MONOTONE METHODS AND STABILITY RESULTS FOR NONLOCAL REACTION-DIFFUSION EQUATIONS WITH TIME DELAY*

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Abstract In this paper, we study the applications of the monotone iteration method for investigating the existence and stability of solutions to nonlocal reaction-diffusion equations with time delay. We emphasize the importance of the idea of monotone iteration schemes for investigating the stability of solutions to such equations. We show that every steady state of such equations obtained by using the monotone iteration method is priori stable and all stable steady states can be obtained by using such method. Finally, we apply our main results to three population models.

Keywords Monotone method, nonlocal reaction term, delay, existence and uniqueness, stability.

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1. Introduction

Nonlocal reaction-diffusion equations with time delay have been studied by many scholars both in theory and in applications, see e.g. [5, 6, 12–14, 38, 39, 41, 42, 44–46, 48, 49] and the references therein. However, the results on the stability (in the sense of Lyapunov), especially on global stability, of steady states of such equations have seldom been found in the references. Most previous researches on such equations focus on spreading speeds and traveling waves in the case of an unbounded domain and the attractivity of steady states in the case of a bounded domain. The reason lies in the fact that it is difficult to employ linearisation methods to study the stability of steady states of such equations, and other methods may not be available for the study of the stability. For example, Lyapunov direct method can be used to study the stability in theory, but for nonlocal reaction-diffusion equations with time delay, the construction of an appropriate Lyapunov function is often difficult.

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Likewise, fixed point theorem can also be used to prove the stability, but it is hard to select an appropriate compact set.

As early as 1969, the method of upper and lower solution was already introduced by Keller in [18], and have been widely adopted since then, see e.g. [2,16,34,36]. Recently, the method of upper and lower solutions and associated monotone iterations have been used to investigate the existence and asymptotic behavior of solutions [9,22,27,30,31,33,35,46,47]. It is obvious that the monotone iteration method is constructive for the proof of existence result and it can be used to compute numerical solutions to the corresponding discretized equations [9,22,23,28,30].

To explore the use of monotone iteration schemes for investigating the existence and stability of solutions, we consider the following nonlocal reaction-diffusion equations with time delay

$$
\begin{cases}
\dfrac{\partial w(t,x)}{\partial t} = Lw(t,x) + f(x, w(t,x), w(t-\tau, x)) \\
\qquad\qquad + \displaystyle\int_\Omega g(x,y,w(t-\tau,y))dy, \ t > 0, \ x \in \Omega, \\
Bw(t,x) = 0, \ t > 0, \ x \in \partial\Omega, \\
w(t,x) = \phi(t,x), \ t \in [-\tau, 0], \ x \in \Omega
\end{cases}
\tag{1.1}
$$

and corresponding boundary value problem for the steady states of (1.1)

$$
\begin{cases}
- Lw(x) - f(x, w(x), w(x)) = \displaystyle\int_\Omega g(x,y,w(y))dy, \ x \in \Omega, \\
Bw(x) = 0, \ x \in \partial\Omega,
\end{cases}
\tag{1.2}
$$

where τ is a positive constant, $\phi(t,x)$ is an initial function to be specified later, Ω is a bounded domain in $\mathbb{R}^m$ with boundary $\partial\Omega$, L is a second order uniformly elliptic operator,

$$
L = \sum_{i,j=1}^m a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^m b_i(x)\frac{\partial}{\partial x_i}, \ x = (x_1, x_2, \cdots, x_m)
$$

and B is one of the following boundary operators

$$
Bw = w
$$

and

$$
Bw = \frac{\partial w}{\partial \mathbf{n}} + \beta(x)w,
$$

where $\partial/\partial\mathbf{n}$ denotes the outward normal derivative on $\partial\Omega$, $\beta \in C^{1+\alpha}(\partial\Omega)$ and $\alpha \in (0,1)$. And we assume that $\beta(x) \geq 0$ for all $x \in \partial\Omega$.

The coefficients of L are assumed to be Hölder continuous and the matrix (a_{ij}) is uniformly positive definite on $\overline{\Omega} = \Omega \bigcup \partial\Omega$. We assume that the boundary $\partial\Omega$ of Ω belongs to the class $C^{2+\alpha}$, though this condition could be relaxed somewhat.

To motivate our discussion, we assume for the moment that $f(x,u,w)$ and $g(x,y,w)$ are Hölder continuous in x and y for $x,y \in \overline{\Omega}$, and continuously differentiable in u and w and monotone nondecreasing in w for some bounded subset Λ of $\mathbb{R}$. The above smoothness and monotonicity assumptions are used to ensure the existence of a classical solution to (1.2) by the monotone iteration scheme.

The purpose of this paper is to study the applications of the monotone iteration method and to emphasize the importance of the idea of monotone iteration schemes for investigating the existence and stability of solutions to (1.1). We will show that every steady state of (1.1) obtained by using the monotone iteration method is priori stable and all stable steady states can be obtained by using such method.

Actually, the theoretical framework about the existence and stability of solutions was established by C. V. Pao in references [27,29–31], where the systems of reaction diffusion equations with time delays were thoroughly studied. They obtained some sufficient conditions of the existence and stability of solutions. However their methods will not be available for (1.1) due to the nonlocal effect of reaction function. To overcome the difficulty, we establish a new maximum principle (see, Lemma 3.1) and use a technique developed in [34]. As far as we know, this is the first attempt to emphasize the importance of the idea of monotone iteration schemes for investigating the existence and stability of solutions to non-local reaction-diffusion equations with time delay.

The rest of this paper is organized as follows. In section 2 and section 3, we will show how to construct solutions to (1.2) and (1.1) respectively. In section 4, we investigate the stability of solutions to (1.2) obtained by monotone iteration methods when considered as steady states of (1.1). It turn out that every steady state of (1.1) obtained by monotone methods is priori stable and all stable steady states can be obtained by these methods. Furthermore, the upper and lower solutions provide an estimate of the region of stability. Finally in section 5, three examples in population dynamics are given to illustrate our main results.

2. Construction of solutions to (1.2)

In this section, we study the construction of solutions to boundary value problem (1.2) by using monotone iteration scheme.

Let $C^\alpha(\overline{\Omega})$ be the space of functions that are Hölder continuous on $\overline{\Omega}$ with exponent $\alpha \in (0,1)$, and by $C^{2+\alpha}(\overline{\Omega})$ the space of functions on $\overline{\Omega}$ which have spatial derivatives up to order two that are continuous on $\overline{\Omega}$, with the derivatives of order two being Hölder continuous with exponent α. Similar notations are used for other function spaces and other domains. Let $\mathbb{X} = C^{2+\alpha}(\overline{\Omega})$ and $\mathbb{Y} = C^\alpha(\overline{\Omega})$. Then we have the following definition.

Definition 2.1. A function $\widetilde{w}_s \in \mathbb{X}$ is said to be an upper solution of (1.2) if it satisfies

$$\begin{cases} -L\widetilde{w}_s(x) - f(x, \widetilde{w}_s(x), \widetilde{w}_s(x)) \geq \int_\Omega g(x, y, \widetilde{w}_s(y))dy, \ x \in \Omega, \\ B\widetilde{w}_s(x) \geq 0, \ x \in \partial\Omega. \end{cases} \tag{2.1}$$

Similarly, $\widehat{w}_s \in \mathbb{X}$ is called a lower solution of (1.2) if it satisfies (2.1) with inequalities being reversed.

The following theorem can be proved by the same arguments as in the proof of Theorem 2.1 of [34] which was first given by H. Amann [2]. We give a proof here for the sake of completeness. The final convergence arguments are based directly on the L_p estimates [1] for regular elliptic boundary value problems.

Theorem 2.1. *Let $\widetilde{w}_s$ and $\widehat{w}_s$ be upper and lower solutions of (1.2) respectively such that $\widehat{w}_s \leq \widetilde{w}_s$. Then there exists a regular solution w_s^* of (1.2) such that $\widehat{w}_s \leq w_s^* \leq \widetilde{w}_s$.*

Proof. Let

$$\Lambda = \left\{ z \in \mathbb{R} \,|\, \min_{x \in \overline{\Omega}} \widehat{w}_s(x) \leq z \leq \max_{x \in \overline{\Omega}} \widetilde{w}_s(x) \right\}. \tag{2.2}$$

In the case that $\min_{x \in \overline{\Omega}} \widehat{w}_s(x) = \max_{x \in \overline{\Omega}} \widetilde{w}_s(x)$, the conclusion holds naturally. If $\min_{x \in \overline{\Omega}} \widehat{w}_s(x) < \max_{x \in \overline{\Omega}} \widetilde{w}_s(x)$, then Λ is a closed interval in $\mathbb{R}$. Thus we can assume that $\partial f(x, u, w)/\partial u$ is bounded from below for $x \in \Omega$ and $u, w \in \Lambda$, which implies

$$\frac{\partial f(x, u, w)}{\partial u} + K > 0 \tag{2.3}$$

for all $x \in \Omega$ and $u, w \in \Lambda$, provided $K \geq 0$ is sufficiently large.

Since for any $w \in \mathbb{Y}$, the linear boundary value problem

$$\begin{cases} -Lu(x) + Ku(x) = Kw(x) + f(x, w(x), w(x)) + \displaystyle\int_{\Omega} g(x, y, w(y))dy, \ x \in \Omega, \\ Bu(x) = 0, \ x \in \partial\Omega \end{cases}$$

has a unique solution $u \in \mathbb{X}$ (see, e.g., [20]), we can define the nonlinear operator $\mathcal{T} : \mathbb{Y} \to \mathbb{X} \subset \mathbb{Y}$ by $\mathcal{T}w = u$. Namely, $\mathcal{T} : \mathbb{Y} \to \mathbb{X} \subset \mathbb{Y}$ is defined by

$$\begin{cases} (\mathcal{T}w)(x) = (-L+K)^{-1}\left[Kw(x) + f(x, w(x), w(x)) + \displaystyle\int_{\Omega} g(x, y, w(y))dy \right], \forall w \in \mathbb{Y}, \\ B\mathcal{T}w(x) = 0, \ x \in \partial\Omega \end{cases}$$

where $(-L+K)^{-1}$ is the inverse of the operator $-L+K$. Therefore, by (2.3), we know that $\mathcal{T}$ is monotone in the sense of Collatz [7] ($w_1 \leq w_2$ implies $\mathcal{T}w_1 \leq \mathcal{T}w_2$) and completely continuous as an operator from $\mathbb{Y}$ to $\mathbb{Y}$ in the order interval $[\widehat{w}_s, \widetilde{w}_s]$.

Now define $\overline{w}_s^{(1)} = \mathcal{T}\widetilde{w}_s$ and $\underline{w}_s^{(1)} = \mathcal{T}\widehat{w}_s$. Let us show that $\overline{w}_s^{(1)} \leq \widetilde{w}_s$ and $\underline{w}_s^{(1)} \geq \widehat{w}_s$. To this end, we note that

$$(-L+K)\overline{w}_s^{(1)}(x) = K\widetilde{w}_s(x) + f(x, \widetilde{w}_s(x), \widetilde{w}_s(x)) + \int_{\Omega} g(x, y, \widetilde{w}_s(y))dy, \ x \in \Omega,$$

then

$$rl(-L+K)(\overline{w}_s^{(1)} - \widetilde{w}_s) = K\widetilde{w}_s + f(\cdot, \widetilde{w}_s, \widetilde{w}_s) + \int_{\Omega} g(\cdot, y, \widetilde{w}_s(y))dy + L\widetilde{w}_s - K\widetilde{w}_s$$

$$= L\widetilde{w}_s + f(\cdot, \widetilde{w}_s, \widetilde{w}_s) + \int_{\Omega} g(\cdot, y, \widetilde{w}_s(y))dy \leq 0, \ x \in \Omega.$$

Therefore, it follows from the strong maximum principle that $\overline{w}_s^{(1)} \leq \widetilde{w}_s$. A similar argument shows that $\underline{w}_s^{(1)} \geq \widehat{w}_s$.

