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ON KINK AND ANTI-KINK WAVE SOLUTIONS OF SCHRÖDINGER EQUATION WITH DISTRIBUTED DELAY*

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Abstract This paper deals with existence problem of traveling wave solutions of a class of nonlinear Schrödinger equation having distributed delay with a strong generic kernal. By using the geometric singular perturbation theory and the Melnikov function method, we establish results of the existence of kink and anti-kink wave solutions of the nonlinear Schrödinger equation with time delay when the average delay is sufficiently small.

Keywords Nonlinear Schrödinger equation, kink and anti-kink wave solutions, geometric singular perturbation theory, Melnikov function, distributed delay.

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1. Introduction

The nonlinear Schrödinger (NLS) equation [1, 25] that describes the propagation of picoseconds light pulses in optical fiber is of great importance to applied mathematics and many fields of physics, including nonlinear quantum field theory, condensed matter, plasma physics, nonlinear optics, quantum electronics, and fluid mechanics [2, 6, 24]. This equation is completely integrable and allows both bright and dark solitons [11] depending on the coefficients of linear GVD and SPM. Lots of methods, especially traveling wave solution methods are employed to the NLS equation and all kinds of generalizations, such as extended direct algebraic method [7], Hirota bilinear method [27, 28], modified simple equation method [3, 20, 26, 38], Backlund transformation method [31], tanh-function method [16], the first integral method [8, 9, 39], the homogeneous balance method [12], extended auxiliary equation method [32, 40], the Jacobi elliptic function expansion method [41], the (G′/G)-expansion method [42], and many more.

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With systems evolving there usually occur aftereffect phenomena, about which many time-delay systems are found and investigated in engineering science fields [17, 21, 23, 33, 34]. In [4, 5, 22], the numerical analysis is applied to study the effect of time delay on the solution of the NLS equation. Yang et al [35, 36] studied the NLS equation with delay term that has much actual significance. The solitary wave solutions are found and related physical problems are also discussed in details. Zhao and Ge [37] investigate the NLS equation with distributed delay and give the conditions that assure existence of the solitary wave and periodic solutions by the homoclinic and periodic orbits. We'll investigate the heteroclinic orbits of the NLS equation with distributed delay, and existence of kink and anti-kink wave solutions in the distributed delay equation will be obtained.

In this paper, we consider the following NLS equation with distributed delay,

$$iU_t + U_{xx} - f * U\,|U|^2 - \tau U(|U|^2)_x = 0, -\infty < t < +\infty, -\infty < x < +\infty, \tag{1.1}$$

where $\tau = \int_0^{+\infty} tf(t)dt > 0$ is time delay, $\tau U(|U|^2)_x$ means the nonlinear response delay term [35, 36], here the convolution $f * U$ is defined by

$$(f * U)(x,t) = \int_{-\infty}^{t} f(t-s)U(x,s)ds, \tag{1.2}$$

and the kernel $f : [0, +\infty) \to [0, +\infty)$, that satisfies the following normalization assumption: $f(t) \geq 0$ for all $t \geq 0$ and $\int_0^{+\infty} f(t)dt = 1, tf(t) \in L^1((0, +\infty), R)$.

Remark 1.1. If the parameter $\tau = 0$ and $f(t) = \delta(t)$, Eq. (1.1) becomes the corresponding undelayed and undisturbed NLS equation. That is the following form

$$iU_t + U_{xx} - U\,|U|^2 = 0. \tag{1.3}$$

Remark 1.2. Here we point out that if the different delay kernels were chosen, then the different types equations can be derived from Eq. (1.3). For example, when we take the kernel to be $f(t) = \delta(t)$, where δ denotes Dirac δ function, then Eq. (1.1) becomes the corresponding NLS equation $iU_t + U_{xx} - U\,|U|^2 - \tau U(|U|^2)_x = 0$.

Usually, Gamma distribution delay kernel is used

$$f(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}, \quad n = 1, 2, ...,$$

where $\lambda > 0$ is a constant, n is a integer, with the average delay $\tau = n/\lambda > 0$.

