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STUDY ON A KIND OF P-LAPLACIAN NEUTRAL DIFFERENTIAL EQUATION WITH MULTIPLE VARIABLE COEFFICIENTS*

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Abstract In this paper, we first discuss some properties of the neutral operator with multiple variable coefficients $(Ax)(t) := x(t) - \sum\limits_{i=1}^{n} c_i(t)x(t-\delta_i)$. Afterwards, by using an extension of Mawhin's continuation theorem, a kind of second order p-Laplacian neutral differential equation with multiple variable coefficients as follows

$$\left(\phi_p\left(x(t) - \sum_{i=1}^{n} c_i(t)x(t-\delta_i)\right)'\right)' = \tilde{f}(t, x(t), x'(t))$$

is studied. Finally, we consider the existence of periodic solutions for two kinds of second-order p-Laplacian neutral Rayleigh equations with singularity and without singularity. Some new results on the existence of periodic solutions are obtained. It is worth noting that c_i $(i = 1, \cdots, n)$ are no longer constants which are different from the corresponding ones of past work.

Keywords Neutral operator with multiple variable coefficients, p-Laplacian, periodic solution, extension of Mawhin's continuation theorem, singularity.

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1. Introduction

The study of properties of neutral operator can be traced back to 1995. In [20], Zhang first investigated the properties of the neutral operator $(A_1x)(t) := x(t) - cx(t-\delta)$, which became an effective tool for the research on differential equations with this prescribed neutral operator, for example [2,3,10,17,18]. Lu and Ge [12] in 2004 investigated an extension of A_1, namely the neutral operator $(A_2x)(t) := x(t) - \sum\limits_{i=1}^{n} c_i x(t-\delta_i)$. Afterwards, Du [5] discussed the neutral operator with variable coefficient $(A_3x)(t) := x(t) - c(t)x(t-\delta)$, here $c(t)$ is T-periodic function.

During the past few years, some good deal of works have been performed on the existence of periodic solutions of second-order p-Laplacian neutral differential equations (see [1,4,6,7,11,13,15,16,19,21]). Zhu and Lu [21] in 2007 first discussed

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the existence of a periodic solution for a kind of p-Laplacian neutral differential equation as follows

$$(\phi_p(x(t)-cx(t-\delta))')'+g(t,x(t-\tau(t)))=p(t),$$

where $\phi_p:\mathbb{R}\to\mathbb{R}$ is given by $\phi_p(s)=|s|^{p-2}s$, here $p>1$ is a constant. Since $(\phi_p(x'(t)))'$ is nonlinear (i.e.quasilinear), Mawhin's continuation theorem [8] did not be apply directly. In order to get around this difficulty, Zhu and Lu translated the p-Laplacian neutral differential equation into a two-dimensional system

$$\begin{cases} (A_1x_1)'(t)=\phi_q(x_2(t))=|x_2(t)|^{q-2}x_2(t),\\ x_2'(t)=-g(t,x_1(t-\tau(t)))+p(t), \end{cases}$$

where $\frac{1}{p}+\frac{1}{q}=1$, for which Mawhin's continuation theorem can be applied. Afterwards, using topological degree theory, Peng [15] discussed the existence of a periodic solution for the following p-Laplacian neutral Rayleigh equation with a deviating argument

$$(\phi_p((x(t)-cx(t-\delta))'))'+f(x'(t))+g(x(t-\tau(t)))=e(t).$$

Lu [13] in 2009 was concerned with the existence of a periodic solution for a kind of p-Laplacian neutral functional differential equation

$$\begin{aligned}\left(\phi_p\left(\left(x(t)-\sum_{j=1}^{n}c_jx(t-\delta_j)\right)'\right)\right)' =& f(x(t))x'(t)+\alpha(t)g(x(t))\\ &+\sum_{j=1}^{n}\beta_j(t)g(x(t-\tau_j(t)))+p(t).\end{aligned}$$

The method of proof used the continuation theorem of coincidence degree theory developed by J. Mawhin. By applications of Mawhins continuation theorem, Du [6] in 2013 obtained some existence results of periodic solutions for a type of p-Laplacian neutral Liénard equation

$$\left(\phi_p\left(\left(x\left(t\right)-c\left(t\right)x\left(t-\delta\right)\right)'\right)\right)'+f\left(x\left(t\right)\right)x'\left(t\right)+g\left(x\left(t-\tau\left(t\right)\right)\right)=e\left(t\right).$$

All the aforementioned results are related to neutral equations or neutral equations with multiple delays or neutral equations with variable coefficient. Naturally, a new question arises: how neutral differential equation works on multiple variable coefficients? Besides practical interests, the topic has obvious intrinsic theoretical significance. To answer this question, in this paper, we focus on a kind of second order p-Laplacian neutral differential equation as follows

$$\left(\phi_p\left(x(t)-\sum_{i=1}^{n}c_i(t)x(t-\delta_i)\right)'\right)'=\tilde{f}(t,x(t),x'(t)), \tag{1.1}$$

where $c_i(t)\in C^1(\mathbb{R},\mathbb{R})$ and $c_i(t+T)=c_i(t)$; δ_i is constant and $0\le\delta_i<T$, $i=1,2,\cdots,n$. $\tilde{f}:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is an L^2-Carathéodory function, i.e. it is measurable in the first variable and continuous in the second variable, and for

every $0 < r < s$ there exists $h_{r,s} \in L^2[0,T]$ such that $|\tilde{f}(t,x(t),x'(t))| \le h_{r,s}$ for all $x \in [r,s]$ and a.e. $t \in [0,T]$.

The techniques used are quite different from that in [6, 13, 15, 21] and our results are more general than those in [6, 13, 15, 21] in two aspects. First, although $(Ax)(t) := x(t) - \sum\limits_{i=1}^{n} c_i(t)x(t-\delta_i)$ is a natural generalization of the operator $(A_jx)(t)$, $j = 1, 2, 3$, the class of neutral differential equation with A typically possesses a more complicated nonlinearity than neutral differential equation with $(A_jx)(t)$. Second, due to $(Ax)'(t) \neq (Ax')(t)$, the work on estimating *a priori bounds* of periodic solutions for (1.1) is more difficult than the corresponding work on neutral equations in [6, 13, 15, 21].

The remaining part of the paper is organized as follows. In section 2, we analyze qualitative properties of the neutral operator $(Ax)(t)$ which will be helpful for further studies of differential equations with this neutral operator. In section 3, by employing an extension of Mawhin's continuation theorem, we prove the existence of a periodic solution for (1.1). In section 4, we investigate the existence of a periodic solution for a kind of p-Laplacian neutral Rayleigh equation with multiple variable coefficients by applications of Theorem 3.1. In comparison to [6, 13, 15, 21], we avoid to translate p-Laplacian neutral Rayleigh equation into the two-dimensional system. In section 5, we discuss the existence of a periodic solution for a kind of p-Laplacian singular neutral equation with multiple variable coefficients by applications of Theorem 3.1. In section 6, some examples are given to show applications of theorems.

2. Analysis of the generalized neutral operator

$\|c_i\| := \max\limits_{t\in[0,T]} |c_i(t)|$, $i = 1, 2, \cdots n$; $\quad\quad \|c_k\| := \max\{\|c_1(t)\|, \|c_2(t)\|, \cdots \|c_n(t)\|\}$.

Set $C_T = \{x \in C(\mathbb{R},\mathbb{R}) : x(t+T) = x(t), t \in \mathbb{R}\}$, then $(x, \|\cdot\|)$ is a Banach space. Define operators $A, B : C_T \to C_T$, by:

$$(Ax)(t) = x(t) - \sum_{i=1}^{n} c_i(t)x(t-\delta_i), \qquad (Bx)(t) = \sum_{i=1}^{n} c_i(t)x(t-\delta_i).$$

Lemma 2.1. *If $\sum\limits_{i=1}^{n} \|c_i\| \neq 1$, then operator A has a continuous inverse A^{-1} on C_T, satisfying*

(1)

$$|(A^{-1}x)(t)| \le \begin{cases} \dfrac{\|x\|}{1-\sum\limits_{i=1}^{n}\|c_i\|}, & \text{for } \sum\limits_{i=1}^{n}\|c_i\| < 1; \\ \dfrac{\frac{1}{\|c_k\|}\|x\|}{1-\frac{1}{\|c_k\|}-\sum\limits_{i=1,i\neq k}^{n}\|\frac{c_i}{c_k}\|}, & \text{for } \sum\limits_{i=1}^{n}\|c_i\| > 1. \end{cases}$$

(2)

$$\int_0^T \left|(A^{-1}x)(t)\right| dt \le \begin{cases} \dfrac{1}{1-\sum\limits_{i=1}^{n}\|c_i\|}\displaystyle\int_0^T |x(t)|\,dt, & \text{for } \sum\limits_{i=1}^{n}\|c_i\| < 1; \\ \dfrac{\frac{1}{\|c_k\|}}{1-\frac{1}{\|c_k\|}-\sum\limits_{i=1,i\neq k}^{n}\|\frac{c_i}{c_k}\|}\displaystyle\int_0^T |x(t)|\,dt, & \text{for } \sum\limits_{i=1}^{n}\|c_i\| > 1. \end{cases}$$

Proof. We have the following two cases.