Due to $\overline{w}_s^{(1)} \leq \widetilde{w}_s$ and the monotonicity of the operator $\mathcal{T}$, we have $\mathcal{T}\overline{w}_s^{(1)} \leq \mathcal{T}\widetilde{w}_s$. Thus, the sequence defined inductively by

$$\overline{w}_s^{(0)} = \widetilde{w}_s, \ \overline{w}_s^{(n)} = \mathcal{T}\overline{w}_s^{(n-1)}, \ \forall n = 1, 2, \cdots$$

is monotone decreasing. Similarly,

$$\underline{w}_s^{(0)} = \widehat{w}_s, \ \underline{w}_s^{(n)} = \mathcal{T}\underline{w}_s^{(n-1)}, \ \forall n = 1, 2, \cdots$$

defines a monotone increasing sequence. Furthermore, we have $\underline{w}_s^{(n)} \leq \overline{w}_s^{(n)}$ for all $n = 1, 2, \cdots$ and

$$\widehat{w}_s = \underline{w}_s^{(0)} \leq \underline{w}_s^{(1)} \leq \cdots \leq \underline{w}_s^{(n)} \leq \cdots \leq \overline{w}_s^{(n)} \leq \cdots \leq \overline{w}_s^{(1)} \leq \overline{w}_s^{(0)} = \widetilde{w}_s.$$

Actually, by assumptions $\widehat{w}_s \leq \widetilde{w}_s$, we have $\underline{w}_s^{(1)} \leq \overline{w}_s^{(1)}$. Suppose that $\underline{w}_s^{(k)} \leq \overline{w}_s^{(k)}$ for given k. Then

$$\underline{w}_s^{(k+1)} = \mathcal{T}\underline{w}_s^{(k)} \leq \mathcal{T}\overline{w}_s^{(k)} = \overline{w}_s^{(k+1)}.$$

The proof follows by induction.

Since the sequences $\left\{\underline{w}_s^{(n)}\right\}_{n=0}^{\infty}$ and $\left\{\overline{w}_s^{(n)}\right\}_{n=0}^{\infty}$ are monotone, both the point-wise limits

$$\overline{w}_s(x) = \lim_{n\to\infty} \overline{w}_s^{(n)}(x) \quad \text{and} \quad \underline{w}_s(x) = \lim_{n\to\infty} \underline{w}_s^{(n)}(x)$$

exist.

The operator $\mathcal{T}$ is a composition of the linear operator $(-L + K)^{-1}$ with the nonlinear operator $\mathcal{S} : \mathbb{Y} \to \mathbb{Y}$ defined by

$$(\mathcal{S}w)(x) = Kw(x) + f(x, w(x), w(x)) + \int_{\Omega} g(x, y, w(y))dy, \forall w \in \mathbb{Y}.$$

For any bounded pointwise convergent sequence $\{w_n\}$, due to the continuity of f, $Kw_n + f(\cdot, w_n(\cdot), w_n(\cdot))$ is also a bounded pointwise convergent sequence. Moreover, for fixed x, $g(x, y, w_n(y)$ converges pointwise in y. By Lebesgue dominated convergence theorem, $\left\{\int_{\Omega} g(x, y, w_n(y))dy\right\}$ converges to a bounded function pointwise in x. Therefore, the operator $\mathcal{S}$ takes bounded pointwise convergent sequences into pointwise convergent sequences.

The operator $(-L + K)^{-1}$ takes $L_p(\Omega)$ continuously into the Sobolev space $W_{2,p}(\Omega)$ for all p, $1 < p < \infty$ by the L_p estimates (see Theorem 15.2 of [1]). Thus, since $\overline{w}_s^{(n)} = \mathcal{T}\overline{w}_s^{(n-1)}$ and since $\left\{\overline{w}_s^{(n)}\right\}_{n=0}^{\infty}$ is a bounded and pointwise convergent sequence, it converges also in $W_{2,p}$. By the embedding lemma (see e.g., Theorem 3.6.6 of [26]), $W_{2,p}$ is embedded continuously into $C^{1+\alpha}$ for $\alpha = 1 - m/p$ when $p > m$. Therefore, $\left\{\overline{w}_s^{(n)}\right\}_{n=0}^{\infty}$ converges in $C^{1+\alpha}$, and by the classical Schauder estimates for regular elliptic boundary value problems, $\left\{\overline{w}_s^{(n)}\right\}_{n=0}^{\infty}$ also converges in $C^{2+\alpha}$. Thus, we have

$$\overline{w}_s = \lim_{n\to\infty} \overline{w}_s^{(n)} = \lim_{n\to\infty} \mathcal{T}\overline{w}_s^{(n-1)} = \mathcal{T} \lim_{n\to\infty} \overline{w}_s^{(n-1)} = \mathcal{T}\overline{w}_s$$

and similarly for $\underline{w}_s$, by the continuity of $\mathcal{T}$. Hence, $\overline{w}_s$ and $\underline{w}_s$ are fixed points of $\mathcal{T}$, and furthermore, they are of class $C^{2+\alpha}(\Omega)$ for $0 < \alpha < 1$. Therefore, they are regular solutions of the elliptic boundary value problem (1.2). The proof is complete. $\square$

Corollary 2.1. *The solutions $\overline{w}_s$ and $\underline{w}_s$ constructed in the proof of Theorem 2.1 are maximal and minimal solutions in the order interval $[\widehat{w}_s, \widetilde{w}_s]$. That is, if w is any solution of (1.2) such that $\widehat{w}_s \leq w \leq \widetilde{w}_s$, then $\underline{w}_s \leq w \leq \overline{w}_s$.*

Proof. Since $w = \mathcal{T}w$, $\overline{w}_s^{(1)} = \mathcal{T}\overline{w}_s^{(0)} = \mathcal{T}\widetilde{w}_s$ and $w \leq \widetilde{w}_s$, it is easy to see that $w \leq \overline{w}_s^{(1)}$. By induction, $w \leq \overline{w}_s^{(n)}$ for all $n = 1, 2, \cdots$. Hence, $w \leq \overline{w}_s$. Similarly, $w \geq \underline{w}_s$. The proof is complete. $\qquad\square$

3. Construction of solutions to (1.1)

In this section, we study the construction of solutions of the initial value problem (1.1) by the same monotone method as used in section 2.

The procedure is as follows. For any finite $T > 0$, denote $Q_T = (0, T] \times \Omega$, $Q_0 = [-\tau, 0] \times \Omega$, $S_T = (0, T] \times \partial\Omega$ and $D_T = [-\tau, T] \times \Omega$. For $\alpha \in (0, 1)$ and functions $w(t, x)$ on $\overline{D_T}$, we define

$$[w]_{\alpha/2,\alpha} = \sup_{\substack{(t,x),(s,y)\in\overline{D_T} \\ (t,x)\neq(s,y)}} \left(\frac{|w(t,x) - w(s,y)|}{|t-s|^{\alpha/2} + |x-y|^\alpha} \right).$$

Functions $w(t, x)$ with $[\ ]_{\alpha/2,\alpha}$ finite form a Banach space $C^{\alpha/2,\alpha}(\overline{D_T})$ under the norm

$$\|w\|_{\alpha/2,\alpha} = \sup_{(t,x)\in\overline{D_T}} |w(t,x)| + [w]_{\alpha/2,\alpha}.$$

Let ∂_x^β denote the derivative with respect to x corresponding to the multi-index $\beta = (\beta_1, \cdots, \beta_m)$, and let ∂_t denote the derivative with respect to t. Let $C^{1+\alpha/2,2+\alpha}(\overline{Q_T})$ be the space of functions on $\overline{Q_T}$ whose derivatives up to order two in x and order one in t are Hölder continuous, with norm

$$\|w\|_{1+\alpha/2,2+\alpha} = \sup_{(t,x)\in\overline{Q_T}} |w(t,x)| + \sum_{|\beta|\leq 2} \sup_{(t,x)\in\overline{Q_T}} |\partial_x^\beta w(t,x)|$$

$$+ \sup_{(t,x)\in\overline{Q_T}} |\partial_t w(t,x)| + \sum_{|\beta|=2} [\partial_x^\beta w(t,x)]_{\alpha/2,\alpha} + [\partial_t w(t,x)]_{\alpha/2,\alpha}.$$

The space $C^{\alpha/2,\alpha}(\overline{D_T})$ and $C^{1+\alpha/2,2+\alpha}(\overline{Q_T})$ are Banach spaces (see, e.g., [4, 10]). we also will need to use the space $C^{1,2}(\overline{Q_T})$ that denotes the set of functions which are continuously differentiable in $t \in [0, T]$ and twice continuously differentiable in $x \in \overline{\Omega}$. Let $\mathbb{X}_T = C^{1+\alpha/2,2+\alpha}(\overline{Q_T}) \bigcap C^{\alpha/2,\alpha}(\overline{D_T})$ and $\mathbb{Y}_T = C^{\alpha/2,\alpha}(\overline{D_T})$. Then we have the following definition and lemmas.

Definition 3.1. A function $\widetilde{w} \in \mathbb{X}_T$ is said to be an upper solution of (1.1) if it satisfies

$$\begin{cases} \dfrac{\partial \widetilde{w}(t,x)}{\partial t} \geq L\widetilde{w}(t,x) + f(x, \widetilde{w}(t,x), \widetilde{w}(t-\tau, x)) \\ \qquad\qquad + \displaystyle\int_\Omega g(x, y, \widetilde{w}(t-\tau, y))dy, \ (t,x) \in Q_T, \\ B\widetilde{w}(t,x) \geq 0, \ (t,x) \in S_T, \\ \widetilde{w}(t,x) \geq \phi(t,x), \ (t,x) \in Q_0. \end{cases} \qquad (3.1)$$

Similarly, $\widehat{w} \in \mathbb{X}_T$ is called a lower solution of (1.1) if it satisfies (3.1) with inequalities being reversed.

Lemma 3.1. *Let $c, d_1 \in C(Q_T)$ and $d_2 \in C(Q_T \times \Omega)$ such that $d_1, d_2 \geq 0$. If $z \in C^{1,2}(\overline{Q_T}) \bigcap C(\overline{D_T})$ satisfies*

$$
\begin{cases}
\dfrac{\partial z(t,x)}{\partial t} - Lz(t,x) \geq c(t,x)z(t,x) + d_1(t,x)z(t-\tau,x) \\
\qquad\qquad\qquad\qquad + \displaystyle\int_\Omega d_2(t,x,y)z(t-\tau,y)dy, \ (t,x) \in Q_T, \\
Bz(t,x) \geq 0, \ (t,x) \in S_T, \\
z(t,x) \geq 0, \ (t,x) \in Q_0,
\end{cases}
\tag{3.2}
$$

then $z(t,x) \geq 0$ for all $(t,x) \in \overline{Q_T}$.

Proof.   Using the hypothesis $d_1, d_2 \geq 0$ and the relation (3.2), we obtain

$$
\begin{cases}
\dfrac{\partial z(t,x)}{\partial t} - Lz(t,x) \geq c(t,x)z(t,x), \ (t,x) \in (0,\tau] \times \Omega, \\
Bz(t,x) \geq 0, \ (t,x) \in (0,\tau] \times \partial\Omega, \\
z(0,x) \geq 0, \ x \in \Omega.
\end{cases}
\tag{3.3}
$$

It follows that $z(t,x) \geq 0$ for all $(t,x) \in [0,\tau] \times \overline{\Omega}$ (see p.564 of [29]). Therefore, $z(t-\tau,x) \geq 0$ for all $(t,x) \in [0,2\tau] \times \overline{\Omega}$. Again by $d_1, d_2 \geq 0$ and (3.2), the inequalities in (3.3) hold when the interval $[0,\tau]$ is replaced by $[0,2\tau]$. This leads to $z(t,x) \geq 0$ for all $(t,x) \in [0,2\tau] \times \overline{\Omega}$. A continuation of the same process yields $z(t,x) \geq 0$ for all $(t,x) \in [0,k\tau] \times \overline{\Omega}$, where $k = 1,2,\cdots$. This proves $z(t,x) \geq 0$ for all $(t,x) \in \overline{Q_T}$. The proof is complete. $\qquad\square$

Remark 3.1. Lemma 3.1 gives a new maximum principle, which enables us to obtain the following important comparison principle for nonlocal reaction-diffusion equations with time delay. And the principle will play a key role in the proof of the existence-comparison theorem and the stability of solutions to (1.1). We point that it is the delay involved in nonlocal term that help us to apply step methods to establish Lemma 3.1. If there is no delay in the nonlocal term, one needs to establish a new comparison theorem, which seems not to be an easy task.

