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GENERALIZED LOCAL MORREY SPACES AND MULTILINEAR COMMUTATORS GENERATED BY MARCINKIEWICZ INTEGRALS WITH ROUGH KERNEL ASSOCIATED WITH SCHRÖDINGER OPERATORS AND LOCAL CAMPANATO FUNCTIONS

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Abstract Let $L = -\Delta + V(x)$ be a Schrödinger operator, where Δ is the Laplacian on $\mathbb{R}^n$, while nonnegative potential $V(x)$ belongs to the reverse Hölder class. In this paper, we consider the behavior of multilinear commutators of Marcinkiewicz integrals with rough kernel associated with Schrödinger operators on generalized local Morrey spaces.

Keywords Marcinkiewicz operator, rough kernel, Schrödinger operator, generalized local Morrey space, multilinear commutator.

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1. Introduction and main results

In this section, we will give some background material that is needed for later chapters. We assume that our readers are familiar with the foundation of real analysis. Since it is impossible to squeeze everything into just a few pages, sometimes we will refer the interested readers to some papers and references.

Notation. Let $x = (x_1, x_2, \ldots, x_n)$, $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \ldots$ etc. be points of the real n-dimensional space $\mathbb{R}^n$. Let $x.\xi = \sum\limits_{i=1}^{n} x_i \xi_i$ stand for the usual dot product in $\mathbb{R}^n$ and $|x| = \left(\sum\limits_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}$ for the Euclidean norm of x.

- By x', we always mean the unit vector corresponding to x, i.e. $x' = \frac{x}{|x|}$ for any $x \neq 0$.
- $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ represents the unit sphere and dx' is its surface measure.

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- By $B(x,r)$, we always mean the open ball centered at x of radius r and by $(B(x,r))^C$, we always mean its complement and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$ and $|B(x,r)| = v_n r^n$, where $v_n = |B(0,1)|$.
- $F \approx G$ means $F \gtrsim G \gtrsim F$; while $F \gtrsim G$ means $F \geq CG$ for a constant $C > 0$; and p' and s' always denote the conjugate index of any $p > 1$ and $s > 1$, that is, $\frac{1}{p'} := 1 - \frac{1}{p}$ and $\frac{1}{s'} := 1 - \frac{1}{s}$.
- C stands for a positive constant that can change its value in each statement without explicit mention.
- The Lebesgue measure of a measurable set E is denoted as $|E|$. Roughly speaking: in one-dimension $|E|$ is the length of E, in two-dimension it is the area of E, and in three dimension (or higher) it is the "volume" of E.
- $\|\Omega\|_{L_s(S^{n-1})} := \left(\int_{S^{n-1}} |\Omega(z')|^s \, d\sigma(z')\right)^{\frac{1}{s}}$.

In this paper we consider the differential Schrödinger operator

$$L = -\Delta + V(x) \text{ on } \mathbb{R}^n, \qquad n \geq 3$$

where $V(x)$ is a nonnegative potential belonging to the reverse Hölder class RH_q, for some exponent $q \geq \frac{n}{2}$; that is, a nonnegative locally L_q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to RH_q $(q > 1)$ if there exists a constant C such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(x)^q \, dx\right)^{\frac{1}{q}} \leq \frac{C}{|B|}\int_B V(x) \, dx, \tag{1.1}$$

holds for every ball $B \subset \mathbb{R}^n$; see [11,12]. Obviously, $RH_{q_2} \subset RH_{q_1}$, if $q_1 < q_2$.

We introduce the definition of the reverse Hölder index of V as $q_0 = \sup\{q : V \in RH_q\}$. It is worth pointing out that the RH_q class is that, if $V \in RH_q$ for some $q > 1$, then there exists $\varepsilon > 0$, which depends only on n and the constant C in (1.1), such that $V \in RH_{q+\varepsilon}$. Therefore, under the assumption $V \in RH_{\frac{n}{2}}$, we may conclude $q_0 > \frac{n}{2}$. Throughout this paper, we always assume that $0 \neq V \in RH_n$.

First of all, we recall some explanations and notations used in the paper.

In 1938, Marcinkiewicz [9] introduced the expression $\mu(f)(x)$ given by

$$\mu(f)(x) = \left(\int_0^{2\pi} \frac{|F(x+t) + F(x-t) - 2F(x)|^2}{t^3} dt\right)^{\frac{1}{2}}, \qquad x \in [0, 2\pi],$$

where $F(x) = \int_0^x f(t) \, dt$. After that, in 1944, Zygmund [15] proved that

$$\|\mu(f)\|_{L_p} \leq C \|f\|_{L_p}, \qquad 1 < p < \infty.$$

The integral $\mu_j(f)$ is called the Marcinkiewicz integral of F and is related in a rather natural way to the Hilbert transform of f. In fact, proceeding formally,

$$\begin{aligned}\int_{-\infty}^{\infty} f(x-t) \frac{dt}{t} &= -\int_0^{\infty} [f(x+t) - f(x-t)] \frac{dt}{t} \\ &= -\int_0^{\infty} \frac{d}{dt} [F(x+t) + F(x-t) - 2F(x)] \frac{dt}{t}\end{aligned}$$

$$= -\int_0^\infty \left(\frac{F(x+t) + F(x-t) - 2F(x)}{t} \right) \frac{dt}{t}.$$

This relation led Stein [13] to define an n-dimensional version of the Marcinkiewicz integral. Let $\Omega(x)$ be a function which is homogenous of degree 0 and which, in addition, satisfies a suitable Lipschitz condition and the zero average condition on S^{n-1}, the unit sphere of $\mathbb{R}^n$. Again proceeding formally, the singular integral

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x-z) \frac{\Omega(z')}{|x-z|^n} dz &= \upsilon_n \int_0^\infty \left(\int_{S^{n-1}} f(x-tz') \Omega(z') d\sigma(z') \frac{dt}{t} \right) \\
&= \int_0^\infty \frac{d}{dt} (F_t(x)) \frac{dt}{t} \\
&= \int_0^\infty \left(\frac{F_t(x)}{t} \right) \frac{dt}{t},
\end{aligned}$$

where

$$F_t(x) = \int_{|z|<t} f(x-z) \frac{\Omega(z)}{|z|^{n-1}} dz.$$

In analogy with the 1-dimensional situation, Stein [13] set

$$\begin{aligned}
\mu_\Omega(f)(x) &= \left(\int_0^\infty \frac{|F_t(x)|^2}{t^3} dt \right)^{\frac{1}{2}} \\
&= \left(\int_0^\infty \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}
\end{aligned}$$

and proved that if $f \in L_p(\mathbb{R}^n)$, then

$$\|\mu_\Omega(f)\|_{L_p} \le C \|f\|_{L_p}, \qquad 1 < p < 2,$$

when $p = 1$,

$$|\{\mu_\Omega(f) > \lambda\}| \le \frac{C}{\lambda} \|f\|_{L_1}, \qquad \text{all } \lambda > 0.$$

Later, Benedek, Calderón and Panzone [3] showed that if Ω is continuously differentiable in $x \neq 0$, then above result holds for $1 < p < \infty$.