Case 1: $\sum\limits_{i=1}^{n}\|c_i\|<1$.

$$(Bx)(t)=\sum_{i=1}^{n}c_i(t)x(t-\delta_i);$$
$$(B^2x)(t)=\sum_{l_1=1}^{n}c_{l_1}(t)\sum_{l_2=1}^{n}c_{l_2}(t-\delta_{l_1})x(t-\delta_{l_1}-\delta_{l_2});$$
$$(B^3x)(t)=\sum_{l_1=1}^{n}c_{l_1}(t)\sum_{l_2=1}^{n}c_{l_2}(t-\delta_{l_1})\sum_{l_3=1}^{n}c_{l_2}(t-\delta_{l_1}-\delta_{l_2})x(t-\delta_{l_1}-\delta_{l_2}-\delta_{l_3}).$$

Therefore, we have

$$(B^jx)(t)=\sum_{l_1=1}^{n}c_{l_1}(t)\sum_{l_2=1}^{n}c_{l_2}(t-\delta_{l_1})\cdots\sum_{l_j=1}^{n}c_{l_j}(t-\delta_{l_1}-\delta_{l_2}\cdots-\delta_{l_{j-1}})x(t-\delta_{l_1}-\delta_{l_2}-\cdots-\delta_{l_j}),$$

and

$$\begin{aligned}\sum_{j=0}^{\infty}(B^jx)(t)=&x(t)+\sum_{j=1}^{\infty}\sum_{l_1=1}^{n}c_{l_1}(t)\sum_{l_2=1}^{n}c_{l_2}(t-\delta_{l_1})\cdots\\&\times\sum_{l_j=1}^{n}c_{l_j}(t-\delta_{l_1}-\delta_{l_2}\cdots-\delta_{l_{j-1}})x(t-\delta_{l_1}-\delta_{l_2}-\cdots-\delta_{l_j}).\end{aligned}$$

Since $A=I-B$ and $\|B\|<1$, we get A has a continuous inverse A^{-1}: $C_T\to C_T$ with

$$A^{-1}=(I-B)^{-1}=I+\sum_{j=1}^{\infty}B^j=\sum_{j=0}^{\infty}B^j,$$

where $B^0=I$. Then, we get

$$\begin{aligned}&|(A^{-1}x)(t)|=\left|\sum_{j=0}^{\infty}(B^jx)(t)\right|=\left|x(t)+\sum_{j=1}^{\infty}(B^jx)(t)\right|\\
=&\left|x(t)+\sum_{j=1}^{\infty}\sum_{l_1=1}^{n}c_{l_1}(t)\cdots\sum_{l_j=1}^{n}c_{l_j}(t-\delta_{l_1}-\delta_{l_2}\cdots-\delta_{l_{j-1}})x(t-\delta_{l_1}-\delta_{l_2}-\cdots-\delta_{l_j})\right|\\
\le&\frac{1}{1-\sum\limits_{i=1}^{n}\|c_i\|}\|x\|\le\frac{\|x\|}{1-\sum\limits_{i=1}^{n}\|c_i\|}.\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}\int_0^T|(A^{-1}x)(t)|dt=&\int_0^T\left|\sum_{j=0}^{\infty}(B^jx)(t)\right|dt\\
\le&\sum_{j=0}^{\infty}\int_0^T\left|(B^jx)(t)\right|dt\\
\le&\sum_{j=0}^{\infty}\int_0^T\left|\sum_{l_1=1}^{n}c_{l_1}(t)\sum_{l_2=1}^{n}c_{l_2}(t-\delta_{l_1})\cdots\sum_{l_j=1}^{n}c_{l_j}(t-\delta_{l_1}-\delta_{l_2}\cdots-\delta_{l_{j-1}})\right.\end{aligned}$$

$$
\begin{aligned}
&\cdot x(t-\delta_{l_1}-\delta_{l_2}-\cdots-\delta_{l_j})\Big|\,dt\\
\leq&\frac{1}{1-\sum\limits_{i=1}^{n}\|c_i\|}\int_0^T|x(t)|dt.
\end{aligned}
$$

Case 2: $\sum\limits_{i=1}^{n}\|c_i\|>1$.

The operator $(Ax)(t)=x(t)-\sum\limits_{i=1}^{n}c_i(t)x(t-\delta_i)$ can be converted to

$$
\begin{aligned}
(Ax)(t)=&x(t)-c_k(t)x(t-\delta_k)-\sum_{i=1,i\neq k}^{n}c_i(t)x(t-\delta_i)\\
=&-c_k(t)\left(-\frac{x(t)}{c_k(t)}+x(t-\delta_k)+\sum_{i=1,i\neq k}^{n}\frac{c_i(t)}{c_k(t)}x(t-\delta_i)\right)\\
=&-c_k(t)\left(x(t-\delta_k)-\frac{x(t)}{c_k(t)}+\sum_{i=1,i\neq k}^{n}\frac{c_i(t)}{c_k(t)}x(t-\delta_i)\right).
\end{aligned}
$$

Let $t_1=t-\delta_k$, we have

$$
(Ax)(t_1+\delta_k)=-c_k(t_1+\delta_k)\left(x(t_1)-\frac{x(t_1+\delta_k)}{c_k(t_1+\delta_k)}+\sum_{i=1,i\neq k}^{n}\frac{c_i(t_1+\delta_k)}{c_k(t_1+\delta_k)}x(t_1+\delta_k-\delta_i)\right).
$$

Define

$$
(Ex)(t)=-c_k(t_1+\delta_k)\left(x(t_1)-\frac{x(t_1+\delta_k)}{c_k(t_1+\delta_k)}+\sum_{i=1,i\neq k}^{n}\frac{c_i(t_1+\delta_k)}{c_k(t_1+\delta_k)}x(t_1+\delta_k-\delta_i)\right),
$$

$$
e_i=\begin{cases}\frac{1}{c_k(t_1+\delta_k)}, & \text{for } \ i=k;\\ -\frac{c_i(t_1+\delta_k)}{c_k(t_1+\delta_k)}, & \text{for } \ i\neq k.\end{cases}\quad
\varepsilon_i=\begin{cases}-\delta_k, & \text{for } \ i=k;\\ \delta_i-\delta_k, & \text{for } \ i\neq k.\end{cases}
$$

So, we get that $(Ex)(t_1+\delta_k)=x(t_1+\delta_k)-\sum\limits_{i=1}^{n}e_i(t_1+\delta_k)x(t_1-\varepsilon_i)$. From Case 1, we get

$$
|(E^{-1}x)(t)|\leq\frac{\|x\|}{1-\sum\limits_{i=1}^{n}\|e_i\|}.
$$

Moreover, since $(A^{-1}x)(t)=-\frac{1}{c_k(t)}(E^{-1}x)(t)$, then we arrive at

$$
|(A^{-1}x)(t)|\leq\left|-\frac{1}{c_k(t)}(E^{-1}x)(t)\right|\leq\frac{\frac{1}{\|c_k\|}\|x\|}{1-\frac{1}{\|c_k\|}-\sum\limits_{i=1,i\neq k}^{n}\|\frac{c_i}{c_k}\|}.
$$

Meanwhile, we can get

$$\int_0^T |(A^{-1}x)(t)|dt \leq \frac{\frac{1}{\|c_k\|}}{1-\frac{1}{\|c_k\|}-\sum\limits_{i=1,i\neq k}^{n}\|\frac{c_i}{c_k}\|}\int_0^T |x'(t)|dt.$$

□

Lemma 2.2 (see [15]). *For $a_i,\ x_i \geq 0$, and $\sum\limits_{i=1}^{n} a_i = 1$, the following inequality holds,*

$$\left(\sum_{i=1}^{n} a_i x_i\right)^p \leq \sum_{i=1}^{n} a_i x_i^p, \quad \textit{for any } p > 1.$$

Lemma 2.3. *If $\sum\limits_{i=1}^{n}\|c_i\| \neq 1$ and $p > 1$, then*

$$\int_0^T |(A^{-1}x)(t)|^p dt \leq \sigma^p \int_0^T |x(t)|^p dt, \quad \forall x \in C_T, \tag{2.1}$$

where

$$\sigma := \begin{cases} \frac{1}{1-\sum\limits_{i=1}^{n}\|c_i\|}, & \text{for} \sum\limits_{i=1}^{n}\|c_i\| < 1, \\ \frac{\frac{1}{\|c_k\|}}{1-\frac{1}{\|c_k\|}-\sum\limits_{i=1,i\neq k}^{n}\|\frac{c_i}{c_k}\|}, & \text{for} \sum\limits_{i=1}^{n}\|c_i\| > 1. \end{cases}$$

Proof. We consider $\sum\limits_{i=1}^{n}\|c_i\| \leq 1$, and the case $\sum\limits_{i=1}^{n}\|c_i\| > 1$ can be treated similarly. From Lemma 2.1, we have

$$|(A^{-1}x)(t)|^p \leq \left(\sum_{j=0}^{\infty}\left(\sum_{i=1}^{n}\|c_i\|\right)^j |x(t-\delta_{l_1}-\delta_{l_2}-\cdots-\delta_{l_j})|\right)^p, \quad \text{for} \sum_{i=1}^{n}\|c_i\| \leq 1.$$

Let $a_j = \frac{\left(1-\sum\limits_{i=1}^{n}\|c_i\|\right)\left(\sum\limits_{i=1}^{n}\|c_i\|\right)^j}{1-\left(\sum\limits_{i=1}^{n}\|c_i\|\right)^n}$, then $a_j \geq 0$ and $\sum\limits_{j=0}^{n-1} a_j = 1$, from Lemma 2.2, we obtain

$$\begin{aligned} &\left(\sum_{j=0}^{\infty}\left(\sum_{i=1}^{n}\|c_i\|\right)^j |x(t-\delta_{l_1}-\delta_{l_2}-\cdots-\delta_{l_j})|\right)^p \\ &= \left(\frac{1-\left(\sum\limits_{i=1}^{n}\|c_i\|\right)^n}{1-\sum\limits_{i=1}^{n}\|c_i\|}\right)^p \left(\sum_{j=0}^{n-1} a_j |x(t-\delta_{l_1}-\delta_{l_2}-\cdots-\delta_{l_j})|\right)^p \\ &\leq \left(\frac{1-\left(\sum\limits_{i=1}^{n}\|c_i\|\right)^n}{1-\sum\limits_{i=1}^{n}\|c_i\|}\right)^p \sum_{j=0}^{n-1} a_j |x(t-\delta_{l_1}-\delta_{l_2}-\cdots-\delta_{l_j})|^p \end{aligned}$$

$$= \left(\frac{1 - \left(\sum_{i=1}^{n} \|c_i\| \right)^n}{1 - \sum_{i=1}^{n} \|c_i\|} \right)^{p-1} \sum_{j=0}^{n-1} \left(\sum_{i=1}^{n} \|c_i\| \right)^j |x(t - \delta_{l_1} - \delta_{l_2} - \cdots - \delta_{l_j})|^p.$$