Lemma 3.2. *Suppose that $w, u \in C^{1,2}(\overline{Q_T}) \bigcap C(\overline{D_T})$ satisfies*

$$
\begin{cases}
\dfrac{\partial w(t,x)}{\partial t} - Lw(t,x) - f(x,w(t,x),w(t-\tau,x)) - \displaystyle\int_\Omega g(x,y,w(t-\tau,y))dy \\
\geq \dfrac{\partial u(t,x)}{\partial t} - Lu(t,x) - f(x,u(t,x),u(t-\tau,x)) - \displaystyle\int_\Omega g(x,y,u(t-\tau,y))dy, (t,x) \in Q_T, \\
Bw(t,x) \geq Bu(t,x), \ (t,x) \in S_T, \\
w(t,x) \geq u(t,x), \ (t,x) \in Q_0.
\end{cases}
$$
$$
\tag{3.4}
$$

Then $w(t,x) \geq u(t,x)$ for all $(t,x) \in Q_T$. Furthermore, if $w(t,x) \not\equiv u(t,x)$ for $(t,x) \in Q_0$, then $w(t,x) > u(t,x)$ for all $(t,x) \in Q_T$.

Proof. Let $z = w - u$. Then we have

$$
\begin{cases}
\dfrac{\partial z(t,x)}{\partial t} - Lz(t,x) \geq c(t,x)z(t,x) + d_1(t,x)z(t-\tau,x) \\
\qquad\qquad\qquad\qquad + \displaystyle\int_\Omega d_2(t,x,y)z(t-\tau,y))dy, \ (t,x) \in Q_T, \\
Bz(t,x) \geq 0, \ (t,x) \in S_T, \\
z(t,x) \geq 0, \ (t,x) \in Q_0,
\end{cases}
$$

where

$$
c(t,x) = \left.\frac{\partial f(x,\xi,\eta)}{\partial \xi}\right|_{\xi=u(t,x)+\theta_1(w(t,x)-u(t,x)), \ \eta=u(t-\tau,x)+\theta_1(w(t-\tau,x)-u(t-\tau,x))},
$$

$$
d_1(t,x) = \left.\frac{\partial f(x,\xi,\eta)}{\partial \eta}\right|_{\xi=u(t,x)+\theta_1(w(t,x)-u(t,x)), \ \eta=u(t-\tau,x)+\theta_1(w(t-\tau,x)-u(t-\tau,x))}
$$

and

$$
d_2(t,x,y) = \left.\frac{\partial g(x,y,\eta)}{\partial \eta}\right|_{\eta=u(t-\tau,y)+\theta_2(w(t-\tau,y)-u(t-\tau,y))},
$$

where $\theta_1, \theta_2 \in (0,1)$. By $d_1, d_2 \geq 0$ and Lemma 3.1, the conclusion of the lemma is obtained. The proof is complete. $\qquad\square$

Lemma 3.3. *Let $\widetilde{w}$ and $\widehat{w}$ be a pair of upper and lower solutions of (1.1). Then $\widehat{w} \leq \widetilde{w}$.*

Proof. By Definition 3.1, $\widehat{w} \leq \widetilde{w}$ follows immediately from Lemma 3.2. The proof is finished. $\qquad\square$

Therefore, we have the following existence-comparison theorem. Note that the existence and uniqueness in the following theorem can be derived directly from the abstract results in [25]. We give a proof here for the sake of completeness.

Theorem 3.1. *Let $\widetilde{w}$ and $\widehat{w}$ be a pair of upper and lower solutions of (1.1). Then (1.1) has a unique regular solution $w^* \in [\widehat{w}, \widetilde{w}]$. Moreover, there exist sequences $\{\underline{w}^{(n)}\}_{n=0}^\infty$ and $\{\overline{w}^{(n)}\}_{n=0}^\infty$ which converge monotonically from above and below, respectively, to w^* as $n \to \infty$.*

Proof. Let

$$
\Lambda_T = \left\{ z \in \mathbb{R} \mid \min_{(t,x)\in D_T} \widehat{w}(t,x) \leq z \leq \max_{(t,x)\in D_T} \widetilde{w}(t,x) \right\}.
$$

Then we can assume that $\partial f(x,u,w)/\partial u$ is bounded below for all $x \in \Omega$ and $u, w \in \Lambda_T$, so that

$$
\frac{\partial f(x,u,w)}{\partial u} + K > 0
$$

for all $x \in \Omega$ and $u, w \in \Lambda_T$, provided K is sufficiently large. Since for any $w \in \mathbb{Y}_T$, the linear problem

$$
\begin{cases}
\dfrac{\partial u(t,x)}{\partial t} - Lu(t,x) + Ku(t,x) = Kw(t,x) + f(x,w(t,x),w(t-\tau,x)) \\
\qquad\qquad\qquad\qquad + \displaystyle\int_\Omega g(x,y,w(t-\tau,y))dy, \ (t,x) \in Q_T, \\
Bu(t,x) = 0, \ (t,x) \in S_T, \\
u(t,x) = \phi(t,x), \ (t,x) \in Q_0
\end{cases}
\tag{3.5}
$$

has a unique solution $u \in \mathbb{X}_T$ (see, e.g., [10,19]), we can define the nonlinear operator $\mathcal{K} : \mathbb{Y}_T \to \mathbb{X}_T \subset \mathbb{Y}_T$ by $\mathcal{K}w = u$. Therefore, we know that $\mathcal{K}$ is monotone ($w_1 \leq w_2$ implies $\mathcal{K}w_1 \leq \mathcal{K}w_2$) and completely continuous as an operator from $\mathbb{Y}_T$ to $\mathbb{Y}_T$ in the order interval $[\widehat{w}, \widetilde{w}]$.

Now define $\overline{w}^{(1)} = \mathcal{K}\widetilde{w}$ and $\underline{w}^{(1)} = \mathcal{K}\widehat{w}$. Let us show that $\overline{w}^{(1)} \leq \widetilde{w}$ and $\underline{w}^{(1)} \geq \widehat{w}$. By (3.1) and

$$
\begin{cases}
\left(\dfrac{\partial}{\partial t} - L + K\right) \overline{w}^{(1)}(t,x) = K\widetilde{w}(t,x) + f(x, \widetilde{w}(t,x), \widetilde{w}(t-\tau, x)) \\
\qquad\qquad\qquad\qquad\qquad + \displaystyle\int_\Omega g(x,y, \widetilde{w}(t-\tau, y)) dy, \ (t,x) \in Q_T, \\
B\overline{w}^{(1)}(t,x) = 0, \ (t,x) \in S_T, \\
\overline{w}^{(1)}(t,x) = \phi(t,x), \ (t,x) \in Q_0,
\end{cases}
$$

we have

$$
\begin{aligned}
&(\frac{\partial}{\partial t} - L + K)(\overline{w}^{(1)}(t,x) - \widetilde{w}(t,x)) \\
=&(-\frac{\partial}{\partial t} + L)\widetilde{w}(t,x) + f(x, \widetilde{w}(t,x), \widetilde{w}(t-\tau, x)) + \int_\Omega g(x,y,\widetilde{w}(t-\tau,y)) dy \\
\leq& 0, \quad (t,x) \in Q_T
\end{aligned}
$$

and

$$
\begin{cases}
B(\overline{w}^{(1)}(t,x) - \widetilde{w}(t,x)) \leq 0, \ (t,x) \in S_T, \\
\overline{w}^{(1)}(t,x) - \widetilde{w}(t,x) \leq 0, \ (t,x) \in Q_0.
\end{cases}
$$

Therefore, by the maximum principle for parabolic equations (see, e.g., [29, 32]), it is easily seen that $\overline{w}^{(1)} \leq \widetilde{w}$. A similar argument shows that $\underline{w}^{(1)} \geq \widehat{w}$.

Since $\overline{w}^{(1)} \leq \widetilde{w}$ and the monotonicity of the operator $\mathcal{K}$, we obtain that $\mathcal{K}\overline{w}^{(1)} \leq \mathcal{K}\widetilde{w}$. Thus, the sequence defined inductively by

$$
\overline{w}^{(0)} = \widetilde{w}, \ \overline{w}^{(n)} = \mathcal{K}\overline{w}^{(n-1)}, \ \forall n = 1, 2, \cdots
$$

is monotone decreasing. Similarly,

$$
\underline{w}^{(0)} = \widehat{w}, \ \underline{w}^{(n)} = \mathcal{K}\underline{w}^{(n-1)}, \ \forall n = 1, 2, \cdots
$$

defines a monotone increasing sequence. Furthermore, we have $\underline{w}^{(n)} \leq \overline{w}^{(n)}$ for all $n = 1, 2, \cdots$ and

$$
\widehat{w} = \underline{w}^{(0)} \leq \underline{w}^{(1)} \leq \cdots \leq \underline{w}^{(n)} \leq \cdots \leq \overline{w}^{(n)} \leq \cdots \leq \overline{w}^{(1)} \leq \overline{w}^{(0)} = \widetilde{w}. \qquad (3.6)
$$

Actually, By Lemma 3.3, $\widehat{w} \leq \widetilde{w}$ and then $\underline{w}^{(1)} \leq \overline{w}^{(1)}$. Suppose that $\underline{w}^{(k)} \leq \overline{w}^{(k)}$ for some $k \geq 1$. Then

$$
\underline{w}^{(k+1)} = \mathcal{K}\underline{w}^{(k)} \leq \mathcal{K}\overline{w}^{(k)} = \overline{w}^{(k+1)}.
$$

Thus, the proof follows by induction.

Since the sequences $\left\{\underline{w}^{(n)}\right\}_{n=0}^\infty$ and $\left\{\overline{w}^{(n)}\right\}_{n=0}^\infty$ are monotone, both the pointwise limits

$$
\overline{w}(t,x) = \lim_{n\to\infty} \overline{w}^{(n)}(t,x) \quad \text{and} \quad \underline{w}(t,x) = \lim_{n\to\infty} \underline{w}^{(n)}(t,x)
$$

exist and satisfy the relation

$$\widehat{w}(t,x) \le \underline{w}(t,x) \le \overline{w}(t,x) = \widetilde{w}(t,x), \quad (t,x) \in Q_T.$$

The same argument as that in [30](Theorem 3.1) or [34](Theorem 3.1) shows that $\underline{w}(t,x)$ and $\overline{w}(t,x)$ are classical solutions to (1.1).

Next, we show the uniqueness of classical solutions to (1.1) in the order interval $[\widehat{w}, \widetilde{w}]$. By (3.6), $\underline{w} \le \overline{w}$. Again by using Lemma 3.2 (since in this case the equalities hold), we obtain $\underline{w} \ge \overline{w}$. Therefore, $\underline{w} = \overline{w}$. If $w(t,x)$ is a classical solution to (1.1) and satisfy $\widehat{w} \le w \le \widetilde{w}$, then $w = \mathcal{K}w$. Thus, it follows from induction that

$$\underline{w}^{(n)} = \mathcal{K}^n \widehat{w} \le \mathcal{K}^n w = w \le \mathcal{K}^n \widetilde{w} = \overline{w}^{(n)}.$$

Then,

$$\lim_{n \to \infty} \overline{w}^{(n)}(t,x) = \overline{w}(t,x) = \lim_{n \to \infty} \underline{w}^{(n)}(t,x) = \underline{w}(t,x).$$

The conclusion follows. This completes the proof. $\qquad\square$

By Theorem 3.1, we immediately obtain the following corollary.

Corollary 3.1. *Let $\widetilde{w}_s$ and $\widehat{w}_s$ be a pair of upper and lower solutions of the boundary value problem (1.2). Then for any $\phi \in C^{\alpha/2,\alpha}(Q_0)$ with $\widehat{w}_s(x) \le \phi(t,x) \le \widetilde{w}_s(x)$ for all $t \in [-\tau, 0)$, we obtain a global regular solution w of the initial boundary value problem (1.1) with initial data ϕ and the solution w satisfies $\widehat{w}_s(x) \le w(t,x) \le \widetilde{w}_s(x)$ for all $t \in [0, +\infty)$.*

Next, we will establish certain monotonicity properties of solutions to (1.1). Actually, if $\widetilde{w}_s$ is an upper solution of (1.2), then, as we have seen, it can be viewed as the first term of a monotone decreasing sequence by iterations. Here we shall also see that when $\widetilde{w}_s$ is taken as initial data for (1.1), the corresponding solution w is monotone decreasing in time.

