, it follows that there are relatively compact sets $V_\varepsilon(t)$ arbitrarily close to $V(t) = \{(\Phi_{21}x)(t) : x \in B_{r^*}\}$, and hence $V(t)$ is also relatively compact in $L^p(\mathbb{P}, \mathbb{H})$ for each $t \in \mathbb{R}$. Since $\{\Phi_{21}x : x \in B_{r^*}\} \subset PC_l^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), q)$, then $\{\Phi_{21}x : x \in B_{r^*}\}$ is a relatively compact set by Lemma 2.6, then Φ_{21} is a compact operator. Similarly, Φ_{22} is a compact operator and Φ_2 is a compact operator. Hence we can conclude that Φ_2 is a completely continuous map.

(5) $\Phi_2 : \mathcal{Y} \to \mathcal{Y}$ is continuous.

Let $\{x^{(n)}\} \subseteq B_{r^*}$ with $x^{(n)} \to x (n \to \infty)$ in $\mathcal{Y}$, then there exists a bounded subset $K \subseteq L^p(\mathbb{P}, \mathbb{K})$ such that $\mathbb{R}(x) \subseteq K, \mathbb{R}(x^n) \subseteq K, n \in \mathbb{N}$. By the assumptions (H4)-(H6), for any $\varepsilon > 0$, there exists $\tilde{\xi} > 0$ such that $x, y \in K$ and $\parallel x - y \parallel_\infty^p < \tilde{\xi}$ implies that

$$E \parallel g(s, x_s) - g(s, y_s) \parallel^p < \varepsilon \quad \text{for all } t \in \mathbb{R},$$

$$E \parallel f(s, x_s) - f(s, y_s) \parallel_{L_2^0}^p < \varepsilon \quad \text{for all } t \in \mathbb{R},$$

and

$$E \parallel I_i(x) - I_i(y) \parallel^p < \varepsilon \quad \text{for all } i \in \mathbb{Z}.$$

For the above $\tilde{\xi}$ there exists n_0 such that $\parallel x^{(n)} - x \parallel_\infty^p < \varepsilon$ and $\parallel x_s^{(n)} - x_s \parallel_\infty^p < \varepsilon$ for $n > n_0$. Then for $n > n_0$, we have

$$E \parallel g(s, x_s^{(n)}) - g(s, x_s) \parallel^p < \varepsilon \quad \text{for all } t \in \mathbb{R},$$

$$E \parallel f(s, x_s^{(n)}) - f(s, x_s) \parallel_{L_2^0}^p < \varepsilon \quad \text{for all } t \in \mathbb{R},$$

and

$$E \parallel I_i(x^{(n)}) - I_i(x) \parallel^p < \varepsilon \quad \text{for all } i \in \mathbb{Z}.$$

Then, by Hölder's inequality, we have that for $p > 2$,

$$E \parallel (\Phi_{21}x^{(n)})(t) - (\Phi_{21}x)(t) \parallel_\alpha^p$$

$$\leq 3^{p-1}E\left\| \int_{-\infty}^{t} T(t-s)P[g(s,x_s^{(n)})-g(s,x_s)]ds \right\|_\alpha^p$$
$$+ 3^{p-1}E\left\| \int_{-\infty}^{t} T(t-s)P[f(s,x_s^{(n)})-f(s,x_s)]dW(s) \right\|_\alpha^p$$
$$+ 3^{p-1}E\left\| \sum_{t_i<t} T(t-t_i)P[I_i(x^{(n)}(t_i))-I_i(x(t_i))] \right\|_\alpha^p$$
$$\leq 3^{p-1}(M(\alpha))^p\left(\int_{-\infty}^{t}(t-s)^{-\frac{p}{p-1}\alpha}e^{-\gamma(t-s)}ds\right)^{p-1}$$
$$\times\left(\int_{-\infty}^{t}e^{-\gamma(t-s)}E\parallel g(s,x_s^{(n)})-g(s,x_s)\parallel^p ds\right)$$
$$+ 3^{p-1}(M(\alpha))^pC_p\left(\int_{-\infty}^{t}(t-s)^{-2\alpha}e^{-2\gamma(t-s)}\right.$$
$$\left.\times E\parallel f(s,x_s^{(n)})-f(s,x_s)\parallel_{L_2^0}^2 ds\right)^{p/2}$$
$$+ 3^{p-1}(M(\alpha))^pE\left[\left(\sum_{t_i<t}(t-t_i)^{-\frac{p}{p-1}\alpha}e^{-\gamma(t-t_i)}\right)^{p-1}\right.$$
$$\left.\times\left(\sum_{t_i<t}e^{-\gamma(t-t_i)}\parallel I_i(x^{(n)}(t_i))-I_i(x(t_i))\parallel^p\right)\right]$$
$$\leq 3^{p-1}(M(\alpha))^p\left(\Gamma(1-\frac{p}{p-1}\alpha)\gamma^{\frac{p}{p-1}\alpha-1}\right)^{p-1}\left(\int_{-\infty}^{t}e^{-\gamma(t-s)}ds\right)\varepsilon$$
$$+ 3^{p-1}(M(\alpha))^pC_p\left(\Gamma(1-\frac{p}{p-2}\alpha)(\frac{p}{p-2}\gamma)^{\frac{p}{p-2}\alpha-1}\right)^{\frac{p-2}{p}}$$
$$\times\left(\int_{-\infty}^{t}e^{-\frac{p}{2}\gamma(t-s)}ds\right)\varepsilon$$
$$+ 3^{p-1}(M(\alpha))^p\varsigma^{-p\alpha}\frac{1}{(1-e^{-\gamma\varsigma})^{p-1}}\left(\sum_{t_i<t}e^{-\gamma(t-t_i)}\right)\varepsilon$$
$$\leq 3^{p-1}(M(\alpha))^p\left[\left(\Gamma(1-\frac{p}{p-1}\alpha)\right)^{p-1}\gamma^{p(\alpha-1)}\right.$$
$$\left.+ C_p\left(\Gamma(1-\frac{p}{p-2}\alpha)(\frac{p}{p-2}\gamma)^{\frac{p}{p-2}\alpha-1}\right)^{\frac{p-2}{p}}\frac{2}{p\gamma}+\frac{\varsigma^{-p\alpha}}{(1-e^{-\gamma\varsigma})^p}\right]\varepsilon.$$

For $p=2$, we have

$$E\parallel(\Phi_{21}x^{(n)})(t)-(\Phi_{21}x)(t)\parallel_\alpha^2$$
$$\leq 3M^2\left[\Gamma(1-2\alpha)\gamma^{2(\alpha-1)}+\Gamma(1-2\alpha)\gamma^{2\alpha-1}+\frac{1}{(1-e^{-\gamma\varsigma})^2}\right]\varepsilon.$$

Thus Φ_{21} is continuous on B_{r^*}. Similarly, we have Φ_{22} is continuous on B_{r^*} and hence Φ_2 is continuous on B_{r^*}.

(6) We shall show the set $G=\{x\in\mathcal{Y}:\lambda\Phi_1(\frac{x}{\lambda})+\lambda\Phi_2(x)=x$ for some $\lambda\in(0,1)\}$ is bounded on $\mathbb{R}$.

To do this, we consider the following nonlinear operator equation

$$x(t) = \lambda(\Phi x)(t), \ 0 < \lambda < 1, \tag{3.5}$$

where Φ is already defined. Next we gives a priori estimate for the solution of the above equation. Indeed, let $x \in \mathcal{Y}$ be a possible solution of $x = \lambda\Phi(x)$ for some $0 < \lambda < 1$. This implies by (3.5) that for each $t \in \mathbb{R}$ we have

$$\begin{aligned} x(t) = \lambda h(t, x_t) &+ \lambda \int_{-\infty}^{t} AT(t-s)Ph(s, x_s)ds \\ &- \lambda \int_{t}^{\infty} AT(t-s)Qh(s, x_s)ds \\ &+ \lambda \int_{-\infty}^{t} T(t-s)Pg(s, x_s)ds - \lambda \int_{t}^{\infty} T(t-s)Qg(s, x_s)ds \\ &+ \lambda \int_{-\infty}^{t} T(t-s)Pf(s, x_s)dW(s) - \lambda \int_{t}^{\infty} T(t-s)Qf(s, x_s)dW(s) \\ &+ \lambda \sum_{t_i<t} T(t-t_i)PI_i(x(t_i)) - \lambda \sum_{t<t_i} T(t-t_i)QI_i(x(t_i)), \quad t \in \mathbb{R}. \end{aligned}$$

From the above equation, we have

$$\begin{aligned} E \parallel x(t) \parallel_\alpha^p \leq &\Bigg[9^{p-1} E \parallel h(t, x_t) \parallel_\alpha^p + 9^{p-1} E \left\| \int_{-\infty}^{t} AT(t-s)Ph(s, x_s)ds \right\|_\alpha^p \\ &+ 9^{p-1} E \left\| \int_{-\infty}^{t} T(t-s)Pg(s, x_s)ds \right\|_\alpha^p \\ &+ 9^{p-1} E \left\| \int_{-\infty}^{t} T(t-s)Pf(s, x_s)dW(s) \right\|_\alpha^p \\ &+ 9^{p-1} E \left\| \sum_{t_i<t} T(t-t_i)PI_i(x(t_i)) \right\|_\alpha^p \Bigg] \\ &+ \Bigg[9^{p-1} E \left\| \int_{-\infty}^{t} AT(t-s)Qh(s, x_s)ds \right\|_\alpha^p \\ &+ 9^{p-1} E \left\| \int_{-\infty}^{t} T(t-s)Qg(s, x_s)ds \right\|_\alpha^p \\ &+ 9^{p-1} E \left\| \int_{-\infty}^{t} T(t-s)Qf(s, x_s)dW(s) \right\|_\alpha^p \\ &+ 9^{p-1} E \left\| \sum_{t_i<t} T(t-t_i)QI_i(x(t_i)) \right\|_\alpha^p \Bigg] \\ := &\tilde{\Phi}_1 + \tilde{\Phi}_2. \end{aligned}$$

By Hölder's inequality and the Ito integral, we have for $p > 2$,

$$\begin{aligned} \tilde{\Phi}_1 \leq &9^{p-1} k(\alpha) L_h (\parallel x_t \parallel_{\mathcal{B}}^p + 1) \\ &+ 9^{p-1} c^p \left(\Gamma(1 + \frac{p(\beta-\alpha-1)}{p-1}) \gamma^{-\frac{p(\beta-\alpha-1)}{p-1} - 1} \right)^{p-1} \end{aligned}$$

$$
\begin{aligned}
&\times\bigg(\int_{-\infty}^{t} e^{-\gamma(t-s)} L_h(\parallel x_s \parallel_{\mathcal{B}}^p +1)ds\bigg)\\
&+9^{p-1}(M(\alpha))^p\bigg(\Gamma(1-\frac{p}{p-1}\alpha)\gamma^{\frac{p}{p-1}\alpha-1}\bigg)^{p-1}\\
&\times\bigg(\int_{-\infty}^{t} e^{-\gamma(t-s)}[(\tilde{\mu}+\epsilon)\parallel x_s \parallel_{\mathcal{B}}^p +\tilde{\nu}]ds\bigg)\\
&+9^{p-1}(M(\alpha))^p C_p\bigg(\Gamma(1-\frac{p}{p-2}\alpha)(\frac{p}{p-2}\gamma)^{\frac{p}{p-2}\alpha-1}\bigg)^{\frac{p-2}{p}}\\
&\times\bigg(\int_{-\infty}^{t} e^{-\frac{p}{2}\gamma(t-s)}[(\tilde{\mu}+\epsilon)\parallel x_s \parallel_{\mathcal{B}}^p +\tilde{\nu}]ds\bigg)\\
&+9^{p-1}(M(\alpha))^p\varsigma^{-p\alpha}\frac{1}{(1-e^{-\gamma\varsigma})^{p-1}}\\
&\times\bigg(\sum_{t_i<t} e^{-\gamma(t-t_i)}[(c_i+\epsilon_i)\parallel x(t_i)\parallel^p +\tilde{\nu}_1]\bigg).
\end{aligned}
$$