Similar to μ_Ω, we define the Marcinkiewicz integral operator with rough kernel $\mu^L_{j,\Omega}$ associated with the Schrödinger operator L by

$$\mu^L_{j,\Omega} f(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} |\Omega(x-y)| K^L_j(x,y) f(y) dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}},$$

where $K^L_j(x,y) = \widetilde{K^L_j}(x,y) |x-y|$ and $\widetilde{K^L_j}(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial x_j} L^{-\frac{1}{2}}$, $j = 1, \ldots, n$. In particular, when

$$\begin{aligned}
V &= 0, \\
K^\Delta_j(x,y) &= \widetilde{K^\Delta_j}(x,y) |x-y|
\end{aligned}$$

$$= \left(\left(x_j - y_j\right) / \left|x-y\right|\right) / \left|x-y\right|^{n-1}$$

and $\widetilde{K_j^{\Delta}}(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial_{x_j}} \Delta^{-\frac{1}{2}}$, $j = 1, \ldots, n$. In this paper, we write $K_j(x,y) = K_j^{\Delta}(x,y)$ and $\mu_{j,\Omega} = \mu_{j,\Omega}^{\Delta}$ and so, $\mu_{j,\Omega}^{\Delta}$ is defined by

$$\mu_{j,\Omega} f(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} |\Omega(x-y)| K_j(x,y) f(y)\, dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}.$$

Now we give the definition of the commutator generalized by μ_Ω and b by

$$\mu_{\Omega,b}(f)(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x) - b(y)] f(y) dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}.$$

Given an operator $\mu_{j,\Omega}^{L}$, and a function b, we define the commutator of $\mu_{j,\Omega}^{L}$ and b by

$$\mu_{j,\Omega,b}^{L} f(x) = [b, \mu_{j,\Omega}^{L}] f(x) = b(x)\, \mu_{j,\Omega}^{L} f(x) - \mu_{j,\Omega}^{L}(bf)(x).$$

If $\mu_{j,\Omega}^{L}$ is defined by integration against a kernel for certain x, such as when $\mu_{j,\Omega}^{L}$ is Marcinkiewicz integral operator with rough kernel associated with the Schrödinger operator L, we have that this becomes

$$\begin{aligned}\mu_{j,\Omega,b}^{L} f(x) &= [b, \mu_{j,\Omega}^{L}] f(x) \\ &= \left(\int_0^\infty \left| \int_{|x-y|\le t} |\Omega(x-y)| K_j^{L}(x,y) [b(x) - b(y)] f(y)\, dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}},\end{aligned}$$

for all x for which the integral representation of $\mu_{j,\Omega}^{L}$ holds. It is worth noting that for a constant C, if $\mu_{j,\Omega}^{L}$ is linear we have,

$$\begin{aligned}[b + C, \mu_{j,\Omega}^{L}] f &= (b + C)\, \mu_{j,\Omega}^{L} f - \mu_{j,\Omega}^{L}((b + C) f) \\ &= b\mu_{j,\Omega}^{L} f + C\mu_{j,\Omega}^{L} f - \mu_{j,\Omega}^{L}(bf) - C\mu_{j,\Omega}^{L} f \\ &= [b, \mu_{j,\Omega}^{L}] f.\end{aligned}$$

This leads one to intuitively look to spaces for which we identify functions which differ by constants, and so it is no surprise that $b \in BMO$ (bounded mean oscillation space) or $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ (local Campanato space) has had the most historical significance.

In this paper, we consider multilinear local Campanato estimates of following multilinear commutator of Marcinkiewicz integral with rough kernel associated with schrödinger operators $\mu_{j,\Omega}^{L}$ on generalized local Morrey space:

$$\begin{aligned}\mu_{j,\Omega,\overrightarrow{b}}^{L} f(x) &= [\overrightarrow{b}, \mu_{j,\Omega}^{L}] f(x) \\ &= \left(\int_0^\infty \left| \int_{|x-y|\le t} \prod_{i=1}^{m} [b_i(x) - b_i(y)]\, |\Omega(x-y)|\, K_j^{L}(x,y) f(y)\, dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}},\end{aligned}$$

where $\Omega \in L_s(S^{n-1})$ $(s>1)$ is homogeneous of degree zero on $\mathbb{R}^n$, let $\overrightarrow{b} = (b_1, \ldots, b_m)$ be a vector-valued locally integrable function such that $b_i \in L_1^{loc}(\mathbb{R}^n)$ for $1 \le i \le m$.

The classical Morrey space was introduced by Morrey in [10], since then a large number of investigations have been given to them by mathematicians. Recently, some authors established the boundedness of some Marcinkiewicz integrals associated with schrödinger operator on the Morrey type spaces from a various point of view provided that the nonnegative potential V belonging to the reverse Hölder class (see [1,6,7,12]). Motivated by these results, our aim in this paper is to establish the boundedness for the multilinear commutators of Marcinkiewicz integrals with rough kernel associated with Schrödinger operators on generalized local Morrey spaces provided that the nonnegative potential V belonging to the reverse Hölder class.

We recall the definition of generalized local (central) Morrey space $LM_{p,\varphi}^{\{x_0\}}$ in the following.

Definition 1.1 (see [2,5,6])**. (generalized local (central) Morrey space)** Let $1 \le p < \infty$, $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$. Then, for any fixed $x_0 \in \mathbb{R}^n$ the generalized local (central) Morrey space $LM_{p,\varphi}^{\{x_0\}} \equiv LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ is defined by

$$\begin{aligned} LM_{p,\varphi}^{\{x_0\}} &\equiv LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n) \\ &= \left\{ \begin{array}{l} f \in L_p^{loc}(\mathbb{R}^n): \\ \|f\|_{LM_{p,\varphi}^{\{x_0\}}} = \sup\limits_{r>0} \varphi(x_0,r)^{-1} |B(x_0,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x_0,r))} < \infty \end{array} \right\}. \end{aligned}$$

According to this definition, we recover the local Morrey space $LM_{p,\lambda}^{\{x_0\}}$:

$$LL_{p,\lambda}^{\{x_0\}} = LM_{p,\varphi}^{\{x_0\}} \Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}} .$$

For the properties and applications of generalized local (central) Morrey spaces $LM_{p,\varphi}^{\{x_0\}}$, see also [2,5,6].

Now, we recall that the definition and some properties of local Campanato space $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ that we use in the following sections.