One can see this as follows. Let $\widetilde{w}_s$ is an upper solution of (1.2) and suppose that w is a regular solution of (1.1) with initial data ϕ and $\phi(t,x) = \widetilde{w}_s(x)$ for all $t \in [-\tau, 0]$. By Corollary 3.1, we know that $w(t,x) \le \widetilde{w}_s(x)$ for all $t \in [0, +\infty)$. Let $h > 0$ and define

$$v_h(t,x) = \frac{w(t+h,x) - w(t,x)}{h}.$$

Then $v_h(t,x) \le 0$ for all $t \in [-\tau, 0]$ and v_h satisfies

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - L\right) v_h(t,x) = c_h(t,x)v_h(t,x) + d_{1,h}(t,x)v_h(t-\tau,x) \\ \qquad\qquad\qquad + \displaystyle\int_\Omega d_{2,h}(t,x,y)v_h(t-\tau,y))dy, (t,x) \in Q_T, \\ Bv_h(t,x) = 0, (t,x) \in S_T, \\ v_h(t,x) \le 0, (t,x) \in Q_0, \end{cases}$$

where

$$c_h(t,x) = \left.\frac{\partial f(x,\xi,\eta)}{\partial \xi}\right|_{\xi=\theta_1 w(t+h,x)+(1-\theta_1)w(t,x),\ \eta=\theta_1 w(t-\tau+h,x)+(1-\theta_1)w(t-\tau,x)},$$

$$d_{1,h}(t,x) = \left.\frac{\partial f(x,\xi,\eta)}{\partial \eta}\right|_{\xi=\theta_1 w(t+h,x)+(1-\theta_1)w(t,x),\ \eta=\theta_1 w(t-\tau+h,x)+(1-\theta_1)w(t-\tau,x)}$$

and

$$d_{2,h}(t,x,y) = \left.\frac{\partial g(x,y,\eta)}{\partial \eta}\right|_{\eta=\theta_2 w(t-\tau+h,y)+(1-\theta_2)w(t-\tau,y)},$$

where $\theta_1, \theta_2 \in (0,1)$. By Lemma 3.1, $v_h(t,x) \le 0$ for all $t \in (0,+\infty)$. Therefore, we obtain

$$\frac{\partial w(t,x)}{\partial t} = \lim_{h \to 0} v_h(t,x) \le 0$$

for all $t \in (0,+\infty)$. Thus, we have the following theorem.

Theorem 3.2. *Every upper solution of (1.2) gives rise to a monotonically nonincreasing solution of (1.1) when taken as initial data, while every lower solution of (1.2) gives rise to a monotonically nondecreasing solution of (1.1).*

Now, we show how the concept of upper and lower solutions can be weakened to correspond to the classical notion of super and sub-harmonic functions in potential theory. Associated with the operator L is the adjoint operator L^*. The domain of L^* is defined by

$$Dom(L^*) = \big\{\, \varphi |\, L^*\varphi \in L^2(\overline{\Omega}) \text{ and exists } \varphi^* \text{ such}$$
$$\text{that } \langle Lw, \varphi \rangle = \langle w, \varphi^* \rangle \text{ for all } w \in Dom(L) \big\},$$

where $\langle \cdot, \cdot \rangle$ denotes inner product on $L^2(\overline{\Omega})$, i.e.,

$$\langle \xi, \eta \rangle = \int_{\overline{\Omega}} \xi(y)\eta(y)dy, \ \ \forall \xi, \eta \in L^2(\overline{\Omega}).$$

If $\varphi \in Dom(L^*)$, then we write $\varphi^* = L^*\varphi$.

Remark 3.2. In general the set of functions on which a differential operator acts is determined partly by the boundary condition. In that sense the boundary conditions are part of the definition of the operator. Usually, the adjoint L^* of a differential operator L can be computed formally by writing the relation

$$\int_\Omega vLudx = \int_\Omega uL^*vdx$$

and determining L^* and the necessary boundary conditions on v by integration by parts via divergence theorem. The operator L^* computed in this way typically will coincide with true adjoint of L defined in terms of duality as long as v is a smooth function. The actual adjoint operator will often have its domain of definition expanded or restricted in some way. For our purpose it is enough to be able to identify L^* as a differential operator (see [4]). For example, let

$$Lu = \sum_{i,j=1}^m a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^m b_i(x)\frac{\partial u}{\partial x_i}, \ \ x = (x_1, x_2, \cdots, x_m)$$

with homogeneous Dirichlet boundary condition $u = 0$, for $x \in \partial\Omega$. Then L^* can be formulated explicitly as follows.

$$L^*v = \sum_{i,j=1}^m a_{ij}(x)\frac{\partial^2 v}{\partial x_i \partial x_j} - \sum_{i=1}^m b_i(x)\frac{\partial v}{\partial x_i}, \ \ x = (x_1, x_2, \cdots, x_m)$$

with homogeneous Dirichlet boundary condition $v = 0$, for $x \in \partial\Omega$. When L takes the form

$$Lu = \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad x = (x_1, x_2, \cdots, x_m)$$

with Robin boundary condition $\frac{\partial u}{\partial \mathbf{n}} + \beta(x)u = 0$ for $x \in \partial\Omega$. Then L^* has exactly the same form as L as well as its boundary condition.

Define the operator $\jmath : L^2(\overline{\Omega}) \to L^2(\overline{\Omega})$ by

$$(\jmath w)(x) = f(x, w(x), w(x)) + \int_{\overline{\Omega}} g(x, y, w(y))dy, \ \forall w \in L^2(\overline{\Omega}).$$

This leads to the following definition of weak upper and lower solutions.

Definition 3.2. A function $\widetilde{w}_s$ is said to be a weak upper solution of (1.2) if $\widetilde{w}_s$ is bounded and measurable on $\overline{\Omega}$ and satisfies

$$\langle \widetilde{w}_s, \varphi^* \rangle + \langle \jmath\widetilde{w}_s, \varphi \rangle \le 0, \ \forall \varphi \in Dom(L^*). \tag{3.7}$$

Similarly, $\widehat{w}_s$ is called a weak lower solution of (1.2) if it satisfies (3.7) in reversed order.

Therefore, we have the following theorems which will be of great importance in investigating the asymptotic behavior of solutions of (1.1).

Theorem 3.3. *Let $\widetilde{w}_s$ ($\widehat{w}_s$) be a weak upper (lower) solution of (1.2) and suppose that w is a regular solution of (1.1) with initial data $\widetilde{w}_s$ ($\widehat{w}_s$). Then $\partial w(t, x)/\partial t \le 0 (\ge 0)$ for all $(t, x) \in (0, +\infty) \times \Omega$.*

Proof. Consider the linear initial boundary value problem

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - L + K \right) w_1(t, x) = K\widetilde{w}_s(x) + (\jmath\widetilde{w}_s)(x), \ (t, x) \in Q_T, \\ Bw_1(t, x) = 0, \ (t, x) \in S_T, \\ w_1(t, x) = \widetilde{w}_s(x), \ (t, x) \in Q_0 \end{cases} \tag{3.8}$$

for a suitable $K > 0$, where K is to be chosen so that $\partial f(x, \xi, \eta)/\partial \xi + K > 0$ for all $x \in \Omega$ and $\xi, \eta \in \Lambda_{1,T}$,

$$\Lambda_{1,T} = \left\{ z \in \mathbb{R} | \min_{(t,x) \in Q_T} w(t, x) \le z \le \max_{x \in \Omega} \widetilde{w}_s(x) \right\}$$

and $w(t, x)$ is the solution of (1.1) with initial data $\widetilde{w}_s(x)$. Let $\psi = \mathcal{T}\widetilde{w}_s$, that is,

$$\psi(x) = (-L + K)^{-1} \left(K\widetilde{w}_s(x) + (\jmath\widetilde{w}_s)(x) \right).$$

And let $u_1 = w_1 - \psi$. Then by (3.8), we have

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - L + K \right) u_1(t, x) = 0, \ (t, x) \in Q_T, \\ Bu_1(t, x) = 0, \ (t, x) \in S_T, \\ u_1(t, x) = \widetilde{w}_s(x) - \psi(x), \ (t, x) \in Q_0. \end{cases} \tag{3.9}$$

Furthermore, for any $\varphi \in Dom(L^*)$ and $\varphi > 0$, we have

$$
\begin{aligned}
\langle \widetilde{w}_s - \psi, (L^* - K)\varphi \rangle &= \langle (L - K)\widetilde{w}_s, \varphi \rangle - \langle (L - K)\psi, \varphi \rangle \\
&= \langle (L - K)\widetilde{w}_s, \varphi \rangle + \langle K\widetilde{w}_s + \jmath\widetilde{w}_s, \varphi \rangle \\
&= \langle \widetilde{w}_s, \varphi^* \rangle + \langle \jmath\widetilde{w}_s, \varphi \rangle \\
&\leq 0.
\end{aligned}
\tag{3.10}
$$

Thus, by (3.9), (3.10) and [34](Lemma 3.5), we obtain

$$
\frac{\partial w_1(t,x)}{\partial t} = \frac{\partial u_1(t,x)}{\partial t} \leq 0, \ (t,x) \in Q_T.
\tag{3.11}
$$

Let $u = w - w_1$, where w is the regular solution of (1.1) with initial data $\widetilde{w}_s$. Then we have

$$
\begin{cases}
\left(\dfrac{\partial}{\partial t} - L \right) u(t,x) = f(x, u(t,x) + w_1(t,x), u(t-\tau,x) + w_1(t-\tau,x)) - (\jmath\widetilde{w}_s)(x) \\
\quad + K(w_1(t,x) - \widetilde{w}_s(x)) + \displaystyle\int_\Omega g(x,y, u(t-\tau,y) + w_1(t-\tau,y))dy, \ (t,x) \in Q_T, \\
Bu(t,x) = 0, \ (t,x) \in S_T, \\
u(t,x) = 0, \ (t,x) \in Q_0.
\end{cases}
\tag{3.12}
$$

As before we form the time differences

$$
u_h(t,x) = \frac{u(t+h,x) - u(t,x)}{h} \ \text{ and } \ w_{1,h}(t,x) = \frac{w_1(t+h,x) - w_1(t,x)}{h}.
$$

Therefore, by (3.12), we obtain

$$
rl \left(\frac{\partial}{\partial t} - L \right) u_h(t,x) - f_1(t,x, u_h(t,x), u_h(t-\tau,x)) - \int_\Omega \eta_{2,h}(t,x,y) u_h(t-\tau,y)dy
$$

$$
= f_2(t,x, w_{1,h}(t,x), w_{1,h}(t-\tau,x)) + \int_\Omega \eta_{2,h}(t,x,y) w_{1,h}(t-\tau,y)dy, (t,x) \in Q_T,
$$

where

$$
f_1(t,x, u_h(t,x), u_h(t-\tau,x)) = \xi_h(t,x) u_h(t,x) + \eta_{1,h}(t,x) u_h(t-\tau,x),
$$

$$
f_2(t,x, w_{1,h}(t,x), w_{1,h}(t-\tau,x)) = (K + \xi_h(t,x)) w_{1,h}(t,x) + \eta_{1,h}(t,x) w_{1,h}(t-\tau,x),
$$

$$
\xi_h(t,x) = \left. \frac{\partial f(x,\xi,\eta)}{\partial \xi} \right|_{\xi = \theta_1 w(t+h,x) + (1-\theta_1)w(t,x), \ \eta = \theta_1 w(t-\tau+h,x) + (1-\theta_1)w(t-\tau,x)},
$$

$$
\eta_{1,h}(t,x) = \left. \frac{\partial f(x,\xi,\eta)}{\partial \eta} \right|_{\xi = \theta_1 w(t+h,x) + (1-\theta_1)w(t,x), \ \eta = \theta_1 w(t-\tau+h,x) + (1-\theta_1)w(t-\tau,x)}
$$

and

$$
\eta_{2,h}(t,x,y) = \left. \frac{\partial g(x,y,\eta)}{\partial \eta} \right|_{\eta = \theta_2 w(t-\tau+h,y) + (1-\theta_2)w(t-\tau,y)},
$$

where $\theta_1, \theta_2 \in (0,1)$. By (3.11) and $w_1(t,x) = \widetilde{w}_s(x)$ for all $(t,x) \in Q_0$, we have

$$
f_2(t,x, w_{1,h}(t,x), w_{1,h}(t-\tau,x)) + \int_\Omega \eta_{2,h}(t,x,y) w_{1,h}(t-\tau,y)dy \leq 0, \ \forall(t,x) \in Q_T.
$$

Therefore,

$$\left(\frac{\partial}{\partial t} - L\right) u_h(t, x) - f_1(t, x, u_h(t, x), u_h(t - \tau, x)) - \int_\Omega \eta_{2,h}(t, x, y) u_h(t - \tau, y) dy \leq 0$$

for all $(t, x) \in Q_T$. Furthermore, $Bu_h(t, x) = 0$ for all $(t, x) \in S_T$ and

$$u_h(t, x) = \frac{u(t + h, x) - u(t, x)}{h} = \frac{w(t + h, x) - w_1(t + h, x)}{h} \leq 0, \ \forall (t, x) \in Q_0.$$