Two special cases $f(t) = \frac{1}{\tau}e^{-t/\tau}$ $(n = 1)$and $f(t) = \frac{t}{\tau^2}e^{-t/\tau}$ $(n = 2)$ are called the weak generic kernel and the strong generic kernel, respectively.

In this paper, the distributed delay kernel $f(t)$ of Eq. (1.1) has the following form

$$f(t) = \frac{t}{\tau^2} e^{-t/\tau - iwt}, \tag{1.4}$$

where the parameter $w > 0$.

Under the assumption that the distributed delay kernel $f(t)$ is the strong generic kernel, our main concern is to ascertain existence of the traveling wave solution for Eq. (1.1). The remaining parts are organized as follows. In Section 2, some preliminary theory and discussion are devoted. The phase portraits of wave equation and traveling waves for the non-delay equation (1.3) are given. In Section 3, by using the linear chain trick, Eq. (1.1) with the strong generic kernel can be transformed into a non-delay four-dimensional ordinary differential system. Since the delay τ is sufficiently small, the four-dimensional ordinary differential system is a standard singular perturbed system. By the singular perturbation theory, the four-dimensional ordinary differential system is reduced to the two-dimensional ordinary differential system. We'll prove that there exist the kink and anti-kink wave solutions of system (1.1) with the Melnikov function method.

2. Preliminaries

In order that Eq. (1.1) can be viewed as perturbation of Eq. (1.3), it is necessary to give some facts about unperturbed nonlinear traveling wave Eq. (1.3).

To study the traveling wave solution of Eq. (1.3), we suppose that $U(x,t) = \varphi(\xi)e^{i\theta}$, $\xi = x - ct$, $\theta = ax - wt$, and $c > 0$, where φ is real valued function and represents the amplitude of the traveling wave with wave number $a > 0$ and frequency $w > 0$.

Now substituting $U(x,t) = \varphi(\xi)e^{i\theta} = \varphi(x-ct)e^{i(ax-wt)}$ into the non-delay Eq. (1.3), we get two equations from the real part and the imaginary part of it.

$$\begin{aligned} &w\varphi + \varphi'' - a^2\varphi + \varphi^3 = 0, \\ &-c\varphi' + 2a\varphi' = 0, \end{aligned} \tag{2.1}$$

where $'$ denotes the derivative with respect to the variable ξ.

Let $a = c/2$ and $\mu = w - c^2/4$, then the Eq. (2.1) becomes

$$\phi'' = -\mu\phi + \phi^3. \tag{2.2}$$

Taking $u = \varphi/\sqrt{\mu}$ and $z = \sqrt{\mu}\xi$ to the Eq. (2.2), it turns to

$$\ddot{u} = -u + u^3, \tag{2.3}$$

where $\cdot$ denotes the derivative with respect to the variable z.

Then we have the following equivalent form

$$\begin{aligned} &\dot{u} = v, \\ &\dot{v} = -u + u^3. \end{aligned} \tag{2.4}$$

In the following lemma, we'll obtain the phase orbit expressions of a kink wave and an anti-kink wave solutions of the non-delay equation (1.3) (The existence of a kink wave and an anti-kink wave of Eq (1.3) can refer to [29, 30]).

Lemma 2.1. *In the (u,v) phase plane, Eq. (2.4) has two heteroclinic orbits connecting the two critical points $(\pm 1, 0)$, so the phase orbit expressions of the kink wave and the anti-kink wave solutions of non-delay equation (1.3) can be obtained.*

Proof. It's easy to see that Eq. (2.4) has three critical points $(0,\ 0)$, $(\pm 1,\ 0)$. The origin is a center and $(\pm 1,\ 0)$ are saddles. Eq. (2.4) is a Hamiltonian system with the Hamiltonian function

$$H(u,v) = u^4 - 2u^2 - 2v^2. \tag{2.5}$$

Let $H(u,v)=k$, and when $k=-1$, it has a heteroclinic loop connected by the two critical points $(\pm 1,\, 0)$, namely $v=\pm\sqrt{2}/2(u^2-1)$, so the corresponding kink wave and anti-kink wave solutions of the non-delay equation (1.3) exist. When $-1<k<0$, system (2.4) has a periodic orbit $v=\pm\sqrt{2}/2\sqrt{u^4-2u^2-k}, -\sqrt{1-\sqrt{1+k}}\le u\le\sqrt{1-\sqrt{1+k}}$, so the corresponding periodic wave solution of non-delay equation (1.3) exists. □

