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THEORY AND APPROXIMATION OF SOLUTIONS TO A HARVESTED HIERARCHICAL AGE-STRUCTURED POPULATION MODEL*

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Abstract This article is concerned with theoretic analysis and numerical approximation of solutions to a hierarchical age-structured population model, in which the vital rates of an individual depend more on the number of older individuals. The well-posedness of the model is rigorously treated by means of fixed point principle, and an algorithm and convergence analysis are presented. An example is used to show the effectiveness of the numerical method.

Keywords Hierarchy of age, integro-differential system, well-posedness, numerical method, convergence.

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1. Introduction

The hierarchy of dominance ranks of individuals is in common and can be observed in many biological populations (e.g., see survey article [9] and references therein). Although hierarchical models for species are realistic, their theoretic analyses and numerical approximations are often challenging tasks, because of the high nonlinearities resulted in the rank structure. Generally speaking, hierarchical models are more difficult to treat than age- and size-structured models incorporating no hierarchy.

Lomnicki proposed the first mathematical model in [16] to describe the relation between resources partition and individuals' rank, he thinks that the resource partition within a population is not even and the competition among individuals is contest. Gurney and Nisbet considered a predator-prey model in which predators are rank-structured [11]. They analyzed the model by stochastic techniques and showed that rank difference was helpful in sustaining stability. Cushing in [5] studied the long-term behaviors of a hierarchical age-structured model, which was reduced to a system of ordinary differential equations. When the vital parameters were determined by energy, produced by food or cannibalism, Cushing and Li's

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investigation displayed that the cannibalism based on ranks was the direct cause to periodic oscillations [7]. In [12] Henson and Cushing examined the effects of the form of competition (contest or scramble) on population evolution via a hierarchical age-structured model. For a kind of nonlinear hierarchical model with parameters partially influenced by the "environment" [6], Calsina and Saldaña treated the existence and uniqueness of solutions and asymptotical behaviors, while this task was completed by Kraev for a forest model [15], where a tree's rank was given by its height. Compared with continuous models, research works on discrete models are much less. Jang and Cushing considered [14] a discrete vision of the model in [12] , and the conclusion is also similar to that in [12] . Ackleh, Deng and Hu obtained the existence and uniqueness of nonnegative solutions for age- and size-structured hierarchical models in [1,4] by means of upper and lower solutions technique. One cannot expect to have analytical solutions to these complicated models, Shen, Shu and Zhang proposed a higher order numerical method for a hierarchical model [18], its performance was effective and non-oscillating. For a very general model, Calsina and Saldaña established existence and uniqueness of solutions by means of characteristic curves and theory in evolution equations [8]. Recently, in [17] Liu and He proved the well-posedness for a kind of size-structured hierarchical population system.

Besides the challenges in theoretic analysis, research on numerical method of hierarchical models is rather rare. In the present paper, we study a hierarchical age-structured model with finite expectancy and human harvest. We pay attention both to the rigorous analysis for existence and uniqueness of solutions, and to an algorithm of approximating them. The convergence analysis and a numerical example are also presented.

The article is organised as follows. The next section presents the model description and basic assumptions, and section 3 is for existence, uniqueness and continuous dependence of solutions on initial distributions. The section 4 contains the numerical algorithm, its convergence analysis and simulation, while some remarks is in the final section.

2. The Model Description

We consider the following infinite dimensional system governing the evolution of an age-specific population with hierarchy and harvesting:

$$\begin{cases} \dfrac{\partial p}{\partial t} + \dfrac{\partial p}{\partial a} = -\mu(a,t,E(p)(a,t))p(a,t) - up(a,t), \qquad (a,t) \in D, \\ p(0,t) = \displaystyle\int_0^A \beta(a,t,E(p)(a,t))p(a,t)da, \qquad 0 < t < T, \\ p(a,0) = p_0(a), \qquad 0 \le a < A, \end{cases} \tag{2.1}$$

where $D = (0, A) \times (0, T)$, and $A > 0$ is the maximum age of individuals and $T > 0$ the control horizon. So called "environment" $E(p)$ is defined as

$$E(p)(a,t) = \alpha \int_0^a p(r,t)dr + \int_a^A p(r,t)dr, \quad 0 \le \alpha < 1. \tag{2.2}$$

Here the constant α shows the weight of individuals with age smaller than a. Functions μ and β are mortality and fertility, respectively. The form of μ and β indicates that, apart from age and time, the vital rates of an individual of age a depend more on the number of individuals with age equal to or larger than a. The positive constant u stands for the human harvesting efforts. Finally, $p_0(a)$ gives the initial age distribution of the individuals.

Throughout this paper, we make the following assumptions:

(An1) $\mu(a,t,x) > 0$, $\beta(a,t,x) \geq 0$, $\forall\ (a,t,x) \in D \times R_+$. For any $x \in R_+$, $\beta(\cdot,\cdot,x) \in L^\infty(D)$, and for $(t,x) \in (0,T) \times R_+$, $\mu(\cdot,t,x) \in L^1_{loc}[0,A)$; $\int_0^A \mu(a,t,x)da = +\infty$.

(An2) β and μ are locally Lipschitz functions w.r.t. the third variable, that is, $\forall\ M > 0$, there exists $L(M) > 0$, such that

$$| \beta(a,t,x_1) - \beta(a,t,x_2) | \leq L(M) \mid x_1 - x_2 \mid,$$

$$| \mu(a,t,x_1) - \mu(a,t,x_2) | \leq L(M) \mid x_1 - x_2 \mid$$

for all $x_1,\ x_2$ with $|x_i| \leq M,\ i = 1,2$.

(An3) For given $(a,t) \in D$, $\beta(a,t,\cdot)$ is nonincreasing and $\mu(a,t,\cdot)$ is nondecreasing.

(An4) There is a positive constant $\overline{p_0}$, such that $0 \leq p_0(a) \leq \overline{p_0}$, $\forall\ a \in [0,A]$.

3. Well-posedness of the model

For $A = +\infty,\ u = 0$, the existence and uniqueness of solutions to (2.1)-(2.2) was treated in [1] via upper and lower solutions method. The approach we employed in this section is Banach fixed point theorem.

Firstly, we deal with the existence of unique solutions to the model (2.1)-(2.2). Then we state a continuity result.