For $p=2$, we have

$$
\begin{aligned}
\tilde{\Phi}_1 \le\ & 9k(\alpha)L_h(\parallel x_t \parallel_{\mathcal{B}}^2 +1)+5c^p\bigg(\Gamma(1+2(\beta-\alpha-1)\gamma^{-2(\beta-\alpha-1)-1}\bigg)\\
&\times\bigg(\int_{-\infty}^{t} e^{-\gamma(t-s)} L_h(\parallel x_s \parallel_{\mathcal{B}}^2 +1)ds\bigg)\\
&+9(M(\alpha))^2\bigg(\Gamma(1-2\alpha)\gamma^{2\alpha-1}\bigg)\bigg(\int_{-\infty}^{t} e^{-\gamma(t-s)}[(\tilde{\mu}+\epsilon)\parallel x_s \parallel_{\mathcal{B}}^2 +\tilde{\nu}]ds\bigg)\\
&+9(M(\alpha))^2\bigg(\int_{-\infty}^{t}(t-s)^{-2\alpha}e^{-2\gamma(t-s)}[(\tilde{\mu}+\epsilon)\parallel x_s \parallel_{\mathcal{B}}^2 +\tilde{\nu}]ds\bigg)\\
&+9(M(\alpha))^2\varsigma^{-p\alpha}\frac{1}{(1-e^{-\gamma\varsigma})}\bigg(\sum_{t_i<t} e^{-\gamma(t-t_i)}[(c_i+\epsilon_i)\parallel x(t_i)\parallel^2 +\tilde{\nu}_1]\bigg).
\end{aligned}
$$

Similarly, we have for $p>2$,

$$
\begin{aligned}
\tilde{\Phi}_2 \le\ & 9^{p-1}c^p\frac{1}{\delta^{p-1}}\bigg(\int_{t}^{\infty} e^{\delta(t-s)} L_h(\parallel x_s \parallel_{\mathcal{B}}^p +1)ds\bigg)\\
&+9^{p-1}(C(\alpha))^p\frac{1}{\delta^{p-1}}\bigg(\int_{t}^{\infty} e^{\delta(t-s)}[(\tilde{\mu}+\epsilon)\parallel x_s \parallel_{\mathcal{B}}^p +\tilde{\nu}]ds\bigg)\\
&+9^{p-1}(C(\alpha))^p C_p\bigg(\frac{p-2}{p\delta}\bigg)^{\frac{p-2}{p}}\bigg(\int_{t}^{\infty} e^{\frac{p}{2}\delta(t-s)}[(\tilde{\mu}+\epsilon)\parallel x_s \parallel_{\mathcal{B}}^p +\tilde{\nu}]ds\bigg)\\
&+9^{p-1}(C(\alpha))^p\frac{1}{(1-e^{\varsigma\delta})^{p-1}}\bigg(\sum_{t<t_i} e^{\delta(t-t_i)}[(c_i+\epsilon_i)\parallel x(t_i)\parallel^p +\tilde{\nu}_1]\bigg).
\end{aligned}
$$

For $p=2$, we have

$$
\begin{aligned}
\tilde{\Phi}_2 \le\ & 9c^2\frac{1}{\delta}\bigg(\int_{t}^{\infty} e^{\delta(t-s)} L_h(\parallel x_s \parallel_{\mathcal{B}}^2 +1)ds\bigg)\\
&+9(C(\alpha))^2\frac{1}{\delta}\bigg(\int_{t}^{\infty} e^{\delta(t-s)}[(\tilde{\mu}+\epsilon)\parallel x_s \parallel_{\mathcal{B}}^2 +\tilde{\nu}]ds\bigg)
\end{aligned}
$$

$$+ 9^{p-1}(C(\alpha))^2\bigg(\int_t^{\infty} e^{2\delta(t-s)}[(\tilde{\mu}+\epsilon) \parallel x_s \parallel_{\mathcal{B}}^2 + \tilde{\nu}]ds\bigg)$$
$$+ 9(C(\alpha))^2\frac{1}{(1-e^{\varsigma\delta})}\bigg(\sum_{t<t_i} e^{\delta(t-t_i)}[(c_i+\epsilon_i) \parallel x(t_i) \parallel^2 + \tilde{\nu}_1]\bigg).$$

By Lemmas 2.1, it follows that

$$\parallel x_s \parallel_{\mathcal{B}}^p \leq 2^{p-1}\mathcal{K}_0^p(\parallel \varphi \parallel_{\mathcal{B}}^p + \sup_{s\in\mathbb{R}} E \parallel x(s) \parallel^p).$$

Then, we have for $p > 2$,

$$E \parallel x(t) \parallel_\alpha^p \leq \tilde{M} + \tilde{L}\sup_{t\in\mathbb{R}} \parallel x(t) \parallel_\alpha^p,$$

where $\tilde{M}$ is a constant. Since $\tilde{L} < 1$, we obtain

$$\sup_{t\in\mathbb{R}} E \parallel x(t) \parallel_\alpha^p \leq \frac{\tilde{M}}{1-\tilde{L}}.$$

This implies that G is bounded on $\mathbb{R}$. Consequently, by Lemma 2.7, we deduce that Φ has a fixed point $x \in \mathcal{Y}$, which is a mild solution of the system (1.1)-(1.2).

Step 3. Pseudo almost periodic in distribution of mild solution.

For given $x \in PAP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), q)$, by the definition of the mapping Φ, we have

$$(\Phi x)(t) = (\Psi x)(t) + (\Upsilon x)(t),$$

where

$$\begin{aligned}(\Psi x)(t) = {} & h_1(t, x_t) + \int_{-\infty}^t AT(t-s)Ph_1(s, x_s)ds \\ & - \int_t^{\infty} AT(t-s)Qh_1(s, x_s)ds \\ & + \int_{-\infty}^t T(t-s)Pg_1(s, x_s)ds - \int_t^{\infty} T(t-s)Qg_1(s, x_s)ds \\ & + \int_{-\infty}^t T(t-s)Pf_1(s, x_s)dW(s) - \int_t^{\infty} T(t-s)Qf_1(s, x_s)dW(s) \\ & + \sum_{t_i<t} T(t-t_i)PI_{i,1}(x(t_i)) - \sum_{t<t_i} T(t-t_i)QI_{i,1}(x(t_i)),\end{aligned}$$

$$\begin{aligned}(\Upsilon x)(t) = {} & h_2(t, x_t) + \int_{-\infty}^t AT(t-s)Ph_2(s, x_s)ds \\ & - \int_t^{\infty} AT(t-s)Qh_2(s, x_s)ds \\ & + \int_{-\infty}^t T(t-s)Pg_2(s, x_s)ds - \int_t^{\infty} T(t-s)Qg_2(s, x_s)ds \\ & + \int_{-\infty}^t T(t-s)Pf_2(s, x_s)dW(s) - \int_t^{\infty} T(t-s)Qf_2(s, x_s)dW(s) \\ & + \sum_{t_i<t} T(t-t_i)PI_{i,2}(x(t_i)) - \sum_{t<t_i} T(t-t_i)QI_{i,2}(x(t_i)).\end{aligned}$$

(1) Ψx is almost periodic in distribution.

Let $t_i < t \le t_{i+1}$. For $\varepsilon > 0$ and $0 < \eta < \min\{\varepsilon, \varsigma/2\}$. Since $h_1 \in AP_T(\mathbb{R} \times \mathcal{B}, L^p(\mathbb{P}, \mathbb{H}_\beta))$, $g_1 \in AP_T(\mathbb{R} \times \mathcal{B}, L^p(\mathbb{P}, \mathbb{H}))$, $f_1 \in AP_T(\mathbb{R} \times \mathcal{B}, L^p(\mathbb{P}, L_2^0))$, thus for every sequence of real numbers $\{s_n'\}$, there exist a subsequence $\{s_n\}$ and a stochastic processes $\tilde{h}_1 \in AP_T(\mathbb{R} \times \mathcal{B}, L^p(\mathbb{P}, \mathbb{H}_\beta))$, $\tilde{g}_1 \in AP_T(\mathbb{R} \times \mathcal{B}, L^p(\mathbb{P}, \mathbb{H}))$, $\tilde{f}_1 \in AP_T(\mathbb{R} \times \mathcal{B}, L^p(\mathbb{P}, L_2^0))$, such that

$$\lim_{n\to\infty} E \parallel h_1(t+s_n, \psi) - \tilde{h}_1(t, \psi) \parallel_\beta^p = 0, \tag{3.6}$$

$$\lim_{n\to\infty} E \parallel g_1(t+s_n, \psi) - \tilde{g}_1(t, \psi) \parallel^p = 0, \tag{3.7}$$

$$\lim_{n\to\infty} E \parallel f_1(t+s_n, \psi) - \tilde{f}_1(t, \psi) \parallel_{L_2^0}^p = 0 \tag{3.8}$$

for each $t \in \mathbb{R}, \psi \in K$, where K is any bounded subset in $\mathcal{B}$. Since $I_{i,1} \in AP_T(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}))$, thus for every sequence of integer numbers $\{\alpha_n'\}$, there exist a subsequence $\{\alpha_n\}$ and a stochastic processes $\tilde{I}_{i,1} \in AP_T(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}))$, such that

$$\lim_{n\to\infty} E \parallel I_{i+\alpha_n,1}(x) - \tilde{I}_{i,1}(x) \parallel^p = 0 \tag{3.9}$$

for each $x \in B$, where B is any bounded subset in $L^p(\mathbb{P}, \mathbb{K})$.

Let $\widetilde{W}_n(s) := W(s+s_n) - W(s_n)$, for each $s \in \mathbb{R}$. It is easy to show that $\widetilde{W}_n$ is a Q-Wiener process with the same distribution as W, then

$$\begin{aligned}
&(\Psi x)(t+s_n)\\
=&h_1(t+s_n, x_{t+s_n}) + \int_{-\infty}^{t+s_n} AT(t+s_n-s)Ph_1(s, x_s)ds\\
&- \int_{t+s_n}^{\infty} AT(t+s_n-s)Qh_1(s, x_s)ds + \int_{-\infty}^{t+s_n} T(t+s_n-s)Pg_1(s, x_s)ds\\
&- \int_{t+s_n}^{\infty} T(t+s_n-s)Qg_1(s, x_s)ds + \int_{-\infty}^{t+s_n} T(t+s_n-s)Pf_1(s, x_s)dW(s)\\
&- \int_{t+s_n}^{\infty} T(t+s_n-s)f_1(s, x_s)dW(s) + \sum_{t_i < t+s_n} T(t+s_n-t_i)PI_{i,1}(x(t_i))\\
&- \sum_{t+s_n < t_i} T(t+s_n-t_i)QI_{i,1}(x(t_i))\\
=&h_1(t+s_n, x_{t+s_n}) + \int_{-\infty}^{t} AT(t-s)Ph_1(s+s_n, x_{s+s_n})ds\\
&- \int_{t}^{\infty} AT(t-s)Qh_1(s+s_n, x_{s+s_n})ds + \int_{-\infty}^{t} T(t-s)Pg_1(s+s_n, x_{s+s_n})ds\\
&- \int_{t}^{\infty} T(t-s)Qg_1(s+s_n, x_{s+s_n})ds + \int_{-\infty}^{t} T(t-s)Pf_1(s+s_n, x_{s+s_n})d\widetilde{W}_n(s)\\
&- \int_{0}^{\infty} T(t-s)Qf_1(s+s_n, x_{s+s_n})d\widetilde{W}_n(s) + \sum_{t_i < t+s_n} U(t+s_n-t_i)PI_{i,1}(x(t_i))\\
&- \sum_{t+s_n < t_i} T(t+s_n-t_i)QI_{i,1}(x(t_i)).
\end{aligned}$$

Consider the process

$$\begin{aligned}
x_n(t) =& h_1(t+s_n,x_{n,t}) + \int_{-\infty}^{t} AT(t-s)Ph_1(s+s_n,x_{n,s})ds \\
&- \int_{t}^{\infty} AT(t-s)Qh_1(s+s_n,x_{n,s})ds + \int_{-\infty}^{t} T(t-s)Pg_1(s+s_n,x_{n,s})ds \\
&- \int_{t}^{\infty} T(t-s)Qg_1(s+s_n,x_{n,s})ds + \int_{-\infty}^{t} T(t-s)Pf_1(s+s_n,x_{n,s})dW(s) \\
&- \int_{0}^{\infty} T(t-s)Qf_1(s+s_n,x_{n,s})dW(s) + \sum_{t_i<t+s_n} T(t+s_n-t_i)PI_{i,1}(x_n(t_i)) \\
&- \sum_{t+s_n<t_i} T(t+s_n-t_i)QI_{i,1}(x_n(t_i)).
\end{aligned}$$

It is easy to see that $(\Psi x)(t+s_n)$ has the same distribution as $x_n(t)$ for each $t \in \mathbb{R}$.