This last inequality follows from the fact that iterations decrease monotonically to w, i.e. $w_0(t, x) \geq w_1(t, x) \geq \cdots \geq w(t, x)$ for $t > 0$, where $w_0(t, x) = \widetilde{w}_s(x)$ for all $(t, x) \in [-\tau, +\infty) \times \overline{\Omega}$. Thus, employing Lemma 3.1, we know that $u_h(t, x) \leq 0$ for all $t \in (0, +\infty)$. Hence,

$$\frac{\partial u(t, x)}{\partial t} = \lim_{h \to 0} u_h(t, x) \leq 0$$

for all $t \in (0, +\infty)$. This shows $\partial w(t, x)/\partial t \leq 0$ for all $t \in (0, +\infty)$. This completes the proof. □

Theorem 3.4. *Let $\widetilde{w}_s(x)$ and $\widehat{w}_s(x)$ be a pair of upper and lower solutions of (1.2). Suppose that $\overline{w}(t, x)$ and $\underline{w}(t, x)$ are the solutions of (1.1) corresponding to $\phi(t, x) = \widetilde{w}_s(x)$ and $\phi(t, x) = \widehat{w}_s(x)$ in Q_0, respectively. Then as $t \to +\infty$, $\overline{w}(t, x)$ converges monotonically from above to $\overline{w}_s(x)$ and $\underline{w}(t, x)$ converges monotonically from below to $\underline{w}_s(x)$, $\underline{w}_s \leq \overline{w}_s$ and $\overline{w}_s$ and $\underline{w}_s$ are regular stationary solutions of (1.2).*

Proof. By Lemma 3.2, we have

$$\widehat{w}_s(x) \leq \underline{w}(t, x) \leq \overline{w}(t, x) \leq \widetilde{w}_s(x), \quad \forall (t, x) \in [0, +\infty) \times \Omega.$$

It follows from Theorem 3.3 that $\partial \overline{w}(t, x)/\partial t \leq 0$ and $\partial \underline{w}(t, x)/\partial t \geq 0$ for all $(t, x) \in (0, +\infty) \times \Omega$. Thus, $\overline{w}(t, x)$ is nonincreasing and $\underline{w}(t, x)$ is nondecreasing in t. Therefore, the pointwise limits

$$\underline{w}_s(x) = \lim_{t \to \infty} \underline{w}(t, x)$$

and

$$\overline{w}_s(x) = \lim_{t \to \infty} \overline{w}(t, x)$$

exist and $\underline{w}_s(x) \leq \overline{w}_s(x)$ for all $x \in \Omega$. It suffice to prove that $\overline{w}_s$ and $\underline{w}_s$ are strong solutions of (1.2).

For all $\varphi \in Dom(L^*)$ and all $t \in (0, +\infty)$, we have

$$\int_{\overline{\Omega}} \frac{\partial \overline{w}(t, x)}{\partial t} \varphi(x) dx$$

$$= \int_{\overline{\Omega}} \left[L\overline{w}(t, x) + f(x, \overline{w}(t, x), \overline{w}(t - \tau, x)) + \int_{\overline{\Omega}} g(x, y, \overline{w}(t - \tau, y)) dy \right] \varphi(x) dx$$

$$= \int_{\overline{\Omega}} \overline{w}(t, x) \varphi^*(x) dx + \int_{\overline{\Omega}} \left[f(x, \overline{w}(t, x), \overline{w}(t - \tau, x)) + \int_{\overline{\Omega}} g(x, y, \overline{w}(t - \tau, y)) dy \right] \varphi(x) dx.$$

Operating on both sides with $T^{-1} \int_0^T dt$, we obtain

$$\int_{\overline{\Omega}} \delta_1(T, x) \varphi(x) dx = \int_{\overline{\Omega}} \varphi^*(x) \delta_2(T, x) dx + \int_{\overline{\Omega}} [\delta_3(T, x) + \delta_4(T, x)] \varphi(x) dx, \quad (3.13)$$

where

$$\delta_1(T,x) = \frac{\overline{w}(T,x) - \overline{w}(0,x)}{T}, \quad \delta_2(T,x) = \frac{1}{T}\int_0^T \overline{w}(t,x)dt,$$

$$\delta_3(T,x) = \frac{1}{T}\int_0^T f(x,\overline{w}(t,x),\overline{w}(t-\tau,x))dt$$

and

$$\delta_4(T,x) = \frac{1}{T}\int_0^T \int_{\overline{\Omega}} g(x,y,\overline{w}(t-\tau,y))dydt.$$

Furthermore,

$$\lim_{T\to\infty}\delta_1(T,x) = 0, \quad \lim_{T\to\infty}\delta_2(T,x) = \overline{w}_s(x), \quad \lim_{T\to\infty}\delta_3(T,x) = f(x,\overline{w}_s(x),\overline{w}_s(x))$$

and

$$\lim_{T\to\infty}\delta_4(T,x) = \int_{\overline{\Omega}} g(x,y,\overline{w}_s(y))dy.$$

And for every $i = 1,2,3,4$, $\delta_i(T,x)$ remains bounded uniformly as $T \to \infty$. Thus, by (3.13) and the Lebesgue dominated convergence theorem, we have

$$0 = \int_{\overline{\Omega}} \varphi^*(x)\overline{w}_s(x)dx + \int_{\overline{\Omega}}\left[f(x,\overline{w}_s(x),\overline{w}_s(x)) + \int_{\overline{\Omega}} g(x,y,\overline{w}_s(y))dy\right]\varphi(x)dx,$$

that is,

$$\langle \overline{w}_s, \varphi^*\rangle + \langle \jmath\overline{w}_s, \varphi\rangle = 0.$$

Now, we claim that if $\langle w_s, \varphi^*\rangle + \langle \jmath w_s, \varphi\rangle = 0$ for all $\varphi \in Dom(L^*)$, then w_s is a classical solution of (1.2). To this end, we note that L and L^* are invertible. Therefore, we set that $\mathcal{L}$ is the inverse of L and that $\mathcal{L}^*$ is the inverse of L^*. Let $u_s = -\mathcal{L}\jmath w_s$. Then we obtain

$$\langle u_s, \varphi^*\rangle = -\langle \mathcal{L}\jmath w_s, L^*\varphi\rangle = -\langle \jmath w_s, \mathcal{L}^*L^*\varphi\rangle = -\langle \jmath w_s, \varphi\rangle.$$

Hence, $\langle w_s - u_s, \varphi^*\rangle = 0$ for all $\varphi \in Dom(L^*)$. Therefore, $w_s = u_s = -\mathcal{L}\jmath w_s$. Thus, w_s is a weak solution to the boundary value problem (1.2). To show that w_s is a strong solution, we need to prove the regularity of w_s. Again, by [1](Theorem 15.2), $w_s \in W_{2,p}(\Omega)$ for any $p \in (1,+\infty)$ since $\mathcal{L}$ takes L_p into $W_{2,p}$ and since $\jmath w_s$ is bounded if w_s is bounded. By the embedding lemma, for any $p \in (m,+\infty)$, $w_s \in C^{1+\alpha}(\overline{\Omega})$. Finally, by the classical Schauder estimates, $w_s \in C^{2+\alpha}(\overline{\Omega})$. The proof is finished. $\qquad\square$

4. Stability of solutions

In this section, we will prove that any solution of (1.2) obtained by the monotone procedures in section 2 is stable without any assumptions upon f and g. We will also show that upper and lower solutions can be used to estimate the extent of stability. Furthermore, from the viewpoint of applications to practical problems, it is important to know whether a given solution is stable. But from the viewpoint of applications of the monotone iteration methods, it is also important to realize that only stable solutions can be obtained by such procedures. Other solutions which

might exist but would be unstable must be obtained by other approaches. We will come back to this point when we discuss examples in section 5.

Let $\mathbb{C} = C([-\tau, 0], \mathbb{Y})$. For any continuous function $w(\cdot) : [-\tau, \sigma) \to \mathbb{Y}$, where $\sigma > 0$, we denote $w_t \in \mathbb{C}$, $t \in [0, \sigma)$ by $w_t(s) = w(t + s)$ for any $s \in [-\tau, 0]$ and its norm

$$\|w_t\|_{\mathbb{C}} = \sup_{s \in [-\tau, 0]} \sup_{x \in \overline{\Omega}} |w(t + s, x)|,$$

where we denote $w(t, x) = w(t)(x)$, $t \in [-\tau, \sigma)$, $x \in \overline{\Omega}$. Define $F : \mathbb{C} \to \mathbb{Y}$ by

$$F(\phi) = f(\cdot, \phi(0), \phi(-\tau)) + \int_{\Omega} g(\cdot, y, \phi(-\tau)) dy, \quad \forall \phi \in \mathbb{C}.$$

Then we can rewrite (1.1) as a nonlinear abstract functional differential equation

$$\begin{cases} \dfrac{dw(t)}{dt} = Aw(t) + F(w_t), & t \geq 0, \\ w_0 = \phi \in \mathbb{C}, \end{cases} \tag{4.1}$$

where A is the infinitesimal generator of a semigroup $\{T(t)\}_{t \geq 0}$ on $\mathbb{Y}$ and its domain

$$Dom(A) = \{w \in \mathbb{Y} \mid Lw \in \mathbb{Y}, Bw(x)|_{x \in \partial\Omega} = 0\},$$
$$Aw = Lw, \quad \forall w \in Dom(A).$$

Therefore, we can give following formal definitions of stability.

Definition 4.1. Let w^* be a solution of the boundary value problem (1.2). It is called stable in the supremum norm if for any $\varepsilon > 0$ there exists $\delta > 0$ such that the solution $w(t, \phi)$ of (1.1) with $\|\phi - w^*\|_{\mathbb{C}} < \delta$ satisfies $\|w(t, \phi) - w^*\|_{\infty} < \varepsilon$ for all $t \geq 0$, where $\|w(t, \phi) - w^*\|_{\infty} = \sup_{x \in \overline{\Omega}} |w(t, \phi)(x) - w^*(x)|$. It is called unstable if it is not stable. It is asymptotically stable if it is stable and there exists $\delta_0 > 0$ such that the solution $w(t, \phi)$ of (1.1) with $\|\phi - w^*\|_{\mathbb{C}} < \delta_0$ satisfies $\lim_{t \to +\infty} \|w(t, \phi) - w^*\|_{\infty} = 0$. It is globally asymptotically stable if it is stable and any solution $w(t, \phi)$ of (1.1) with arbitrary $\phi \in \mathbb{C}$ satisfies $\lim_{t \to +\infty} \|w(t, \phi) - w^*\|_{\infty} = 0$.