Definition 1.2 (see [2,5,6])**.** Let $1 \le p < \infty$ and $0 \le \lambda < \frac{1}{n}$. A function $b \in L_p^{loc}(\mathbb{R}^n)$ is said to belong to the $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, if

$$\|b\|_{LC_{p,\lambda}^{\{x_0\}}} = \sup_{r>0} \left(\frac{1}{|B(x_0,r)|^{1+\lambda p}} \int_{B(x_0,r)} \left| b(y) - b_{B(x_0,r)} \right|^p dy \right)^{\frac{1}{p}} < \infty, \qquad (1.2)$$

where

$$b_{B(x_0,r)} = \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} b(y)\, dy.$$

Define

$$LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n) = \left\{ b \in L_p^{loc}(\mathbb{R}^n) : \|b\|_{LC_{p,\lambda}^{\{x_0\}}} < \infty \right\}.$$

Remark 1.1 (see [2,5,6]). If two functions which differ by a constant are regarded as a function in the space $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, then $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ becomes a Banach space. The space $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ when $\lambda = 0$ is just the $LC_p^{\{x_0\}}(\mathbb{R}^n)$. Apparently, (1.2) is equivalent to the following condition:

$$\sup_{r>0} \inf_{c\in\mathbb{C}} \left(\frac{1}{|B(x_0,r)|^{1+\lambda p}} \int_{B(x_0,r)} |b(y) - c|^p \, dy \right)^{\frac{1}{p}} < \infty.$$

Also, in [8], Lu and Yang introduced some new Hardy space $H\dot{A}_p$ related to the homogeneous Herz space $\dot{A}_p$, and obtained that dual space of $H\dot{A}_p$ was the central BMO space $CBMO_p(\mathbb{R}^n) = LC_{p,0}^{\{0\}}(\mathbb{R}^n)$. Note that $BMO(\mathbb{R}^n) \subset \bigcap\limits_{p>1} LC_p^{\{x_0\}}(\mathbb{R}^n)$, $1 \le p < \infty$.

Lemma 1.1 (see [2,5,6]). *Let b be function in $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, $1 \le p < \infty$, $0 \le \lambda < \frac{1}{n}$ and r_1, $r_2 > 0$. Then*

$$\left(\frac{1}{|B(x_0,r_1)|^{1+\lambda p}} \int_{B(x_0,r_1)} \left|b(y) - b_{B(x_0,r_2)}\right|^p dy \right)^{\frac{1}{p}} \le C \left(1 + \left|\ln \frac{r_1}{r_2}\right|\right) \|b\|_{LC_{p,\lambda}^{\{x_0\}}}, \tag{1.3}$$

where $C > 0$ is independent of b, r_1 and r_2.

From this inequality (1.3), we have

$$\left|b_{B(x_0,r_1)} - b_{B(x_0,r_2)}\right| \le C \left(1 + \ln \frac{r_1}{r_2}\right) |B(x_0,r_1)|^{\lambda} \|b\|_{LC_{p,\lambda}^{\{x_0\}}}, \tag{1.4}$$

and it is easy to see that

$$\|b - (b)_B\|_{L_p(B)} \le C r^{\frac{n}{p}+n\lambda} \|b\|_{LC_{p,\lambda}^{\{x_0\}}}. \tag{1.5}$$

Remark 1.2. From Lemma 1.1, it is easy to see that

$$\|b_i - (b_i)_B\|_{L_{p_i}(B)} \le C|B(x,r)|^{\frac{1}{p_i}+\lambda_i} \|b_i\|_{LC_{p_i,\lambda_i}^{\{x_0\}}}, \tag{1.6}$$

and

$$\begin{aligned} \|b_i - (b_i)_B\|_{L_{p_i}(2B)} &\le \left(\|b_i - (b_i)_{2B}\|_{L_{p_i}(2B)} + \|(b_i)_B - (b_i)_{2B}\|_{L_{p_i}(2B)} \right) \\ &\lesssim |B(x,r)|^{\frac{1}{p_i}+\lambda_i} \|b_i\|_{LC_{p_i,\lambda_i}^{\{x_0\}}}. \end{aligned} \tag{1.7}$$

Gao and Tang [4] have shown that Marcinkiewicz integral μ_j^L is bounded on $L_p(\mathbb{R}^n)$, for $1 < p < \infty$, and is bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. Later, Akbulut and Kuzu [1] have shown that the Marcinkiewicz integral operators with rough kernel $\mu_{j,\Omega}^L$, $j = 1,\dots,n$, associated with the Schrödinger operator L are bounded on $L_p(\mathbb{R}^n)$, for $1 < p < \infty$, and are bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$ that we need. Their results can be formulated as follows.

Theorem 1.1 (see [1]). *Let $1 < p < \infty$, $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, $\Omega(\mu x) = \Omega(x)$ for any $\mu > 0, x \in \mathbb{R}^n \setminus \{0\}$ and $V \in RH_n$. If $s' \le p$, $p \ne 1$ or $p < s$, then the* *operator* $\mu_{j,\Omega}^{L}$ *is bounded on* $L_p(\mathbb{R}^n)$. *Also the operator* $\mu_{j,\Omega}^{L}$ *is bounded from* $L_1(\mathbb{R}^n)$ *to* $WL_1(\mathbb{R}^n)$. *Moreover, we have for* $p>1$

$$\left\|\mu_{j,\Omega}^{L} f\right\|_{L_p} \le C \left\|f\right\|_{L_p},$$

and for $p=1$

$$\left\|\mu_{j,\Omega}^{L} f\right\|_{WL_1} \le C \left\|f\right\|_{L_1},$$

where C *does not depend on* f.

Our main result can be formulated as follows.

Theorem 1.2. *Suppose that* $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1<s\le\infty$, $\Omega(\mu x)=\Omega(x)$ *for any* $\mu>0, x\in\mathbb{R}^n\setminus\{0\}$ *and* $V\in RH_n$. *Let* $1<p,q,p_1,\dots,p_m<\infty$ *with* $\frac{1}{q}=\sum\limits_{i=1}^{m}\frac{1}{p_i}+\frac{1}{p}$ *and* $b_i\in LC_{p_i,\lambda_i}^{\{x_0\}}(\mathbb{R}^n)$ *for* $0\le\lambda_i<\frac{1}{n}$, $i=1,\dots,m$.

Let also, for $s'\le q$ *the pair* (φ_1,φ_2) *satisfies the condition*

$$\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)^m \frac{ess\inf\limits_{t<\tau<\infty}\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{n\left(\frac{1}{p}-\sum\limits_{i=1}^{m}\lambda_i\right)+1}}dt \le C\,\varphi_2(x_0,r), \tag{1.8}$$

and for $p<s$ *the pair* (φ_1,φ_2) *satisfies the condition*

$$\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)^m \frac{ess\inf\limits_{t<\tau<\infty}\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{n\left(\frac{1}{p}-\frac{1}{s}-\sum\limits_{i=1}^{m}\lambda_i\right)+1}}dt \le C\,\varphi_2(x_0,r)r^{\frac{n}{s}},$$

where C *does not depend on* r.