Let $q(a,t) \in L^\infty(0,T;L^1(0,A))$ be arbitrary but fixed, so is the function

$$E(q)(a,t) = \alpha \int_0^a q(r,t)dr + \int_a^A q(r,t)dr.$$

We consider the linear system

$$\begin{cases} \dfrac{\partial p}{\partial t} + \dfrac{\partial p}{\partial a} = -\mu^*(a,t,E(q)(a,t))p(a,t), \quad (a,t) \in D, \\ p(0,t) = \displaystyle\int_0^A \beta(a,t,E(q)(a,t))p(a,t)da, \quad 0 < t < T, \\ p(a,0) = p_0(a), \quad 0 \leq a < A, \end{cases} \tag{3.1}$$

where $\mu^* := \mu + u$. It is well-known that (3.1) has a unique non-negative solution, which is of the form (see [2]):

$$p(a,t;q) = \begin{cases} p_0(a-t)\Pi(a,t,t;q), \quad a \geq t, \\ b(t-a;q)\Pi(a,t,a;q), \quad a < t, \end{cases} \tag{3.2}$$

where $b(t;q) := p(0,t;q)$, and

$$\Pi(a,t,s;q) := \exp\{-\int_0^s \mu^*(a-\tau,t-\tau,E(q)(a-\tau,t-\tau))d\tau\}. \tag{3.3}$$

Using (3.2) and the second equation in (3.1), we claim that $b(t;q)$ satisfies

$$b(t;q) = F(t;q) + \int_0^t K(t,s;q)b(t-s;q)ds, \ t \in (0,T), \tag{3.4}$$

with

$$K(t,s;q) = \beta(s,t,E(q))\Pi(s,t,s;q), \tag{3.5}$$

$$F(t;q) = \int_0^\infty \beta(a+t,t,E(q))p_0(a)\Pi(a+t,t,t;q))da. \tag{3.6}$$

One may extend the values of p_0, β, and Π outside their domains to 0 if necessary.

In this section, we assume $T > A$. Otherwise the process is similar.

Define

$$M_T = \max\{\|F(\cdot;0)\|_{L^\infty(0,T)} e^{T\|\beta(\cdot,\cdot,0)\|_{L^\infty(D)}}, \|p_0\|_{L^1(0,A)}\},$$
$$H := \{v \in L^\infty(0,T,L^1(0,A)) : \ v(a,t) \geq 0, \quad (a,t) \in D; \ \|v(\cdot,t)\|_{L^1(0,A)} \leq M_T\},$$
$$\mathcal{H} := \{h \in L^\infty(0,T) : \ 0 \leq h \leq M_T, \quad t \in (0,T)\}.$$

Lemma 3.1. *There are M_{1T}, $M_{2T} > 0$, depending on T only, such that for $\forall\, q_1, q_2 \in H, \ t > 0$,*

$$\begin{aligned}&| F(t;q_1) - F(t;q_2) | \\ \leq & M_{1T}\left(\|E(q_1)(\cdot,t) - E(q_2)(\cdot,t)\|_{L^\infty(0,A)} + \int_0^t \|q_1(\cdot,s) - q_2(\cdot,s)\|_{L^1(0,A)}ds\right)\end{aligned} \tag{3.7}$$

and

$$\begin{cases} 0 \leq b(t;q) \leq M_{2T}; \\ |b(t;q_1) - b(t;q_2)| \leq M_{2T}\big(\|E(q_1)(\cdot,t) - E(q_2)(\cdot,t)\|_{L^\infty(0,A)} \\ \qquad\qquad + \int_0^t \|q_1(\cdot,s) - q_2(\cdot,s)\|_{L^1(0,A)}ds\big). \end{cases} \tag{3.8}$$

Proof. When $t \in [A,T)$, $\beta(a+t,t,E(q)) \equiv 0$, then $F(t;q) \equiv 0$. Clearly (3.7) holds.

When $t \in (0,A)$, by (3.3), (3.6) and (An1)-(An4), we have

$$\begin{aligned}
&|F(t;q_1) - F(t;q_2)| \\
\leq & \int_0^\infty |\beta(a+t,t,E(q_1)) - \beta(a+t,t,E(q_2))|p_0(a)\Pi(a+t,t,t;q_1)da \\
&+ \int_0^\infty \beta(a+t,t,E(q_2))p_0(a)|\Pi(a+t,t,t;q_1) - \Pi(a+t,t,t;q_2)|da \\
\leq & L(M_T)\|E(q_1)(\cdot,t) - E(q_2)(\cdot,t)\|_{L^\infty(0,A)} \quad \|p_0\|_{L^1(0,A)} \\
&+ L(M_T)\int_0^A \beta(a,t,0)p_0(a)\int_0^t |\mu^*(a-\tau,t-\tau,E(q_1)(a-\tau,t-\tau)) \\
&- \mu^*(a-\tau,t-\tau,E(q_2)(a-\tau,t-\tau))|d\tau da \\
\leq & L(M_T)\|E(q_1)(\cdot,t) - E(q_2)(\cdot,t)\|_{L^\infty(0,A)} \quad \|p_0\|_{L^1(0,A)} \\
&+ L(M_T)\int_0^A \beta(a,t,0)p_0(a)\int_0^t |E(q_1)(a-\tau,t-\tau) - E(q_2)(a-\tau,t-\tau)|d\tau da.
\end{aligned}$$

Notice that:

$$\begin{aligned}
&\int_0^t |E(q_1)(a-\tau,t-\tau)-E(q_2)(a-\tau,t-\tau)|d\tau \\
=&\int_0^t \left|\alpha\int_0^{a-\tau}[q_1(r,t-\tau)-q_2(r,t-\tau)]dr+\int_{a-\tau}^A[q_1(r,t-\tau)-q_2(r,t-\tau)]dr\right|d\tau \\
\leq& 2\int_0^t \|q_1(\cdot,s)-q_2(\cdot,s)\|_{L^1(0,A)}ds, \qquad (3.9)
\end{aligned}$$

it suffices for (3.7) to take $M_{1T}=L(M_T)\|p_0\|_{L^1(0,A)}(1+2\|\beta(\cdot,\cdot,0)\|_{L^\infty(D)})$.