Let $\tilde{x}(t)$ satisfy the integral equation

$$\begin{aligned}
\tilde{x}(t) = & \tilde{h}_1(t,\tilde{x}_t) + \int_{-\infty}^{t} AT(t-s)P\tilde{h}_1(s,\tilde{x}_s)ds - \int_{t}^{\infty} AT(t-s)Q\tilde{h}_1(s,\tilde{x}_s)ds \\
&+ \int_{-\infty}^{t} T(t-s)P\tilde{g}_1(s,\tilde{x}_s)ds - \int_{t}^{\infty} T(t-s)Q\tilde{g}_1(s,\tilde{x}_s)ds \\
&+ \int_{-\infty}^{t} T(t-s)P\tilde{f}_1(s,\tilde{x}_s)dW(s) - \int_{t}^{\infty} T(t-s)Q\tilde{f}_1(s,\tilde{x}_s)dW(s) \\
&+ \sum_{t_i<t} T(t-t_i)P\tilde{I}_{i,1}(\tilde{x}(t_i)) - \sum_{t<t_i} T(t-t_i)Q\tilde{I}_{i,1}(\tilde{x}(t_i)).
\end{aligned}$$

Then we have

$$\begin{aligned}
& E \parallel x_n(t) - \tilde{x}(t) \parallel_\alpha^p \\
\leq & 9^{p-1} E \parallel h_1(t+s_n,x_{n,t}) - \tilde{h}_1(t,\tilde{x}_t) \parallel_\alpha^p \\
&+ 9^{p-1} E \left\| \int_{-\infty}^{t} AT(t-s)P[h_1(s+s_n,x_{n,s}) - \tilde{h}_1(s,\tilde{x}_s)]ds \right\|_\alpha^p \\
&+ 9^{p-1} E \left\| \int_{t}^{\infty} AT(t-s)Q[h_1(s+s_n,x_{n,s}) - \tilde{h}_1(s,\tilde{x}_s)]ds \right\|_\alpha^p \\
&+ 9^{p-1} E \left\| \int_{-\infty}^{t} T(t-s)P[g_1(s+s_n,x_{n,s}) - \tilde{g}_1(s,\tilde{x}_s)]ds \right\|_\alpha^p \\
&+ 9^{p-1} E \left\| \int_{t}^{\infty} T(t-s)Q[g_1(s+s_n,x_{n,s}) - \tilde{g}_1(s,\tilde{x}_s)]ds \right\|_\alpha^p \\
&+ 9^{p-1} E \left\| \int_{-\infty}^{t} T(t-s)P[f_1(s+s_n,x_{n,s}) - \tilde{f}_1(s,\tilde{x}_s)]dW(s) \right\|_\alpha^p \\
&+ 9^{p-1} E \left\| \int_{t}^{\infty} T(t-s)Q[f_1(s+s_n,x_{n,s}) - \tilde{f}_1(s,\tilde{x}_s)]dW(s) \right\|_\alpha^p \\
&+ 9^{p-1} E \left\| \sum_{t_i<t+s_n} T(t+s_n-t_i)PI_{i,1}(x_n(t_i)) - \sum_{t_i<t} T(t-t_i)P\tilde{I}_{i,1}(\tilde{x}(t_i)) \right\|_\alpha^p
\end{aligned}$$

$$+ 9^{p-1}E\Big\| \sum_{t+s_n<t_i} T(t+s_n-t_i)QI_{i,1}(x_n(t_i)) - \sum_{t<t_i} T(t-t_i)Q\tilde{I}_{i,1}(\tilde{x}(t_i))\Big\|_\alpha^p$$
$$:= \sum_{j=1}^{9} \Psi_j.$$

By (H4), we have

$$\begin{aligned}\Psi_1 &\le 18^{p-1}k(\alpha)[E \parallel h_1(t+s_n, x_{n,t}) - h_1(t+s_n, \tilde{x}_t) \parallel_\beta^p \\ &\quad + E \parallel h_1(t+s_n, \tilde{x}_t) - \tilde{h}_1(t, \tilde{x}_t) \parallel_\beta^p] \\ &\le 18^{p-1}k(\alpha)[L_h \parallel x_{n,t} - \tilde{x}_t \parallel_{\mathcal{D}}^p + \varepsilon_n^{(1)}(t)] \\ &\le 18^{p-1}k(\alpha)[L_h \parallel x_n - \tilde{x} \parallel_\infty^p + \varepsilon_n^{(1)}(t)],\end{aligned}$$

where $\varepsilon_n^1(t) = E \parallel h_1(t+s_n, \tilde{x}_t) - \tilde{h}_1(t, \tilde{x}_t) \parallel_\beta^p$. By (3.6), we have $\lim_{n\to\infty} \varepsilon_n^{(1)} = 0$. Using (H4) and Hölder's inequality, we have

$$\begin{aligned}\Psi_2 &\le 18^{p-1}c^p\Big(\Gamma(1+\frac{p(\beta-\alpha-1)}{p-1})\gamma^{-\frac{p(\beta-\alpha-1)}{p-1}-1}\Big)^{p-1} \\ &\quad\times\Big[\Big(\int_{-\infty}^t e^{-\gamma(t-s)}E \parallel h_1(s+s_n, x_{n,s}) - h_1(s+s_n, \tilde{x}_s) \parallel_\beta^p ds\Big) \\ &\quad+\Big(\int_{-\infty}^t e^{-\gamma(t-s)}E \parallel h_1(s+s_n, \tilde{x}_s) - \tilde{h}_1(s, \tilde{x}_s) \parallel_\beta^p ds\Big)\Big] \\ &\le 18^{p-1}c^p\Big(\Gamma(1+\frac{p(\beta-\alpha-1)}{p-1})\gamma^{-\frac{p(\beta-\alpha-1)}{p-1}-1}\Big)^{p-1} \\ &\quad\times\Big[\Big(\int_{-\infty}^t e^{-\gamma(t-s)}L \parallel x_{n,s} - \tilde{x}_s \parallel_{\mathcal{B}}^p ds\Big) + \sup_{t\in\mathbb{R}} \varepsilon_n^{(2)}(t)\Big] \\ &\le 18^{p-1}c^p\Big(\Gamma(1+\frac{p(\beta-\alpha-1)}{p-1})\gamma^{-\frac{p(\beta-\alpha-1)}{p-1}-1}\Big)^{p-1} \\ &\quad\times[\frac{1}{\gamma}L_h \parallel x_n - \tilde{x} \parallel_\infty^p + \sup_{t\in\mathbb{R}} \varepsilon_n^{(2)}(t)],\end{aligned}$$

where $\varepsilon_n^{(2)}(t) = \sum_{j=1}^4 \varepsilon_n^{(2j)}(t)$, and

$$\varepsilon_n^{(21)}(t) = \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\gamma(t-s)}E \parallel h_1(s+s_n, \tilde{x}_s) - \tilde{h}_1(s, \tilde{x}_s) \parallel_\beta^p ds,$$
$$\varepsilon_n^{(22)}(t) = \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\gamma(t-s)}E \parallel h_1(s+s_n, \tilde{x}_s) - \tilde{h}_1(s, \tilde{x}_s) \parallel_\beta^p ds,$$
$$\varepsilon_n^{(23)}(t) = \sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\gamma(t-s)}E \parallel h_1(s+s_n, \tilde{x}_s) - \tilde{h}_1(s, \tilde{x}_s) \parallel_\beta^p ds,$$
$$\varepsilon_n^{(24)}(t) = \int_{t_i}^{t} e^{-\gamma(t-s)}E \parallel h_1(s+s_n, \tilde{x}_s) - \tilde{h}_1(s, \tilde{x}_s) \parallel_\beta^p ds.$$

By (3.6), there exists $N_1 \in \mathbb{N}$ such that

$$E \parallel h_1(s+s_n, \tilde{x}_s) - \tilde{h}_1(s, \tilde{x}_s) \parallel_\beta^p < \varepsilon$$

for all $s \in [t_j+\eta, t_{j+1}-\eta]$, $j \in \mathbb{Z}, j \le i$, and $t - s \ge t - t_i + t_i - (t_{j+1} - \eta) \ge s - t_i + \gamma(i-1-j) + \eta$, whenever $n \ge N_1$. Then,

$$\begin{aligned}
\varepsilon_n^{(21)}(t) &\le \varepsilon \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\gamma(t-s)} ds \\
&\le \frac{\varepsilon}{\gamma} \sum_{j=-\infty}^{i-1} e^{-\gamma(t-t_{j+1}+\eta)} \\
&\le \frac{\varepsilon}{\gamma} \sum_{j=-\infty}^{i-1} e^{-\gamma\varsigma(i-j-1)} \\
&\le \frac{\varepsilon}{\gamma(1-e^{-\gamma\varsigma})}, \\
\varepsilon_n^{(22)}(t) &\le 2^{p-1}[\parallel h_1 \parallel_{\beta,\infty}^p + \parallel \tilde{h}_1 \parallel_{\beta,\infty}^p] \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\gamma(t-s)} ds \\
&\le 2^{p-1}[\parallel h_1 \parallel_{\beta,\infty}^p + \parallel \tilde{h}_1 \parallel_{\beta,\infty}^p] \varepsilon e^{\gamma\eta} \sum_{j=-\infty}^{i-1} e^{-\gamma(t-t_j)} \\
&\le 2^{p-1}[\parallel h_1 \parallel_{\beta,\infty}^p + \parallel \tilde{h}_1 \parallel_{\beta,\infty}^p] \varepsilon e^{\gamma\eta} e^{-\gamma(t-t_i)} \sum_{j=-\infty}^{i-1} e^{-\gamma\varsigma(i-j)} \\
&\le \frac{2^{p-1}[\parallel h_1 \parallel_{\beta,\infty}^p + \parallel \tilde{h}_1 \parallel_{\beta,\infty}^p] e^{(\gamma\varsigma)/2}\varepsilon}{1-e^{-\gamma\varsigma}}.
\end{aligned}$$

Similarly, one has

$$\varepsilon_n^{(23)}(t) \le \tilde{M}_1 \varepsilon, \quad \varepsilon_n^{(24)}(t) \le \tilde{M}_2 \varepsilon,$$

where $\tilde{M}_1, \tilde{M}_2$ are some positive constants. Therefore, we get that $\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \varepsilon_n^{(2)}(t) = 0$. By a similar argument, we can show that

$$\Psi_3 \le 18^{p-1} c^p \frac{1}{\delta^{p-1}} [\frac{1}{\delta} L_h \parallel x_n - \tilde{x} \parallel_\infty^p + \sup_{t\in\mathbb{R}} \varepsilon_n^{(3)}(t)],$$

where $\varepsilon_n^{(3)}(t) = \int_t^\infty e^{(\delta/2)(t-s)} E \parallel h_1(s+s_n, \tilde{x}_s) - \tilde{h}_1(s, \tilde{x}_s) \parallel_\beta^p ds$ and $\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \varepsilon_n^{(3)}(t) = 0$.