Then, the operators $\mu_{j,\Omega,\vec{b}}^{L}$, $j=1,\dots,n$ *are bounded from* $LM_{p,\varphi_1}^{\{x_0\}}$ *to* $LM_{q,\varphi_2}^{\{x_0\}}$. *Moreover,*

$$\left\|\mu_{j,\Omega,\vec{b}}^{L} f\right\|_{LM_{q,\varphi_2}^{\{x_0\}}} \lesssim \prod_{i=1}^{m}\|b_i\|_{LC_{p_i,\lambda_i}^{\{x_0\}}}\|f\|_{LM_{p,\varphi_1}^{\{x_0\}}}. \tag{1.9}$$

Corollary 1.1. *(see [6]) Suppose that* $x_0\in\mathbb{R}^n$, $1<p<\infty$, $\Omega\in L_s(S^{n-1})$,$1<s\le\infty$, $\Omega(\mu x)=\Omega(x)$ *for any* $\mu>0, x\in\mathbb{R}^n\setminus\{0\}$ *and* $V\in RH_n$. *Let* $b\in LC_{p_2,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ *and* $0\le\lambda<\frac{1}{n}$.

Let also, for $s'\le p$ *the pair* (φ_1,φ_2) *satisfies the condition*

$$\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{ess\inf\limits_{t<\tau<\infty}\varphi_1\left(x_0,\tau\right)\tau^{\frac{n}{p_1}}}{t^{\frac{n}{p_1}+1-n\lambda}}dt\le C\varphi_2\left(x_0,r\right),$$

and for $p_1<s$ *the pair* (φ_1,φ_2) *satisfies the condition*

$$\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{ess\inf\limits_{t<\tau<\infty}\varphi_1\left(x_0,\tau\right)\tau^{\frac{n}{p_1}}}{t^{\frac{n}{p_1}-\frac{n}{s}+1-n\lambda}}dt\le C\varphi_2\left(x_0,r\right)r^{\frac{n}{s}},$$

where C *does not depend on* r.

Then, the operators $\mu_{j,\Omega,b}^{L}$, $j=1,\dots,n$ *are bounded from* $LM_{p_1,\varphi_1}^{\{x_0\}}$ *to* $LM_{p,\varphi_2}^{\{x_0\}}$. *Moreover,*

$$\left\|\mu_{j,\Omega,b}^{L} f\right\|_{LM_{p,\varphi_2}^{\{x_0\}}}\lesssim\|b\|_{LC_{p_2,\lambda}^{\{x_0\}}}\|f\|_{LM_{p_1,\varphi_1}^{\{x_0\}}}.$$

2. A Key Lemma

In order to prove the main result (Theorem 1.2), we need the following lemma with its proof.

Lemma 2.1. *Suppose that $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, $\Omega(\mu x) = \Omega(x)$ for any $\mu > 0, x \in \mathbb{R}^n \setminus \{0\}$ and $V \in RH_n$. Let $1 < p, q, p_1, \dots, p_m < \infty$ with $\frac{1}{q} = \sum\limits_{i=1}^{m} \frac{1}{p_i} + \frac{1}{p}$ and $b_i \in LC^{\{x_0\}}_{p_i,\lambda_i}(\mathbb{R}^n)$ for $0 \leq \lambda_i < \frac{1}{n}$, $i = 1, \dots, m$. Then, for $s' \leq q$ the inequality*

$$\|\mu^L_{j,\Omega,\overrightarrow{b}} f\|_{L_q(B(x_0,r))} \lesssim \prod_{i=1}^{m} \|b_i\|_{LC^{\{x_0\}}_{p_i,\lambda_i}} r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^m \frac{\|f\|_{L_p(B(x_0,t))}}{t^{n\left(\frac{1}{p} - \sum\limits_{i=1}^{m}\lambda_i\right)+1}} dt \quad (2.1)$$

holds for any ball $B(x_0, r)$ and for all $f \in L^{loc}_p(\mathbb{R}^n)$. Also, for $p < s$ the inequality

$$\|\mu^L_{j,\Omega,\overrightarrow{b}} f\|_{L_q(B(x_0,r))} \lesssim \prod_{i=1}^{m} \|b_i\|_{LC^{\{x_0\}}_{p_i,\lambda_i}} r^{\frac{n}{q} - \frac{n}{s}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^m \frac{\|f\|_{L_p(B(x_0,t))}}{t^{n\left(\frac{1}{p} - \frac{1}{s} - \sum\limits_{i=1}^{m}\lambda_i\right)+1}} dt$$

holds for any ball $B(x_0, r)$ and for all $f \in L^{loc}_p(\mathbb{R}^n)$.

Proof. Without loss of generality, it is sufficient to show that the conclusion holds for $\mu^L_{j,\Omega,\overrightarrow{b}} f = \mu^L_{j,\Omega,(b_1,b_2)} f$. Let $1 < q, p_1, p_2, p < \infty$ with $\frac{1}{q} = \sum\limits_{i=1}^{2} \frac{1}{p_i} + \frac{1}{p}$ and $b_i \in LC^{\{x_0\}}_{p_i,\lambda_i}(\mathbb{R}^n)$ for $0 \leq \lambda_i < \frac{1}{n}$, $i = 1, 2$. Set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r and $2B = B(x_0, 2r)$. We represent f as

$$f = f_1 + f_2, \qquad f_1(y) = f(y)\chi_{2B}(y), \qquad f_2(y) = f(y)\chi_{(2B)^C}(y), \qquad r > 0$$

and thus we have

$$\left\|\mu^L_{j,\Omega,(b_1,b_2)} f\right\|_{L_q(B)} \leq \left\|\mu^L_{j,\Omega,(b_1,b_2)} f_1\right\|_{L_q(B)} + \left\|\mu^L_{j,\Omega,(b_1,b_2)} f_2\right\|_{L_q(B)} =: F + G.$$

Let us estimate F and G, respectively.

For $\mu^L_{j,\Omega,(b_1,b_2)} f_1(x)$, we have the following decomposition,

$$\begin{aligned}
\mu^L_{j,\Omega,(b_1,b_2)} f_1(x) = {} & (b_1(x) - (b_1)_B)(b_2(x) - (b_2)_B)\,\mu^L_{j,\Omega} f_1(x) \\
& - (b_1(x) - (b_1)_B)\,\mu^L_{j,\Omega}((b_2(\cdot) - (b_2)_B) f_1)(x) \\
& + (b_2(x) - (b_2)_B)\,\mu^L_{j,\Omega}((b_1(\cdot) - (b_1)_B) f_1)(x) \\
& - \mu^L_{j,\Omega}((b_1(\cdot) - (b_1)_B)(b_2(\cdot) - (b_2)_B) f_1)(x).
\end{aligned}$$

Hence, we get

$$\begin{aligned}
F &= \left\|\mu^L_{j,\Omega,(b_1,b_2)} f_1\right\|_{L_q(B)} \\
&\lesssim \left\|(b_1 - (b_1)_B)(b_2 - (b_2)_B)\,\mu^L_{j,\Omega} f_1\right\|_{L_q(B)} \\
&\quad + \left\|(b_1 - (b_1)_B)\,\mu^L_{j,\Omega}((b_2 - (b_2)_B) f_1)\right\|_{L_q(B)}
\end{aligned}$$

$$+\left\|\left(b_2-(b_2)_B\right)\mu_{j,\Omega}^L\left(\left(b_1-(b_1)_B\right)f_1\right)\right\|_{L_q(B)}$$
$$+\left\|\mu_{j,\Omega}^L\left(\left(b_1-(b_1)_B\right)\left(b_2-(b_2)_B\right)f_1\right)\right\|_{L_q(B)}$$
$$\equiv F_1+F_2+F_3+F_4. \tag{2.2}$$

One observes that the estimate of F_2 is analogous to that of F_3. Thus, we will only estimate F_1, F_2 and F_4.