Let $n \to +\infty$, we get

$$\begin{aligned} |(A^{-1}x)(t)|^p &\leq \left(\sum_{j=0}^{\infty} \left(\sum_{i=1}^{n} \|c_i\| \right)^j |x(t - \delta_{l_1} - \delta_{l_2} - \cdots - \delta_{l_j})| \right)^p \\ &\leq \frac{1}{\left(1 - \sum_{i=1}^{n} \|c_i\|\right)^{p-1}} \sum_{j=0}^{\infty} \left(\sum_{i=1}^{n} \|c_i\| \right)^j |x(t - \delta_{l_1} - \delta_{l_2} - \cdots - \delta_{l_j})|^p. \end{aligned} \tag{2.2}$$

Since $\int_0^T |x(t - \delta_{l_1} - \delta_{l_2} - \cdots - \delta_{l_j})|^p dt = \int_0^T |x(t)|^p dt$, for any $j > 0$, and from (2.2), we get

$$\int_0^T |(A^{-1}x)(t)|^p dt \leq \frac{1}{\left(1 - \sum_{i=1}^{n} \|c_i\|\right)^p} \int_0^T |x(t)|^p dt.$$

Similarly, it is clear

$$\int_0^T |(A^{-1}x)(t)|^p dt \leq \left(\frac{\frac{1}{\|c_k\|}}{1 - \frac{1}{\|c_k\|} - \sum_{i=1, i\neq k}^{n} \|\frac{c_i}{c_k}\|} \right)^p \int_0^T |x(t)|^p dt.$$

□

3. Periodic solution for (1.1)

We first recall the extension of Mawhin's continuation theorem [9].

Let X and Z be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Z$, respectively. A continuous operator $M : X \bigcap \text{dom } M \to Z$ is said to be $quasi-linear$ if

(1) $\text{Im } M := M(X \bigcap \text{dom } M)$ is a closed subset of Z;

(2) $\ker M := \{x \in X \bigcap \text{dom } M : Mx = 0\}$ is a subspace of X with $\dim \ker M < +\infty$.

Let $X_1 = \ker M$ and X_2 be the complement space of X_1 in X, then $X = X_1 \oplus X_2$. In the meanwhile, Z_1 is a subspace of Z and Z_2 is the complement space of Z_1 in Z, so $Z = Z_1 \oplus Z_2$. Suppose that $P : X \to X_1$ and $Q : Z \to Z_1$ two projects and $\Omega \subset X$ is an open and bounded set with the origin $\theta \in \Omega$.

Let $N_\lambda : \bar{\Omega} \to Z,\ \lambda \in [0,1]$ is a continuous operator. Denote N_1 by N, and let $\sum_\lambda = \{x \in \bar{\Omega} : Mx = N_\lambda x\}$. N_λ is said to be $M-$compact in $\bar{\Omega}$ if

(3) there is a vector subspace Z_1 of Z with $\dim Z_1 = \dim X_1$ and an operator $R : \bar{\Omega} \times X_2$ being continuous and compact such that for $\lambda \in [0,1]$,

$$(I - Q)N_\lambda(\bar{\Omega}) \subset \text{Im}M \subset (I - Q)Z, \tag{3.1}$$

$$QN_\lambda x = 0, \quad \lambda \in (0,1) \Leftrightarrow \; QNx = 0, \tag{3.2}$$

$$R(\cdot, 0) \text{ is the zero operator and } R(\cdot,\lambda)|_{\sum_\lambda} = (I-P)|_{\sum_\lambda}, \tag{3.3}$$

and

$$M[P + R(\cdot,\lambda)] = (I-Q)N_\lambda. \tag{3.4}$$

Let $J : Z_1 \to X_1$ be a homeomorphism with $J(\theta) = \theta$.

Lemma 3.1 (see [9])**.** *Let X and Z be Banach space with norm $\|\cdot\|_X$ and $\|\cdot\|_Z$, respectively, and $\Omega \subset X$ be an open and bounded set with $\theta \in \Omega$. Suppose that $M : X \cap domM \to Z$ is a quasi-linear operator and*

$$N_\lambda : \bar{\Omega} \to Z, \;\; \lambda \in (0,1)$$

is an M-compact mapping. In addition, if
(a) $Mx \neq N_\lambda x, \;\; \lambda \in (0,1), \;\; x \in \partial\Omega$,
(b) $\deg\{JQN, \Omega \cap kerM, 0\} \neq 0$,
where $N = N_1$, then the abstract equation $Mx = Nx$ has at least one solution in $\bar{\Omega}$.

Next, we investigative the existence of a periodic solution for (1.1) by applications of the extension of Mawhin's continuation theorem.

Theorem 3.1. *Assume $\sum_{i=1}^{n} \|c_i\| \neq 1$, Ω be open bounded set in C_T^1. Suppose the following conditions hold:*
(i) For each $\lambda \in (0,1)$, the equation

$$(\phi_p(Ax)'(t))' = \lambda \tilde{f}(t, x(t), x'(t)) \tag{3.5}$$

has no solution on $\partial\Omega$;
(ii) The equation

$$F(a) := \frac{1}{T}\int_0^T \tilde{f}(a, x(a), 0)dt = 0$$

has no solution on $\partial\Omega \cap \mathbb{R}$;
(iii) The Brouwer degree

$$\deg\{F, \Omega \cap \mathbb{R}, 0\} \neq 0.$$

Then (1.1) *has at least one periodic solution on $\bar{\Omega}$.*

Proof. In order to use Lemma 3.1 studying the existence of a periodic solution to (1.1). Set $X := \{x \in C[0,T] : \; x(0) = x(T)\}$ and $Z := C[0,T]$,

$$M : \;\; X \cap \text{dom } M \to Z, \qquad (Mx)(t) = (\phi_p(Ax)'(t))', \tag{3.6}$$

where $\text{dom } M := \{u \in X : \phi_p(Au)' \in C^1(\mathbb{R},\mathbb{R})\}$. Then $\ker \; M = \mathbb{R}$. In fact

$$\begin{aligned}\ker \;\; M =& \{x \in X : (\phi_p(Ax)'(t))' = 0\} \\ =& \{x \in X : \phi_p(Ax)' \equiv c\} \\ =& \{x \in X : (Ax)' \equiv \phi_q(c) := c_1\} \\ =& \{x \in X : (Ax)(t) \equiv c_1 t + c_2\},\end{aligned}$$

where $q > 1$ is a constant with $\frac{1}{p} + \frac{1}{q} = 1$ and $c, \; c_1, \; c_2$ are constants in $\mathbb{R}$. Since $(Ax)(0) = (Ax)(T)$, then we get $\ker M = \{x \in X : (Ax)(t) \equiv c_2\}$. In addition

$$\text{Im } M = \{y \in Z, \; \text{for } x(t) \in X \cap \text{dom } M, \; (\phi_p(Ax)')'(t) = y(t),$$

$$\int_0^T y(t)dt = \int_0^T (\phi_p((Ax)')'(t)dt = 0\}.$$

So M is *quasi-linear*. Let

$$X_1 = \ker\ M, \qquad X_2 = \{x \in X :\ x(0) = x(T) = 0\},$$
$$Z_1 = \mathbb{R}, \qquad Z_2 = \mathrm{Im}\ M.$$

Clearly, dim $X_1 =$ dim $Z_1 = 1$, and $X = X_1 \oplus X_2$, $P : X \to X_1$, $Q : Z \to Z_1$, be defined by

$$Px = x(0), \qquad Qy = \frac{1}{T}\int_0^T y(s)ds.$$

For $\forall\ \bar{\Omega} \subset X$, define $N_\lambda : \bar{\Omega} \to Z$ by

$$(N_\lambda x)(t) = \lambda \tilde{f}(t, x(t), x'(t)).$$

We claim $(I-Q)N_\lambda(\bar{\Omega}) \subset \mathrm{Im}M = (I-Q)Z$ holds. In fact, for $x \in \bar{\Omega}$, we have

$$\begin{aligned}
&\int_0^T (I-Q)N_\lambda x(t)dt \\
=&\int_0^T (I-Q)\lambda\tilde{f}(t, x(t), x'(t))dt \\
=&\int_0^T \lambda\tilde{f}(t, x(t), x'(t))dt - \int_0^T \frac{\lambda}{T}\int_0^T \tilde{f}(s, x(s), x'(s))dsdt \\
=&0.
\end{aligned}$$

Therefore, we have $(I-Q)N_\lambda(\bar{\Omega}) \subset \mathrm{Im}\ M$.

Moreover, for any $x \in Z$, we see that

$$\begin{aligned}
&\int_0^T (I-Q)x(t)dt \\
=&\int_0^T \left(x(t) - \int_0^T \frac{1}{T}\int_0^T x(t)dt\right)dt \\
=&0.
\end{aligned}$$

So, we have $(I-Q)Z \subset \mathrm{Im}M$.