By (H6), for any $\varepsilon > 0$, there exist $\delta_1 > 0$ and a bounded subset $K \subset \mathcal{B}$ such that $\varphi, \psi \subset K$ and $E \parallel \varphi - \psi \parallel_{\mathcal{B}}^p < \delta_1$ imply that

$$\begin{gathered}
E \parallel g_1(t,\varphi) - g_1(t,\psi) \parallel^p < \varepsilon, \\
E \parallel f_1(t,\varphi) - f_1(t,\psi) \parallel_{L_2^0}^p < \varepsilon
\end{gathered}$$

for each $t \in \mathbb{R}$. For the above $\delta_1 > 0$, there exists $N_1 > 0$ such that $\parallel x_{n,t} - \tilde{x}_t \parallel_{\mathcal{B}}^p < \delta_1$ for all $n > N_1$ and all $t \in \mathbb{R}$. Therefore,

$$E \parallel g_1(s+s_n, x_{n,s}) - g_1(s+s_n, \tilde{x}_s) \parallel^p < \varepsilon, \tag{3.10}$$
$$E \parallel f_1(s+s_n, x_{n,s}) - f_1(s+s_n, \tilde{x}_s) \parallel_{L_2^0}^p < \varepsilon \tag{3.11}$$

for all $n > N_1$ and all $s + s_n \in \mathbb{R}$. Using (3.10) and Hölder's inequality, we have

$$\begin{aligned}\Psi_4 \le\ & 18^{p-1}(M(\alpha))^p\left(\Gamma(1-\frac{p}{p-1}\alpha)\gamma^{\frac{p}{p-1}\alpha-1}\right)^{p-1}\\ & \times\Big[\int_{-\infty}^{t} e^{-\gamma(t-s)}E\parallel g_1(s+s_n,x_{n,s})-g_1(s+s_n,\tilde{x}_s)\parallel^p ds\\ & +\int_{-\infty}^{t} e^{-\gamma(t-s)}E\parallel g_1(s+s_n,\tilde{x}_s)-\tilde{g}_1(s,\tilde{x}_s)ds\Big]\\ \le\ & 18^{p-1}(M(\alpha))^p\left(\Gamma(1-\frac{p}{p-1}\alpha)\gamma^{\frac{p}{p-1}\alpha-1}\right)^{p-1}[\frac{1}{\gamma}\varepsilon+\sup_{t\in\mathbb{R}}\varepsilon_n^4(t)],\end{aligned}$$

where $\varepsilon_n^{(4)}(t)=\sum_j^4\varepsilon_n^{(4j)}(t)$, and

$$\begin{aligned}\varepsilon_n^{41}(t) &= \sum_{j=-\infty}^{i-1}\int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\gamma(t-s)}E\parallel g_1(s+s_n,\tilde{x}_s)-\tilde{g}_1(s,\tilde{x}_s)\parallel^p ds,\\ \varepsilon_n^{(42)}(t) &= \sum_{j=-\infty}^{i-1}\int_{t_j}^{t_j+\eta} e^{-\gamma(t-s)}E\parallel g_1(s+s_n,\tilde{x}_s)-\tilde{g}_1(s,\tilde{x}_s)\parallel^p ds,\\ \varepsilon_n^{(43)}(t) &= \sum_{j=-\infty}^{i-1}\int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\gamma(t-s)}E\parallel g_1(s+s_n,\tilde{x}_s)-\tilde{g}_1(s,\tilde{x}_s)\parallel^p ds,\\ \varepsilon_n^{(44)}(t) &= \int_{t_i}^{t} e^{-\gamma(t-s)}E\parallel g_1(s+s_n,\tilde{x}_s)-\tilde{g}_1(s,\tilde{x}_s)\parallel^p ds.\end{aligned}$$

By (3.7), there exists $N_2 \in \mathbb{N}$ such that

$$E\parallel g_1(s+s_n,\tilde{x}_s)-\tilde{g}_1(s,\tilde{x}_s)\parallel^p<\varepsilon$$

for all $s \in [t_j+\eta, t_{j+1}-\eta]$, $j \in \mathbb{Z}, j \le i$, and $t-s \ge t-t_i+t_i-(t_{j+1}-\eta) \ge s-t_i+\gamma(i-1-j)+\eta$, whenever $n \ge N_2$. Then,

$$\begin{aligned}\varepsilon_n^{(41)}(t) &\le \varepsilon\sum_{j=-\infty}^{i-1}\int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\gamma(t-s)}ds\\ &\le \frac{\varepsilon}{\gamma}\sum_{j=-\infty}^{i-1} e^{-\gamma(t-t_{j+1}+\eta)}\\ &\le \frac{\varepsilon}{\gamma}\sum_{j=-\infty}^{i-1} e^{-\gamma\varsigma(i-j-1)}\\ &\le \frac{\varepsilon}{\gamma(1-e^{-\gamma\varsigma})},\\ \varepsilon_n^{(42)}(t) &\le 2^{p-1}[\parallel g_1\parallel_\infty^p+\parallel\tilde{g}_1\parallel_\infty^p]\sum_{j=-\infty}^{i-1}\int_{t_j}^{t_j+\eta} e^{-\gamma(t-s)}ds\\ &\le 2^{p-1}[\parallel g_1\parallel_\infty^p+\parallel\tilde{g}_1\parallel_\infty^p]\varepsilon e^{\gamma\eta}\sum_{j=-\infty}^{i-1} e^{-\gamma(t-t_j)}\end{aligned}$$

$$\le 2^{p-1}[\| g_1 \|_\infty^p + \| \tilde{g}_1 \|_\infty^p]\varepsilon e^{\gamma\eta} e^{-\gamma(t-t_i)} \sum_{j=-\infty}^{i-1} e^{-\gamma\varsigma(i-j)}$$

$$\le \frac{2^{p-1}[\| g_1 \|_\infty^p + \| \tilde{g}_1 \|_\infty^p]e^{(\gamma\varsigma)/2}\varepsilon}{1-e^{-\gamma\varsigma}}.$$

Similarly, one has

$$\varepsilon_n^{(43)}(t) \le \tilde{M}_3\varepsilon, \quad \varepsilon_n^{(44)}(t) \le \tilde{M}_4\varepsilon,$$

where $\tilde{M}_3, \tilde{M}_4$ are some positive constants. Therefore, we get that $\lim_{n\to\infty}\sup_{t\in\mathbb{R}} \varepsilon_n^{(4)}(t) = 0$. By a similar argument, we can show that

$$\Psi_5 \le 18^{p-1}(C(\alpha))^p \frac{1}{\delta^{p-1}}[\frac{1}{\delta}\varepsilon + \sup_{t\in\mathbb{R}} \varepsilon_n^{(5)}(t)],$$

where $\varepsilon_n^{(5)}(t) = \int_t^\infty e^{\delta(t-s)} E \parallel g_1(s+s_n, \tilde{x}_s) - \tilde{g}_1(s, \tilde{x}_s) \parallel^p ds$ and $\lim_{n\to\infty}\sup_{t\in\mathbb{R}} \varepsilon_n^{(5)}(t) = 0$.

Using (3.11) and the Ito integral, we have for $p > 2$,

$$\Psi_6 \le 18^{p-1}(M(\alpha))^p C_p \left(\Gamma(1-\frac{p}{p-2}\alpha)(\frac{p}{p-2}\gamma)^{\frac{p}{p-2}\alpha-1}\right)^{\frac{p-2}{p}}$$
$$\times \Big[\int_{-\infty}^t e^{-\frac{p}{2}\gamma(t-s)} E \parallel f_1(s+s_n, x_{n,s}) - f_1(s+s_n, \tilde{x}_s) \parallel_{L_2^0}^p ds$$
$$+ \int_{-\infty}^t e^{-\frac{p}{2}\gamma(t-s)} E \parallel f_1(s+s_n, \tilde{x}_s - \tilde{f}_1(s, \tilde{x}_s) \parallel_{L_2^0}^p ds\Big]$$
$$\le 18^{p-1}(M(\alpha))^p C_p \left(\Gamma(1-\frac{p}{p-2}\alpha)(\frac{p}{p-2}\gamma)^{\frac{p}{p-2}\alpha-1}\right)^{\frac{p-2}{p}}$$
$$\times [\frac{2}{p\gamma}\varepsilon + \sup_{t\in\mathbb{R}} \varepsilon_n^{(6)}(t)],$$

where $\varepsilon_n^{(6)}(t) = \sum_{j=1}^4 \varepsilon_n^{(6j)}(t)$, and

$$\varepsilon_n^{(61)}(t) = \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\frac{p}{2}\gamma(t-s)} E \parallel f_1(s+s_n, \tilde{x}_s) - \tilde{f}_1(s, \tilde{x}_s) \parallel_{L_2^0}^p ds,$$
$$\varepsilon_n^{(62)}(t) = \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\frac{p}{2}\gamma(t-s)} E \parallel f_1(s+s_n, \tilde{x}_s) - \tilde{f}_1(s, \tilde{x}_s) \parallel_{L_2^0}^p ds,$$
$$\varepsilon_n^{(63)}(t) = \sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\frac{p}{2}\gamma(t-s)} E \parallel f_1(s+s_n, \tilde{x}_s) - \tilde{f}_1(s, \tilde{x}_s) \parallel_{L_2^0}^p ds,$$
$$\varepsilon_n^{(64)}(t) = \int_{t_i}^t e^{-\frac{p}{2}\gamma(t-s)} E \parallel f_1(s+s_n, \tilde{x}_s) - \tilde{f}_1(s, \tilde{x}_s) \parallel_{L_2^0}^p ds.$$

By (3.8), there exists $N_3 \in \mathbb{N}$ such that

$$E \parallel f_1(s+s_n, \tilde{x}_s) - \tilde{f}_1(s, \tilde{x}_s) \parallel_{L_2^0}^p < \varepsilon$$

for all $s \in [t_j+\eta, t_{j+1}-\eta]$, $j \in \mathbb{Z}, j \leq i$, and $t-s \geq t-t_i+t_i-(t_{j+1}-\eta) \geq s-t_i+\gamma(i-1-j)+\eta$, whenever $n \geq N_3$. Then,

$$\begin{aligned}
\varepsilon_n^{(61)}(t) &\leq \varepsilon \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\frac{p}{2}\gamma(t-s)} ds \\
&\leq \frac{2}{\gamma p} \sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\varsigma(t-t_{j+1}+\eta)} \\
&\leq \frac{2\varepsilon}{\gamma p} \sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\gamma\varsigma(i-j-1)} \\
&\leq \frac{2\varepsilon}{\gamma p(1-e^{-(\varsigma\gamma/2)})}, \\
\varepsilon_n^{(62)}(t) &\leq 2^{p-1}[\| f_1 \|_\infty^p + \| \tilde{f}_1 \|_\infty^p] \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_{j+1}+\eta} e^{-\frac{p}{2}\gamma(t-s)} ds \\
&\leq 2^{p-1}[\| f_1 \|_\infty^p + \| \tilde{f}_1 \|_\infty^p]\varepsilon e^{\frac{p}{2}\gamma\eta} e^{-\frac{p}{4}\varsigma(t-t_i)} \sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\gamma\varsigma(i-j)} \\
&\leq 2^{p-1}[\| f_1 \|_\infty^p + \| \tilde{f}_1 \|_\infty^p]\varepsilon e^{\frac{p}{2}\gamma\eta} e^{-\frac{p}{2}\gamma(t-t_i)} \sum_{j=-\infty}^{i-1} e^{-\frac{p}{2}\gamma\varsigma(i-j)} \\
&\leq \frac{2^{p-1}[\| f_1 \|_\infty^p + \| \tilde{f}_1 \|_\infty^p] e^{(\gamma\varsigma)/4}\varepsilon}{1-e^{-\frac{p}{2}\gamma\varsigma}}.
\end{aligned}$$

Similarly, one has

$$\varepsilon_n^{(63)}(t) \leq \tilde{M}_5\varepsilon, \quad \varepsilon_n^{(64)}(t) \leq \tilde{M}_6\varepsilon,$$

where $\tilde{M}_5, \tilde{M}_6$ are some positive constants. Therefore, we get that $\lim_{n\to\infty} \varepsilon_n^{(6)}(t) = 0$. By a similar argument, we can show that

$$\Psi_7 \leq 18^{p-1}(C(\alpha))^p C_p \left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{p}} [\frac{2}{p\delta}\varepsilon + \sup_{t\in\mathbb{R}} \varepsilon_n^{(7)}(t)],$$

where $\varepsilon_n^{(7)}(t) = \int_t^\infty e^{(\delta/2)(t-s)} E \parallel f_1(s+s_n, \tilde{x}_s) - \tilde{f}_1(s, \tilde{x}_s) \parallel_{L_2^0}^p ds$ and $\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \varepsilon_n^{(7)}(t) = 0$.