To estimate F_1, let $1<q,\tau<\infty$, such that $\frac{1}{q}=\frac{1}{\tau}+\frac{1}{p}$, $\frac{1}{\tau}=\frac{1}{p_1}+\frac{1}{p_2}$. Then, using Hölder's inequality and by the boundedness of $\mu_{j,\Omega}^L$ on L_p (see Theorem 1.1) it follows that:

$$\begin{aligned}F_1&=\left\|\left(b_1-(b_1)_B\right)\left(b_2-(b_2)_B\right)\mu_{j,\Omega}^L f_1\right\|_{L_q(B)}\\&\lesssim\left\|\left(b_1-(b_1)_B\right)\left(b_2-(b_2)_B\right)\right\|_{L_\tau(B)}\left\|\mu_{j,\Omega}^L f_1\right\|_{L_p(B)}\\&\lesssim\prod_{i=1}^{2}\left\|b_i-(b_i)_B\right\|_{L_{p_i}(B)}\|f\|_{L_p(2B)}\\&\lesssim\prod_{i=1}^{2}\left\|b_i-(b_i)_B\right\|_{L_{p_i}(B)}r^{\frac{n}{p}}\int_{2r}^{\infty}\|f\|_{L_p(B(x_0,t))}\frac{dt}{t^{\frac{n}{p}+1}}.\end{aligned}$$

Hence, by (1.6) we get

$$\begin{aligned}F_1&\lesssim\prod_{i=1}^{2}\|b_i\|_{LC_{p_i,\lambda_i}^{\{x_0\}}}r^{n\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p}\right)}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2t^{-\frac{n}{p}+n(\lambda_1+\lambda_2)-1}\|f\|_{L_p(B(x_0,t))}dt\\&\lesssim\prod_{i=1}^{2}\|b_i\|_{LC_{p_i,\lambda_i}^{\{x_0\}}}r^{\frac{n}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\frac{\|f\|_{L_p(B(x_0,t))}}{t^{n\left(\frac{1}{p}-(\lambda_1+\lambda_2)\right)+1}}dt.\end{aligned}$$

To estimate F_2, let $1<\tau<\infty$, such that $\frac{1}{q}=\frac{1}{p_1}+\frac{1}{\tau}$. Then, similar to the estimates for F_1, we have

$$\begin{aligned}F_2&\lesssim\left\|b_1-(b_1)_B\right\|_{L_{p_1}(B)}\left\|\mu_{j,\Omega}^L\left(\left(b_2(\cdot)-(b_2)_B\right)f_1\right)\right\|_{L_\tau(B)}\\&\lesssim\left\|b_1-(b_1)_B\right\|_{L_{p_1}(B)}\left\|\left(b_2(\cdot)-(b_2)_B\right)f_1\right\|_{L_k(\mathbb{R}^n)}\\&\lesssim\left\|b_1-(b_1)_B\right\|_{L_{p_1}(B)}\left\|b_2-(b_2)_B\right\|_{L_{p_2}(2B)}\|f\|_{L_p(2B)},\end{aligned}$$

where $1<k<\infty$, such that $\frac{1}{k}=\frac{1}{p_2}+\frac{1}{p}=\frac{1}{\tau}$. By (1.6) and (1.7), we get

$$F_2\lesssim\prod_{i=1}^{2}\|b_i\|_{LC_{p_i,\lambda_i}^{\{x_0\}}}r^{\frac{n}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\frac{\|f\|_{L_p(B(x_0,t))}}{t^{n\left(\frac{1}{p}-(\lambda_1+\lambda_2)\right)+1}}dt.$$

In a similar way, F_3 has the same estimate as above, so we omit the details. Then we have that

$$F_3\lesssim\prod_{i=1}^{2}\|b_i\|_{LC_{p_i,\lambda_i}^{\{x_0\}}}r^{\frac{n}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\frac{\|f\|_{L_p(B(x_0,t))}}{t^{n\left(\frac{1}{p}-(\lambda_1+\lambda_2)\right)+1}}dt.$$

Now let us consider the term F_4. Let $1<q,\tau<\infty$, such that $\frac{1}{q}=\frac{1}{\tau}+\frac{1}{p}$, $\frac{1}{\tau}=\frac{1}{p_1}+\frac{1}{p_2}$. Then by the boundedness of $\mu_{j,\Omega}^L$ on L_p (see Theorem 1.1), Hölder's inequality and (1.7), we obtain

$$F_4=\left\|\mu_{j,\Omega}^L\left(\left(b_1-(b_1)_B\right)\left(b_2-(b_2)_B\right)f_1\right)\right\|_{L_q(B)}$$

$$\lesssim \left\| (b_1 - (b_1)_B)\,(b_2 - (b_2)_B)\, f_1 \right\|_{L_q(\mathbb{R}^n)}$$
$$\lesssim \prod_{i=1}^{2} \|b_i - (b_i)_B\|_{L_{p_i}(2B)} \|f\|_{L_p(2B)}$$
$$\lesssim \prod_{i=1}^{2} \|b_i\|_{LC^{\{x_0\}}_{p_i,\lambda_i}} r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \frac{\|f\|_{L_p(B(x_0,t))}}{t^{n\left(\frac{1}{p}-(\lambda_1+\lambda_2)\right)+1}} dt.$$

Combining all the estimates of F_1, F_2, F_3, F_4; we get

$$F = \left\| \mu^L_{j,\Omega,(b_1,b_2)} f_1 \right\|_{L_q(B)} \lesssim \prod_{i=1}^{2} \|b_i\|_{LC^{\{x_0\}}_{p_i,\lambda_i}} r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \frac{\|f\|_{L_p(B(x_0,t))}}{t^{n\left(\frac{1}{p}-(\lambda_1+\lambda_2)\right)+1}} dt.$$