On the other hand, form $x \in \mathrm{Im}M$ and $\int_0^T x(t)dt = 0$, we have $x(t) = x(t) - \int_0^T x(t)dt$. Hence, we get $x(t) \in (I-Q)Z$. Then $\mathrm{Im}M = (I-Q)Z$.

From $QN_\lambda x = 0$, we get $\frac{\lambda}{T}\int_0^T \tilde{f}(t, x(t), x'(t))dt = 0$. Since $\lambda \in (0,1)$, then we have $\frac{1}{T}\int_0^T \tilde{f}(t, x(t), x'(t))dt = 0$. Therefore, we obtain $QNx = 0$, then, (3.4) also holds.

Let $J : Z_1 \to X_1, J(x) = x$, then $J(0) = 0$. Define $R : \bar{\Omega} \times [0,1] \to X_2$,

$$R(x,\lambda)(t) = A^{-1}\int_0^t \phi_p^{-1}\left(a + \int_0^s \lambda\tilde{f}(u, x(u), x'(u))du - \frac{\lambda s}{T}\int_0^T \tilde{f}(u, x(u), x'(u))du\right)ds, \tag{3.7}$$

where $a \in R$ is a constant such that

$$R(x,\lambda)(T)=A^{-1}\int_0^T\phi_p^{-1}\left(a+\int_0^s\lambda\tilde{f}(u,x(u),x'(u))du-\frac{\lambda s}{T}\int_0^T\tilde{f}(u,x(t),x'(u))du\right)ds$$
$$=0. \tag{3.8}$$

From Lemma 3.1 of [14], we know that a is uniquely defined by

$$a=\tilde{a}(x,\lambda),$$

where $\tilde{a}(x,\lambda)$ is continuous on $\bar{\Omega}\times[0,1]$ and bounded sets of $\bar{\Omega}\times[0,1]$ into bounded sets of $\mathbb{R}$.

From (3.4), we can find that

$$\mathrm{R}:\bar{\Omega}\times[0,1]\to X_2.$$

Now, for any $x\in\sum_\lambda=\{x\in\bar{\Omega}:Mx=N_\lambda x\}=\{x\in\bar{\Omega}:(\phi_p(Ax)'(t))'=\lambda\tilde{f}(t,x(t),x'(t))\}$, we have $\int_0^T\tilde{f}(t,x(t),x'(t))dt=0$, together with (3.7) gives

$$\begin{aligned}R(x,\lambda)(t)=&A^{-1}\int_0^t\phi_p^{-1}\left(a+\int_0^s\lambda\tilde{f}(u,x(u),x'(u)du\right)ds\\=&A^{-1}\int_0^t\phi_p^{-1}\left(a+\int_0^s(\phi_p(Ax)'(u))'du\right)ds\\=&A^{-1}\int_0^t\phi_p^{-1}\left(a+\phi_p(Ax)'(s)-\phi_p(Ax)'(0)\right)ds.\end{aligned}$$

Take $a=\phi_p(Ax)'(0)$, then we get

$$\begin{aligned}R(x,\lambda)(T)=&A^{-1}\int_0^T\left(\phi_p^{-1}(\phi_p(Ax)'(s))\right)ds\\=&A^{-1}\int_0^T(Ax)'(t)ds\\=&A^{-1}\left((Ax)(T)-(Ax)(0)\right)\\=&x(T)-x(0)\\=&0,\end{aligned}$$

where a is unique, we see that

$$a=\tilde{a}(x,\lambda)=\phi_p(Ax)'(0),\qquad\forall\ \lambda\in[0,1].$$

So, we have

$$\begin{aligned}R(x,\lambda)(t)|_{x\in\sum_\lambda}=&A^{-1}\int_0^t\left(\phi_p^{-1}\left(\phi_p(Ax)'(0)+\int_0^s\lambda\tilde{f}(t,u,x(u),x'(u))du\right)\right)ds\\=&A^{-1}\int_0^t\left(\phi_p^{-1}(\phi_p(Ax)'(s))\right)ds\\=&A^{-1}\int_0^t(Ax)'(s)ds\end{aligned}$$

$$
\begin{aligned}
&=x(t)-x(0)\\
&=(I-P)x(t),
\end{aligned}
$$

which yields the second part of (3.8). Meanwhile, if $\lambda = 0$, we deduce

$\sum_\lambda = \{x \in \bar{\Omega} : Mx = N_\lambda x\} = \{x \in \bar{\Omega} : (\phi_p(Ax)'(t))' = \lambda \tilde{f}(t, x(t), x'(t))\} = c_3$, where $c_3 \in \mathbb{R}$ is a constant. Thus, by the continuity of $\tilde{a}(x, \lambda)$ with respect to (x, λ), $a = \tilde{a}(x, 0) = \phi_p(Ac)'(0) = 0$, we arrive at

$$R(x,0)(t) = A^{-1}\int_0^t \phi_p^{-1}(0)ds = 0, \quad \forall\ x \in \bar{\Omega},$$

which yields the first part of (3.8). Furthermore, we consider

$$M(P+R) = (I-Q)N_\lambda.$$

In fact,

$$\frac{d}{dt}\phi_p(A(P+R))' = (I-Q)N_\lambda. \tag{3.9}$$

Integrating both side of (3.9) over $[0, s]$, we get

$$\int_0^s \frac{d}{dt}\phi_p(A(P+R))'ds = \int_0^s (I-Q)N_\lambda ds.$$

Therefore, we obtain

$$
\begin{aligned}
\phi_p(A(P+R))'(s) - a =& \lambda\int_0^s \tilde{f}(u, x(u), x'(u))du - \int_0^s \frac{\lambda}{T}\int_0^T \tilde{f}(u, x(u), x'(u))dudt\\
=& \lambda\int_0^s \tilde{f}(u, x(u), x'(u))du - \frac{\lambda s}{T}\int_0^T \tilde{f}(u, x(u), x'(u))du,
\end{aligned}
$$

where $a := \phi_p(A(P+R))'(0)$. Then, we see that

$$(A(P+R))'(s) = \phi_p^{-1}\left(a + \lambda\int_0^s \tilde{f}(u, x(u), x'(u))du - \frac{\lambda s}{T}\int_0^T \tilde{f}(u, x(u), x'(u))du\right). \tag{3.10}$$

Integrating both side of (3.10) over $[0, t]$, we have

$$
\begin{aligned}
\int_0^t (A(P+R))'(s)ds = \int_0^t \phi_p^{-1}\bigg(&a + \lambda\int_0^s \tilde{f}(u, x(u), x'(u))du\\
&- \frac{\lambda s}{T}\int_0^T \tilde{f}(u, x(u), x'(u))du\bigg)ds,
\end{aligned}
$$

then

$$
\begin{aligned}
(P+R)(t) - (P+R)(0) =& A^{-1}\bigg(\int_0^t (\phi_p^{-1}((a + \lambda\int_0^s \tilde{f}(u, x(u), x'(u))du\\
&- \frac{\lambda s}{T}\int_0^T \tilde{f}(u, x(u), x'(u))du)))ds\bigg).
\end{aligned}
$$

Since $R(x,\lambda)(0)=0$, $P(t)=P(0)=0$, we get

$$\begin{aligned} R(x,\lambda)(t) =& A^{-1}\bigg(\int_0^t \phi_p^{-1}\Big(a+\lambda \int_0^s \tilde{f}(u,x(u),x'(u))du \\ & - \frac{\lambda s}{T}\int_0^T \tilde{f}(u,x(u),x'(u))du \Big) dt \bigg). \end{aligned}$$

Hence, we have N_λ is M-compact on $\bar{\Omega}$. Obviously, the equation

$$(\phi_p(Ax)'(t))' = \lambda \tilde{f}(t,x(t),x'(t))$$

can be converted to

$$Mx = N_\lambda x, \quad \lambda \in (0,1),$$

where M and N_λ are defined by (3.6) and (3.7), respectively. As proved above,

$$N_\lambda : \bar{\Omega} \to Z, \quad \lambda \in (0,1)$$

is an M-compact mapping. From assumption (i), one finds

$$Mx \neq N_\lambda x, \quad \lambda \in (0,1),\ x \in \partial\Omega,$$

and assumptions (ii) and (iii) imply that $\deg\{JQN, \Omega \cap \ker M, \theta\}$ is valid and

$$\deg\{JQN, \Omega \cap \ker M, \theta\} \neq 0.$$

So, applying Lemma 3.1, (1.1) has at least one T-periodic solution. □

4. Application of Theorem 3.1: p-Laplacian equation

As an application, we consider the following p-Laplacian neutral Rayleigh equation

$$\left(\phi_p\left(x(t) - \sum_{i=1}^n c_i(t)x(t-\delta_i)\right)'\right)' + f(t,x'(t)) + g(t,x(t)) = e(t), \tag{4.1}$$

where $f,\ g \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ are T-periodic function about t and $f(t,0)=0$, $e \in C(\mathbb{R},\mathbb{R})$ is a T-periodic function and $\int_0^T e(t)\,dt = 0$. Next, by applications of Theorem 3.1, we investigate the existence of a periodic solution for (4.1) in the case that $\sum\limits_{i=1}^n \|c_i\| \neq 1$.

Theorem 4.1. *Suppose* $\sum\limits_{i=1}^n \|c_i\| \neq 1$ *holds. Furthermore, assume that the following conditions are satisfied:*

(H_1) *There exists a constant* $K>0$ *such that* $|f(t,u)| \leq K$, *for* $(t,u) \in [0,T]\times\mathbb{R}$.

(H_2) *There exists a positive constant* D *such that* $xg(t,x) > 0$ *and* $|g(t,x)| > K$, *for* $|x| > D$ *and* $t \in [0,T]$.