To study the problem of existence of traveling solution of Eq. (1.1), we will transform Eq. (1.1) into a four-dimensional singular perturbed ordinary differential system and study the existence problem of heteroclinic orbit of the singular perturbed ODE, so we need to use the following Geometric Singular Perturbation Theorem [15, 18].

Lemma 2.2 (Geometric Singular Perturbation Theorem)**.** *For the system*

$$\begin{aligned} x'(t) &= f(x,y,\varepsilon),\\ y'(t) &= \varepsilon g(x,y,\varepsilon), \end{aligned} \tag{2.6}$$

where $x\in R^n$, $y\in R^l$ and ε is a real parameter, f, g are C^∞ on the set $V\times I$, where $V\in R^{n+l}$ and I is an open interval, containing 0. If when $\varepsilon=0$, the system has a compact, normally hyperbolic manifold of critical points M_0, which is contained in the set $\{f(x,y,0)=0\}$. Then for any $0<r<+\infty$, if $\varepsilon>0$, but sufficiently small, there exists a manifold M_ε:

(i) which is locally invariant under the flow of (2.6);

(ii) which is C^r in x,y and ε;

(iii) $M_\varepsilon=\{(x,y): x=h^\varepsilon(y)\}$ for some C^r function $h^\varepsilon(y)$ and y in some compact K;

(iv) there exist locally invariant stable and unstable manifolds $W^s(M_\varepsilon)$ and $W^u(M_\varepsilon)$ that lie within $O(\varepsilon)$, and are diffeomorphic to $W^s(M_0)$ and $W^u(M_0)$ respectively.

3. Existence of solitary wave of the equation with delay

3.1. Existence of a locally invariant two-dimensional manifold M_τ

Now we consider Eq. (1.1) with distribution delay. With traveling wave transformation, let $U(x,t)=\varphi(\xi)e^{i\theta}=\varphi(x-ct)e^{i(ax-wt)}$, and substituting it into Eq. (1.1), we get a real and an imaginary component of Eq. (1.1) with (1.4) respectively

$$\begin{aligned} &w\varphi+\varphi''-a^2\varphi-(g*\varphi)\varphi^2-2\tau\varphi^2\varphi'=0,\\ &-c\varphi'+2a\varphi'=0, \end{aligned} \tag{3.1}$$

where

$$(g*\varphi)(\xi)=\int_0^\infty \frac{s}{\tau^2}e^{-\frac{s}{\tau}}\varphi(\xi+cs)ds. \tag{3.2}$$

Let $a = c/2$ and $\mu = w - c^2/4$, the system (3.1) with (3.2) is rewritten as

$$\varphi'' - \mu\varphi - (g * \varphi)\varphi^2 - 2\tau\varphi^2\varphi' = 0, \tag{3.3}$$

where $'$ denotes the derivative with respect to the variable ξ.

Taking $u = \varphi/\sqrt{\mu}$ and $z = \sqrt{\mu}\xi$ to Eq. (3.3) with (3.2), it becomes

$$\ddot{u} = -u + (g * u)u^2 + 2\tau\sqrt{\mu}u^2\dot{u}, \tag{3.4}$$

where $\cdot$ denotes the derivative with respect to the variable z and

$$(g * \varphi)(z) = \int_0^\infty \frac{s}{\tau^2} e^{-\frac{s}{\tau}} \varphi(\frac{z}{\sqrt{\mu}} + cs)ds. \tag{3.5}$$