Now, let us estimate $G = \left\| \mu^L_{j,\Omega,(b_1,b_2)} f_2 \right\|_{L_q(B)}$. For G, it's similar to (2.2) we also write

$$\begin{aligned} G &= \left\| \mu^L_{j,\Omega,(b_1,b_2)} f_2 \right\|_{L_q(B)} \\ &\lesssim \left\| (b_1 - (b_1)_B)\,(b_2(x) - (b_2)_B)\, \mu^L_{j,\Omega} f_2 \right\|_{L_q(B)} \\ &\quad + \left\| (b_1 - (b_1)_B)\, \mu^L_{j,\Omega}\left((b_2 - (b_2)_B)\, f_2\right) \right\|_{L_q(B)} \\ &\quad + \left\| (b_2 - (b_2)_B)\, \mu^L_{j,\Omega}\left((b_1 - (b_1)_B)\, f_2\right) \right\|_{L_q(B)} \\ &\quad + \left\| \mu^L_{j,\Omega}\left((b_1 - (b_1)_B)\,(b_2 - (b_2)_B)\, f_2\right) \right\|_{L_q(B)} \\ &\equiv G_1 + G_2 + G_3 + G_4. \end{aligned}$$

To estimate G_1, let $1 < q, \tau < \infty$, such that $\frac{1}{q} = \frac{1}{\tau} + \frac{1}{p}$, $\frac{1}{\tau} = \frac{1}{p_1} + \frac{1}{p_2}$. Then, using Hölder's inequality and by (1.6), we have

$$\begin{aligned} G_1 &= \left\| (b_1 - (b_1)_B)\,(b_2(x) - (b_2)_B)\, \mu^L_{j,\Omega} f_2 \right\|_{L_q(B)} \\ &\lesssim \left\| (b_1 - (b_1)_B)\,(b_2 - (b_2)_B) \right\|_{L_\tau(B)} \left\| \mu^L_{j,\Omega} f_2 \right\|_{L_p(B)} \\ &\lesssim \prod_{i=1}^{2} \|b_i - (b_i)_B\|_{L_{p_i}(B)} r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} t^{-\frac{n}{p}-1} dt \\ &\lesssim \prod_{i=1}^{2} \|b_i\|_{LC^{\{x_0\}}_{p_i,\lambda_i}} r^{n\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p}\right)} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{-\frac{n}{p}+n(\lambda_1+\lambda_2)-1} \|f\|_{L_p(B(x_0,t))} dt \\ &\lesssim \prod_{i=1}^{2} \|b_i\|_{LC^{\{x_0\}}_{p_i,\lambda_i}} r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \frac{\|f\|_{L_p(B(x_0,t))}}{t^{n\left(\frac{1}{p}-(\lambda_1+\lambda_2)\right)+1}} dt, \end{aligned}$$

where in the second inequality we have used the following fact:

It's clear that $x \in B$, $y \in (2B)^C$ implies

$$\frac{1}{2}|x_0 - y| \leq |x - y| \leq \frac{3}{2}|x_0 - y|.$$

Hence, we get

$$\left| \mu^L_{j,\Omega} f_2(x) \right| \leq 2^n c_0 \int_{(2B)^C} \frac{|f(y)|\,|\Omega(x-y)|}{|x_0 - y|^n} dy.$$

By Fubini's theorem, we have

$$\begin{aligned}\int_{(2B)^C}\frac{|f(y)|\,|\Omega(x-y)|}{|x_0-y|^n}dy &\approx \int_{(2B)^C}|f(y)|\,|\Omega(x-y)|\int_{|x_0-y|}^{\infty}\frac{dt}{t^{n+1}}dy\\ &\approx \int_{2r}^{\infty}\int_{2r\le|x_0-y|\le t}|f(y)|\,|\Omega(x-y)|\,dy\frac{dt}{t^{n+1}}\\ &\lesssim \int_{2r}^{\infty}\int_{B(x_0,t)}|f(y)|\,|\Omega(x-y)|\,dy\frac{dt}{t^{n+1}}.\end{aligned}$$

Applying Hölder's inequality, we get

$$\begin{aligned}&\int_{(2B)^C}\frac{|f(y)|\,|\Omega(x-y)|}{|x_0-y|^n}dy\\ &\lesssim\int_{2r}^{\infty}\|f\|_{L_p(B(x_0,t))}\,\|\Omega(x-\cdot)\|_{L_s(B(x_0,t))}\,|B(x_0,t)|^{1-\frac{1}{p}-\frac{1}{s}}\,\frac{dt}{t^{n+1}}.\end{aligned}\tag{2.3}$$

Note that $t>2r$ and $|x-x_0|<r$, we have $t+|x-x_0|<t+r<\frac{3}{2}t<2t$. Moreover, noticing that Ω is homogenous of degree zero and $\Omega\in L_s(S^{n-1})$, $s>1$, we obtain

$$\begin{aligned}\left(\int_{B(x_0,t)}|\Omega(x-y)|^s\,dy\right)^{\frac{1}{s}} &= \left(\int_{B(x-x_0,t)}|\Omega(z)|^s\,dz\right)^{\frac{1}{s}}\\ &\le \left(\int_{B(0,t+|x-x_0|)}|\Omega(z)|^s\,dz\right)^{\frac{1}{s}}\\ &\le \left(\int_{B(0,2t)}|\Omega(z)|^s\,dz\right)^{\frac{1}{s}}\\ &= \left(\int_{S^{n-1}}\int_0^{2t}|\Omega(z')|^s\,d\sigma(z')\,r^{n-1}dr\right)^{\frac{1}{s}}\\ &= C\,\|\Omega\|_{L_s(S^{n-1})}\,|B(x_0,2t)|^{\frac{1}{s}}.\end{aligned}\tag{2.4}$$

Thus, by (2.4), it follows that:

$$\left|\mu_{j,\Omega}^L f_2(x)\right|\lesssim\int_{2r}^{\infty}\|f\|_{L_p(B(x_0,t))}\,\frac{dt}{t^{\frac{n}{p}+1}}.$$

Moreover, for all $p\in[1,\infty)$ the inequality

$$\left\|\mu_{j,\Omega}^L f_2\right\|_{L_p(B)}\lesssim r^{\frac{n}{p}}\int_{2r}^{\infty}\|f\|_{L_p(B(x_0,t))}\,\frac{dt}{t^{\frac{n}{p}+1}}$$

is valid.