By a standard approach, it is not difficult to show that $b(t;q)\geq 0$ for $t>0$. On the other hand, (3.4) derives that

$$b(t;q)\leq\| F(\cdot;0)\|_\infty+\int_0^t \|\beta(\cdot,\cdot,0)\| b(t-s;q)ds,$$

Bellman's inequality tells us that

$$b(t;q)\leq\|F(\cdot,0)\|_{L^\infty(0,T)}e^{T\|\beta(\cdot,\cdot,0)\|_{L^\infty(D)}}=:\widetilde{M}_{2T}. \qquad (3.10)$$

From the assumptions and (3.4)-(3.6), it follows that

$$\begin{aligned}
&|b(t;q_1)-b(t;q_2)| \\
\leq&|F(t;q_1)-F(t;q_2)|+\int_0^t|K(t,t-s;q_1))-K(t,t-s;q_2))|b(s;q_1)ds \\
&+\int_0^t K(t,t-s;q_2))|b(s;q_1)-b(s;q_2)|ds \\
\leq& M_{1T}\left(\|E(q_1)(\cdot,t)-E(q_2)(\cdot,t)\|_{L^\infty(0,A)}+\int_0^t \| q_1(\cdot,s)-q_2(\cdot,s)\|_{L^1}\,ds\right) \\
&+\widetilde{M}_{2T}(L(M_T)T\|E(q_1)(\cdot,t)-E(q_2)(\cdot,t)\|_{L^\infty(0,A)} \\
&+L(M_T)T\|\beta(\cdot,\cdot,0)\|_{L^\infty(D)}\int_0^t|E(q_1)(a-\tau,t-\tau)-E(q_2)(a-\tau,t-\tau)|d\tau) \\
&+\|\beta(\cdot,\cdot,0)\|_{L^\infty(D)}\int_0^t|b(\tau;q_1)-b(\tau;q_2)|d\tau \\
\leq& M_{3T}\left(\|E(q_1)(\cdot,t)-E(q_2)(\cdot,t)\|_{L^\infty(0,A)}+\int_0^t\|q_1(\cdot,s)-q_2(\cdot,s)\|_{L^1(0,A)}ds\right) \\
&+\|\beta(\cdot,\cdot,0)\|_{L^\infty(D)}\int_0^t|b(\tau;q_1)-b(\tau;q_2)|d\tau,
\end{aligned}$$

where M_{3T} is a constant depending on T only. Gronwall's inequality enables us to obtain (3.8). □

Define the mapping

$$\mathcal{T}:\ H\to L^\infty(0,T;L^1(0,A)),\ (\mathcal{T}q)(a,t)=p(a,t;q),\ q\in H,$$

where $p(a,t;q)$ is the solution to (3.1) and given by (3.2).

Lemma 3.2. *$\mathcal{T}$ maps H into itself, and there exists constant $\overline{M} > 0$ (depending on T only), such that for $\forall\, q_1,\, q_2 \in H$,*

$$\|(\mathcal{T}q_1)(\cdot,t) - (\mathcal{T}q_2)(\cdot,t)\|_{L^1(0,A)} \le \overline{M}\int_0^t \|q_1(\cdot,s) - q_2(\cdot,s)\|_{L^1(0,A)}ds. \tag{3.11}$$

Proof. When $t \in (0, A)$, by Lemma 3.1, we have

$$\begin{aligned}
&\|(\mathcal{T}q_1)(\cdot,t) - (\mathcal{T}q_2)(\cdot,t)\| \\
=&\|p(a,t;q_1) - p(a,t;q_2)\|_{L^1(0,A)} \\
\le&\int_0^t |b(t-a;q_1) - b(t-a;q_2)|\Pi(a,t,a;q_1)da \\
&+ \int_0^t b(t-a;q_2|\Pi(a,t,a;q_1) - \Pi(a,t,a;q_2)|da \\
&+ \int_t^A p_0(a-t)|\Pi(a,t,t;q_1) - \Pi(a,t,t;q_2)|da \\
\le&\int_0^t M_{2T}\left(\|E(q_1)(\cdot,t-a) - E(q_2)(\cdot,t-a)\|_{L^\infty(0,A)}\right. \\
&\left.+ \int_0^t \|q_1(\cdot,s) - q_2(\cdot,s)\|_{L^1(0,A)}ds\right) da \\
&+ M_{2T}L(M_T)\int_0^t\int_0^a |E(q_1)(a-\tau,t-\tau) - E(q_2)(a-\tau,t-\tau)|d\tau da \\
&+ L(M_T)\int_t^A p_0(a-t)\int_0^t |E(q_1)(a-\tau,t-\tau) - E(q_2)(a-\tau,t-\tau)|d\tau da \\
\le& M_{2T}\int_0^t \|E(q_1)(\cdot,t-a) - E(q_2)(\cdot,t-a)\|_{L^\infty(0,A)}da \\
&+ M_{2T}T\int_0^t \|q_1(\cdot,s) - q_2(\cdot,s)\|_{L^1(0,A)}ds \\
&+ 2M_TL(M_T)T\int_0^t \|q_1(\cdot,s) - q_2(\cdot,s)\|_{L^1(0,A)}ds \\
&+ 2L(M_T)\|p_0\|_{L^1(0,A)}\int_0^t \|q_1(\cdot,s) - q_2(\cdot,s)\|_{L^1(0,A)}ds,
\end{aligned}$$

in which

$$\begin{aligned}
&\int_0^t \|E(q_1)(\cdot,t-a) - E(q_2)(\cdot,t-a)\|_{L^\infty(0,A)}da \\
=&\int_0^t \|\alpha\int_0^{\cdot}[q_1(r,t-a) - q_2(r,t-a)]dr + \int_{\cdot}^A [q_1(r,t-a) - q_2(r,t-a)]dr\|_{L^\infty(0,A)}da \\
\le&\int_0^t\int_0^A |q_1(r,t-a) - q_2(r,t-a)|drda \\
=&\int_0^t \|q_1(\cdot,t-a) - q_2(\cdot,t-a)\|_{L^1(0,A)}da \\
=&\int_0^t \|q_1(\cdot,s) - q_2(\cdot,s)\|_{L^1(0,A)}ds.
\end{aligned} \tag{3.12}$$

This completes the proof. □

Theorem 3.1. *The model (2.1)-(2.2) has a unique solution, which is nonnegative and bounded.*

Proof. Choose constant $\lambda > \overline{M}$, and define the equivalent norm on $L^\infty(0,T;L^1(0,A))$:

$$\|q\|_* = Ess \sup_{t\in(0,T)} \{e^{-\lambda t}\|q(\cdot,t)\|_{L^1(0,A)}\}, \quad \forall\, q \in L^\infty(0,T;L^1(0,A)).$$