For $p = 2$, we have

$$\begin{aligned}
\Psi_6 &\leq 9(M(\alpha))^2 E \int_{-\infty}^t (t-s)^{-2\alpha} e^{-2\gamma(t-s)} \parallel f_1(s+s_n, x_{n,s}) - \tilde{f}_1(s, \tilde{x}_s) \parallel_{L_2^0}^2 ds \\
&\leq 18(M(\alpha))^2 \varsigma^{-2\alpha} \Big[\int_{-\infty}^t e^{-2\gamma(t-s)} \parallel f_1(s+s_n, x_{n,s}) - f_1(s+s_n, \tilde{x}_s) \parallel_{L_2^0}^2 ds \\
&\quad + \int_{-\infty}^t e^{-2\gamma(t-s)} \parallel f_1(s+s_n, \tilde{x}_s) - \tilde{f}_1(s, \tilde{x}_s) \parallel_{L_2^0}^2 ds \Big] \\
&\leq 18(M(\alpha))^2 \varsigma^{-2\alpha} [\frac{1}{2\gamma}\varepsilon + \sup_{t\in\mathbb{R}} \varepsilon_n^{(6)}],
\end{aligned}$$

where $\varepsilon_n^{(6)} = \sum_{j=1}^{4} \varepsilon_n^{(6j)}$, and

$$\varepsilon_n^{(61)} = \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-2\gamma(t-s)} E \parallel f_1(s+s_n, \tilde{x}_s) - \tilde{f}_1(s, \tilde{x}_s) \parallel_{L_2^0}^2 ds,$$

$$\varepsilon_n^{(62)} = \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-2\gamma(t-s)} E \parallel f_1(s+s_n, \tilde{x}_s) - \tilde{f}_1(s, \tilde{x}_s) \parallel_{L_2^0}^2 ds,$$

$$\varepsilon_n^{(63)} = \sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} e^{-2\gamma(t-s)} E \parallel f_1(s+s_n, \tilde{x}_s) - \tilde{f}_1(s, \tilde{x}_s) \parallel_{L_2^0}^2 ds,$$

$$\varepsilon_n^{(64)} = \int_{t_i}^{t} e^{-2\gamma(t-s)} E \parallel f_1(s+s_n, \tilde{x}_s) - \tilde{f}_1(s, \tilde{x}_s) \parallel_{L_2^0}^2 ds.$$

Similarly, we get that $\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \varepsilon_n^{(6)}(t) = 0$. By a similar argument, we can show that

$$\Psi_7 \le 18(C(\alpha))^2 [\frac{1}{2\delta}\varepsilon + \sup_{t\in\mathbb{R}} \varepsilon_n^{(7)}],$$

where $\varepsilon_n^{(7)}(t) = \int_t^{\infty} e^{2\delta(t-s)} E \parallel f_1(s+s_n, \tilde{x}_s) - \tilde{f}_1(s, \tilde{x}_s) \parallel_{L_2^0}^2 ds$ and $\lim_{n\to\infty} \sup_{t\in\mathbb{R}}$ $\varepsilon_n^{(7)}(t) = 0$.

Since $\{t_i^j\}, i \in_Z, j = 0, 1, ...,$ are equipotentially almost periodic, then for $\varepsilon > 0$, there exist the sequence of real numbers $\{s_n\}$ and sequence of integer numbers $\{\alpha_n\}$, such that $t_i < t \le t_{i+1}$, $|t-t_i| > \varepsilon, |t-t_{i+1}| > \varepsilon, i \in \mathbb{Z}$, one has $t+s_n > t+s_n+\varepsilon > t_{i+\alpha_n}$ and $t_{i+\alpha_n+1} > t_{i+1} + s_n - \varepsilon > t + s_n$, that is $t_{i+\alpha_n} < t + s_n < t_{i+\alpha_n+1}$. By (H6), for any $\varepsilon > 0$, there exist $\delta_2 > 0$ and a bounded subset $\tilde{K} \subset L^p(\mathbb{P}, \mathbb{K})$ such that $x, y \subset \tilde{K}$ and $E \parallel x - y \parallel^p < \delta_2$ imply that

$$E \parallel I_{i,1}(x(t_i)) - I_{i,1}(y(t_i)) \parallel^p < \varepsilon,$$

for $i \in \mathbb{Z}$. For the above $\delta_2 > 0$, there exists $N_4 > 0$ such that $E \parallel x_n(t_{i+\alpha_i}) - \tilde{x}(t_{i+\alpha_i}) \parallel^p$, $E \parallel \tilde{x}(t_{i+\alpha_n}) - \tilde{x}(t_i) \parallel^p < \delta_2$ for all $n > N_4$ and all $i \in \mathbb{Z}$. Therefore,

$$E \parallel I_{i+\alpha_n,1}(x_n(t_{i+\alpha_i})) - I_{i+\alpha_n,1}(\tilde{x}(t_{i+\alpha_i})) \parallel^p < \varepsilon, \tag{3.12}$$

$$E \parallel I_{i+\alpha_n,1}(\tilde{x}(t_{i+\alpha_n})) - I_{i+\alpha_n,1}(\tilde{x}(t_i)) \parallel^p < \varepsilon \tag{3.13}$$

for all $n > N_4$ and all $s + s_n \in \mathbb{R}$. Then by (3.12), (3.13) and Hölder's inequality, we have

$$\begin{aligned}\Psi_8 \le\ & 54^{p-1}(M(\alpha))^p \varsigma^{-p\alpha} \left(\sum_{t_i<t} e^{-\gamma(t-t_i)} \right)^{p-1} \\ & \times \left(\sum_{t_i<t} e^{-\gamma(t-t_i)} [E \parallel I_{i+\alpha_n,1}(x_n(t_i+\alpha_n)) - I_{i+\alpha_n,1}(\tilde{x}(t_i+\alpha_n)) \parallel^p \right. \\ & + E \parallel I_{i+\alpha_n,1}(\tilde{x}(t_{i+\alpha_n})) - I_{i+\alpha_n,1}(\tilde{x}(t_i)) \parallel^p \\ & \left. + E \parallel I_{i+\alpha_n,1}(\tilde{x}(t_i)) - \tilde{I}_{i,1}(\tilde{x}(t_i)) \parallel^p] \right) \\ \le\ & \frac{54^{p-1}(M(\alpha))^p \varsigma^{-p\alpha}}{(1-e^{-\gamma\varsigma})^p} [2\varepsilon + \varepsilon_n^{(8)}],\end{aligned}$$

where $\varepsilon_n^{(8)} = E \parallel I_{i+\alpha_n,1}(\tilde{x}(t_i+\alpha_n)) - \tilde{I}_{i,1}(\tilde{x}(t_i)) \parallel^p$. By (3.10), we have $\lim_{n\to\infty} \varepsilon_n^{(8)} = 0$. By a similar argument, we can show that

$$\Psi_9 \le \frac{54^{p-1}(C(\alpha))^p}{(1-e^{-\delta\varsigma})^p}[2\varepsilon + \varepsilon_n^{(9)}],$$

where $\varepsilon_n^{(9)} = E \parallel I_{i+\alpha_n,1}(\tilde{x}(t_i+\alpha_n)) - \tilde{I}_{i,1}(\tilde{x}(t_i)) \parallel^p$ and $\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \varepsilon_n^{(9)}(t) = 0$.

By above estimations, we have for all $t \in \mathbb{R}$,

$$E \parallel x_n(t) - \tilde{x}(t) \parallel_\alpha^p \le \tilde{\vartheta}_1 \varepsilon_n(t) + \tilde{\vartheta}\varepsilon,$$

where $\varepsilon_n(t) = \sum_{j=1}^9 \varepsilon_n^{(j)}(t)$, and $\tilde{\vartheta}_1, \tilde{\vartheta}$ are given constants. Therefore,

$$\sup_{t\in\mathbb{R}} E \parallel x_n(t) - \tilde{x}(t) \parallel_\alpha^p \le \tilde{\vartheta}_1 \sup_{t\in\mathbb{R}} \varepsilon_n(t) + \tilde{\vartheta}\varepsilon.$$

By $\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \varepsilon_n(t) = 0$ with ε sufficiently small, it follows that

$$\sup_{t\in\mathbb{R}} \parallel x_n(t) - \tilde{x}(t) \parallel_\alpha^p \to \infty \quad \text{as } n \to \infty$$

for all $t \in \mathbb{R}$. Since $(\Psi x)(t+s_n)$ has the the same distribution as $x_n(t)$, it follows that $(\Psi x)(t+s_n) \to \tilde{x}(t)$ in distribution as $n \to \infty$. Hence Ψx has almost periodic in one-dimensional distributions. Note that the sequence $(E \parallel x_n(t) \parallel^p)$ is uniformly integrable, thus $(E \parallel \Psi x(t+s_n) \parallel^p)$ is also uniformly integrable, so the family $(E \parallel x_n(t) \parallel^p)_{t\in\mathbb{R}}$ is uniformly integrable. Next, we prove that Ψx is almost periodic in distribution. For fixed $\tau \in \mathbb{R}$, let $\xi_n = \Psi x(\tau + s_n)$, $h_1^n = h_1(t+s_n,\psi)$, $g_1^n = g_1(t+s_n,\psi)$, $f_1^n = f_1(t+s_n,\psi)$ and $I_{i,1}^n = I_{i+\alpha_n,1}(x), i \in \mathbb{Z}$. By the foregoing, (ξ_n) converges in distribution to some variable $\Psi x(\tau)$. We deduce that (ξ_n) is tight, so (ξ_n, W) is tight also. We can thus choose a subsequence (still noted s_n for simplicity) such that (ξ_n, W) converges in distribution to $(\Psi x(\tau), W)$. Similarly as the proof of Properties 3.1 in [?], for every $T \ge \tau, \Psi x(\cdot + s_n)$ converges in distribution on $PC_T([\tau,T], L^p(\mathbb{P},\mathbb{H}))$ to the solution to

$$\begin{aligned} x(t) = {} & T(t-\tau)[\varphi(\tau) - h_1(\tau,\varphi)] + h_1(t,x_t) \\ & + \int_\tau^t AT(t-s)h_1(s,x_s)ds + \int_\tau^t T(t-s)g_1(s,x_s)ds \\ & + \int_\tau^t T(t-s)f_1(s,x_s)dW(s) + \sum_{\tau<t_i<t} T(t-t_i)I_{i,1}(x(t_i)). \end{aligned}$$

Note that Ψx does not depend on the chosen interval $[\tau, T]$, thus the convergence takes place on $PC_T([\tau,T], L^p(\mathbb{P},\mathbb{H}))$. Hence Ψx is almost periodic in distribution.

(2) $\Upsilon x \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}), q)$.

In fact, for $r > 0$, one has

$$\begin{aligned} & \frac{1}{2r}\int_{-r}^r \sup_{\theta\in[t-q,t]} E \parallel \Upsilon(\theta) \parallel_\alpha^p dt \\ \le & 9^{p-1}\frac{1}{2r}\int_{-r}^r \sup_{\theta\in[t-q,t]} E \parallel h_2(\theta,x_\theta) \parallel_\alpha^p dt \\ & + 9^{p-1}\frac{1}{2r}\int_{-r}^r \sup_{\theta\in[t-q,t]} E \left\| \int_{-\infty}^\theta AT(\theta-s)Ph_2(s,x_s)ds \right\|_\alpha^p dt \end{aligned}$$

$$
\begin{aligned}
&+9^{p-1}\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]}E\left\|\int_{\theta}^{\infty}AT(\theta-s)Qh_2(s,x_s)ds\right\|_{\alpha}^{p}dt\\
&+9^{p-1}\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]}E\left\|\int_{-\infty}^{\theta}T(\theta-s)Pg_2(s,x_s)ds\right\|_{\alpha}^{p}dt\\
&+9^{p-1}\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]}E\left\|\int_{\theta}^{\infty}T(\theta-s)Qg_2(s,x_s)ds\right\|_{\alpha}^{p}dt\\
&+9^{p-1}\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]}E\left\|\int_{-\infty}^{\theta}T(\theta-s)Pf_2(s,x_s)dW(s)\right\|_{\alpha}^{p}dt\\
&+9^{p-1}\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]}E\left\|\int_{\theta}^{\infty}T(\theta-s)Qf_2(s,x_s)dW(s)\right\|_{\alpha}^{p}dt\\
&+9^{p-1}\frac{1}{2r}\int_{-r}^{r}E\left\|\sum_{t_i<t}T(t-t_i)PI_{i,2}(x(t_i))\right\|_{\alpha}^{p}\\
&+9^{p-1}\frac{1}{2r}\int_{-r}^{r}E\left\|\sum_{t<t_i}T(t-t_i)QI_{i,2}(x(t_i))\right\|_{\alpha}^{p}\\
:=&\sum_{j=1}^{9}\Pi_j.
\end{aligned}
$$

By (H3), we have

$$
\Pi_1\le 9^{p-1}\parallel A^{\alpha-\beta}\parallel^p\ \frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]}E\parallel h_2(\theta,x_\theta)\parallel_\beta^p dt\to 0\quad\text{as } r\to\infty.
$$

As to Π_2, we have

$$
\begin{aligned}
\Pi_2&=\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]}E\left\|\int_{0}^{\infty}AT(s)Ph_2(\theta-s,x_{\theta-s})ds\right\|_{\alpha}^{p}dt\\
&\le c^p\frac{1}{2r}\int_{-r}^{r}\left(\int_0^\infty s^{\frac{p}{p-1}(\beta-\alpha-1)}e^{-\gamma s}ds\right)^{p-1}\\
&\quad\times\int_0^\infty e^{-\gamma s}\sup_{\theta\in[t-q,t]}E\parallel h_2(\theta-s,x_{\theta-s})\parallel_\beta^p dsdt\\
&=c^p\left(\int_0^\infty s^{\frac{p}{p-1}(\beta-\alpha-1)}e^{-\gamma s}ds\right)^{p-1}\int_0^\infty e^{-\gamma s}ds\\
&\quad\times\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]}E\parallel h_2(\theta-s,x_{\theta-s})\parallel_\beta^p dt.
\end{aligned}
$$

Since $h_2\in PAP_T^0(\mathbb{R}\times\mathcal{B},L^p(\mathbb{P},\mathbb{H}_\beta),q)$, it follows that

$$
\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]}E\left\|\int_{-\infty}^{\theta}AT(\theta-s)Ph_2(s,x_s)ds\right\|_{\alpha}^{p}dt\to 0\quad\text{as } r\to\infty
$$

for all $s\in\mathbb{R}$. Using the Lebesgue's dominated convergence theorem, we have $\Pi_2\to 0$ as $r\to\infty$. Similarly, we can show that $\Pi_3\to 0$ as $r\to\infty$.