We introduce a new variable p given by

$$p(z) = (g * u)(z).$$

Differentiating p with respect to z, we get that

$$\frac{dp}{dz} = \frac{1}{\sqrt{\mu}c\tau}(p - q), \tag{3.6}$$

where

$$q(z) = \int_0^{+\infty} \frac{1}{\tau} e^{-\frac{s}{\tau}} u(\frac{z}{\sqrt{\mu}} + cs)ds.$$

Differentiating q with respect to z, we get that

$$\frac{dq}{dz} = \frac{1}{\sqrt{\mu}c\tau}(q - u). \tag{3.7}$$

If let $v = \dot{u}$ and noting (3.6), (3.7), then Eq. (3.4) can be rewritten into the following system

$$\begin{aligned} &\dot{u} = v, \\ &\dot{v} = -u + u^2 p + 2\tau\sqrt{\mu}u^2 v, \\ &\sqrt{\mu}c\tau\dot{p} = p - q, \\ &\sqrt{\mu}c\tau\dot{q} = q - u. \end{aligned} \tag{3.8}$$

When $\tau = 0$, the above-mentioned system (3.8) is transformed into the following differential-algebraic system

$$\begin{aligned} &\dot{u} = v, \\ &\dot{v} = -u + u^2 p, \\ &0 = p - q, \\ &0 = q - u. \end{aligned}$$

Namely

$$\ddot{u} = -u + u^3. \tag{3.9}$$

When $\tau > 0$, system (3.8) determines a system of ODEs and its solutions exist in the four-dimensional (u, v, p, q) phase space in which system (3.9) has three critical points: $(0,0,0,0), (1,0,1,1)$ and $(-1,0,-1,-1)$.

By introducing a new variable η defined by $z = \tau\eta$, system (3.8) becomes the following fast system:

$$\begin{aligned} u' &= \tau v, \\ v' &= \tau(-u + u^2 p + 2\tau\sqrt{\mu}u^2 v), \\ \sqrt{\mu}cp' &= p - q, \\ \sqrt{\mu}cq' &= q - u. \end{aligned} \tag{3.10}$$

where $'$ denotes the derivative by η. If $\tau > 0$, the slow system (3.8) and the fast system (3.10) are equivalent.

In the slow system (3.8), if $\tau = 0$, the flow of this system is confined to the following set

$$M_0 = \{(u, v, p, q) \in R^4,\ p = q = u\},$$

which is a two-dimensional invariant manifold for system (3.8). It's easy to obtain that M_0 is normally hyperbolic by the method of the linearization matrix [10,15,37]. According to the Geometric Singular Perturbation Theorem, there exists a locally invariant two-manifold M_τ of system (3.8) with sufficiently small $\tau > 0$, which can be expressed as

$$M_\tau = \{(u, v, p, q) \in R^4 :\ p = q + \phi(u, v, \tau),\ q = u + \psi(u, v, \tau)\}, \tag{3.11}$$

where ϕ, ψ depend smoothly on τ and satisfy

$$\phi(u, v, 0) = \psi(u, v, 0) = 0.$$

The functions ϕ and ψ can be expanded into the form of Taylor series about τ

$$\begin{aligned} \phi(u, v, \tau) &= \tau\phi_1(u, v) + \tau^2\phi_2(u, v) + \cdots, \\ \psi(u, v, \tau) &= \tau\psi_1(u, v) + \tau^2\psi_2(u, v) + \cdots. \end{aligned} \tag{3.12}$$

Substituting (3.12) into the slow system (3.8), we get

$$\begin{aligned} &\sqrt{\mu}c\tau\left[v + (\tfrac{\partial\phi}{\partial u} + \tfrac{\partial\psi}{\partial u})v + (\tfrac{\partial\phi}{\partial v} + \tfrac{\partial\psi}{\partial v})(-u + u^2(u + \phi + \psi) + 2\tau\sqrt{\mu}u^2 v)\right] = \phi, \\ &\sqrt{\mu}c\tau\left[v + \tfrac{\partial\psi}{\partial u}v + \tfrac{\partial\psi}{\partial v}(-u + u^2(u + \phi + \psi) + 2\tau\sqrt{\mu}u^2 v)\right] = \psi. \end{aligned} \tag{3.13}$$