Using Lemma 3.2, we obtain

$$\begin{aligned}\|\mathcal{T}q_1 - \mathcal{T}q_2\|_* &= Ess\ \sup_{t\in(0,T)}\{e^{-\lambda t}\|(\mathcal{T}q_1)(\cdot,t) - (\mathcal{T}q_2)(\cdot,t)\|_{L^1(0,A)}\}\\ &\le \overline{M}Ess\ \sup_{t\in(0,T)}\{e^{-\lambda t}\textstyle\int_0^t \|q_1(\cdot,s) - q_2(\cdot,s)\|_{L^1(0,A)}ds\}\\ &\le \overline{M}Ess\ \sup_{t\in(0,T)}\{e^{-\lambda t}\textstyle\int_0^t e^{\lambda s}e^{-\lambda s}\|q_1(\cdot,s) - q_2(\cdot,s)\|_{L^1(0,A)}ds\}\\ &\le \tfrac{\overline{M}}{\lambda}\|q_1 - q_2\|_*.\end{aligned}$$

According to Banach fixed point theorem, the mapping $\mathcal{T}$ has only one fixed point q^*, which is the solution of (2.1)-(2.2).

Since $b(t) \ge 0$, $t \in [0,T]$, relation (3.2) implies that $p(a,t) \ge 0$, $\forall\,(a,t) \in D$.

Finally, let $p^0(a,t)$ be the bounded solution to the following linear system

$$\begin{cases} \dfrac{\partial p}{\partial t} + \dfrac{\partial p}{\partial a} = -\mu(a,t,0)p(a,t) - up(a,t), \quad (a,t) \in D,\\ p(0,t) = \displaystyle\int_0^A \beta(a,t,0)p(a,t)da, \quad 0 < t < T,\\ p(a,0) = p_0(a), \quad 0 \le a < A. \end{cases} \tag{3.13}$$

By the monotonicity (An3) of μ, β, and the comparison principle, we claim that

$$p(a,t) \le p^0(a,t),\ \forall\,(a,t) \in D.$$

□

Next, we state a result to describe the continuous dependence of solutions upon the initial distributions. Since the proof is similar to that in the well-posedness, we omit the details.

Let p_i, $i = 1,2$, be the solution of the following

$$\begin{cases} \dfrac{\partial p}{\partial t} + \dfrac{\partial p}{\partial a} = -\mu(a,t,E(p)(a,t))p(a,t) - up(a,t), \quad (a,t) \in D,\\ p(0,t) = \displaystyle\int_0^A \beta(a,t,E(p)(a,t))p(a,t)da, \quad 0 < t < T,\\ p(a,0) = p_{0i}(a), \quad 0 \le a < A;\ i = 1,2. \end{cases} \tag{3.14}$$

Theorem 3.2. *There is a constant $M > 0$, such that*

$$\|p_1 - p_2\|_{L^\infty(D)} \le M\|p_{01} - p_{02}\|_{L^\infty(0,A)}.$$

4. Algorithm and Its Convergence

In this section, we follow the spirits in [3] and develop a numerical scheme to approximate the solution of (2.1)-(2.2).

We need some additional assumptions:

(H1) $\beta \in \mathcal{C}^2([0,A]\times[0,T])$, and $\mu^* \in \mathcal{C}^2([0,A)\times[0,T])$;

(H2) $\exists\, 0 < A^* < A$, such that $\mu^* \in L^\infty[0,A^*]$, and $\int_{A^*}^A \mu^*(a,t,E(p)(a,t))da = +\infty$;

(H3) $p_0 \in \mathcal{C}^2[0,A)$ and $\lim_{s\to A} p_0(s)\exp\left(\int_0^s \mu^*(a,t,E(p)(a,t))da\right) < +\infty$.

Under the above assumptions, one can show that the solution to the model satisfies (similar to Theorem 4.2 in Chapter I, [13])

$$p \in \mathcal{C}^2([0,A]\times[0,T]),\quad p(a,t)\ge 0,\ \text{for}\,\forall\ a\in[0,A), t>0;\ p(A,t)=0. \tag{4.1}$$

Along characteristic lines $a - t = c$, c is a constant, the equation (2.1) can be rewritten as

$$\frac{d}{dt}p(t+c,t) = -\mu^*(t+c,t,E(p)(t+c,t))p(t+c,t).$$

For each $(a_0,t_0)\in D$, there is $h>0$, such that $(a_0+h,t_0+h)\in D$, and

$$p(a_0+h,t_0+h) = p(a_0,t_0)\exp\left(-\int_0^h \mu^*(a_0+\tau,t_0+\tau,E(p)(a_0+\tau,t_0+\tau))d\tau\right). \tag{4.2}$$

We introduce the following notations:

(1) $f(a,t) = \int_{A^*}^a \mu^*(r,t,E(p)(r,t))dr$, $f(A,t) = +\infty$. For a given A^*, let the step size $h = A^*/J^*$, where J^* is a positive integer. The total number of steps is $J = [A/h]$.

(2) $a_j = jh,\quad 0\le j\le J\quad (a_{J^*} = A^*,\ a_J \le A)$;

(3) $t_n = nh,\quad t_{n+\frac12} = t_n + \frac{h}{2}, 0\le n\le N;\ N = [T/h]$;

(4) $P^n = [P_0^n, P_1^n, ..., P_J^n]$, where P_j^n is the numerical approximation of $p(a_j,t_n)$, $0\le j\le J,\ 1\le n\le N$;

(5) Denote the grid values of the initial distribution by $P^0 = [P_0^0, P_1^0, ..., P_J^0]$; $P_j^0 = p_0(a_j),\quad 0\le j\le J$.