(H_3) *There exist positive constants* $a,\ b$ *and* B *such that*

$$|g(t,x)| \leq a|x|^{p-1} + b, \quad \text{for } |x| > B \text{ and } t \in [0,T].$$

Then (4.1) *has at least one solution with period* T *if*

$$\sigma T\frac{\left(a\left(1+\sum\limits_{i=1}^{n}\|c_i\|\right)\right)^{\frac{1}{p}}}{2}+\frac{\sigma T\sum\limits_{i=1}^{n}\|c_i'\|}{2}<1.$$

Proof. Consider the homotopic equation

$$\left(\phi_p\left(x(t)-\sum_{i=1}^{n}c_i(t)x(t-\delta_i)\right)'\right)'+\lambda f(t,x'(t))+\lambda g(t,x(t))=\lambda e(t). \tag{4.2}$$

Firstly, we claim that the set of all T-periodic solutions of (4.2) is bounded. Let $x(t)\in C_T$ be an arbitrary T-periodic solution of (4.2). Integrating both side of (4.2) over $[0,T]$, we have

$$\int_0^T (f(t,x'(t))+g(t,x(t)))dt=0. \tag{4.3}$$

By mean-value theorem of integrals, there is a constant $\xi\in(0,T)$ such that

$$g(\xi,x(\xi))+f(\xi,x'(\xi))=0,$$

then by (H_1), it is clear

$$|g(\xi,x(\xi))|=|-f(\xi,x'(\xi))|\le K.$$

From (H_2), we obtain

$$|x(\xi)|\le D.$$

Then, we have

$$\begin{aligned}\|x\|&=\max_{t\in[0,T]}|x(t)|=\max_{t\in[\xi,\xi+T]}|x(t)|\\&=\frac{1}{2}\max_{t\in[\xi,\xi+T]}(|x(t)|+|x(t-T)|)\\&=\frac{1}{2}\max_{t\in[\xi,\xi+T]}\left(\left|x(\xi)+\int_{\xi}^{T}x'(s)ds\right|+\left|x(\xi)-\int_{t-T}^{\xi}x'(s)ds\right|\right)\\&\le D+\frac{1}{2}\left(\int_{\xi}^{t}|x'(s)|ds+\int_{t-T}^{\xi}|x'(s)|ds\right)\\&\le D+\frac{1}{2}\int_0^T|x'(s)|ds.\end{aligned} \tag{4.4}$$

Multiplying both sides of (4.2) by $(Ax)(t)$ and integrating over the interval $[0,T]$, we get

$$\begin{aligned}\int_0^T(\phi_p(Ax)'(t))'(Ax)(t)dt+\lambda\int_0^T f(t,x'(t))(Ax)(t)dt+\lambda\int_0^T g(t,x(t))(Ax)(t)dt\\=\lambda\int_0^T e(t)(Ax)(t)dt.\end{aligned} \tag{4.5}$$

Substituting $\int_0^T (\phi_p(Ax)'(t))'(Ax)(t)dt = -\int_0^T |(Ax)'(t)|^p dt$ into (4.5), we have

$$\begin{aligned}
-\int_0^T |(Ax)'(t)|^p dt =& -\lambda \int_0^T f(t,x'(t))(Ax)(t)dt - \lambda \int_0^T g(t,x(t))(Ax)(t)dt \\
&+ \lambda \int_0^T e(t)(Ax)(t)dt.
\end{aligned}$$

So, we arrive at

$$\begin{aligned}
\int_0^T |(Ax)'(t)|^p dt \leq & \int_0^T |f(t,x'(t))| \left| x(t) - \sum_{i=1}^n c_i(t)x(t-\delta_i) \right| dt \\
&+ \int_0^T |g(t,x(t))| \left| x(t) - \sum_{i=1}^n c_i(t)x(t-\delta_i) \right| dt \\
&+ \int_0^T |e(t)| \left| x(t) - \sum_{i=1}^n c_i(t)x(t-\delta_i) \right| dt \\
\leq & \left(1 + \sum_{i=1}^n \|c_i\|\right) \|x\| \int_0^T |f(t,x'(t))|dt \\
&+ \left(1 + \sum_{i=1}^n \|c_i\|\right) \|x\| \int_0^T |g(t,x(t))|dt \\
&+ \left(1 + \sum_{i=1}^n \|c_i\|\right) \|x\| \int_0^T |e(t)|dt.
\end{aligned} \tag{4.6}$$

Define

$$E_1 := \{t \in [0,T] |\ |x(t)| \leq B\}, \quad E_2 := \{t \in [0,T] |\ |x(t)| > B\}.$$

From (H_2) and (H_3), we get

$$\begin{aligned}
\int_0^T |(Ax)'(t)|^p dt \leq & \left(1 + \sum_{i=1}^n \|c_i\|\right) \|x\| \int_0^T |f(t,x'(t))|dt \\
&+ \left(1 + \sum_{i=1}^n \|c_i\|\right) \|x\| \int_{E_1+E_2} |g(t,x(t))|dt \\
&+ \left(1 + \sum_{i=1}^n \|c_i\|\right) \|x\| \int_0^T |e(t)|dt \\
\leq & \left(1 + \sum_{i=1}^n \|c_i\|\right) \|x\| KT \\
&+ \left(1 + \sum_{i=1}^n \|c_i\|\right) \|x\| (a\|x\|^{p-1}T + bT + \|g_B\|T) \\
&+ \left(1 + \sum_{i=1}^n \|c_i\|\right) \|e\| T \|x\| \\
=& aT \left(1 + \sum_{i=1}^n \|c_i\|\right) \|x\|^p + N_1 \|x\|,
\end{aligned} \tag{4.7}$$

where $\|g_B\| := \max_{|x|\le B} |g(t,x)|$, $\|e\| := \max_{t\in[0,T]} |e(t)|$ and $N_1 := \left(1+\sum_{i=1}^{n}\|c_i\|\right)(KT + bT + \|g_B\|T + \|e\|T)$. Substituting (4.4) into (4.7), we obtain

$$\int_0^T |(Ax)'(t)|^p dt \le aT\left(1+\sum_{i=1}^{n}\|c_i\|\right)\left(D+\frac{1}{2}\int_0^T |x'(t)|dt\right)^p + N_1\left(D+\frac{1}{2}\int_0^T |x'(t)|dt\right). \tag{4.8}$$

Since $(Ax)(t) = x(t) - \sum_{i=1}^{n} x(t-\delta_i)$, we have

$$\begin{aligned}(Ax)'(t) &= \left(x(t) - \sum_{i=1}^{n} c_i(t)x(t-\delta_i)\right)' \\ &= x'(t) - \sum_{i=1}^{n} c_i'(t)x(t-\delta_i) - \sum_{i=1}^{n} c_i(t)x'(t-\delta_i) \\ &= (Ax')(t) - \sum_{i=1}^{n} c_i'(t)x(t-\delta_i),\end{aligned}$$

and

$$(Ax')(t) = (Ax)'(t) + \sum_{i=1}^{n} c_i'(t)x(t-\delta_i).$$

Applying Lemma 2.1 and the Hölder inequality, we see that

$$\begin{aligned}\int_0^T |x'(t)|dt &= \int_0^T |(A^{-1}Ax')(t)|dt \\ &\le \sigma \int_0^T |(Ax')(t)|dt \\ &= \sigma \int_0^T \left|(Ax)'(t) + \sum_{i=1}^{n} c_i'(t)x(t-\delta_i)\right| dt \\ &\le \sigma \int_0^T |(Ax)'(t)|dt + \sigma \int_0^T \left|\sum_{i=1}^{n} c_i'(t)x(t-\delta_i)\right| dt \\ &\le \sigma T^{\frac{1}{q}} \left(\int_0^T |(Ax)'(t)|^p dt\right)^{\frac{1}{p}} + \sigma T \sum_{i=1}^{n} \|c_i'\|\|x\|,\end{aligned} \tag{4.9}$$

where $\|c_i'\| = \max_{t\in[0,T]} |c_i'(t)|$, for $i = 1, 2, \cdots n$. Substituting (4.8) into (4.9), applying a classical inequality,

$$(a+b)^k \le a^k + b^k,\ 0 < k < 1,$$

we have

$$\begin{aligned}\int_0^T |x'(t)|dt \le &\sigma T^{\frac{1}{q}} \left(aT\left(1+\sum_{i=1}^{n}\|c_i\|\right)\right)^{\frac{1}{p}} \left(D+\frac{1}{2}\int_0^T |x'(t)|dt\right) \\ &+ \sigma T^{\frac{1}{q}} (N_1)^{\frac{1}{p}} \left(D+\frac{1}{2}\int_0^T |x'(t)|dt\right)^{\frac{1}{p}}\end{aligned}$$

$$
\begin{aligned}
&+\sigma\sum_{i=1}^{n}\|c_i'\|T\left(D+\frac{1}{2}\int_0^T|x'(t)|dt\right)\\
\le&\left(\sigma T^{\frac{1}{q}}\frac{\left(aT\left(1+\sum_{i=1}^{n}\|c_i\|\right)\right)^{\frac{1}{p}}}{2}+\frac{\sigma\sum_{i=1}^{n}\|c_i'\|T}{2}\right)\int_0^T|x'(t)|dt \qquad (4.10)\\
&+\sigma T^{\frac{1}{q}}\left(aT\left(1+\sum_{i=1}^{n}\|c_i\|\right)\right)^{\frac{1}{p}}D+\sigma T^{\frac{1}{q}}(N_1D)^{\frac{1}{p}}\\
&+\sigma T^{\frac{1}{q}}(N_1)^{\frac{1}{p}}\left(\frac{1}{2}\int_0^T|x'(t)|dt\right)^{\frac{1}{p}}+\sigma\sum_{i=1}^{n}\|c_i'\|TD.
\end{aligned}
$$