On the other hand, for the estimates used in G_2, G_3, we have to prove the below inequality:

$$\left|\mu_{j,\Omega}^L\left(\left(b_2-(b_2)_B\right)f_2\right)(x)\right|\lesssim\|b_2\|_{LC_{p_2,\lambda_2}^{\{x_0\}}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\|f\|_{L_p(B(x_0,t))}\,\frac{dt}{t^{n\left(\frac{1}{p}-\lambda_2\right)+1}}.\tag{2.5}$$

Indeed, when $s' \leq q$, for $x \in B$, by Fubini's theorem and applying Hölder's inequality and from (1.4), (1.5), (2.4) we have

$$\left|\mu_{j,\Omega}^{L}\left(\left(b_2(\cdot)-(b_2)_B\right)f_2\right)(x)\right| \lesssim \int_{(2B)^C}\left|b_2(y)-(b_2)_B\right|\left|\Omega(x-y)\right|\frac{|f(y)|}{|x-y|^n}dy$$
$$\lesssim \int_{(2B)^C}\left|b_2(y)-(b_2)_B\right|\left|\Omega(x-y)\right|\frac{|f(y)|}{|x_0-y|^n}dy$$
$$\approx \int_{2r}^{\infty}\int_{2r<|x_0-y|<t}\left|b_2(y)-(b_2)_B\right|\left|\Omega(x-y)\right||f(y)|\,dy\frac{dt}{t^{n+1}}$$
$$\lesssim \int_{2r}^{\infty}\int_{B(x_0,t)}\left|b_2(y)-(b_2)_{B(x_0,t)}\right|\left|\Omega(x-y)\right||f(y)|\,dy\frac{dt}{t^{n+1}}$$
$$+\int_{2r}^{\infty}\left|(b_2)_{B(x_0,r)}-(b_2)_{B(x_0,t)}\right|\int_{B(x_0,t)}\left|\Omega(x-y)\right||f(y)|\,dy\frac{dt}{t^{n+1}}$$
$$\lesssim \int_{2r}^{\infty}\left\|b_2(\cdot)-(b_2)_{B(x_0,t)}\right\|_{L_{p_2}(B(x_0,t))}\left\|\Omega(x-\cdot)\right\|_{L_s(B(x_0,t))}\|f\|_{L_p(B(x_0,t))}$$
$$\times\left|B(x_0,t)\right|^{1-\frac{1}{p_2}-\frac{1}{s}-\frac{1}{p}}\frac{dt}{t^{n+1}}$$
$$+\int_{2r}^{\infty}\left|(b_2)_{B(x_0,r)}-(b_2)_{B(x_0,t)}\right|\|f\|_{L_p(B(x_0,t))}\left\|\Omega(x-\cdot)\right\|_{L_s(B(x_0,t))}$$
$$\times\left|B(x_0,t)\right|^{1-\frac{1}{p}-\frac{1}{s}}\frac{dt}{t^{n+1}}$$
$$\lesssim \int_{2r}^{\infty}\left\|b_2(\cdot)-(b_2)_{B(x_0,t)}\right\|_{L_{p_2}(B(x_0,t))}\|f\|_{L_p(B(x_0,t))}\frac{dt}{t^{n\left(\frac{1}{p}+\frac{1}{p_2}\right)+1}}$$
$$+\|b\|_{LC_{p_2,\lambda}^{\{x_0\}}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\|f\|_{L_p(B(x_0,t))}\frac{dt}{t^{n\left(\frac{1}{p}-\lambda_2\right)+1}}$$
$$\lesssim \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\|f\|_{L_p(B(x_0,t))}\frac{dt}{t^{n\left(\frac{1}{p}-\lambda_2\right)+1}}.$$

This completes the proof of inequality (2.5).

Let $1<\tau<\infty$, such that $\frac{1}{q}=\frac{1}{p_1}+\frac{1}{\tau}$. Then, using Hölder's inequality and from (2.5) and (1.5), we get

$$G_2=\left\|\left(b_1-(b_1)_B\right)\mu_{j,\Omega}^{L}\left(\left(b_2-(b_2)_B\right)f_2\right)\right\|_{L_q(B)}$$
$$\lesssim\left\|b_1-(b_1)_B\right\|_{L_{p_1}(B)}\left\|\mu_{j,\Omega}^{L}\left(\left(b_2(\cdot)-(b_2)_B\right)f_2\right)\right\|_{L_\tau(B)}$$
$$\lesssim\prod_{i=1}^{2}\|b_i\|_{LC_{p_i,\lambda_i}^{\{x_0\}}}r^{\frac{n}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\frac{\|f\|_{L_p(B(x_0,t))}}{t^{n\left(\frac{1}{p}-(\lambda_1+\lambda_2)\right)+1}}dt.$$

Similarly, G_3 has the same estimate above, so here we omit the details. Then the inequality

$$G_3=\left\|\left(b_2-(b_2)_B\right)\mu_{j,\Omega}^{L}\left(\left(b_1-(b_1)_B\right)f_2\right)\right\|_{L_q(B)}$$
$$\lesssim\prod_{i=1}^{2}\|b_i\|_{LC_{p_i,\lambda_i}^{\{x_0\}}}r^{\frac{n}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\frac{\|f\|_{L_p(B(x_0,t))}}{t^{n\left(\frac{1}{p}-(\lambda_1+\lambda_2)\right)+1}}dt$$

is valid.

Now, let us estimate $G_4=\left\|\mu_{j,\Omega}^L\left(\left(b_1-(b_1)_B\right)\left(b_2-(b_2)_B\right)f_2\right)\right\|_{L_q(B)}$. It's similar to the estimate of (2.5), for any $x\in B$, we also write

$$\begin{aligned}
&\left|\mu_{j,\Omega}^L\left(\left(b_1-(b_1)_B\right)\left(b_2-(b_2)_B\right)f_2\right)(x)\right|\\
\lesssim&\int_{2r}^{\infty}\int_{B(x_0,t)}\left|b_1(y)-(b_1)_{B(x_0,t)}\right|\left|b_2(y)-(b_2)_{B(x_0,t)}\right||\Omega(x-y)||f(y)|\,dy\frac{dt}{t^{n+1}}\\
&+\int_{2r}^{\infty}\int_{B(x_0,t)}\left|b_1(y)-(b_1)_{B(x_0,t)}\right|\left|(b_2)_{B(x_0,t)}-(b_2)_{B(x_0,r)}\right||\Omega(x-y)||f(y)|\,dy\frac{dt}{t^{n+1}}\\
&+\int_{2r}^{\infty}\int_{B(x_0,t)}\left|(b_1)_{B(x_0,t)}-(b_1)_{B(x_0,r)}\right|\left|b_2(y)-(b_2)_{B(x_0,t)}\right||\Omega(x-y)||f(y)|\,dy\frac{dt}{t^{n+1}}\\
&+\int_{2r}^{\infty}\int_{B(x_0,t)}\left|(b_1)_{B(x_0,t)}-(b_1)_{B(x_0,r)}\right|\left|(b_2)_{B(x_0,t)}-(b_2)_{B(x_0,r)}\right||\Omega(x-y)||f(y)|dy\frac{dt}{t^{n+1}}\\
\equiv&G_{41}+G_{42}+G_{43}+G_{44}.
\end{aligned}$$

Let us estimate G_{41}, G_{42}, G_{43}, G_{44}, respectively.