For $0\le n\le N-1$, we approximate $p(0,t_n)$ and the other grid values of solutions at (a_j,t_n) via midpoint and trapezoidal quadrature rules, respectively. Define the following recursive algorithm

$$P_{j+1}^{n+1} = P_j^n \exp\left\{-h\mu^*(a_{j+\frac12}, t_{n+\frac12}, g_h^*(P_{j+\frac12}^{n+\frac12}))\right\},\quad 0\le j\le J^*-1, \tag{4.3}$$

$$P_{j+1}^{n+1} = P_j^n \exp\left\{f(a_j,t_n) - f(a_{j+1},t_n)\right\},\quad J^*\le j\le J-1, \tag{4.4}$$

$$P_0^{n+1} = g_h(\beta(P^{n+1})P^{n+1}), \tag{4.5}$$

where $\beta(P^n)P^n = (\beta(P_1^n)P_1^n,\cdots,\beta(P_J^n)P_J^n)$, $\beta(P_j^n) = \beta(a_j,t_n,g_h^{**}(P_j^n))$, $1\le j\le J,\ 1\le n\le N$;

$$g_h(U^n) = hU_1^n + \sum_{i=1}^{J-1}\frac{h}{2}(U_i^n + U_{i+1}^n); \tag{4.6}$$

$$g_h^*(U_{j+\frac{1}{2}}^n) = \alpha \sum_{i=0}^{J-1} hU_{i+\frac{1}{2}}^n + (1-\alpha)[\sum_{i=j}^{J-2} \frac{h}{2}(U_{i+\frac{1}{2}}^n + U_{i+\frac{3}{2}}^n) + \frac{h}{2}U_{j+\frac{1}{2}}^n]; \tag{4.7}$$

$$g_h^{**}(U_j^n) = \alpha g_h(U^n) + (1-\alpha)\sum_{i=j}^{J-1} \frac{h}{2}(U_i^n + U_{i+1}^n); \tag{4.8}$$

$$P^{n+\frac{1}{2}} = [P_{\frac{1}{2}}^{n+\frac{1}{2}}, P_{\frac{3}{2}}^{n+\frac{1}{2}}, ..., P_{J-\frac{1}{2}}^{n+\frac{1}{2}}]; \tag{4.9}$$

$$P_{j+\frac{1}{2}}^{n+\frac{1}{2}} = P_j^n \exp\{-\frac{h}{2}\mu^*(a_j, t_n, g_h^{**}(P_j^n))\}, \quad 0 \le j \le J^*-1; \tag{4.10}$$

$$P_{j+\frac{1}{2}}^{n+\frac{1}{2}} = P_j^n \exp\{f(a_j, t_n) - f(a_{j+\frac{1}{2}}, t_n)\}, \quad J^* \le j \le J-1. \tag{4.11}$$

In the remainder of this section, we establish the convergence of above algorithm.

Let $H^* = \{A^*/J^*,\ J^* \in \mathbb{N}\}$. For $\forall h \in H^*$, we define the space $X_h = (\mathbb{R}^{J+1})^{N+1}$. Here $\mathbb{R}^{J+1}$is used to consider the approximation of the theoretical solution at each point. And we define the space $Y_h = \mathbb{R}^{J+1} \times \mathbb{R}^N \times (\mathbb{R}^J)^N$, which is used to consider errors due to initial values, approximation of boundary points, and approximation of other points.

To describe the errors, we define

$$\|x\|_{\infty,q} = \max_{1\le i\le q} |x_i|, \text{ for } x = (x_1, x_2, ..., x_q) \in \mathbb{R}^q;$$

$$\|U\|_{1,J+1} = \sum_{i=0}^{J} h|U_i|, \text{ for } U = (U_0, U_1, ..., U_J) \in \mathbb{R}^{J+1},$$

and $B_{\infty,q}(x_h, r)$ denotes the open ball with center x_h and radius r, defined by the norm $\|\cdot\|_{\infty,q}$.

For $(U^0, U^1, ..., U^N) \in X_h$, let $\|(U^0, U^1, ..., U^N)\|_{X_h} = \max_{0\le n\le N} \|U^n\|_{\infty,J+1}$; and for $(Q^0, Q_0, Q^1, ..., Q^N) \in Y_h$, let

$$\|(Q^0, Q_0, Q^1, Q^2..., Q^N)\|_{Y_h} = \|Q^0\|_{\infty,J+1} + \|Q_0\|_{\infty,N} + \sum_{n=1}^{N} \|Q^n\|_{\infty,J}.$$

Let p denote the solution of the system (2.1)-(2.2). $\forall h\in H^*, p_h = (p^0, p^1, ..., p^N) \in X_h$, where

$$p^n = (p_0^n, p_1^n, ..., p_J^n)^T \in \mathbb{R}^{J+1},\ p_j^n = p(a_j, t_n),\ 0 \le j \le J,\ 0 \le n \le N.$$

For $\varepsilon > 0$, define the mapping $\phi_h : B_{X_h}(p_h, \varepsilon) \subset X_h \to Y_h, \phi_h(U^0, U^1, ..., U^N) = (Q^0, Q_0, Q^1, Q^2..., Q^N)$, concrete computations are given by

$$Q^0 = U^0 - P^0 \in \mathbb{R}^{J+1}, \tag{4.12}$$

$$Q_0^n = U_0^n - g_h(\beta(P^n)P^n),\ 1 \le n \le N, \tag{4.13}$$

and for $0 \le n \le N-1$,

$$Q_{j+1}^{n+1} = \frac{U_{j+1}^{n+1} - U_j^n \exp\left\{-h\mu^*(a_{j+\frac{1}{2}}, t_{n+\frac{1}{2}}, g_h^*(U_{j+\frac{1}{2}}^{n+\frac{1}{2}}))\right\}}{h},\ 0 \le j \le J^*-1, \tag{4.14}$$

$$Q_{j+1}^{n+1} = \frac{U_{j+1}^{n+1} - U_j^n \exp\{f(a_j, t_n) - f(a_{j+1}, t_n)\}}{h},\ J^* \le j \le J-1, \tag{4.15}$$

where

$$U_{j+\frac{1}{2}}^{n+\frac{1}{2}} = U_j^n \exp\{-\frac{h}{2}\mu^*(a_j, t_n, g_h^{**}(U_j^n))\}, \quad 0 \le j \le J^* - 1, \tag{4.16}$$

$$U_{j+\frac{1}{2}}^{n+\frac{1}{2}} = U_j^n \exp\{f(a_j, t_n) - f(a_{j+\frac{1}{2}}, t_n)\}, \quad J^* \le j \le J - 1. \tag{4.17}$$

It is obvious that $(P^0, P^1, ..., P^N) \in X_h$ is a solution of the system (4.3)-(4.5) if and only if $\phi_h(P^0, P^1, ..., P^N) = 0$.

Hereafter, C will be a positive constant, which may take different values in different situations.