As a preliminary result, we have

Theorem 4.1. *Let $\widetilde{w}_s(x)$ and $\widehat{w}_s(x)$ be a pair of upper and lower solutions of (1.2) and let $\overline{w}(t, x)$ and $\underline{w}(t, x)$ be the solutions of (1.1) corresponding to $\phi(t, x) = \widetilde{w}_s(x)$ and $\phi(t, x) = \widehat{w}_s(x)$ in Q_0, respectively. If $w(t, x)$ is a solution of (1.1) corresponding to $\phi(t, x) = w_s(x)$ in Q_0 with $\widehat{w}_s \leq w_s \leq \widetilde{w}_s$, then $\underline{w}(t, x) \leq w(t, x) \leq \overline{w}(t, x)$ for all $(t, x) \in (0, +\infty) \times \Omega$. Moreover, if $w^* \in [\widehat{w}_s, \widetilde{w}_s]$ is a solution of (1.2) and sequences $\{\mathcal{T}^n \widetilde{w}_s\}_{n=1}^{\infty}$ and $\{\mathcal{T}^n \widehat{w}_s\}_{n=1}^{\infty}$ converge monotonically from above and below, respectively, to w^* as $n \to \infty$, then w^* is asymptotically stable and $w(t, x) \to w^*$ as $t \to +\infty$.*

Proof. It follows immediately from Lemma 3.2 that $\underline{w}(t, x) \leq w(t, x) \leq \overline{w}(t, x)$ for all $(t, x) \in (0, +\infty) \times \Omega$. By Theorem 3.4, we know that $\overline{w}(t, x)$ converges monotonically from above to $\overline{w}_s(x)$ and $\underline{w}(t, x)$ converges monotonically from below to $\underline{w}_s(x)$ as t tends to infinity. If $\overline{w}_s(x) = \underline{w}_s(x)$, then necessarily $w^*(x) = \overline{w}_s(x) = \underline{w}_s(x)$ and $\lim_{t \to +\infty} w(t, x) = w^*(x)$. This will be the case if $\overline{w}_s(\underline{w}_s)$ generates a monotone decreasing(increasing) sequence which converges to w^*. In particular, it follows from Corollary 2.1 that if $\{\mathcal{T}^n \widetilde{w}_s\}_{n=1}^{\infty}$ and $\{\mathcal{T}^n \widehat{w}_s\}_{n=1}^{\infty}$ converge monotonically from

above and below, respectively, to w^* as $n \to \infty$, then w^* is asymptotically stable, and any solution of (1.1) corresponding to $\phi(t,x) = w_s(x)$ in Q_0 and $\widehat{w}_s \leq w_s \leq \widetilde{w}_s$ tends to w^* as $t \to +\infty$. The proof is complete. $\square$

We point out that the converse of Theorem 4.1 also holds. Namely, if w^* is a stable solution of (1.2), then it can be obtained as a limit of some upper and lower solutions. To prove this, we consider the linear eigenvalue problem

$$\begin{cases} Lw(x) + \pounds_w(w^*)w(x) = \lambda w(x), & x \in \Omega, \\ Bw(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4.2}$$

where

$$\pounds_w(w^*)w = f_{1,w}(\cdot, w^*, w^*)w + f_{2,w}(\cdot, w^*, w^*)w + \int_{\overline{\Omega}} g_w(\cdot, y, w^*(y))w(y)dy,$$

$$f_{1,w}(\cdot, w^*, w^*) = \left. \frac{\partial f(\cdot, u, w)}{\partial u} \right|_{u=w^*,\ w=w^*},$$

$$f_{2,w}(\cdot, w^*, w^*) = \left. \frac{\partial f(\cdot, u, w)}{\partial w} \right|_{u=w^*,\ w=w^*}$$

and

$$g_w(\cdot, y, w^*) = \left. \frac{\partial g(\cdot, y, w)}{\partial w} \right|_{w=w^*}.$$

The linear equation (4.2) can be rewritten as $\mathfrak{T}w = \lambda w$, where $\mathfrak{T} = L + \pounds_w(w^*)$: $\mathbb{X} \to \mathbb{Y}$. By the famous Krein-Rutman theorem (see Theorem 3.2 of [3]) and the perturbation theory for linear operators (see Sections IV–3.5 and VII–6 of [17]), we know that the simple eigenvalue λ_0 of $\mathfrak{T}$ has an associated positive eigenfunction $\varphi_0 > 0$. In fact, for each $\varepsilon \in \mathbb{R}$, we consider the linear eigenvalue problem

$$\begin{cases} \pounds_1(\varepsilon)w(x) = \lambda(\varepsilon)\mathcal{K}_1(\varepsilon)w(x), & x \in \Omega, \\ Bw(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4.3}$$

where

$$\pounds_1(\varepsilon)w = \begin{cases} -Lw + (M - f_{1,w}(\cdot, w^*, w^*) - f_{2,w}(\cdot, w^*, w^*))w + \varepsilon w, & \text{if } \varepsilon \geq 0, \\ -Lw + (M - f_{1,w}(\cdot, w^*, w^*) - f_{2,w}(\cdot, w^*, w^*))w, & \text{if } \varepsilon < 0, \end{cases}$$

$$\mathcal{K}_1(\varepsilon)w = \begin{cases} Mw + \displaystyle\int_{\overline{\Omega}} g_w(\cdot, y, w^*(y))w(y)dy, & \text{if } \varepsilon \geq 0, \\ Mw - \varepsilon w + \displaystyle\int_{\overline{\Omega}} g_w(\cdot, y, w^*(y))w(y)dy, & \text{if } \varepsilon < 0 \end{cases}$$

and

$$M = \max_{x \in \Omega} |f_{1,w}(x, w^*(x), w^*(x))| + \max_{x \in \Omega} |f_{2,w}(x, w^*(x), w^*(x))|.$$

The linear equation (4.3) can be rewritten as $\mathfrak{T}_1(\varepsilon)w = \frac{1}{\lambda(\varepsilon)}w$, where $\mathfrak{T}_1(\varepsilon) = (\pounds_1(\varepsilon))^{-1}\mathcal{K}_1(\varepsilon) : \mathbb{Y} \to \mathbb{X} \subset \mathbb{Y}$. By the Krein-Rutman theorem, the simple eigenvalue $\lambda^*(\varepsilon)$ of $\mathfrak{T}_1(\varepsilon)$ has an associated positive eigenfunction $\varphi^*(\varepsilon) > 0$. Therefore,

$$\pounds_1(\varepsilon)\varphi^*(\varepsilon) = \lambda^*(\varepsilon)\mathcal{K}_1(\varepsilon)\varphi^*(\varepsilon). \tag{4.4}$$

By (4.4) and the perturbation theory for linear operators, $\lambda^*(\varepsilon)$ is continuous in ε and $(0, +\infty) \subset \{\lambda^*(\varepsilon) | \varepsilon \in \mathbb{R}\}$ (see Sections IV–3.5 and VII–6 of [17]). Thus, there exists a real number ε_0 such that $\lambda^*(\varepsilon_0) = 1$. Let $\lambda_0 = \varepsilon_0$ and $\varphi_0 = \varphi^*(\varepsilon_0)$. Then λ_0 is a simple eigenvalue of $\mathfrak{T}$ and have a positive eigenfunction φ_0.

Therefore, we have the following theorem.

Theorem 4.2. *If $\lambda_0 < 0$, then w^* is stable and is the limit of a sequence of upper solutions from above and lower solutions from below. If $\lambda_0 > 0$, then w^* is unstable and is the limit of a sequence of lower solutions from above and upper solutions from below.*

Proof. For sufficiently small $\varepsilon^2 > 0$, we have

$$L(w^* + \varepsilon\varphi_0) + \jmath(w^* + \varepsilon\varphi_0) = Lw^* + \jmath w^* + \varepsilon\mathfrak{T}\varphi_0 + o(1)\varepsilon = \varepsilon\lambda_0\varphi_0 + o(1)\varepsilon$$

and

$$B(w^* + \varepsilon\varphi_0) = Bw^* = 0,$$

where $o(1) \to 0$ as $\varepsilon \to 0$. If $\lambda_0 < 0$, then $w^* + \varepsilon\varphi_0$ is an upper solution for $\varepsilon > 0$ and a lower solution for $\varepsilon < 0$ since $\varphi_0 > 0$ and $\varepsilon\lambda_0\varphi_0$ dominates the term $o(1)\varepsilon$ for sufficiently small $\varepsilon^2 > 0$. By Theorem 4.1, we know that this establishes the first statement above. If $\lambda_0 > 0$, then $w^* + \varepsilon\varphi_0$ is an upper solution for $\varepsilon < 0$ and a lower solution for $\varepsilon > 0$. To establish the instability of w^*, let w_δ be a solution of the initial value problem (1.1) with $w_\delta(t, x) = w^*(x) + \delta\varphi_0(x)$ in Q_0 and $\delta > 0$. Assuming δ is sufficiently small so that $w^* + \delta\varphi_0$ is a lower solution of (1.2), then $w_\delta(t, x)$ is increasing for $t > 0$. Consequently, we have solutions with small initial data which do not remain small, and this amounts to a statement of instability. The proof is finished. $\square$

Remark 4.1. Actually, we only proved that $\partial w_\delta/\partial t \geq 0$ in Theorem 3.3. However, either $w_\delta(t, w)$ is bounded above for all $t > 0$, in which case it tends to a steady state $\underline{w}(x)$, or $w_\delta(t, x) \to +\infty$ as $t \to +\infty$. In both cases, it must be growing as t increasing.

Remark 4.2. Suppose we know only that w^* is the limit from above of upper solutions. What can we conclude about the stability of w^* in this case? Firstly, it is clear from the previous arguments that w^* is stable to sufficiently small perturbations from above. Moreover, we claim have $\lambda_0 \leq 0$. Suppose to the contrary, let $\lambda_0 > 0$ and $\left\{w_s^{(n)}\right\}_{n=1}^{\infty}$ be a sequence of upper solutions converging downward to w^*. We can construct a lower solution $w^* + \delta\varphi_0$ for small $\delta > 0$ with $w^* + \delta\varphi_0 \leq w_s^{(k)}$ for some fixed integer k. Let $w(t, x)$ and $w^{(k)}(t, x)$ be solutions of the initial value problem (1.1) with $w(t, x) = w^*(x) + \delta\varphi_0(x)$ and $w^{(k)}(t, x) = w_s^{(k)}(x)$ in Q_0. Then as $t \to +\infty$, $w^{(k)}(t, x)$ converges monotonically from above to $w^*(x)$ while $w(t, x)$ increases. This is a contradiction to Lemma 3.2.

5. Examples

In this section, we present three examples to illustrate the feasibility of our main results. These examples have been investigated by many researchers in the current literature (see [11, 15, 42–44, 46, 49]).

Example 5.1. Consider the following reaction-diffusion population model with stage structure

$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} = d\Delta w(t,x) - \mu w^2(t,x) + \eta p \displaystyle\int_{\overline{\Omega}} \Gamma(\alpha,x,y)w(t-\tau,y)dy, t > 0, x \in \Omega, \\ Bw(t,x) = 0, t > 0, x \in \partial\Omega, \\ w(t,x) = \phi(t,x), t \in [-\tau,0], x \in \Omega, \end{cases}$$

(5.1)

where $\tau > 0$ is the maturation time for the species and $w(t,x)$ represents the density of the mature population at time t and location x, $d > 0$ denotes the diffusion rate; the indirect parameters η and α are defined by $\eta = e^{-\int_0^\tau \mu(a)da}$ and $\alpha = \int_0^\tau d(a)da$ where $\mu(a)$ and $d(a)$ denote the death rate and the diffusion rate of the immature population with age $a \geq 0$, respectively; μw^2 and pw represent the death function of the mature population and the birth function, respectively, where $\mu > 0$ and $p > 0$; Δ is the Laplacian operator on $\mathbb{R}^m$, $\phi(t,x)$ is a positive initial function to be specified later; Ω and B can be referred to section 1; $\Gamma(\alpha,x,y)$ is given by

$$\Gamma(\alpha,x,y) = \begin{cases} \displaystyle\sum_{n=1}^{+\infty} e^{-\lambda_n \alpha}\varphi_n(x)\varphi_n(y), & \text{if } \alpha > 0, \\ \delta(x-y), & \text{if } \alpha = 0. \end{cases}$$

(5.2)

Here, $0 \leq \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ with $\lim\limits_{n\to\infty} \lambda_n = +\infty$ are the eigenvalues of the linear operator $-\Delta$ subject to the boundary condition $Bw = 0$ on $\partial\Omega$, φ_n is the eigenvector corresponding to λ_n, $\{\varphi_n\}_{n=1}^\infty$ is a complete orthonormal system in the space $L^2(\overline{\Omega})$, $\varphi_1(x) > 0$ for all $x \in \Omega$, and $\delta(x)$ is the Dirac function on $\mathbb{R}^m$ [8, 49].