Since $\sigma T\frac{\left(a\left(1+\sum_{i=1}^{n}\|c_i\|\right)\right)^{\frac{1}{p}}}{2}+\frac{\sigma T\sum_{i=1}^{n}\|c_i'\|}{2}<1$, it is easy to see that there exists a constant $M_1'>0$ (independent of λ) such that

$$
\int_0^T|x'(t)|dt\le M_1'. \tag{4.11}
$$

From (4.4), we see that

$$
\|x\|\le D+\frac{1}{2}\int_0^T|x'(s)|ds\le D+\frac{1}{2}M_1':=M_1. \tag{4.12}
$$

As $(Ax)(0)=(Ax)(T)$, there exists a point $t_0\in(0,T)$ such that $(Ax)'(t_0)=0$, while $\phi_p(0)=0$, from (4.2), we have

$$
\begin{aligned}
|\phi_p(Ax)'(t)|=&\left|\int_{t_0}^{t}(\phi_p(Ax)'(s))'ds\right|\\
\le&\lambda\int_0^T|f(t,x'(t))|dt+\lambda\int_0^T|g(t,x(t))|dt+\lambda\int_0^T|e(t)|dt\\
\le&KT+T\|g_{M_1}\|+T\|e\|\\
\le&KT+T\|g_{M_1}\|+T\|e\|:=M_2',
\end{aligned}
$$

where $\|g_{M_1}\|:=\max\limits_{|x(t)|\le M_1}|g(t,x(t))|$.

Next, we claim that there exists a positive constant $M_2^*>M_2'+1$ such that for all $t\in\mathbb{R}$, we get

$$
\|(Ax)'\|\le M_2^*. \tag{4.13}
$$

In fact, if $(Ax)'$ is not bounded, there exists a positive constant M_2'' such that $\|(Ax)'\|>M_2''$ for some $(Ax)'\in\mathbb{R}$. Therefore, we have $\|\phi_p(Ax)'\|=\|(Ax)'\|^{p-1}\ge(M_2'')^{p-1}$. Then, it is a contradiction. So, (4.13) holds. From Lemma 2.1 and (4.13), we deduce

$$
\begin{aligned}
\|x'\| =&\|A^{-1}Ax'\| \\
=&\|A^{-1}(Ax')(t)\| \\
\leq&\sigma \left\| (Ax)'(t) + \sum_{i=1}^{n} c_i'(t)x(t-\delta_i) \right\| \\
\leq&\sigma \|(Ax)'\| + \sigma \left(\sum_{i=1}^{n} \|c_i'\|\|x\| \right) \\
\leq&\sigma M_2^* + \sigma \sum_{i=1}^{n} \|c_i'\| M_1 := M_2.
\end{aligned}
\tag{4.14}
$$

Set $M = \sqrt{M_1^2 + M_2^2} + 1$, we have

$$\Omega = \{x \in C_T^1(\mathbb{R}, \mathbb{R}) | \ \|x\| \leq M+1, \ \|x'\| \leq M+1\},$$

and we know that (4.1) has no solution on $\partial\Omega$ as $\lambda \in (0,1)$ and when $x(t) \in \partial\Omega \cap \mathbb{R}$, $x(t) = M+1$ or $x(t) = -M-1$, from (4.4) we know that $M+1 > D$. So, from (H_2), we see that

$$
\begin{aligned}
\frac{1}{T}\int_0^T g(t, M+1)dt \neq 0, \\
\frac{1}{T}\int_0^T g(t, -M-1)dt \neq 0,
\end{aligned}
$$

since $\int_0^T e(t)dt = 0$. So condition (ii) of Theorem 3.1 is also satisfied. Set

$$H(x,\mu) = \mu x + (1-\mu)\frac{1}{T}\int_0^T g(t,x)dt, \quad x \in \partial\Omega \cap \mathbb{R}, \quad \mu \in [0,1].$$

Obviously, from (H_1), we can get $xH(x,\mu) > 0$ and thus $H(x,\mu)$ is a homotopic transformation and

$$
\begin{aligned}
\deg\{F, \Omega \cap \mathbb{R}, 0\} &= \deg \left\{ \frac{1}{T}\int_0^T g(t,x)dt, \Omega \cap \mathbb{R}, 0 \right\} \\
&= \deg\{x, \Omega \cap \mathbb{R}, 0\} \neq 0.
\end{aligned}
$$

So condition (iii) of Theorem 3.1 is satisfied. Applying Theorem 3.1, (4.1) has at least one T-periodic solution. □

5. Application of Theorem 3.1: p-Laplacian equation with singularity

In this section, we consider the existence of a periodic solution for (4.1) with singularity, where $g(t,x) = g_0(x) + g_1(t,x)$, $g_1 \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is a T-periodic function, $g_0 \in C((0,\infty); \mathbb{R})$ has a strong singularity at $x = 0$, i.e.

$$\int_0^1 g_0(x)dx = -\infty. \tag{5.1}$$

Theorem 5.1. *Suppose conditions* $\sum_{i=1}^{n}\|c_i\|\neq 1$ *and* (H_1) *hold. Assume that the following conditions are satisfied:*

(H_4) *There exist positive constants* $D_1,\ D_2$ *with* $D_2<D_1$ *such that* $g(t,x)<-K$ *for* $(t,x)\in[0,T]\times(0,D_2)$ *and* $g(t,x)>K$ *for* $(t,x)\in[0,T]\times(D_1,+\infty)$.

(H_5) *There exist positive constants* α *and* β *such that*

$$g(t,x)\leq \alpha x^{p-1}+\beta,\quad \textit{for } (t,x)\in[0,T]\times(0,+\infty).$$

Then (4.1) *has at least one* T*-periodic solution if*

$$\frac{1}{2}\sigma T\left(2^{\frac{1}{p}}\alpha^{\frac{1}{p}}\left(1+\sum_{i=1}^{n}\|c_i\|\right)^{\frac{1}{p}}+\sum_{i=1}^{n}\|c_i'\|\right)<1.$$

Proof. Consider the homotopic equation

$$(\phi_p(Ax)'(t))'+\lambda f(t,x'(t))+\lambda g(t,x(t))=\lambda e(t). \tag{5.2}$$

We follow same strategy and notation as in the proof of Theorem 4.1. Integrating both sides of (5.2), we get

$$\int_0^T (f(t,x'(t))+g(t,x(t)))dt=0, \tag{5.3}$$

since $\int_0^T e(t)dt=0$. Therefore, form (H_1), we deduce

$$-KT\leq\int_0^T g(t,x(t))dt\leq KT.$$

From (H_4), we know that there exist two point $\tau,\ \eta\in(0,T)$ such that

$$x(\tau)\geq D_2,\quad 0<x(\eta)\leq D_1, \tag{5.4}$$

since $x(t)$ is a T-periodic function and $x(t)>0$. Hence, from (4.4) and (5.4), we get

$$|x(t)|\leq D_1+\frac{1}{2}\int_0^T|x'(t)|dt. \tag{5.5}$$

From (5.3), (H_1) and (H_5), we get

$$\begin{aligned}\int_0^T|g(t,x(t))|dt=&\int_{g(t,x(t))\geq 0}g(t,x(t))dt-\int_{g(t,x(t))\leq 0}g(t,x(t))dt\\ =&2\int_{g(t,x(t))\geq 0}g(t,x(t))dt+\int_0^T f(t,x'(t))dt\\ \leq&2\int_0^T(\alpha x^{p-1}+\beta)dt+\int_0^T|f(t,x'(t))|dt\\ \leq&2\alpha T\|x\|^{p-1}+2\beta T+KT.\end{aligned} \tag{5.6}$$

From (H_1), (4.6), (5.5) and (5.6), we have

$$\begin{aligned}\int_0^T |(Ax)'(t)|^p dt \le & \left(1+\sum_{i=1}^n \|c_i\|\right)\|x\|\int_0^T |f(t,x'(t))|dt \\ & + \left(1+\sum_{i=1}^n \|c_i\|\right)\|x\|\int_0^T |g(t,x(t))|dt \\ & + \left(1+\sum_{i=1}^n \|c_i\|\right)\|x\|\int_0^T |e(t)|dt \\ \le & \left(1+\sum_{i=1}^n \|c_i\|\right)\|x\|\left(2\alpha T\|x\|^{p-1}+2\beta T+KT\right) \\ & + \left(1+\sum_{i=1}^n \|c_i\|\right)KT\|x\| + \left(1+\sum_{i=1}^n \|c_i\|\right)\|e\|T\|x\| \\ \le & 2\alpha T\left(1+\sum_{i=1}^n \|c_i\|\right)\|x\|^p + N_2\|x\| \\ \le & 2\alpha T\left(1+\sum_{i=1}^n \|c_i\|\right)\left(D_1+\frac{1}{2}\int_0^T |x'(t)|dt\right)^p \\ & + N_2\left(D_1+\frac{1}{2}\int_0^T |x'(t)|dt\right),\end{aligned} \tag{5.7}$$

where $N_2 := \left(1+\sum_{i=1}^n \|c_i\|\right)(2\beta T+2KT+\|e\|T)$. From (4.9) and (5.7), we see that

$$\begin{aligned}\int_0^T |x'(t)|dt \le & \sigma T^{\frac{1}{q}}\left(\int_0^T |(Ax')(t)|^p dt\right)^{\frac{1}{p}} + \sigma T\sum_{i=1}^n \|c_i'\|\|x\| \\ \le & \sigma T^{\frac{1}{q}}\left(2\alpha T\left(1+\sum_{i=1}^n \|c_i\|\right)\right)^{\frac{1}{p}}\left(D_1+\frac{1}{2}\int_0^T |x'(t)|dt\right) \\ & + \sigma T^{\frac{1}{q}}N_2^{\frac{1}{p}}\left(D_1+\frac{1}{2}\int_0^T |x'(t)|dt\right)^{\frac{1}{p}} + \sigma T\sum_{i=1}^n \|c_i'\|\left(D_1+\frac{1}{2}\int_0^T |x'(t)|dt\right) \\ = & \frac{1}{2}\sigma T\left(2^{\frac{1}{p}}\alpha^{\frac{1}{p}}\left(1+\sum_{i=1}^n \|c_i\|\right)^{\frac{1}{p}} + \sum_{i=1}^n \|c_i'\|\right)\int_0^T |x'(t)|dt \\ & + \sigma T^{\frac{1}{q}}N_2^{\frac{1}{p}}2^{-\frac{1}{p}}\left(\int_0^T |x'(t)|dt\right)^{\frac{1}{p}} + \sigma T D_1\left(2^{\frac{1}{p}}\alpha^{\frac{1}{p}}\left(1+\sum_{i=1}^n \|c_i\|\right)\right).\end{aligned}$$