Lemma 4.1. *Under assumptions (H1)-(H3), if $U^n, W^n \in B_{\infty,J+1}(p^n, \varepsilon), 1 \le n \le N-1$, then for h small enough, we have*

$$|g_h^{**}(U^n) - g_h^{**}(W^n)| \le C\|U^n - W^n\|_1, \tag{4.18}$$
$$|g_h(\beta(U^n)U^n) - g_h(\beta(W^n)W^n)| \le C\|U^n - W^n\|_1, \tag{4.19}$$
$$|U_{j+\frac{1}{2}}^{n+\frac{1}{2}} - W_{j+\frac{1}{2}}^{n+\frac{1}{2}}| \le |U_j^n - W_j^n| + Ch\|U^n - W^n\|_{1,J+1}, \tag{4.20}$$
$$|g_h^*(U_{j+\frac{1}{2}}^{n+\frac{1}{2}}) - g_h^*(W_{j+\frac{1}{2}}^{n+\frac{1}{2}})| \le C\|U^n - W^n\|_{1,J+1}. \tag{4.21}$$

Proof. The first two inequalities are clearly true.

When $0 \le j \le J^* - 1$, by (4.16) we have

$$\begin{aligned} U_{j+\frac{1}{2}}^{n+\frac{1}{2}} - W_{j+\frac{1}{2}}^{n+\frac{1}{2}} =& (U_j^n - W_j^n)\exp\{-\frac{h}{2}\mu^*(a_j, t_n, g_h^{**}(U_j^n))\} \\ &+ W_j^n\left[\exp\{-\frac{h}{2}\mu^*(a_j, t_n, g_h^{**}(U_j^n))\} - \exp\{-\frac{h}{2}\mu^*(a_j, t_n, g_h^{**}(W_j^n))\}\right], \end{aligned}$$

when $J^* \le j \le J - 1$, relation (4.17) gives

$$U_{j+\frac{1}{2}}^{n+\frac{1}{2}} - W_{j+\frac{1}{2}}^{n+\frac{1}{2}} = (U_j^n - W_j^n)\exp\{f(a_j, t_n) - f(a_{j+\frac{1}{2}}, t_n)\}.$$

Assumptions in the lemma implies that $\|W^n\|_{\infty,J+1} \le C$. So, for $0 \le j \le J-1,\ 0 \le n \le N-1$, we have

$$\begin{aligned} |U_{j+\frac{1}{2}}^{n+\frac{1}{2}} - W_{j+\frac{1}{2}}^{n+\frac{1}{2}}| &\le |U_j^n - W_j^n| + Ch|g_h^{**}(U_j^n) - g_h^{**}(W_j^n)| \\ &\le |U_j^n - W_j^n| + Ch\|U^n - W^n\|_{1,J+1}, \end{aligned}$$

which is relation (4.20).

Using (4.7) and (4.20), we have that, for $0 \le n \le N-1$,

$$\begin{aligned} |g_h^*(U_{j+\frac{1}{2}}^{n+\frac{1}{2}}) - g_h^*(W_{j+\frac{1}{2}}^{n+\frac{1}{2}})| &= |g_h^*(U_{j+\frac{1}{2}}^{n+\frac{1}{2}} - W_{j+\frac{1}{2}}^{n+\frac{1}{2}})| \\ &\le C\sum_{i=0}^{J-1} h|U_{j+\frac{1}{2}}^{n+\frac{1}{2}} - W_{j+\frac{1}{2}}^{n+\frac{1}{2}}| \\ &\le C\sum_{i=0}^{J-1} h(|U_j^n - W_j^n| + Ch|U^n - W^n|_{1,J+1}) \\ &\le C\sum_{i=0}^{J} h(|U_j^n - W_j^n| + C^2h^2\|U^n - W^n\|_{1,J+1} \\ &\le (C + C^2A)\|U^n - W^n\|_{1,J+1}. \end{aligned}$$

The proof is complete. □

For $p_h \in X_h$, the local error of the discretization (4.3)-(4.5) is given by $\phi_h(p_h)$, and the discretization is said to be consistent if $\lim_{h\to 0} \|\phi_h(p_h)\|_{Y_h} = 0$. Next result describes the consistence of the discretization.

Lemma 4.2. *Under assumptions (H1)-(H3), if h is small enough, then the local discretization error satisfies $\|\phi(p_h)\|_{Y_h} = \|p^0 - P^0\|_{\infty,J+1} + O(h)$, for $p_h = (p^0, p^1, ..., p^N) \in X_h$.*

Proof. Let $\phi_h(p_h) = (L^0, L_0, L^1, L^2, ..., L^N)$. First we derive the bounds of L^{n+1}, $0 \le n \le N-1$.

Note that $p_{j+1}^{n+1} = p(a_{j+1}, t_{n+1})$, from assumptions, (4.14) and the standard error bound of the mid-point quadrature rule, it follows that, for $0 \le j \le J^* - 1$,

$$
\begin{aligned}
|L_{j+1}^{n+1}| \le & \frac{p_j^n}{h}\Bigg\{\Bigg|\exp\left(-\int_0^h \mu^*(a_j+\tau, t_n+\tau, E(p)(a_j+\tau, t_n+\tau))d\tau\right) \\
& - \exp(-h\mu^*(a_{j+\frac{1}{2}}, t_{n+\frac{1}{2}}, E(p)(a_{j+\frac{1}{2}}, t_{n+\frac{1}{2}})))\Bigg| \\
& + |\exp(-h\mu^*(a_{j+\frac{1}{2}}, t_{n+\frac{1}{2}}, E(p)(a_{j+\frac{1}{2}}, t_{n+\frac{1}{2}}))) \\
& - \exp(-h\mu^*(a_{j+\frac{1}{2}}, t_{n+\frac{1}{2}}, g_h^*(p_{j+\frac{1}{2}}^{n+\frac{1}{2}})))|\Bigg\} \\
\le & C(h + |\mu^*(a_{j+\frac{1}{2}}, t_{n+\frac{1}{2}}, E(p)(a_{j+\frac{1}{2}}, t_{n+\frac{1}{2}})) - \mu^*(a_{j+\frac{1}{2}}, t_{n+\frac{1}{2}}, g_h^*(p_{j+\frac{1}{2}}^{n+\frac{1}{2}}))|) \\
\le & C\left(h + |E(p)(a_{j+\frac{1}{2}}, t_{n+\frac{1}{2}}) - g_h^*(p_{j+\frac{1}{2}}^{n+\frac{1}{2}})|\right).
\end{aligned}
$$

On the other hand, the definitions of $E(p)$ and g_h^* lead to

$$
|E(p)(a_{j+\frac{1}{2}}, t_{n+\frac{1}{2}}) - g_h^*(p_{j+\frac{1}{2}}^{n+\frac{1}{2}})| \le Ch,
$$

where $p_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ is calculated by (4.10)-(4.11), not the grid value of solution at $(a_{j+\frac{1}{2}}, t_{n+\frac{1}{2}})$.