As to Π_4, we have

$$
\Pi_4=9^{p-1}\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]}E\left\|\int_{0}^{\infty}T(s)Pg_2(\theta-s,x_{\theta-s})ds\right\|_{\alpha}^{p}dt
$$

$$\leq 9^{p-1}(M(\alpha))^p \frac{1}{2r}\int_{-r}^{r}\left(\int_0^\infty s^{-\frac{p}{p-1}\alpha}e^{-\gamma s}ds\right)^{p-1}$$
$$\times \int_0^\infty e^{-\gamma s}\sup_{\theta\in[t-q,t]} E\parallel g_2(\theta-s,x_{\theta-s})\parallel^p dsdt$$
$$= 9^{p-1}(M(\alpha))^p\left(\int_0^\infty s^{-\frac{p}{p-1}\alpha}e^{-\gamma s}ds\right)^{p-1}$$
$$\times \int_0^\infty e^{-\gamma s}ds\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]} E\parallel g_2(\theta-s,x_{\theta-s})\parallel^p dt.$$

Since $g_2 \in PAP_T^0(\mathbb{R}\times\mathcal{B}, L^p(\mathbb{P},\mathbb{H}_\beta), q)$, it follows that

$$\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-p,t]} E\left\|\int_{-\infty}^{\theta}T(\theta-s)Pg_2(s,x_s)ds\right\|_\alpha^p dt \to 0 \quad \text{as } r\to\infty$$

for all $s\in\mathbb{R}$. Using the Lebesgue's dominated convergence theorem, we have $\Pi_4\to 0$ as $r\to\infty$. Similarly, we can show that $\Pi_5 \to 0$ as $r\to\infty$.

As to Π_6, we have for $p>2$,

$$\Pi_6 = 9^{p-1}\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]} E\left\|\int_0^\infty T(s)Pf_2(\theta-s,x_{\theta-s})dW(s)\right\|_\alpha^p dt$$
$$\leq 9^{p-1}(M(\alpha))^pC_p$$
$$\times \frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]} E\left[\int_0^\infty s^{-2\alpha}e^{-2\gamma s}\parallel f_2(\theta-s,x_{\theta-s})\parallel_{L_2^0}^2 ds\right]^{p/2}dt$$
$$\leq 9^{p-1}(M(\alpha))^pC_p\frac{1}{2r}\int_{-r}^{r}\left(\int_0^\infty s^{-\frac{p}{p-2}\alpha}e^{-\frac{p}{p-2}\gamma s}ds\right)^{\frac{p-2}{p}}$$
$$\times\int_0^\infty e^{-\frac{p}{2}\gamma s}\sup_{\theta\in[t-q,t]} E\parallel f_2(\theta-s,x_{\theta-s})\parallel_{L_2^0}^p dsdt$$
$$= 9^{p-1}(M(\alpha))^pC_p\left(\int_0^\infty s^{-\frac{p}{p-2}\alpha}e^{-\frac{p}{p-2}\gamma s}ds\right)^{\frac{p-2}{p}}\int_0^\infty e^{-\frac{p}{2}\gamma s}ds$$
$$\times\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]} E\parallel f_2(\theta-s,x_{\theta-s})\parallel_{L_2^0}^p dt.$$

For $p=2$, we have

$$\Pi_6 \leq 9(M(\alpha))^2\frac{1}{2r}\int_{-r}^{r}\int_0^\infty s^{-2\alpha}e^{-2\gamma s}\sup_{\theta\in[t-q,t]} E\parallel f_2(\theta-s,x_{\theta-s})\parallel_{L_2^0}^2 dsdt$$
$$= 9(M(\alpha))^2\left(\int_0^\infty s^{-2\alpha}e^{-2\gamma s}ds\right)\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]} E\parallel f_2(\theta-s,x_{\theta-s})\parallel_{L_2^0}^2 dt.$$

Since $f_2\in PAP_T^0(\mathbb{R}\times\mathcal{B}, L^p(\mathbb{P},L_2^0), q)$, it follows that

$$\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[t-q,t]} E\left\|\int_{-\infty}^{\theta}T(\theta-s)Pf_2(s,x_s)dW(s)\right\|_\alpha^2 dt\to 0 \quad \text{as } r\to\infty$$

for all $s\in\mathbb{R}$. Using the Lebesgue's dominated convergence theorem, we have $\Pi_6\to 0$ as $r\to\infty$. Similarly, we can show that $\Pi_7\to 0$ as $r\to\infty$.

For a given $i \in \mathbb{Z}$, define the function $(\mathcal{V}x)(t)$ by $(\mathcal{V}x)(t) = T(t-t_i)PI_{i,2}(x(t_i)), t_i < t \leq t_{i+1}$, then

$$\begin{aligned}\lim_{t\to\infty} \sup_{\theta\in[t-q,t]} E \parallel (\mathcal{V}x)(\theta) \parallel_\alpha^p &= \lim_{t\to\infty} \sup_{\theta\in[t-q,t]} E \parallel T(\theta - t_i)PI_{i,2}(x(t_i)) \parallel_\alpha^p \\ &\leq \lim_{t\to\infty} (M(\alpha))^p (t-t_i)^{-p\alpha} e^{-p\gamma(t-t_i)} \sup_{i\in\mathbb{Z}} E \parallel I_{i,2} \parallel_\infty^p \\ &\leq \lim_{t\to\infty} (M(\alpha))^p \varsigma^{-p\alpha} e^{-p\gamma(t-t_i)} \sup_{i\in\mathbb{Z}} E \parallel I_{i,2} \parallel_\infty^p = 0.\end{aligned}$$

Thus $\mathcal{V}x \in PC_T^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}), q) \subset PAP_T^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}), q)$. Define $\mathcal{V}_j x : \mathbb{R} \to L^p(\mathbb{P}, \mathbb{H})$ by

$$(\mathcal{V}_j x)(t) = T(t-t_{i-j})PI_{i-j,2}(x(t_i)), t_i < t \leq t_{i+1}, j \in \mathbb{N}.$$

So $\mathcal{V}_j x \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}), q)$. Moreover,

$$\begin{aligned}\sup_{\theta\in[t-q,t]} E \parallel (\mathcal{V}_j x)(\theta) \parallel^p &= \sup_{\theta\in[t-q,t]} E \parallel T(\theta - t_{i-j})PI_{i-j,2}(x(t_i)) \parallel^p \\ &\leq (M(\alpha))^p (t-t_{i-j})^{-p\alpha} e^{-p\gamma(t-t_{i-j})} \sup_{i\in\mathbb{Z}} E \parallel I_{i-j,2} \parallel_\infty^p \\ &\leq (M(\alpha))^p \varsigma^{-p\alpha} e^{-p\gamma(t-t_i)} e^{-p\gamma\varsigma j} \sup_{i\in\mathbb{Z}} E \parallel I_{i-j,2} \parallel_\infty^p .\end{aligned}$$

Therefore, the series $\sum_{j=0}^{\infty} \mathcal{V}_j x$ is uniformly convergent on $\mathbb{R}$. By Lemma 2.3, one has

$$\sum_{t_i<t} T(t-t_i)I_{i,2}(x(t_i)) = \sum_{j=0}^{\infty} (\mathcal{V}_j x)(t) \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}), q),$$

that is

$$\frac{1}{2r}\int_{-r}^{r} \sup_{\theta\in[t-q,t]} E \left\| \sum_{t_i<t} T(\theta-t_i)PI_{i,2}(x(t_i)) \right\|_\alpha^p dt \to 0 \quad \text{as } r\to\infty.$$

Using the Lebesgue's dominated convergence theorem, we have $\Pi_8 \to 0$ as $r \to \infty$. Similarly, we can show that $\Pi_9 \to 0$ as $r \to \infty$.

Hence

$$\lim_{n\to\infty} \frac{1}{2r}\int_{-r}^{r} \sup_{\theta\in[t-q,t]} E \parallel \Upsilon(\theta) \parallel_\alpha^p dt = 0,$$

which is mean that $\Upsilon x \in PAP_T^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}_\alpha), q)$. Therefore, x is piecewise pseudo almost periodic in distribution mild solution to system (1.1)-(1.2). □

4. Existence of optimal mild solutions

In this section we give the existence of optimal mild solution for the system (1.1)-(1.2). Next, denote by Ω_f the set of all mild solutions $x(t)$ to the system (1.1)-(1.2) which are bounded over $\mathbb{R}$, that is $\mu^*(x) = \sup_{t\in\mathbb{R}} E \parallel x(t) \parallel_\alpha^p$. Assume here that $\Omega_f \neq \emptyset$, and recall the following definition.

Definition 4.1. A bounded mild solution $x^*(t)$ to the system (1.1)-(1.2) is called an optimal mild solution to the systems (1.1)-(1.2) if $\mu^*(x^*) \equiv \mu^{**} = \inf_{x\in\Omega_f} \mu^*(x)$.

Our proof is based on the following lemma.

Lemma 4.1 ([15]). *If D is a nonempty convex and closed subset of a uniformly convex Banach space X and $v \notin D$, then there exists a unique $k_0 \in D$ such that* $\| v - k_0 \| = \inf_{k \in D} \| v - k \|$.

To study the optimal mild solutions to the system (1.1)-(1.2), we require the following assumption.

(S1) The functions $h, g : \mathbb{R} \times \mathcal{B} \to L^p(\mathbb{P}, \mathbb{H})$, $f : \mathbb{R} \times \mathcal{B} \to L^p(\mathbb{P}, L_2^0)$, $I_i : L^p(\mathbb{P}, \mathbb{K}) \to L^p(\mathbb{P}, \mathbb{H})$, are nontrivial functions. Moreover h, g, f, I_i are convex in $\psi \in \mathcal{B}, x \in L^p(\mathbb{P}, \mathbb{K})$ for all $i \in \mathbb{Z}$.