Since

$$\frac{1}{2}\sigma T\left(2^{\frac{1}{p}}\alpha^{\frac{1}{p}}\left(1+\sum_{i=1}^n \|c_i\|\right)^{\frac{1}{p}} + \sum_{i=1}^n \|c_i'\|\right) < 1,$$

it is easy to see that there exists a constant $M_3'>0$ (independent of λ) such that

$$\int_0^T |x'(t)|dt \le M_3'. \tag{5.8}$$

From (5.5) and (5.8), we have

$$\|x\| \le D_1 + \frac{1}{2}\int_0^T |x'(s)|ds \le D_1 + \frac{1}{2}M_3' := M_3. \tag{5.9}$$

From (4.13), (4.14) and (5.9), we can get there exists a constant M_3^*, such that, for all $t\in\mathbb{R}$, we have

$$\|x'\| \le M_3^*. \tag{5.10}$$

On the other hand, multiplying both sides of (5.4) by $x'(t)$, we get

$$(\phi_p(Ax)'(t))'x'(t) + \lambda f(t,x'(t))x'(t) + \lambda(g_1(t,x)+g_0(x))x'(t) = \lambda e(t)x'(t), \tag{5.11}$$

since $g(t,x) = g_0(x)+g_1(t,x)$. Let $\tau\in[0,T]$ be as in (5.4), for any $\tau\le t\le T$, we integrate (5.11) on $[\tau,t]$ and get

$$\begin{aligned}\lambda\int_\tau^t g_0(x)x'(t)dt = &-\int_\tau^t (\phi_p(Ax)'(t))'x'(t)dt - \lambda\int_\tau^t f(t,x'(t))x'(t)dt\\ &-\lambda\int_\tau^t g_1(t,x)x'(t)dt + \lambda\int_\tau^t e(t)x'(t)dt.\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}\lambda\left|\int_{x(\tau)}^{x(t)} g_0(u)du\right| \le &\int_\tau^t |(\phi_p(Ax)'(t))'||x'(t)|dt + \lambda\int_\tau^t |f(t,x'(t))||x'(t)|dt\\ &+\lambda\int_\tau^t |g_1(t,x)||x'(t)|dt + \lambda\int_\tau^t |e(t)||x'(t)|dt.\end{aligned}$$

From (5.6), (5.9) and (5.10), and applying (H_1), we have

$$\begin{aligned}&\int_\tau^t |(\phi_p(Ax)'(t))'||x'(t)|dt\\ =&\|x'\|\int_\tau^t |(\phi_p(Ax)'(t))'|dt\\ \le&\|x'\|\left(\lambda\int_0^T |f(t,x'(t))|dt + \lambda\int_0^T |g(t,x)|dt + \lambda\int_0^T |e(t)|dt\right)\\ \le&\lambda M_3^*\left(2KT + 2\alpha T\|x\|^{p-1} + 2\beta T + T\|e\|\right)\\ \le&\lambda M_3^*(2KT + 2\alpha T(M_3)^{p-1} + 2\beta T + \|e\|T).\end{aligned}$$

From (5.9) and (5.10), applying (H_1), we see that

$$\int_\tau^t |f(t,x'(t))||x'(t)|dt \le M_3^*\int_0^T |f(t,x'(t))|dt \le M_3^*KT;$$

$$\int_{\tau}^{t} |g_1(t,x)||x'(t)|dt \le \int_{0}^{T} |g_1(t,x)||x'(t)|dt \le \|g_{M_3}\| M_3^* T,$$

where $\|g_{M_3}\| := \max\limits_{0<x(t)\le M_3} |g_1(t,x)|$.

$$\int_{\tau}^{t} |e(t)||x'(t)|dt \le \int_{0}^{T} |e(t)||x'(t)|dt \le \|e\| M_3^* T.$$

From the above inequalities, we get

$$\left| \int_{x(\tau)}^{x(t)} g_0(u)du \right| \le M_3^*(2\alpha T(M_3)^{p-1} + 2\beta T + 3KT + 2T\|e\| + \|g_{M_1}\|T).$$

In view of strong condition (5.1), we know there exists a constant $M_4 > 0$ such that

$$x(t) \ge M_4, \qquad \forall t \in [\tau, T]. \tag{5.12}$$

The case $t \in [0, \tau]$ can be treated similarly.

From (5.9), (5.10) and (5.12), we have

$$\Omega = \{x \in C_T^1(\mathbb{R}, \mathbb{R}) |\ G_1 \le x \le G_2, \|x'\| \le G_3,\ \forall\ t \in [0, T]\},$$

where $0 < G_1 < \min(M_4, D_2), G_2 > \max(M_3, D_1),\ G_3 > M_3^*$.

The proof left is the same as Theorem 4.1. $\square$

6. Examples

Example 6.1. Consider the p-Laplacian neutral Rayleigh equation in the case that $\sum\limits_{i=1}^{n} \|c_i\| < 1$,

$$\begin{aligned} &\left(\phi_p\left(x(t) - \left(\frac{1}{40}\sin(4t - \frac{\pi}{3})x(t-\delta_1) + \frac{1}{60}\cos\left(4t + \frac{\pi}{4}\right)x(t-\delta_2)\right)\right)'\right)' \\ &+ \cos^2(2t)\sin x'(t) + \frac{1}{40}\left(2 + \cos 4t\right)x^2(t) = \cos^2(2t), \end{aligned} \tag{6.1}$$

where $p = 3$, δ_1, δ_2 are constants and $0 < \delta_1, \delta_2 < T$.

Comparing (6.1) to (4.1), it is easy to see that $c_1(t) = \frac{1}{40}\sin(4t - \frac{\pi}{3})$, $c_2(t) = \frac{1}{60}\cos(4t + \frac{\pi}{4})$, $f(t,u) = \cos^2(2t)\sin u$, $g(t, x(t)) = \frac{1}{40}(2 + \cos 4t)\, x^2(t)$, $e(t) = \cos^2(2t)$, $T = \frac{\pi}{2}$. Choose $K = 1$, $D = 1$, it is obvious that (H_1) and (H_2) hold. Consider $|g(t, x(t))| = |\frac{1}{40}(2 + \cos 4t)x^2(t)| \le \frac{3}{40}|x|^2(t) + 1$, here $a = \frac{3}{40}$, $b = 1$. So, condition (H_3) is satisfied. $\|c_1\| = \frac{1}{40}$, $\|c_2\| = \frac{1}{60}$. So, we have $\sum\limits_{i=1}^{2} \|c_i\| = \|c_1\| + \|c_2\| = \frac{5}{12} < 1$. $\sigma = \frac{1}{1 - \|c_1\| - \|c_2\|} = \frac{12}{7}$. $\|c_1'\| = \frac{1}{10}$ and $\|c_2'\| = \frac{1}{15}$. Next, we consider the condition

$$\sigma T^{\frac{1}{q}} \frac{\left(aT\left(1 + \sum\limits_{i=1}^{n}\|c_i\|\right)\right)^{\frac{1}{p}}}{2} + \frac{\sigma T \sum\limits_{i=1}^{n}\|c_i'\|}{2}$$

$$
\begin{aligned}
&= \frac{12}{7} \times (\frac{\pi}{2})^{\frac{2}{3}} \times \frac{(\frac{3}{40} \times \frac{\pi}{2} \times \frac{17}{12})^{\frac{1}{3}}}{2} + \frac{\frac{12}{7} \times \frac{\pi}{2} \times \frac{1}{6}}{2} \\
&\approx 0.8621 < 1.
\end{aligned}
$$

Therefore, by Theorem 4.1, we know that (6.1) has at least one $\frac{\pi}{2}$-periodic solution.