Similarly, we derive that, for $J^* \le j \le J-1$,

$$
\begin{aligned}
|L_{j+1}^{n+1}| \le & \frac{p_j^n}{h}\Bigg\{ \,|\exp\Big(-\int_0^h \mu^*(a_j+\tau, t_n+\tau, E(p)(a_j+\tau, t_n+\tau))d\tau\Big) \\
& - \exp\{f(a_j, t_n) - f(a_{j+1}, t_n)\}\,|\Bigg\} \\
\le & Ch.
\end{aligned}
$$

Therefore, $|L_{j+1}^{n+1}| \le Ch, \quad 0 \le j \le J-1, \ 0 \le n \le N-1$.

Finally we consider the bound of L_0. For $1 \le n \le N$,

$$
\begin{aligned}
|L_0^n| = & |\int_0^A \beta(a, t_n, E(p)(a, t_n))p(a, t_n)da - g_h(\beta(p^n)p^n)| \\
\le & |\int_0^h \beta(a, t_n, E(p)(a, t_n))p(a, t_n)da - h\beta(h, t_n, g_h^{**}(p_1^n))p_1^n| \\
& + \sum_{i=1}^{J-1} |\int_{a_i}^{a_{i+1}} \beta(a, t_n, E(p)(a, t_n))p(a, t_n)da
\end{aligned}
$$

$$-\frac{h}{2}[\beta(a_i,t_n,g_h^{**}(p_i^n))p_i^n+\beta(a_{i+1},t_n,g_h^{**}(p_{i+1}^n))p_{i+1}^n\,|$$
$$\leq Ch,$$

which ends the proof. □

To establish the convergence, we still need the concept of stability.

Definition 4.1 ([3]). The discretization mapping ϕ_h is stable for p_h, restricted to the thresholds ε, if there exist positive constants h_0, S, such that, for $\forall\, h \in H^*$, $h \leq h_0$, the open ball $B_{X_h}(p_h,\varepsilon)$ is contained in the domain of ϕ_h, and

$$\|V_h-W_h\|_{X_h}\leq S\|\phi_h(V_h)-\phi_h(W_h)\|_{Y_h},\ \forall\, V_h,\, W_h\in B_{X_h}(p_h,\varepsilon).$$

Lemma 4.3. *Let $\varepsilon > 0$ be a fixed constant. Then the discretization (4.3)-(4.5) is stable for p_h with thresholds ε.*

Proof. Let $U_h = (U^0, U^1, ..., U^N)$, $W_h = (W^0, W^1, ..., W^N) \in B_{X_h}(p_h,\varepsilon)$, and $E^n = U^n - W^n \in \mathbb{R}^{J+1}$, $0 \leq n \leq N$, $\phi_h(U^0,U^1,...,U^N) = (Q^0,Q_0,Q^1,Q^2...,Q^N)$, $\phi_h(W^0,W^1,...,W^N) = (R^0,R_0,R^1,R^2...,R^N)$.

For $0 \leq j \leq J^* - 1$, by (4.14) we have

$$\begin{aligned}(Q_{j+1}^{n+1}-R_{j+1}^{n+1})h=&\left(U_{j+1}^{n+1}-U_j^n\exp\left\{-h\mu^*(a_{j+\frac12},t_{n+\frac12},g_h^*(U_{j+\frac12}^{n+\frac12}))\right\}\right)\\&-\left(W_{j+1}^{n+1}-W_j^n\exp\left\{-h\mu^*(a_{j+\frac12},t_{n+\frac12},g_h^*(W_{j+\frac12}^{n+\frac12}))\right\}\right).\end{aligned}$$

Consequently,

$$\begin{aligned}E_{j+1}^{n+1}=&(Q_{j+1}^{n+1}-R_{j+1}^{n+1})h+E_j^n\exp\left\{-h\mu^*(a_{j+\frac12},t_{n+\frac12},g_h^*(U_{j+\frac12}^{n+\frac12}))\right\}\\&+W_j^n\Bigg(\exp\left\{-h\mu^*(a_{j+\frac12},t_{n+\frac12},g_h^*(U_{j+\frac12}^{n+\frac12}))\right\}\\&-\exp\left\{-h\mu^*(a_{j+\frac12},t_{n+\frac12},g_h^*(W_{j+\frac12}^{n+\frac12}))\right\}\Bigg).\end{aligned}\tag{4.22}$$

By (4.15), we have, for $J^* \leq j \leq J-1$,

$$E_{j+1}^{n+1}=(Q_{j+1}^{n+1}-R_{j+1}^{n+1})h+E_j^n\exp\{f(a_j,t_n)-f(a_{j+1},t_n)\}.\tag{4.23}$$

Using relations (4.21)-(4.23) and $\|W\|_{\infty J+1}\leq C$, we obtain that for $0 \leq j \leq J-1$,

$$\begin{aligned}|E_{j+1}^{n+1}|\leq&|E_j^n|+Ch|g_h^*(U_{j+\frac12}^{n+\frac12})-g_h^*(W_{j+\frac12}^{n+\frac12})|+h|Q_{j+1}^{n+1}-R_{j+1}^{n+1}|\\\leq&|E_j^n|+2Ch|\sum_{i=0}^{J-1}hU_{i+\frac12}^{n+\frac12}-\sum_{i=0}^{J-1}hW_{i+\frac12}^{n+\frac12}|+h|Q_{j+1}^{n+1}-R_{j+1}^{n+1}|\\\leq&|E_j^n|+Ch\|E^n\|_{1,J+1}+h|Q_{j+1}^{n+1}-R_{j+1}^{n+1}|.\end{aligned}\tag{4.24}$$

When $N \geq n > j \geq 1$, (4.24) implies

$$|E_j^n|\leq|E_0^{n-j}|+Ch\sum_{l=1}^{j}\|E^{n-l}\|_{1,J+1}+h\sum_{l=0}^{j-1}|Q_{j-l}^{n-l}-R_{j-l}^{n-l}|$$

$$\leq |E_0^{n-j}| + Ch\sum_{l=0}^{n-1}\|E^l\|_{1,J+1} + h\sum_{l=1}^{n}\|Q^l - R^l\|_{\infty,J}. \tag{4.25}$$