Theorem 4.1. *If the assumption (S1) and the assumptions of Theorem 3.1 hold. Then the system* (1.1)-(1.2) *has an optimal mild solution.*

Proof. It suffices to prove that Ω_f is a convex and closed set because the trivial solution $0 \notin \Omega_f$, then we use Lemma 4.1 to deduce the uniqueness of the optimal mild solution. For the convexity of Ω_f, we consider two distinct bounded pseudo almost periodic in distribution mild solution $x_1(t), x_2(t) \in \Omega_f$ and a real number $0 \leq \lambda \leq 1$, let $x(t) = \lambda x_1(t) + (1 - \lambda) x_2(t), t \in \mathbb{R}$. By (H1-(H6) and (S1), we get that $x(t)$ is continuous for every $t \in \mathbb{R}$, and for any $t \in \mathbb{R}$, we have

$$
\begin{aligned}
x(t) = {} & [\lambda h(t, x_{1,t}) + (1 - \lambda) h(t, x_{2,t})] \\
& + \int_{-\infty}^{t} AT(t-s) P[\lambda h(s, x_{1,s}) + (1 - \lambda) h(s, x_{2,s})] ds \\
& + \int_{t}^{\infty} AT(t-s) Q[\lambda h(s, x_{1,s}) + (1 - \lambda) h(s, x_{2,s})] ds \\
& + \int_{-\infty}^{t} T(t-s) P[\lambda g(s, x_{1,s}) + (1 - \lambda) g(s, x_{2,s})] ds \\
& + \int_{t}^{\infty} T(t-s) Q[\lambda g(s, x_{1,s}) + (1 - \lambda) g(s, x_{2,s})] ds \\
& + \int_{-\infty}^{t} T(t-s) P[\lambda f(s, x_{1,s}) + (1 - \lambda) f(s, x_{2,s})] dW(s) \\
& + \int_{t}^{\infty} T(t-s) Q[\lambda g(s, x_{1,s}) + (1 - \lambda) f(s, x_{2,s})] dW(s) \\
& + \sum_{t_i < t} T(t - t_i) P[\lambda I_i(x_1(t_i)) + (1 - \lambda) I_i(x_2(t_i))] \\
& + \sum_{t < t_i} T(t - t_i) Q[\lambda I_i(x_1(t_i)) + (1 - \lambda) I_i(x_2(t_i))] \\
= {} & h(s, \lambda x_{1,s} + (1 - \lambda) x_{2,s}) \\
& + \int_{-\infty}^{t} AT(t-s) P h(s, (\lambda x_{1,s} + (1 - \lambda) x_{2,s})) ds \\
& + \int_{t}^{\infty} AT(t-s) Q h(s, (\lambda x_{1,s} + (1 - \lambda) x_{2,s})) ds \\
& + \int_{-\infty}^{t} T(t-s) P g(s, (\lambda x_{1,s} + (1 - \lambda) x_{2,s})) ds \\
& + \int_{t}^{\infty} T(t-s) Q g(s, (\lambda x_{1,s} + (1 - \lambda) x_{2,s})) ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{-\infty}^{t} T(t-s)Pf(s,(\lambda x_{1,s}+(1-\lambda)x_{2,s}))dW(s)\\
&+\int_{t}^{\infty} T(t-s)Qf(s,(\lambda x_{1,s}+(1-\lambda)x_{2,s}))dW(s)\\
&+\sum_{t_i<t} T(t-t_i)PI_i(\lambda x_1(t_i)+(1-\lambda)x_2(t_i))\\
&+\sum_{t<t_i} T(t-t_i)QI_i(\lambda x_1(t_i)+(1-\lambda)x_2(t_i))\\
=&\, h(t,x_t)+\int_{-\infty}^{t} AT(t-s)Ph(s,x_s)ds-\int_{t}^{\infty} AT(t-s)Qh(s,x_s)ds\\
&+\int_{-\infty}^{t} T(t-s)Pg(s,x_s)ds-\int_{t}^{\infty} T(t-s)Qg(s,x_s)ds\\
&+\int_{-\infty}^{t} T(t-s)Pf(s,x_s)dW(s)-\int_{t}^{\infty} T(t-s)Qf(s,x_s)dW(s)\\
&+\sum_{t_i<t} T(t-t_i)PI_i(x(t_i))-\sum_{t<t_i} T(t-t_i)QI_i(x(t_i)).
\end{aligned}
$$

Then $x(t)$ is a pseudo almost periodic in distribution mild solution to the system (1.1)-(1.2). Note that $x(t)$ is bounded over $\mathbb{R}$ since

$$\mu^*(x)=\sup_{t\in\mathbb{R}} E \parallel x(t) \parallel_\alpha^p \leq \mu^*(x_1)+(1-\lambda)\mu^*(x_2)<\infty.$$

Thus $x(t)\in\Omega_f$.

Now we show that Ω_f is closed. Let a sequence $x_n\in\Omega_f$ such that $\lim_{n\to\infty} x_n(t) = x(t), t\in\mathbb{R}$, and

$$
\begin{aligned}
x_n(t) =&\, h(t,x_{n,t})+\int_{-\infty}^{t} AT(t-s)Ph(s,x_{n,s})ds-\int_{t}^{\infty} AT(t-s)Qh(s,x_{n,s})ds\\
&+\int_{-\infty}^{t} T(t-s)Pg(s,x_{n,s})ds-\int_{t}^{\infty} T(t-s)Qg(s,x_{n,s})ds\\
&+\int_{-\infty}^{t} T(t-s)Pf(s,x_{n,s})dW(s)-\int_{t}^{\infty} T(t-s)Qf(s,x_{n,s})dW(s)\\
&+\sum_{t_i<t} T(t-t_i)PI_i(x_n(t_i))-\sum_{t<t_i} T(t-t_i)QI_i(x_n(t_i)).
\end{aligned}
\tag{4.1}
$$

By (H1)-(H6), taking limits in (4.1), we have for every $t\in\mathbb{R}$,

$$
\begin{aligned}
x(t) =&\, h(t,x_t)+\int_{-\infty}^{t} AT(t-s)Ph(s,x_s)ds-\int_{t}^{\infty} AT(t-s)Qh(s,x_s)ds\\
&+\int_{-\infty}^{t} T(t-s)Pg(s,x_s)ds-\int_{t}^{\infty} T(t-s)Qg(s,x_s)ds\\
&+\int_{-\infty}^{t} T(t-s)Pf(s,x_s)dW(s)-\int_{t}^{\infty} T(t-s)Qf(s,x_s)dW(s)\\
&+\sum_{t_i<t} T(t-t_i)PI_i(x(t_i))-\sum_{t<t_i} T(t-t_i)QI_i(x(t_i)),
\end{aligned}
\tag{4.2}
$$

which means that $x(t)$ is a pseudo almost periodic in distribution mild solution to the system (1.1)-(1.2). Finally we show that $x(t)$ is bounded over $\mathbb{R}$. Indeed, we can write (4.2) as $x(t) = [x(t) - x_n(t)] + x_n(t)$ for all $t \in \mathbb{R}$. Then, for any $t \in \mathbb{R}$, we have

$$\begin{aligned} E \parallel x(t) \parallel_\alpha^p &\le 2^{p-1} E \parallel x(t) - x_n(t) \parallel_\alpha^p + 2^{p-1} E \parallel x_n(t) \parallel_\alpha^p \\ &\le \sum_{i=1}^{2} Z_i + 2^{p-1} E \parallel x_n(t) \parallel_\alpha^p . \end{aligned}$$

By Hölder's inequality and the Ito integral, we have for $p > 2$,

$$\begin{aligned} Z_1 \le\ & 18^{p-1} k(\alpha) E \parallel h(t, x_{n,t}) - h(t, x_t) \parallel_\beta^p \\ & + 18^{p-1} c^p \left(\Gamma(1 + \frac{p(\beta-\alpha-1)}{p-1}) \gamma^{-\frac{p(\beta-\alpha-1)}{p-1}-1} \right)^{p-1} \\ & \times \left(\int_{-\infty}^t e^{-\gamma(t-s)} E \parallel h(s, x_{n,s}) - h(s, x_s) \parallel_\beta^p ds \right) \\ & + 18^{p-1} (M(\alpha))^p \left(\Gamma(1 - \frac{p}{p-1}\alpha) \gamma^{\frac{p}{p-1}\alpha - 1} \right)^{p-1} \\ & \times \left(\int_{-\infty}^t e^{-\gamma(t-s)} E \parallel g(s, x_{n,s}) - g(s, x_s) \parallel^p ds \right) \\ & + 18^{p-1} (M(\alpha))^p C_p \left(\Gamma(1 - \frac{p}{p-2}\alpha) (\frac{p}{p-2}\gamma)^{\frac{p}{p-2}\alpha - 1} \right)^{\frac{p-2}{p}} \\ & \times \left(\int_{-\infty}^t e^{-\frac{p}{2}\gamma(t-s)} E \parallel f(s, x_{n,s}) - f(s, x_s) \parallel_{L_2^0}^p ds \right) \\ & + 18^{p-1} (M(\alpha))^p \varsigma^{-p\alpha} \frac{1}{(1 - e^{-\gamma\varsigma})^{p-1}} \\ & \times \left(\sum_{t_i < t} e^{-\gamma(t-t_i)} E \parallel I_i(x_n(t_i)) - I_i(x(t_i)) \parallel^p \right). \end{aligned}$$

For $p = 2$, we have

$$\begin{aligned} Z_1 \le\ & 18 k(\alpha) E \parallel h(t, x_{n,t}) - h(t, x_t) \parallel_\beta^2 \\ & + 5 c^p \left(\Gamma(1 + 2(\beta - \alpha - 1) \gamma^{-2(\beta-\alpha-1)-1} \right) \\ & \times \left(\int_{-\infty}^t e^{-\gamma(t-s)} E \parallel h(s, x_{n,s}) - h(s, x_s) \parallel_\beta^2 ds \right) \\ & + 18 (M(\alpha))^2 \left(\Gamma(1 - 2\alpha) \gamma^{2\alpha - 1} \right) \\ & \times \left(\int_{-\infty}^t e^{-\gamma(t-s)} E \parallel g(s, x_{n,s}) - g(s, x_s) \parallel^2 ds \right) \\ & + 18 (M(\alpha))^2 \Bigg(\int_{-\infty}^t (t-s)^{-2\alpha} e^{-2\gamma(t-s)} \\ & \times E \parallel f(s, x_{n,s}) - f(s, x_s) \parallel_{L_2^0}^2 ds \Bigg) \end{aligned}$$

$$+ 18(M(\alpha))^2 \varsigma^{-2\alpha} \frac{1}{(1-e^{-\gamma\varsigma})}$$
$$\times \left(\sum_{t_i < t} e^{-\gamma(t-t_i)} E \parallel I_i(x_n(t_i)) - I_i(x(t_i)) \parallel^2 \right).$$

Similarly, we have for $p > 2$,

$$\begin{aligned} Z_2 \leq\ & 18^{p-1} c^p \frac{1}{\delta^{p-1}} \left(\int_t^\infty e^{\delta(t-s)} E \parallel h(s, x_{n,s}) - h(s, x_s) \parallel_\beta^p ds \right) \\ & + 18^{p-1} (C(\alpha))^p \frac{1}{\delta^{p-1}} \left(\int_t^\infty e^{\delta(t-s)} [E \parallel g(s, x_{n,s}) - g(s, x_s) \parallel_\beta^p] ds \right) \\ & + 18^{p-1} (C(\alpha))^p C_p \left(\frac{p-2}{p\delta} \right)^{\frac{p-2}{p}} \\ & \times \left(\int_t^\infty e^{\frac{p}{2}\delta(t-s)} [E \parallel f(s, x_{n,s}) - f(s, x_s) \parallel_\beta^p] ds \right) \\ & + 18^{p-1} (C(\alpha))^p \frac{1}{(1-e^{\varsigma\delta})^{p-1}} \left(\sum_{t < t_i} e^{\delta(t-t_i)} [E \parallel I_i(x_n(t_i)) - I_i(x(t_i)) \parallel^p] \right). \end{aligned}$$

For $p = 2$, we have

$$\begin{aligned} Z_2 \leq\ & 18 c^2 \frac{1}{\delta} \left(\int_t^\infty e^{\delta(t-s)} E \parallel h(s, x_{n,s}) - h(s, x_s) \parallel_\beta^2 ds \right) \\ & + 18 (C(\alpha))^2 \frac{1}{\delta} \left(\int_t^\infty e^{\delta(t-s)} E \parallel g(s, x_{n,s}) - g(s, x_s) \parallel^2 ds \right) \\ & + 18^{p-1} (C(\alpha))^2 \left(\int_t^\infty e^{2\delta(t-s)} E \parallel f(s, x_{n,s}) - f(s, x_s) \parallel_{L_2^0}^2 ds \right) \\ & + 18 (C(\alpha))^2 \frac{1}{(1-e^{\varsigma\delta})} \left(\sum_{t < t_i} e^{\delta(t-t_i)} E \parallel I_i(x_n(t_i)) - I_i(x(t_i)) \parallel^2 \right). \end{aligned}$$

Choose n large enough and combine Step 2 in Theorem 3.1, for every $\varepsilon > 0$ we get

$$E \parallel x(t) \parallel_\alpha^p \leq N^* \varepsilon + 2^{p-1} E \parallel x_n(t) \parallel_\alpha^p$$

for a constant N^* and for all $t \in \mathbb{R}$. Then one has

$$\mu^*(x) \leq N^* \varepsilon + 2^{p-1} \mu^*(x_n) < \infty.$$

Thus $x \in \Omega_f$. □

5. An example

Consider following partial stochastic differential equations of the form

$$\begin{aligned} & d\left[z(t,x) - a_0(t) \int_{-\infty}^0 \int_0^\pi b(s,\tau,x) \sin z(t+s,x) d\tau ds \right] \\ = & \frac{\partial^2}{\partial x^2} z(t,x) dt + a_1(t) \int_{-\infty}^0 a_2(s) \sin z(t+s,x) ds dt \end{aligned}$$

$$+\, a_1(t)\int_{-\infty}^{0} a_3(s)\sin z(t+s,x)dsdW(t), t\in\mathbb{R}, t\neq t_i, i\in\mathbb{Z}, x\in[0,\pi], \quad (5.1)$$

$$z(t,0)=z(t,\pi)=0,\ t\in\mathbb{R}, \quad (5.2)$$

$$\triangle z(t_i,x)=\beta_i\sin z(t_i,x),\ i\in\mathbb{Z}, x\in[0,\pi], \quad (5.3)$$

where $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega,\mathcal{F},\mathbb{P},\mathcal{F}_t)$. In this system, $t_i = i+\frac{1}{4}|\sin i+\sin\sqrt{2}i|$, $\{t_i^j\}, i\in\mathbb{Z}, j\in\mathbb{Z}$ are equipotentially almost periodic and $\varsigma=\inf_{i\in\mathbb{Z}}(t_{i+1}-t_i)>0$, one can see [22] for more details.