Example 6.2. Consider the p-Laplacian neutral Rayleigh equation in the case that $\sum_{i=1}^{n} \|c_i\| > 1$,

$$
\begin{aligned}
&\left(\phi_p\left(x(t) - \left(\left(\frac{1}{8}\cos(8t+\frac{\pi}{6}) + \frac{15}{8}\right)x(t-\delta_3) + \frac{1}{64}\sin(8t)x(t-\delta_4)\right)\right)'\right)' \\
&+ 2\sin^2(4t)\sin x'(t) + \frac{1}{16}(2-\cos 8t)x^4(t) = \sin\left(8t - \frac{\pi}{4}\right),
\end{aligned}
\tag{6.2}
$$

where $p = 5$, δ_3, δ_4 are constants and $0 < \delta_3, \delta_4 < T$. Comparing (6.2) to (4.1), it is easy to see that $c_1(t) = (\frac{1}{8}\cos 8t + \frac{\pi}{6}) + \frac{15}{8}$, $c_2 = \frac{1}{64}\sin(8t)$ $f(t,u) = 2\sin^2(4t)\sin u$, $g(t,x(t)) = \frac{1}{16}(2+\sin 8t)x^4(t)$, $e(t) = \cos(8t - \frac{\pi}{4})$. $T = \frac{\pi}{4}$. It is easy to see that there exist a constant $K = 1$ and $D = 1$ such that (H_1) and (H_2) hold. Consider $|g(t,x(t))| = |\frac{1}{16}(2+\sin 8t)x^4(t)| \le \frac{3}{16}|x|^4(t) + 1$, here $a = \frac{3}{16}$, $b = 1$. So, condition (H_3) is satisfied. $\|c_1\| = \frac{1}{8} + \frac{15}{8} = 2$, $\|c_2\| = \frac{1}{64}$, so, we have $\sum_{i=1}^{2} \|c_i\| = \|c_1\| + \|c_2\| = \frac{129}{64} > 1$, $\sigma = \frac{\frac{1}{\|c_k\|}}{1 - \frac{1}{\|c_k\|} - \sum_{i=1, i\neq k}^{n} \|\frac{c_i}{c_k}\|} \approx \frac{\frac{1}{2}}{1-\frac{1}{2}-0.0086} \approx 1.0175$. $\|c_1'\| = 1$ and $\|c_2'\| = \frac{1}{8}$. Next, we consider the condition

$$
\begin{aligned}
&\sigma T^{\frac{1}{q}} \frac{\left(aT\left(1 + \sum_{i=1}^{n}\|c_i\|\right)\right)^{\frac{1}{p}}}{2} + \frac{\sigma T \sum_{i=1}^{n}\|c_i'\|}{2} \\
&= 1.0175 \times (\frac{\pi}{4})^{\frac{4}{5}} \times \frac{(\frac{3}{16} \times \frac{\pi}{4} \times \frac{193}{64})^{\frac{1}{5}}}{2} + \frac{1.0175 \times \frac{\pi}{4} \times (1 + \frac{1}{8})}{2} \\
&\approx 0.8060 < 1.
\end{aligned}
$$

Therefore, by Theorem 4.1, we know that (6.2) has at least one $\frac{\pi}{4}$-periodic solution.

Example 6.3. Consider the following p-Laplacian singular neutral Rayleigh equation in the case that $\sum_{i=1}^{n} \|c_i\| < 1$

$$
\begin{aligned}
&\left(\phi_p\left(x(t) - \left(\left(\frac{1}{16}\cos\left(16t + \frac{\pi}{4}\right)\right)x(t-\delta_1) + \frac{1}{48}\sin(16t)x(t-\delta_2)\right.\right.\right. \\
&\left.\left.\left. + \frac{1}{48}\cos(16t - \frac{\pi}{48})x(t-\delta_3)\right)\right)'\right)' + \frac{1}{100\pi}(1+\sin(16t))\sin x'(t) \\
&+ \frac{1}{16}\left(\frac{3}{4} + \frac{1}{2}\cos 16t\right)x^3(t) - \frac{1}{x^{\mu}} = \cos^2(8t),
\end{aligned}
\tag{6.3}
$$

where $p = 4$, $\mu \ge 1$, δ_1, δ_2 and δ_3 are constants and $0 < \delta_1, \delta_2, \delta_3 < T$.

Comparing (6.3) to (4.1), it is easy to see that $c_1(t) = \frac{1}{16}\cos(16t + \frac{\pi}{4})$, $c_2(t) = \frac{1}{48}\sin(16t)$, $c_3 = \frac{1}{48}\cos(16t - \frac{\pi}{48})$, $f(t,u) = \frac{1}{100\pi}(1 + \sin(8t))\sin u$, $g(t,x(t)) = \frac{1}{16}\left(\frac{3}{4} + \cos 8t\right)x^3(t) - \frac{1}{x^\mu}$, $e(t) = \cos^2(4t)$. $T = \frac{\pi}{8}$. $\|c_1\| = \frac{1}{16}$, $\|c_2\| = \frac{1}{48}$ and $\|c_3\| = \frac{1}{48}$, so, we have $\sum\limits_{i=1}^{3}\|c_i\| = \frac{5}{48} < 1$, $\sigma = \frac{1}{1-\sum\limits_{i=1}^{3}\|c_i\|} = \frac{1}{1-\frac{1}{16}-\frac{1}{48}-\frac{1}{48}} = \frac{48}{43}$. $\|c_1'\| = 1$, $\|c_2'\| = \frac{1}{3}$ and $\|c_3'\| = \frac{1}{3}$. It is easy to see that there exist a constant $K = 1$, $D_2 = \frac{1}{2}$ and $D_1 = 5$ such that (H_1) and (H_4) hold. Besides, $g(t,x(t)) = \frac{1}{16}\left(\frac{3}{4} + \cos 16t\right)x^3(t) - \frac{1}{x^\mu}$, we obtain $\alpha = \frac{5}{64}$ and $\beta = 1$. Then, condition (H_5) holds. Next, we consider the condition

$$\begin{aligned}
&\frac{1}{2}\sigma T\left(2^{\frac{1}{p}}\alpha^{\frac{1}{p}}\left(1+\sum_{i=1}^{n}\|c_i\|\right)^{\frac{1}{p}}+\sum_{i=1}^{n}\|c_i'\|\right)\\
=&\frac{1}{2}\times\frac{\pi}{8}\times\frac{48}{43}\left(2^{\frac{1}{4}}\times\left(\frac{5}{64}\right)^{\frac{1}{4}}\times\left(\frac{53}{48}\right)^{\frac{1}{4}}+1+\frac{1}{3}+\frac{1}{3}\right)\\
\approx&0.5065 < 1.
\end{aligned}$$

Therefore, by Theorem 5.1, we know that (6.3) has at least one positive $\frac{\pi}{8}$-periodic solution.

Example 6.4. Consider the following p-Laplacian singular neutral Rayleigh equation in the case that $\sum\limits_{i=1}^{n}\|c_i\| > 1$

$$\begin{aligned}
&\left(\phi_p\left(x(t)-\left(2x(t-\delta_1)+\frac{1}{36}\cos^2(3t)x(t-\delta_2)-\frac{1}{24}\cos\left(6t+\frac{\pi}{18}\right)x(t-\delta_3)\right)\right)'\right)'\\
&+\frac{1}{325}\left(\frac{1}{4}+\frac{1}{8}\cos(6t)\right)\sin(x'(t))^4+\frac{1}{125\pi}\left(\frac{1}{2}+\frac{1}{2}\sin(6t)\right)x^4(t)-\frac{1}{x^\mu}=\cos\left(6t+\frac{\pi}{4}\right),
\end{aligned}\tag{6.4}$$

where $p = 5$, $\mu \geq 1$, δ_1, δ_2, and δ_3 are constants and $0 < \delta_1, \delta_2, \delta_3 < T$.

Comparing (6.4) to (4.1), it is easy to see that $c_1(t) = 2$, $c_2 = \frac{1}{36}\cos^2(3t)x(t-\delta_2)$, $c_3(t) = -\frac{1}{24}\cos\left(6t + \frac{\pi}{18}\right)x(t-\delta_3)$. $f(t,u) = \frac{1}{40\pi}(\frac{1}{4} + \frac{5}{2}\cos(6t))\sin u^4$, $g(t,x(t)) = \frac{1}{125\pi}(\frac{1}{2} + \frac{1}{2}\sin(6t))x^4(t) - \frac{1}{x^\mu}$, $e(t) = \sin(6t - \frac{\pi}{4})$. $T = \frac{\pi}{3}$. $\|c_1\| = 2$, $\|c_2\| = \frac{1}{36}$ and $\|c_3\| = \frac{1}{24}$, so we have $\sum\limits_{i=1}^{3}\|c_i\| = \frac{149}{72} > 1$, $\sigma = \frac{\frac{1}{\|c_k\|}}{1-\frac{1}{\|c_k\|}-\sum\limits_{i=1,i\neq k}^{n}\|\frac{c_i}{c_k}\|} = \frac{\frac{1}{2}}{1-\frac{1}{2}-\frac{\frac{1}{36}}{2}-\frac{\frac{1}{24}}{2}} = \frac{72}{67}$. $\|c_1'\| = 0$, $\|c_2'\| = \frac{1}{12}$ and $\|c_3'\| = \frac{1}{4}$. It is easy to see that there exist a constant $K = 1$, $D_2 = \frac{1}{4}$ and $D_1 = 7$ such that (H_1) and (H_4) hold. Besides, $g(t,x(t)) = \frac{1}{125\pi}(\frac{1}{2} + \frac{1}{2}\sin(6t))x^4(t) - \frac{1}{x^\mu}$, we obtain $\alpha = \frac{1}{125\pi}$ and $\beta = 1$. Then condition (H_5) is satisfied. Next, we consider the condition

$$\begin{aligned}
&\frac{1}{2}\sigma T\left(2^{\frac{1}{p}}\alpha^{\frac{1}{p}}\left(1+\sum_{i=1}^{n}\|c_i\|\right)^{\frac{1}{p}}+\sum_{i=1}^{n}\|c_i'\|\right)\\
=&\frac{1}{2}\times\frac{\pi}{3}\times\frac{72}{67}\left(2^{\frac{1}{5}}\times\left(\frac{1}{125\pi}\right)^{\frac{1}{5}}\times\left(\frac{221}{72}\right)^{\frac{1}{5}}+0+\frac{1}{12}+\frac{1}{4}\right)
\end{aligned}$$

$$\approx 0.4303 < 1.$$

Therefore, by Theorem 5.1, we know that (6.4) has at least one positive $\frac{\pi}{3}$-periodic solution.

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