Similarly, one derives for $j > n \geq 1$,

$$\begin{aligned}|E_j^n| &\leq |E_{j-n}^0| + Ch\sum_{l=1}^{n}\|E^{n-l}\|_{1,J+1} + h\sum_{l=0}^{n}|Q_{j-l}^{n-l} - R_{j-l}^{n-l}|\\ &\leq |E_{j-n}^0| + Ch\sum_{l=0}^{n-1}\|E^l\|_{1,J+1} + h\sum_{l=1}^{n}\|Q^l - R^l\|_{\infty,J}.\end{aligned} \tag{4.26}$$

By (4.13), we see

$$\begin{aligned}E_0^n &= g_h(\beta(U^n)U^n) - g_h(\beta(W^n)W^n) + (Q_0^n - R_0^n)\\ &= g_h(\beta(U^n)E^n) - g_h([\beta(U^n) - \beta(W^n)]W^n) + (Q_0^n - R_0^n).\end{aligned} \tag{4.27}$$

Thus, by (4.19)

$$\begin{aligned}|E_0^n| &\leq C\|E^n\|_{1,J+1} + C|g_h(\beta(U^n) - \beta(W^n))| + |Q_0^n - R_0^n|\\ &\leq C\|E^n\|_{1,J+1} + |Q_0^n - R_0^n|.\end{aligned} \tag{4.28}$$

Using the definition of $\|E^n\|_{1,J+1}$ and (4.24)-(4.28), we derive that, for $1 \leq n \leq N$,

$$\begin{aligned}\|E^n\|_{1,J+1} =& h|E_0^n| + \sum_{j=1}^{n-1} h|E_j^n| + \sum_{j=n}^{J} h|E_j^n|\\ \leq& h(C\|E^n\|_{1,J+1} + |Q_0^n - R_0^n|)\\ &+ \sum_{j=1}^{n-1} h\left(|E_0^{n-j}| + Ch\sum_{l=0}^{n-1}\|E^l\|_{1,J+1} + h\sum_{l=1}^{n}\|Q^l - R^l\|_{\infty,J}\right)\\ &+ \sum_{j=n}^{J} h\left(|E_{j-n}^0| + Ch\sum_{l=0}^{n-1}\|E^l\|_{1,J+1} + h\sum_{l=1}^{n}\|Q^l - R^l\|_{\infty,J}\right)\\ \leq& h(C\|E^n\|_{1,J+1} + |Q_0^n - R_0^n|) + \sum_{j=1}^{n-1} h(C\|E^{n-j}\|_{1,J+1} + |Q_0^{n-j} - R_0^{n-j}|)\\ &+ C\|E^0\|_{1,J+1} + Ch\sum_{l=0}^{n-1}\|E^l\|_{1,J+1} + Ch\sum_{l=1}^{n}\|Q^l - R^l\|_{\infty,J}\\ \leq& C\|E^0\|_{1,J+1} + Ch\sum_{l=0}^{n}\|E^l\|_{1,J+1}\\ &+ Ch\sum_{l=1}^{n}\|Q^l - R^l\|_{\infty,J} + C\sum_{l=1}^{n} h|Q_0^l - R_0^l|.\end{aligned}$$

By means of the discrete Gronwall inequality (see [10]), we have

$$\|E^n\|_{1,J+1} \leq C\left(\|E^0\|_{1,J+1} + \sum_{l=1}^{n}\|Q^l - R^l\|_{\infty,J} + \|Q_0 - R_0\|\right). \tag{4.29}$$

Substituting (4.29) into inequlities (4.24), and (4.27)-(4.28), we conclude that

$$\|U_h - W_h\|_{X_h} \leq C\|\phi_h(U_h) - \phi_h(W_h)\|_{Y_h}.$$

□

To finish the convergence analysis, we cite the following result.

Lemma 4.4 ([3]). *Assume that the discretization ϕ_h is consistent and stable for p_h with thresholds ε. If ϕ_h is continuous in $B(p_h, \varepsilon)$,and $\|\phi_h(p_h)\|_{Y_h} = o(\varepsilon)(h \to 0)$, then for h small enough,*

(i) *The discretization system has a unique solution in $B(p_h, \varepsilon)$;*

(ii) *When $h \to 0$, the solution of discrete system (4.3)-(4.5) converges to the solution of the continuous system (2.1)-(2.2).*

Combining the above lemma with lemmas 4.2 and 4.3, we obtain the main result in this section:

Theorem 4.1. *When $h \to 0$, the solution obtained by the numerical method (4.3)-(4.5) converges to the solution of original model (2.1)-(2.2).*

Example 4.1. Choose the parameters in the model as follows:

$A = 10,\ A^* = 8.75,\ \alpha = 0.5, h = 0.25,\ J = 40,$

$\mu^* = 2(0.02a + 0.02 * \cos(t) + 0.01E(p))/(10 - a), \quad \beta = 0.2(\sin(a) + 1);$

$p_0(a) = 0.3(10 - a)(\sin(a) + 1).$

By means of the algorithm (4.3)-(4.5) and MATLAB, we compute the values of the density and draw the following surface of population:

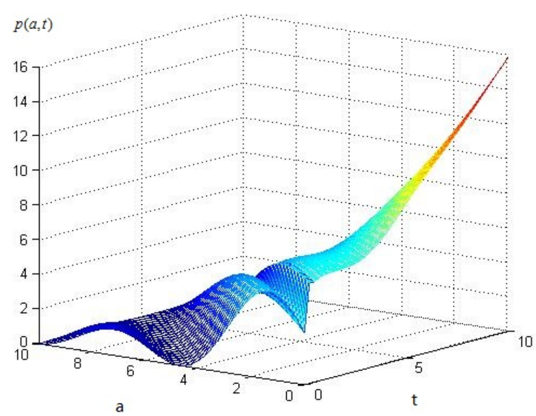

Figure 1. The surface of population density $p(a, t)$.

5. Concluding Remarks

The assumptions that guarantee the well-posedness of system (2.1)-(2.2) in section 2 are biologically meaningful. Particularly, $\int_0^A \mu(a, t, x)da = +\infty$, for arbitrarily fixed (t, x), is used to assure $p(A, t) = 0$, which is exactly what the maximum age A means. On the other hand, the assumptions in section 4 are more restrictive since we need the higher regularity of the solutions to establish the convergence of the numerical scheme. A number of simulations we did show that the algorithm is applicable and efficient.

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