It is easy to see that model (5.1) is a special case of equation (1.1) with $f(x,w,u) = -\mu w^2$, $g(x,y,u) = \eta p\Gamma(\alpha,x,y)u$ (as $\alpha > 0$) or $f(x,w,u) = -\mu w^2 + \eta pu$, $g(x,y,u) = 0$ (as $\alpha = 0$) and $L = d\Delta$. Thus, by Corollary 2.1, Theorems 4.1 and 4.2, we obtain

Theorem 5.1. (i) *If $d\lambda_1 \geq \eta pe^{-\lambda_1 \alpha}$, then there is no positive steady state to (5.1).*
   (ii) *If $d\lambda_1 < \eta pe^{-\lambda_1 \alpha}$ the zero solution of (5.1) is unstable, and if $d\lambda_1 > \eta pe^{-\lambda_1 \alpha}$ it is globally asymptotically stable.*
   (iii) *If $d\lambda_1 < \eta pe^{-\lambda_1 \alpha}$, then (5.1) has a unique positive steady state w^* which is globally asymptotically stable.*

Proof.   (i)   Assume for the sake of contradiction that $w = w^*(x)$ is a positive steady state to (5.1). Then

$$-d\Delta w^*(x) = -\mu w^{*2}(x) + \eta p \int_{\overline{\Omega}} \Gamma(\alpha,x,y)w^*(y)dy.$$

(5.3)

Let $k_1 = \max_{x\in\Omega} w^*(x)$ and consider the linear eigenvalue problem

$$\begin{cases} -d\Delta w(x) + \mu k_1 w(x) = \lambda\left(\mu k_1 w(x) + \eta p\displaystyle\int_{\overline{\Omega}} \Gamma(\alpha,x,y)w(y)dy\right), & x \in \Omega, \\ Bw(x) = 0, & x \in \partial\Omega. \end{cases}$$

(5.4)

Define $\mathcal{L}_1 : \mathbb{X} \to \mathbb{Y}$ by

$$(\mathcal{L}_1 w)(x) = -d\Delta w(x) + \mu k_1 w(x), \ \forall w \in \mathbb{X}$$

and $\mathcal{S}_1 : \mathbb{Y} \to \mathbb{Y}$ by

$$(\mathcal{S}_1 w)(x) = \mu k_1 w(x) + \eta p \int_{\overline{\Omega}} \Gamma(\alpha, x, y) w(y) dy, \ \forall w \in \mathbb{Y}.$$

Then the linear equation (5.4) can be rewritten as $\mathfrak{T}_1 w = \frac{1}{\lambda} w$ where $\mathfrak{T}_1 = \mathcal{L}_1^{-1} \mathcal{S}_1 :$ $\mathbb{Y} \to \mathbb{X} \subset \mathbb{Y}$. By [46](Lemma 2.3) and the property of the operator $\mathcal{L}_1$, it is known (see [3]) that $\mathfrak{T}_1$ is a strongly positive compact endomorphism in $C_{\mathbf{e}}(\overline{\Omega})$, where $\mathbf{e}$ is the unique solution of

$$\begin{cases} -d\Delta w(x) + \mu k_1 w(x) = 1, \ x \in \Omega, \\ Bw(x) = 0, \ x \in \partial\Omega, \end{cases} \tag{5.5}$$

and $C_{\mathbf{e}}(\overline{\Omega})$ is the Banach space generated by the order unit $\mathbf{e} \in \mathbb{X}$ with order unit norm $\|\cdot\|_{\mathbf{e}}$ (see [3]). By the famous Krein-Rutman theorem and its sharper version for strongly positive linear operators (see [3], Theorem 3.2), the spectral radius $r(\mathfrak{T}_1)$ is a simple positive eigenvalue of $\mathfrak{T}_1$ having a positive eigenvector. Indeed, one can easily determine $r(\mathfrak{T}_1)$ as

$$r(\mathfrak{T}_1) = \frac{\mu k_1 + \eta p e^{-\lambda_1 \alpha}}{d\lambda_1 + \mu k_1}.$$

Now, define $\mathcal{S}_2 : \mathbb{Y} \to \mathbb{Y}$ by

$$(\mathcal{S}_2 w)(x) = \mu k_1 w(x) - \mu w^*(x) w(x) + \eta p \int_{\overline{\Omega}} \Gamma(\alpha, x, y) w(y) dy, \ \forall w \in \mathbb{Y},$$

and let $\mathfrak{T}_2 = \mathcal{L}_1^{-1} \mathcal{S}_2 : \mathbb{Y} \to \mathbb{X} \subset \mathbb{Y}$. Clearly, $\mathfrak{T}_2$ is also a strongly positive compact endomorphism of $C_{\mathbf{e}}(\overline{\Omega})$ and for any $w \in C_{\mathbf{e}}^+(\overline{\Omega})$, $\mathfrak{T}_2 w < \mathfrak{T}_1 w$. Again by [3] (Theorem 3.2), $r(\mathfrak{T}_2) < r(\mathfrak{T}_1)$, where $r(\mathfrak{T}_2)$ is the spectral radius of $\mathfrak{T}_2$. It follows that

$$r(\mathfrak{T}_2) < r(\mathfrak{T}_1) = \frac{\mu k_1 + \eta p e^{-\lambda_1 \alpha}}{d\lambda_1 + \mu k_1} \leq 1.$$

On the other hand, (5.3) implies that 1 is an eigenvalue of $\mathfrak{T}_2$ corresponding to a positive eigenvector w^*, contradicting $r(\mathfrak{T}_2) < 1$. This contradiction proves that (5.1) has no positive steady state when $d\lambda_1 \geq \eta p e^{-\lambda_1 \alpha}$.

(ii) Consider the boundary value problem

$$\begin{cases} -d\Delta w(x) + \mu w^2(x) = \eta p \int_{\overline{\Omega}} \Gamma(\alpha, x, y) w(y) dy, \ x \in \Omega, \\ Bw(x) = 0, \ x \in \partial\Omega. \end{cases} \tag{5.6}$$

To construct upper solutions of (5.6), we try $\widetilde{w}_s(x) \equiv M$, where M is a sufficiently large positive constant. Let

$$\gamma = \max_{x \in \overline{\Omega}} \int_{\Omega} \Gamma(\alpha, x, y) dy. \tag{5.7}$$

Then

$$-d\Delta\widetilde{w}_s(x) + \mu\widetilde{w}_s^2(x) - \eta p \int_{\overline{\Omega}} \Gamma(\alpha, x, y) \widetilde{w}_s(y) dy$$

$$=\mu M^2 - \eta p M \int_{\overline{\Omega}} \Gamma(\alpha, x, y) dy \geq M(\mu M - \eta p \gamma) > 0,$$

which implies that $\widetilde{w}_s$ is an upper solution of (5.6). On the other hand, $\widehat{w}_{s,0} \equiv 0$ is a solution of (5.6). Therefore, by Corollary 2.1, we know that (5.6) has a maximal solution and a minimal solution in the order interval $[\widehat{w}_{s,0}, \widetilde{w}_s]$, denoted by $\overline{w}_{s,0}(x)$ and $\underline{w}_{s,0}(x)$ respectively. If $d\lambda_1 > \eta p e^{-\lambda_1 \alpha}$, then it follows from part (i) that $\overline{w}_{s,0}(x) = \underline{w}_{s,0}(x) \equiv 0$. Thus, by Theorem 4.1, we know that the zero solution of (5.1) is globally asymptotically stable when $d\lambda_1 > \eta p e^{-\lambda_1 \alpha}$.

Next, we show that the zero solution of (5.1) is unstable while $d\lambda_1 < \eta p e^{-\lambda_1 \alpha}$. By (4.2), we need consider the following linear eigenvalue problem

$$\begin{cases} d\Delta w(x) + \eta p \displaystyle\int_{\overline{\Omega}} \Gamma(\alpha, x, y)w(y)dy = \lambda w(x), \ x \in \Omega, \\ Bw(x) = 0, \ x \in \partial\Omega. \end{cases} \tag{5.8}$$

Let $\mathfrak{T}_3 : \mathbb{X} \to \mathbb{Y}$ by

$$(\mathfrak{T}_3 w)(x) = d\Delta w(x) + \eta p \int_{\overline{\Omega}} \Gamma(\alpha, x, y)w(y)dy, \ \forall w \in \mathbb{X}.$$

Then the linear equation (5.8) can be rewritten as $\mathfrak{T}_3 w = \lambda w$. By [3] (Theorem 3.2) and [17] (Sections IV–3.5 and VII–6), we know that the simple eigenvalue λ_0 of $\mathfrak{T}_3$ has associated with it a positive eigenfunction. In fact, we can easily determine λ_0 as

$$\lambda_0 = -d\lambda_1 + \eta p e^{-\lambda_1 \alpha}.$$

By Theorem 4.2, we obtain that if $\lambda_0 > 0$, i.e., $d\lambda_1 < \eta p e^{-\lambda_1 \alpha}$, the zero solution of (5.1) is unstable.

(iii) Since $d\lambda_1 < \eta p e^{-\lambda_1 \alpha}$, for sufficiently small σ, we have $d\lambda_1 + \mu\sigma h < \eta p e^{-\lambda_1 \alpha}$, where $h = \max_{x \in \overline{\Omega}} \varphi_1(x)$. Let $\widehat{w}_s(x) = \sigma\varphi_1(x)$, $\sigma > 0$. Thus, we have

$$\begin{aligned} &- d\Delta\widehat{w}_s(x) + \mu\widehat{w}_s^2(x) - \eta p \int_{\overline{\Omega}} \Gamma(\alpha, x, y)\widehat{w}_s(y)dy \\ =&\sigma \left(d\lambda_1 + \mu\sigma\varphi_1(x) - \eta p e^{-\lambda_1 \alpha} \right) \varphi_1(x) \\ \leq&\sigma \left(d\lambda_1 + \mu\sigma h - \eta p e^{-\lambda_1 \alpha} \right) \varphi_1(x) \\ <&0 \end{aligned}$$

for all $x \in \Omega$, which implies that $\widehat{w}_s$ is a lower solution of (5.6). Therefore, by Corollary 2.1, we know that (5.6) has a maximal solution and a minimal solution in the order interval $[\widehat{w}_s, \widetilde{w}_s]$, denoted by $\overline{w}_s(x)$ and $\underline{w}_s(x)$ respectively.

Next, we prove the uniqueness of positive solution to (5.6) in the order interval $[\widehat{w}_s, \widetilde{w}_s]$. In fact, let $w_s^{(0)}$ be any positive solution to (5.6) satisfying $\widehat{w}_s \leq w_s^{(0)} \leq \widetilde{w}_s$. Then $w_s^{(0)}(x) \leq \overline{w}_s(x)$ for $x \in \overline{\Omega}$. If $w_s^{(0)} \neq \overline{w}_s$, then $w_s^{(0)} < \overline{w}_s$ in the sense of ordering in Banach space $\mathbb{X}$.

Let $k = \max_{x \in \Omega} \overline{w}_s(x)$ and consider the eigenvalue problem

$$\begin{cases} - d\Delta w(x) + \mu k w(x) = \lambda \left(\mu k w(x) - \mu\overline{w}_s(x)w(x) + \eta p \displaystyle\int_{\overline{\Omega}} \Gamma(\alpha, x, y)w(y)dy \right), x \in \Omega, \\ w(x) = 0, \ x \in \partial\Omega. \end{cases}$$
$$\tag{5.9}$$

Define the linear operator $\mathcal{L} : \mathbb{X} \to \mathbb{Y}$ by

$$(\mathcal{L}w)(x) = -d\Delta w(x) + \mu k w(x), \ \forall w \in \mathbb{X}$$

and $\mathcal{S}_4 : \mathbb{Y} \to \mathbb{Y}$ by

$$(\mathcal{S}_4 w)(x) = \mu k w(x) - \mu \overline{w}_s(x) w(x) + \eta p \int_{\overline{\Omega}} \Gamma(\alpha, x, y) w(y) dy, \ \forall w \in \mathbb{Y}. \qquad (5.10)$$

And let $\mathfrak{T}_4 = \mathcal{L}^{-1} \mathcal{S}_4 : \mathbb{Y} \to \mathbb{X} \subset \mathbb{Y}$. Clearly, $\mathfrak{T}_4$ is a strongly positive compact endomorphism of $C_{\mathbf{e}}(\overline{\Omega})$. By [3] (Theorem 3.2), the spectral radius $r(\mathfrak{T}_4)$ is the only eigenvalue having positive eigenvector. It follows that $r(\mathfrak{T}_4) = 1$ since $\overline{w}_s$ is a positive eigenvector corresponding to the eigenvalue 1 of the eigenvalue problem (5.9).