Substituting (3.12) into (3.13) and comparing coefficients of τ, we get

$$\phi_1 = \sqrt{\mu}cv,\ \ \psi_1 = \sqrt{\mu}cv.$$

Thus the first order approximation of the invariant manifold M_τ of system (3.8) with the small $\tau > 0$ is given by

$$M_\tau = \{(u, v, p, q) \in R^4 :\ p = u + 2\tau\sqrt{\mu}cv + O(\tau^2),\ q = u + \tau\sqrt{\mu}cv + O(\tau^2)\}. \tag{3.14}$$

Then the slow system (3.8) restricted to M_τ is given by

$$
\begin{aligned}
&u' = v,\\
&v' = -u + u^3 + 2\tau\sqrt{\mu}(c+1)u^2 v + O(\tau^2).
\end{aligned}
\tag{3.15}
$$

Note that when $\tau = 0$, system (3.15) reduces to the wave equation (2.4) of the corresponding non-delay system (1.3).

When $\tau = 0$, the system (3.15) has a heteroclinic loop $L(L = L_1 + L_2)$(see Figure 1). Generally speaking, the heteroclinic orbits will break as $\tau \neq 0$ and small. Consider the saddle points O_1, O_2 of system (3.15) whose connection is L_i for $\tau = 0$, $i = 1, 2$. It's easy to check that as $\tau \neq 0$ and is small the two saddle points of system (3.15) O_1, O_2 are well kept. Let $L^s_{1,\tau}$ be a stable manifold of O_2 and let $L^u_{1,\tau}$ be an unstable manifold of O_1 for $0 < \tau << 1$. To study whether system (3.15) has saddle connection near L_1 for $0 < \tau << 1$, we choose $M_1 \in L_1$ and let l_1 be a segment normal to L_1 at point M_1. For $0 < \tau << 1$ suppose that the line l_1 intersects L^s_τ, L^u_τ transversally at points M^s_τ, M^u_τ respectively.

Let $d(\tau, L_1) = -\overrightarrow{n_1} \cdot \overrightarrow{M^s_\tau M^u_\tau}$, where $\overrightarrow{n_1} = (H_u(M_1), H_v(M_1))/|(H_v(M_1), -H_u(M_1))|$. The distance between $L^s_{1,\tau}$ and $L^u_{2,\tau}$ can be measured by $d(\tau, L_1)$, and if $d(\tau, L_1) = 0$, then we conclude that system (3.15) has a saddle connection starting from the saddle point O_1 and ending at the saddle point O_2 for $0 < \tau << 1$. Similar process can be applied to study whether system (3.15) have saddle connection near L_2 for $0 < \tau << 1$.

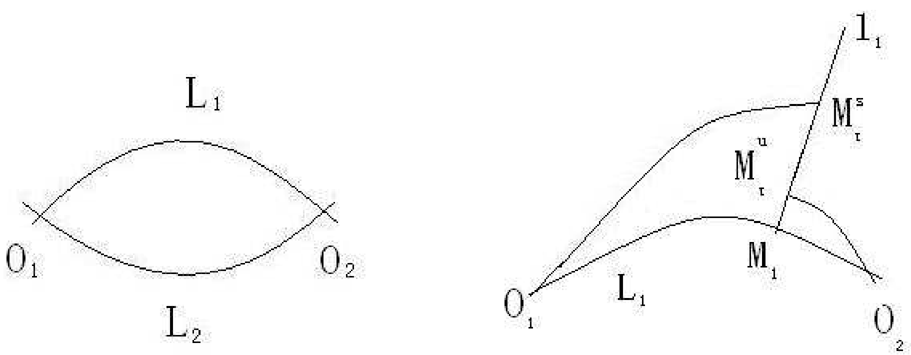

Figure 1. Saddle connections of system (3.15) for the case $\tau = 0$ and $0 < \tau << 1$.

3.2. Maintenance of the heteroclinic orbits and existence of the kink and anti-kink wave solutions

As regards the expression for $d(\tau, L_i)$, $i = 1, 2$, we have the following lemma.