Let $\mathbb{H}=L^2([0,\pi])$ with the norm $\|\cdot\|$ and define the operators $A: D(A)\subseteq L^p(\mathbb{P},\mathbb{H})\to L^p(\mathbb{P},\mathbb{H})$ by $A\omega=\omega''$ with the domain

$$D(A) := \{\omega\in\mathbb{H} : \omega,\omega' \text{ are absolutely continuous}, \omega''\in\mathbb{H}, \omega(0)=\omega(\pi)=0\}.$$

It is well known that A generates a strongly continuous semigroup $T(\cdot)$ which is compact, analytic and self-adjoint. Furthermore, A has a discrete spectrum; the eigenvalues are $-n^2, n\in\mathbb{N}$, with the corresponding normalized eigenvectors $z_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx)$. Then the following properties hold:

(a) If $\omega\in D(A)$, then

$$A\omega=\sum_{n=1}^{\infty} n^2\langle\omega,\omega_n\rangle\omega_n.$$

(b) For each $\omega\in H$,

$$A^{-\frac{1}{2}}\omega=\sum_{n=1}^{\infty}\frac{1}{n}\langle\omega,\omega_n\rangle\omega_n.$$

(c) The operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}\omega=\sum_{n=1}^{\infty} n\langle\omega,\omega_n\rangle\omega_n$$

on the space $D(A^{\frac{1}{2}})=\{\omega(\cdot)\in H, \sum\limits_{n=1}^{\infty} n\langle\omega,\omega_n\rangle\omega_n\in H\}$ and $\| A^{-\frac{1}{2}}\|=1$.

Moreover, $\| A^{\frac{1}{2}}T(t)\|\leq\frac{1}{\sqrt{2e}}t^{-\frac{1}{2}}$ for all $t>0$, and satisfy (H1)-(H2).

We assume that the following conditions hold.

(i) The functions $b(\cdot), \frac{\partial^j}{\partial x^j}b(s,\tau,x), j=1,2$ are (Lebesgue) measurable, $b(s,\tau 0)=b(s,\tau\pi)$ for every (s,τ) and

$$L_b=\max\left\{\left(\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi} e^{-2s}\left(\frac{\partial^j}{\partial x^j}b(s,\tau,x)\right)^2 d\tau dsdx\right)^{1/2} : j=0,1,2\right\}<\infty.$$

(ii) The functions $a_j:\mathbb{R}\to\mathbb{R}$, $j=0,1,2,3$, are continuous.

(iii) The functions $a_j:\mathbb{R}\to\mathbb{R}$, $j=0,1$, are pseudo almost periodic with $a_j=\vartheta_j+\kappa_j$, where $\vartheta_j\in PAP_T(\mathbb{R},\mathbb{R})$ and $\kappa_j\in PAP_T^0(\mathbb{R},\mathbb{R}), j=0,1$.

(iv) There exist $l_0,l_1>0$, such that $|a_0(t)-a_0(s)|^p\leq l_0|t-s|$ and $|\vartheta_1(t)-\vartheta_1(s)|^p\leq l_1|t-s|$ for all $t,s\in\mathbb{R}$.

(v) $\beta_i = \zeta_i + \eta_i \in PAP(\mathbb{Z}, \mathbb{R})$, where $\zeta_i \in PA(\mathbb{Z}, \mathbb{R})$, $\eta_i \in PAP_0(\mathbb{Z}, \mathbb{R})$.

Let the phase space $\mathcal{B}$ be $BUC(\mathbb{R}^-, L^p(\mathbb{P}, \mathbb{H}))$, the space of bounded uniformly continuous functions endowed with the following norm:

$$\| \psi \|_{\mathcal{B}} = \sup_{-\infty < \theta \leq 0} \| \psi(\theta) \|, \ \psi \in \mathcal{B}.$$

It is well known that $\mathcal{B}$ satisfies the axioms (A) and (B) with $\tilde{H} = 1, K(t) = 1$ and $M(t) = 1$.

Let $\varphi(\theta)(x) = \varphi(\theta, x) \in \mathcal{B} \times [0, \pi]$. Taking

$$\begin{aligned}
h(t, \psi)(x) &= a_0(t) \int_{-\infty}^{0} \int_{0}^{\pi} b(s, \tau, x) \sin(\psi(s)(x)) d\tau ds, \\
h_1(t, \psi)(x) &= \vartheta_0 \int_{-\infty}^{0} \int_{0}^{\pi} b(s, \tau, x) \sin(\psi(s)(x)) d\tau ds, \\
g(t, \psi)(x) &= a_1(t) \int_{-\infty}^{0} a_2(s) \sin(\psi(s)(x)) ds, \\
g_1(t, \psi)(x) &= \vartheta_1(t) \int_{-q}^{0} a_2(s) \sin(\psi(s)(x)) ds, \\
f(t, \psi)(x) &= a_1(t) \int_{-\infty}^{0} a_3(s) \sin(\psi(s)(x)) ds, \\
f_1(t, \psi)(x) &= \vartheta_1(t) \int_{-\infty}^{0} a_3(s) \sin(\psi(s)(x)) ds,
\end{aligned}$$

and

$$I_i(z)(x) = \beta_i \sin(z(t_i, x)), \quad I_{i,1}(z)(x) = \zeta_i \sin(z(t_i, x)), i \in \mathbb{Z}.$$

Then, the above equation (5.1)-(5.3) can be written in the abstract form as the system (1.1)-(1.2). Since $a_0(t), a_1(t)$ are the pseudo almost periodic component. It follows that h, g, f are uniformly pseudo almost periodic of infinite class. Moreover, assumption (i) implies that h is $D(A^{\frac{1}{2}})$-valued. In fact, for any $\psi \in \mathcal{B}$, we have by assumption (i)

$$\begin{aligned}
&\langle h(t, \psi), \omega_n \rangle \\
=& \int_0^{\pi} \omega_n(x) \Big(a_0(t) \int_{-\infty}^{0} \int_0^{\pi} b(s, \tau, x) \psi(s)(x) d\tau ds \Big) dx \\
=& \frac{a_0(t)}{n} \Big\langle \int_0^{\pi} \frac{\partial}{\partial x} \Big(\int_{-\infty}^{0} \int_0^{\pi} b(s, \tau, x) \psi(s)(x) d\tau ds \Big), \sqrt{\frac{2}{\pi}} \cos(nx) \Big\rangle,
\end{aligned}$$

and

$$\begin{aligned}
&E \| A^{\frac{1}{2}} h(t, \psi) - A^{\frac{1}{2}} h(t_1, \psi_1) \|^p \\
=& E \Big\| \sum_{n=1}^{\infty} n \langle h(t, \psi) - h(t_1, \psi_1), \omega_n \rangle \omega_n \Big\|^p \\
\leq& \Big[\sqrt{\pi} \Big(\int_0^{\pi} \int_{-\infty}^{0} \int_0^{\pi} e^{-2s} \Big(\frac{\partial}{\partial x} b(s, \tau, x) \Big)^2 d\tau ds dx \Big)^{1/2} \Big]^p
\end{aligned}$$

$$\times E\bigg[|a_0(t)-a_0(t_1)|\int_{-\infty}^{0}e^{2s}ds+\parallel a_0\parallel_\infty\int_{-\infty}^{0}e^{2s}\parallel\psi(s)-\psi_1(s)\parallel ds\bigg]^p$$
$$\le 2^{p-1}(\sqrt{\pi}L_b)^p\bigg[\frac{l_0}{2^p}|t-t_1|+\parallel a_0\parallel_\infty^p\frac{1}{2^p}\parallel\psi-\psi_1\parallel_{\mathcal{B}}^p\bigg]$$
$$\le L_h[|t-t_1|+\parallel\psi-\psi_1\parallel_{\mathcal{B}}^p],$$

and $E\parallel A^{\frac{1}{2}}h(t,\psi)\parallel^p\le L_h\parallel\psi\parallel_{\mathcal{B}}^p$,

$$E\parallel A^{\frac{1}{2}}h_1(t,\psi)-A^{\frac{1}{2}}h_1(t,\psi_1)\parallel^p\le L_h\parallel\psi-\psi_1\parallel_{\mathcal{B}}^p$$

for all $t,t_1\in\mathbb{R},\psi,\psi_1\in\mathcal{B}$, where $L_h=\frac{1}{2}(\sqrt{\pi}L_b)^p\max\{l_0,\parallel a_0\parallel_\infty^p\}$. From assumption (ii), we have

$$\begin{aligned}
&E\parallel g(t,\psi)\parallel^p+E\parallel f(t,\psi)\parallel^p\\
&=E\bigg[\bigg(\int_0^\pi\bigg(a_1(t)\int_{-\infty}^0 a_2(s)\sin(\psi(s)(x))ds\bigg)^2dx\bigg)^{1/2}\bigg]^p\\
&\quad+E\bigg[\bigg(\int_0^\pi\bigg(a_1(t)\int_{-\infty}^0 a_3(s)\sin(\psi(s)(x))ds\bigg)^2dx\bigg)^{1/2}\bigg]^p\\
&\le E\bigg[\parallel a_1\parallel_\infty\int_{-\infty}^0 a_2(s)\parallel\psi(s)\parallel ds\bigg]^p\\
&\quad+E\bigg[\parallel a_1\parallel_\infty\int_{-\infty}^0 a_3(s)\parallel\psi(s)\parallel ds\bigg]^p\\
&\le\parallel a_1\parallel_\infty^p\bigg(\int_{-\infty}^0 a_2(s)ds\bigg)^p\parallel\psi\parallel_{\mathcal{B}}^p\\
&\quad+\parallel a_1\parallel_\infty^p\bigg(\int_{-\infty}^0 a_3(s)ds\bigg)^p\parallel\psi\parallel_{\mathcal{B}}^p\\
&\le L_g\parallel\psi\parallel_{\mathcal{B}}^p,
\end{aligned}$$

and

$$\begin{aligned}
&E\parallel g_1(t,\psi)-g_1(t_1,\psi_1)\parallel^p+E\parallel f_1(t,\psi)-f_1(t_1,\psi_1)\parallel^p\\
&\quad\le L_{g_1}[|t-t_1|+\parallel\psi-\psi_1\parallel_{\mathcal{B}}^p]
\end{aligned}$$

for all $t,t_1\in\mathbb{R},\psi,\psi_1\in\mathcal{B}$, where $L_g=\|a_1\|_\infty^p\max\{(\int_{-\infty}^0 a_2(s)ds)^p,(\int_{-\infty}^0 a_3(s)ds)^p\}$, $L_{g_1}=\max\{l_1(\int_{-\infty}^0 a_2(s)ds)^p,\parallel\vartheta_1\parallel_\infty^p(\int_{-\infty}^0 a_3(s)ds)^p\}$. Further, $\beta_i\in PAP(\mathbb{Z},\mathbb{R})$ implies that $I_i\in PAP(\mathbb{Z},L^p(\mathbb{P},\mathbb{H})),i\in\mathbb{Z}$, and $E\parallel I_i(y)\parallel^p\le\sup_{i\in\mathbb{Z}}|\beta_i|^p\parallel y\parallel^p$,

$$E\parallel I_{i,1}(y)-I_{i,1}(y_1)\parallel^p\le\sup_{i\in\mathbb{Z}}|\zeta_i|^p\parallel y-y_1\parallel^p$$

for all $y,y_1\in L^p(\mathbb{P},\mathbb{H}),i\in\mathbb{Z}$. Then, it satisfies all the assumptions given in Theorems 3.1. Therefore, the system (5.1)-(5.3) has a pseudo almost periodic in distribution mild solution. Further, we can impose some suitable conditions on the above-defined functions to verify the assumptions on Theorem 4.1. Hence by Theorems 4.1, the system (5.1)-(5.3) has an optimal mild solution on $\mathbb{R}$.

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