Similarly, consider the eigenvalue problem

$$\begin{cases} -d\Delta w(x) + \mu k w(x) = \lambda \left(\mu k w(x) - \mu w_s^{(0)}(x) w(x) + \eta p \int_{\overline{\Omega}} \Gamma(\alpha, x, y) w(y) dy \right), x \in \Omega, \\ w(x) = 0, \ x \in \partial\Omega. \end{cases}$$
$$(5.11)$$

Let $\mathcal{S}_5 : \mathbb{Y} \to \mathbb{Y}$ be a linear operator defined by

$$(\mathcal{S}_5 w)(x) = \mu k w(x) - \mu w_s^{(0)}(x) w(x) + \eta p \int_{\overline{\Omega}} \Gamma(\alpha, x, y) w(y) dy, \forall w \in \mathbb{Y}, \qquad (5.12)$$

and $\mathfrak{T}_5 : \mathbb{Y} \to \mathbb{Y}$ defined by $\mathfrak{T}_5 = \mathcal{L}^{-1} \mathcal{S}_5$. Then $\mathfrak{T}_5$ is also a strongly positive compact endomorphism of $C_{\mathbf{e}}(\overline{\Omega})$. Since $w_s^{(0)}$ is a positive eigenvector corresponding to the eigenvalue 1 of the eigenvalue problem (5.11), we get $r(\mathfrak{T}_5) = 1$. However, since $w_s^{(0)} < \overline{w}_s$, we obtain $\mathcal{S}_5 w > \mathcal{S}_4 w$ for any $w \in \mathbb{Y}$, implying that $\mathfrak{T}_5 w > \mathfrak{T}_4 w$ for any $w \in \mathbb{Y}$. From the monotonicity of the spectral radius, it follows that $1 = r(\mathfrak{T}_5) > r(\mathfrak{T}_4) = 1$, which is a contradiction. Thus, we have $w_s^{(0)}(x) \equiv \overline{w}_s(x)$ for all $x \in \overline{\Omega}$, i.e., $w_s^{(0)} = \overline{w}_s$. Similarly, $w_s^{(0)} = \underline{w}_s$. It follows the uniqueness of positive solution of (5.6) in the order interval $[\widehat{w}_s, \widetilde{w}_s]$. Because $\sigma > 0$ and $M > 0$ is arbitrary, we have actually shown that (5.1) has a unique positive steady state, denoted by w^*. Again by Theorem 4.1, we know that w^* is globally asymptotically stable. The proof is completed. $\square$

Remark 5.1. By the proof of Theorem 5.1, we know that $\sigma \varphi_1$ is a lower solution of (5.6) for sufficiently small $\sigma > 0$ when $d\lambda_1 < \eta p e^{-\lambda_1 \alpha}$. Therefore, the zero solution of (5.1) is unstable in this case.

Example 5.2. Consider the following Nicholson's blowfly model

$$\begin{cases} \dfrac{\partial w(t, x)}{\partial t} = d\Delta w(t, x) - \mu w(t, x) + \eta \int_{\overline{\Omega}} \Gamma(\alpha, x, y) b_1(w(t - \tau, y)) dy, t > 0, x \in \Omega, \\ Bw(t, x) = 0, t > 0, x \in \partial\Omega, \\ w(t, x) = \phi(t, x), t \in [-\tau, 0], x \in \Omega, \end{cases}$$
$$(5.13)$$

where d, μ, η, τ, α, B, Ω, Γ and ϕ can be referred to Example 5.1, $b_1(w) = pwe^{-qw}$ which is referred to as the Ricker's birth function in population dynamics, where $p > 0$, $q > 0$ (see, [21, 37]).

It is easy to see that model (5.13) is a special case of equation (1.1) with $f(x, w, u) = -\mu w$, $g(x, y, u) = \eta \Gamma(\alpha, x, y) b_1(u)$ or $f(x, w, u) = -\mu w + \eta b_1(u)$, $g(x, y, u) = 0$ and $L = d\Delta$. Therefore, by Corollary 2.1, Theorems 2.1, 4.1 and 4.2, we have

Theorem 5.2. **(i)** *If $\eta p e^{-\lambda_1 \alpha} \leq \mu + d\lambda_1$, then there is no positive steady state to (5.13).*

(ii) *If $\eta p e^{-\lambda_1 \alpha} > \mu + d\lambda_1$ the zero solution of (5.13) is unstable, and if $\eta p e^{-\lambda_1 \alpha} < \mu + d\lambda_1$ it is globally asymptotically stable.*

(iii) *If $\mu + d\lambda_1 < \eta p e^{-\lambda_1 \alpha}$ and $\mu \geq \eta p \gamma e^{-1}$, where γ is given by (5.7), then (5.13) has a unique positive steady state w^* which is asymptotically stable.*

Proof. **(i)** The result of this part can follow from [46](Theorem 2.6) or a similar argument as that in part (i) of Theorem 5.1.

(ii) By (4.2), we need consider the following linear eigenvalue problem

$$\begin{cases} d\Delta w(x) - \mu w(x) + \eta p \int_{\overline{\Omega}} \Gamma(\alpha, x, y) w(y) dy = \lambda w(x), \ x \in \Omega, \\ Bw(x) = 0, \ x \in \partial\Omega. \end{cases} \tag{5.14}$$

Let $\mathfrak{T}_6 : \mathbb{X} \to \mathbb{Y}$ by

$$(\mathfrak{T}_6 w)(x) = d\Delta w(x) - \mu w(x) + \eta p \int_{\overline{\Omega}} \Gamma(\alpha, x, y) w(y) dy, \ \forall w \in \mathbb{X}.$$

Then the linear equation (5.14) can rewritten as $\mathfrak{T}_6 w = \lambda w$. By [3] (Theorem 3.2) and [17] (Sections IV–3.5 and VII–6), we know that the simple eigenvalue λ_0 of $\mathfrak{T}_6$ has associated with it a positive eigenfunction. In fact, we can easily determine λ_0 as

$$\lambda_0 = -d\lambda_1 - \mu + \eta p e^{-\lambda_1 \alpha}.$$

By Theorem 4.2, we obtain that if $\lambda_0 > 0$, i.e., $d\lambda_1 + \mu < \eta p e^{-\lambda_1 \alpha}$, the zero solution of (5.13) is unstable, and if $\lambda_0 < 0$, i.e., $d\lambda_1 + \mu > \eta p e^{-\lambda_1 \alpha}$, it is stable.

Next, we show that the zero solution of (5.13) is globally asymptotically stable while $\eta p e^{-\lambda_1 \alpha} < \mu + d\lambda_1$. In fact, we consider the follwing reaction-diffusion equation with time delay

$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} = d\Delta w(t,x) - \mu w(t,x) + \eta p \int_{\overline{\Omega}} \Gamma(\alpha, x, y) w(t-\tau, y) dy, t > 0, x \in \Omega, \\ Bw(t,x) = 0, t > 0, x \in \partial\Omega, \\ w(t,x) = \phi(t,x), t \in [-\tau, 0], x \in \Omega \end{cases} \tag{5.15}$$

and its corresponding boundary value problem

$$\begin{cases} -d\Delta w(x) + \mu w(x) = \eta p \int_{\overline{\Omega}} \Gamma(\alpha, x, y) w(y) dy, x \in \Omega, \\ Bw(x) = 0, x \in \partial\Omega. \end{cases} \tag{5.16}$$

A similar argument as that in the proof of part (i) of Theorem 5.1 shows that (5.16) has no positive solution when $\eta p e^{-\lambda_1 \alpha} < \mu + d\lambda_1$. On the other hand, if $\eta p e^{-\lambda_1 \alpha} < \mu + d\lambda_1$, then for any $\sigma > 0$, we have

$$-d\sigma\Delta\varphi_1(x) + \mu\sigma\varphi_1(x) - \eta p \int_{\overline{\Omega}} \Gamma(\alpha, x, y)\sigma\varphi_1(y) dy$$
$$= \sigma \left(d\lambda_1 + \mu - \eta p e^{-\lambda_1 \alpha} \right) \varphi_1(x) > 0,$$

which implies that $\sigma\varphi_1(x)$ is an upper solution of (5.16). Therefore, by Corollary 2.1, Theorem 4.1 and [46](Lemma 3.4), the zero solutions of (5.15) and (5.13) is globally asymptotically stable when $\eta p e^{-\lambda_1\alpha} < \mu + d\lambda_1$.

(iii) If $\mu + d\lambda_1 < \eta p e^{-\lambda_1\alpha}$, then for sufficiently small σ, we have $\mu + d\lambda_1 < \eta p e^{-\lambda_1\alpha - \sigma qh}$, where $h = \max_{x\in\overline{\Omega}} \varphi_1(x)$. Let $\widehat{w}_s(x) = \sigma\varphi_1(x)$, $\sigma > 0$. Then when σ is sufficiently small, we obtain

$$
- d\Delta\widehat{w}_s(x) + \mu\widehat{w}_s(x) - \eta \int_{\overline{\Omega}} \Gamma(\alpha, x, y) b_1(\widehat{w}_s(y)) dy
$$
$$
= - d\sigma\Delta\varphi_1(x) + \mu\sigma\varphi_1(x) - \eta p \int_{\overline{\Omega}} \Gamma(\alpha, x, y)\sigma\varphi_1(y) e^{-q\sigma\varphi_1(y)} dy
$$
$$
\leq \sigma \left(d\lambda_1 + \mu - \eta p e^{-\lambda_1\alpha - \sigma qh} \right) \varphi_1(x) < 0,
$$

which implies that $\widehat{w}_s(x)$ is a lower solution of the following boundary value problem

$$
\begin{cases}
- d\Delta w(x) + \mu w(x) = \eta \int_{\overline{\Omega}} \Gamma(\alpha, x, y) b_1(w(y)) dy, x \in \Omega, \\
Bw(x) = 0, x \in \partial\Omega.
\end{cases}
\tag{5.17}
$$

Next, we show that $\widetilde{w}_s(x) \equiv q^{-1}$ is an upper solution of (5.17) when $\mu \geq \eta p\gamma e^{-1}$. Indeed,

$$
- d\Delta\widetilde{w}_s(x) + \mu\widetilde{w}_s(x) - \eta \int_{\overline{\Omega}} \Gamma(\alpha, x, y) b_1(\widetilde{w}_s(y)) dy
$$
$$
= \mu q^{-1} - \eta p q^{-1} e^{-1} \int_{\overline{\Omega}} \Gamma(\alpha, x, y) dy
$$
$$
\geq q^{-1} \left(\mu - \eta p\gamma e^{-1} \right) \geq 0.
$$

Therefore, by Theorem 2.1, we conclude that there is a positive steady state w^* to (5.13) satisfying $\widehat{w}_s(x) \leq w^*(x) \leq \widetilde{w}_s(x)$ for $x \in \overline{\Omega}$. Thus, by Theorem 4.1 and the similar arguments as that in Theorem 5.1 and [46](Corollary 2), it follows that (5.13) has a unique positive steady state w^* which is asymptotically stable while $\mu + d\lambda_1 < \eta p e^{-\lambda_1\alpha}$ and $\mu \geq \eta p\gamma e^{-1}$. The proof is completed. $\square$

Example 5.3. Consider the following Mackey-Glass model

$$
\begin{cases}
\dfrac{\partial w(t, x)}{\partial t} = d\Delta w(t, x) - \mu w(t, x) + \eta \int_{\overline{\Omega}} \Gamma(\alpha, x, y) b_2(w(t - \tau, y)) dy, t > 0, x \in \Omega, \\
Bw(t, x) = 0, t > 0, x \in \partial\Omega, \\
w(t, x) = \phi(t, x), t \in [-\tau, 0], x \in \Omega,
\end{cases}
\tag{5.18}
$$

where d, μ, η, τ, α, B, Ω, Γ and ϕ can be referred to Example 5.1, $b_2(w) = \frac{pw}{q + w^l}$, $l > 0$, $p > 0$ and $q > 0$.

This nonlinear function $b_2(w)$ was used as the production function for blood cells in [24], and has since been widely adopted. It is easy to see that model (5.18) is a special case of equation (1.1) with $f(x, w, u) = -\mu w$, $g(x, y, u) = \eta\Gamma(\alpha, x, y) b_2(u)$ or $f(x, w, u) = -\mu w + \eta b_2(u)$, $g(x, y, u) = 0$ and $L = d\Delta$. Therefore, by Corollary 2.1, Theorems 2.1, 4.1 and 4.2, we have

Theorem 5.3. **(i)** *If $q(\mu + d\lambda_1) \geq \eta p e^{-\lambda_1\alpha}$, there is no positive steady state to (5.18).*

(ii) *If $q(\mu + d\lambda_1) < \eta p e^{-\lambda_1 \alpha}$ the zero solution of (5.18) is unstable, and if $q(\mu + d\lambda_1) > \eta p e^{-\lambda_1 \alpha}$ it is globally asymptotically stable.*

(iii) *If $\eta p \gamma (1 - l^{-1}) + q d\lambda_1 < q(\mu + d\lambda_1) < \eta p e^{-\lambda_1 \alpha}$, where γ is given by (5.7), then (5.18) has a unique positive steady state w^* which is asymptotically stable. Furthermore, if $l \le 1$, then this positive steady state w^* is globally asymptotically stable.*

Proof.   The proof is similar to that for Theorem 5.1 in the case of $l \le 1$ and that for Theorem 5.2 in the case of $l > 1$ and is omitted. The proof is completed. $\square$

Acknowledgements

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