Lemma 3.1. *For $\tau > 0$, but sufficiently small, we have*

$$
d(\tau, L) = \tau \cdot N \cdot M(L) + O(\tau^2), \tag{3.16}
$$

where the Melnikov function $M(L) = \frac{4\sqrt{2\mu}}{15}(c+1) + O(\tau)$, and $N > 0$ is a constant.

Proof. Rewrite system (3.15) into

$$\begin{aligned} u' &= v + \tau P(u, v), \\ v' &= -u + u^3 + \tau Q(u, v), \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} P(u, v) &= 0, \\ Q(u, v) &= 2\sqrt{\mu}(c + 1)u^2 v + O(\tau). \end{aligned}$$

Form [10,13,19,29] and noticing that system $(3.17)|_{\tau=0}$ is Hamiltonian, we have (3.16), and

$$M(L) = \int_L [H_v Q - (-H_u P)] X_i(s) ds = \int_L Q(u, v(u)) du - \int_L P(u(v), v) dv,$$

where $X_i(s)$, $-\infty < s < +\infty$, is a parametric expression for L_i, $i = 1, 2$.

By lemma 2.2, we have the expression for L_1, L_2:

$$\begin{aligned} L_1 : v^+(u) &= \frac{\sqrt{2}}{2}(u^2 - 1), \quad -1 \le u \le 1, \\ L_2 : v^-(u) &= -\frac{\sqrt{2}}{2}(u^2 - 1), \quad -1 \le u \le 1. \end{aligned}$$

Hence,

$$\begin{aligned} M(L_1) &= \int_{-1}^{1} Q(u, v^+(u)) du \\ &= \int_{-1}^{1} \sqrt{2\mu}(c + 1)u^2(-u^2 + 1) du + O(\tau) \\ &= \frac{4\sqrt{2\mu}}{15}(c + 1) + O(\tau), \end{aligned}$$

where $\mu = w - \frac{c^2}{4}$. □

Theorem 3.1. *Consider the NLS equation with distributed delay having form Eq. (1.1), as $0 < \tau << 1$. The following conclusions hold.*

1. *There exists $\phi_1(\tau) = -1 + O(\tau)$, and as the wave speed satisfies $c = \phi_1(\tau)$, then Eq. (1.1) has a kink wave solution.*
2. *There exists $\phi_2(\tau) = -1 + O(\tau)$, and as the wave speed satisfies $c = \phi_2(\tau)$, then Eq. (1.4) has an anti-kink wave solution.*

Proof. Noting that as $c = -1$, $\tau = 0$ and $M(L_1) = 0$, we get

$$\frac{\partial M(L_1)}{\partial c}\Big|_{(c,\tau)=(-1,0)} = \frac{4}{15}\sqrt{2}\mu_0, \; \mu_0 = w - \frac{1}{4} \neq 0.$$

For $0 < \tau << 1$, when $c \in U(-1)$, according to the Implicit Function Theorem, we have that there exists a function $c = \phi_1(\tau) = -1 + O(\tau)$ such that $d(\tau, L_1) = 0$. From the definition of the function $d(\tau, L_1)$, we conclude that system (3.15) has a heteroclinic orbit for $0 < \tau << 1$. In other words, system (1.1) has a kink wave solution for $0 < \tau << 1$.

Similarly, For $0 < \tau << 1$, there exists a function $c = \phi_2(\tau) = -1 + O(\tau)$ such that $d(\tau, L_2) = 0$. And we conclude that system (3.15) has a heteroclinic orbit for $0 < \tau << 1$. In other words, system (1.1) has a anti-kink wave solution for $0 < \tau << 1$. □

4. Conclusion

In this work, we establish existence of kink and anti-kink wave solutions for the NLS equation with distributed delay having form (1.1) when $0 < \tau << 1$, $c = -1+O(\tau)$. Our methods are geometrical singular perturbation theory and Melnikov function. There is also other method to investigate the equation, but by the Melnikov function plenty of calculation is decreased.

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