Firstly, to estimate G_{41}, similar to the estimate of (2.5), we get

$$G_{41}\lesssim\prod_{i=1}^{2}\|b_i\|_{LC_{p_i,\lambda_i}^{\{x_0\}}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\frac{\|f\|_{L_p(B(x_0,t))}}{t^{n\left(\frac{1}{p}-(\lambda_1+\lambda_2)\right)+1}}dt.$$

Secondly, to estimate G_{42} and G_{43}, from (2.5), (1.4) and (1.5), it follows that

$$G_{42}\lesssim\prod_{i=1}^{2}\|b_i\|_{LC_{p_i,\lambda_i}^{\{x_0\}}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\frac{\|f\|_{L_p(B(x_0,t))}}{t^{n\left(\frac{1}{p}-(\lambda_1+\lambda_2)\right)+1}}dt,$$

and

$$G_{43}\lesssim\prod_{i=1}^{2}\|b_i\|_{LC_{p_i,\lambda_i}^{\{x_0\}}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\frac{\|f\|_{L_p(B(x_0,t))}}{t^{n\left(\frac{1}{p}-(\lambda_1+\lambda_2)\right)+1}}dt.$$

Finally, to estimate G_{44}, similar to the estimate of (2.5) and from (1.4) and (1.5), we have

$$G_{44}\lesssim\prod_{i=1}^{2}\|b_i\|_{LC_{p_i,\lambda_i}^{\{x_0\}}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\frac{\|f\|_{L_p(B(x_0,t))}}{t^{n\left(\frac{1}{p}-(\lambda_1+\lambda_2)\right)+1}}dt.$$

By the estimates of G_{4j} above, where $j=1, 2, 3$, we know that

$$\begin{aligned}
&\left|\mu_{j,\Omega}^L\left(\left(b_1-(b_1)_B\right)\left(b_2-(b_2)_B\right)f_2\right)(x)\right|\\
\lesssim&\prod_{i=1}^{2}\|b_i\|_{LC_{p_i,\lambda_i}^{\{x_0\}}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\frac{\|f\|_{L_p(B(x_0,t))}}{t^{n\left(\frac{1}{p}-(\lambda_1+\lambda_2)\right)+1}}dt.
\end{aligned}$$

Then, we have

$$\begin{aligned}
G_4&=\left\|\mu_{j,\Omega}^L\left(\left(b_1-(b_1)_B\right)\left(b_2-(b_2)_B\right)f_2\right)\right\|_{L_q(B)}\\
&\lesssim\prod_{i=1}^{2}\|b_i\|_{LC_{p_i,\lambda_i}^{\{x_0\}}}r^{\frac{n}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\frac{\|f\|_{L_p(B(x_0,t))}}{t^{n\left(\frac{1}{p}-(\lambda_1+\lambda_2)\right)+1}}dt.
\end{aligned}$$

So, combining all the estimates for G_1, G_2, G_3, G_4, we get

$$G = \left\| \mu^L_{j,\Omega,(b_1,b_2)} f_2 \right\|_{L_q(B)} \lesssim \prod_{i=1}^{2} \|b_i\|_{LC^{\{x_0\}}_{p_i,\lambda_i}} r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \frac{\|f\|_{L_p(B(x_0,t))}}{t^{n\left(\frac{1}{p}-(\lambda_1+\lambda_2)\right)+1}} dt.$$

Thus, putting estimates F and G together, we get the desired conclusion

$$\left\| \mu^L_{j,\Omega,(b_1,b_2)} f \right\|_{L_q(B)} \lesssim \prod_{i=1}^{2} \|b_i\|_{LC^{\{x_0\}}_{p_i,\lambda_i}} r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \frac{\|f\|_{L_p(B(x_0,t))}}{t^{n\left(\frac{1}{p}-(\lambda_1+\lambda_2)\right)+1}} dt.$$

For the case of $p < s$, we can also use the same method, so we omit the details. This completes the proof of Lemma 2.1. $\square$

3. Proof of the main result

Now we are ready to return to the proof of Theorem 1.2.

3.1. Proof of Theorem 1.2.

Proof. To prove Theorem 1.2, we will use the following relationship between essential supremum and essential infimum

$$ess \inf_{E} f = \left(ess \sup_{E} 1/f \right)^{-1}, \tag{3.1}$$

where f is any real-valued nonnegative function and measurable on E (see [14], page 143). Since $f \in LM^{\{x_0\}}_{p,\varphi_1}$, by (3.1) and it is also non-decreasing, with respect to t, of the norm $\|f\|_{L_p(B(x_0,t))}$, we get

$$\begin{aligned} \frac{\|f\|_{L_p(B(x_0,t))}}{ess \inf\limits_{0<t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}} &\le ess \sup_{0<t<\tau<\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}} \\ &\le ess \sup_{0<\tau<\infty} \frac{\|f\|_{L_p(B(x_0,\tau))}}{\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}} \le \|f\|_{LM^{\{x_0\}}_{p,\varphi_1}}. \end{aligned} \tag{3.2}$$

For $s' \le q < \infty$, since (φ_1, φ_2) satisfies (1.8) and by (3.2), we have

$$\begin{aligned} &\int_{r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^m \frac{\|f\|_{L_p(B(x_0,t))}}{t^{n\left(\frac{1}{p}-\sum\limits_{i=1}^{m}\lambda_i\right)+1}} dt \\ &\le \int_{r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^m \frac{\|f\|_{L_p(B(x_0,t))}}{ess \inf\limits_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}} \frac{ess \inf\limits_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{n\left(\frac{1}{p}-\sum\limits_{i=1}^{m}\lambda_i\right)+1}} dt \\ &\le C\|f\|_{LM^{\{x_0\}}_{p,\varphi_1}} \int_{r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^m \frac{ess \inf\limits_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{n\left(\frac{1}{p}-\sum\limits_{i=1}^{m}\lambda_i\right)+1}} dt \end{aligned}$$

$$\leq C\|f\|_{LM^{\{x_0\}}_{p,\varphi_1}}\varphi_2(x_0,r). \tag{3.3}$$

Then by (2.1) and (3.3), we get

$$\begin{aligned}\left\|\mu^L_{j,\Omega,\vec{b}}f\right\|_{LM^{\{x_0\}}_{q,\varphi_2}} &= \sup_{r>0}\varphi_2\,(x_0,r)^{-1}\,|B(x_0,r)|^{-\frac{1}{q}}\left\|\mu^L_{j,\Omega,\vec{b}}f\right\|_{L_q(B(x_0,r))}\\ &\leq C\prod_{i=1}^m\|b_i\|_{LC^{\{x_0\}}_{p_i,\lambda_i}}\sup_{r>0}\varphi_2\,(x_0,r)^{-1}\int_r^\infty\left(1+\ln\frac{t}{r}\right)^m\frac{\|f\|_{L_p(B(x_0,t))}}{t^{n\left(\frac{1}{p}-\sum\limits_{i=1}^m\lambda_i\right)+1}}dt\\ &\leq C\prod_{i=1}^m\|b_i\|_{LC^{\{x_0\}}_{p_i,\lambda_i}}\|f\|_{LM^{\{x_0\}}_{p,\varphi_1}}.\end{aligned}$$

For the case of $p < s$, we can also use the same method, so we omit the details. Thus, we finish the proof of (1.9). □

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