

OSCILLATORY BEHAVIOR OF A FRACTIONAL PARTIAL DIFFERENTIAL EQUATION*

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Abstract In this paper, a fractional partial differential equation subject to the Robin boundary condition is considered. Based on the properties of Riemann-Liouville fractional derivative and a generalized Riccati technique, we obtained sufficient conditions for oscillation of the solutions of such equation. Examples are given to illustrate the main results.

Keywords Oscillation, partial differential equation, fractional derivative.

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1. Introduction

Fractional differential equations are generalizations of classical differential equations of integer order and have gained increasing attention due to their various applications in various fields of engineering, chemical physics, electrical networks, control theory of dynamical systems, industrial robotics, economics and so on. The research of fractional differential equations and their applications have received more and more attention very recently, see the monographs [7, 14, 17].

Recently, the oscillation behavior of solutions for partial differential equation has been developed rapidly and some results are established. However, to the best of our knowledge very little is known about the oscillatory behavior of fractional partial differential equations up to now. Some articles about oscillation theory of partial differential equations and fractional partial differential equations have been published, such as [1–6, 8–13, 15, 16, 18–21].

A few paper studied the oscillation of fractional partial differential equations which involve the Riemann-Liouville fractional derivative. Prakash et al. [15] considered the oscillation of the fractional differential equation

$$\frac{\partial}{\partial t} (r(t)D_{+,t}^\alpha u(x,t)) + q(t)f\left(\int_0^t (t-v)^{-\alpha} u(x,v)dv\right) = a(t)\Delta u(x,t), \quad (1.1)$$

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where $(x, t) \in G = \Omega \times \mathbb{R}_+$.

Harikrishnan et al. [6] established the oscillation of the fractional differential equation of the form

$$D_{+,t}^\alpha (r(t)D_{+,t}^\alpha u(x, t)) + q(x, t)f(u(x, t)) = a(t)\Delta u(x, t) + g(x, t), (x, t) \in G. \quad (1.2)$$

Li [11] studied the forced oscillation of fractional partial differential equations with the damping term of the form

$$\frac{\partial}{\partial t} (D_{+,t}^\alpha u(x, t)) + p(x, t)D_{+,t}^\alpha u(x, t) = a(t)\Delta u(x, t) - q(x, t)u(x, t) + f(x, t), (x, t) \in G. \quad (1.3)$$

However, to the best of our knowledge, very little is known regarding the oscillatory behavior of fractional differential equations. To develop the qualitative properties of fractional partial differential equation, it is of great interest to study the oscillatory behavior of fractional partial differential equation. In this paper, we establish several oscillation criteria for fractional partial differential equation by applying a generalized Riccati transformation technique and by using the properties of the Riemann-Liouville fractional derivative. These results are considered essentially new. Examples are given to illustrate the main results.

In this paper, we consider the oscillatory properties of solutions to the fractional partial differential equations of the form

$$\begin{aligned} & D_{+,t}^\alpha (r(t)D_{+,t}^\alpha u(x, t)) + p(t)D_{+,t}^\alpha u(x, t) + q(x, t)f\left(\int_0^t (t-v)^{-\alpha}u(x, v)dv\right) \\ & = a(t)\Delta u(x, t), \quad (x, t) \in G = \Omega \times \mathbb{R}_+, \end{aligned} \quad (1.4)$$

with the Robin boundary condition

$$\frac{\partial u(x, t)}{\partial N} + g(x, t)u(x, t) = 0, \quad (x, t) \in \partial\Omega \times \mathbb{R}_+, \quad (1.5)$$

where $\alpha \in (0, 1)$ is a constant, $D_{+,t}^\alpha$ is the Riemann-Liouville fractional derivative of order α of u with respect t , Ω is a bounded domain in \mathbb{R}^n with piecewise smooth boundary $\partial\Omega$, Δ is the Laplacian operator and N is the unit exterior normal vector to $\partial\Omega$, and $g(x, t)$ is a nonnegative continuous function on $\partial\Omega \times \mathbb{R}_+$.

Throughout this paper, we assume that:

(A₁) $r(t) \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, $p(t) \in C(\mathbb{R}_+, \mathbb{R})$, $a(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$;

(A₂) $q(x, t) \in C(\bar{G}, \mathbb{R}_+)$ and $\min_{x \in \Omega} q(x, t) = q(t)$

(A₃) $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that $f(x)/x > \mu$ for certain constant $\mu > 0$ and for all $x \neq 0$.

By a solution of (1.4) we mean a nontrivial function $u(x, t) \in C^{1+\alpha}(\bar{\Omega} \times [0, \infty))$ such that $\int_0^t (t-v)^{-\alpha}u(x, v)dv \in C^1(\bar{G}; \mathbb{R})$, $D_{+,t}^\alpha u(x, t) \in C^1(\bar{G}; \mathbb{R})$ and satisfies (1.4) on \bar{G} .

A solution $u(x, t)$ of (1.4) is called oscillatory in G if it is neither eventually positive nor eventually negative. Otherwise, it is called non-oscillatory. Equation (1.4) is said to be oscillatory if all its solutions are oscillatory.

2. Preliminaries and lemmas

In this section, we recall several definitions of fractional integral and fractional derivative, which will be used in the following proof. There are several kinds of

definitions of fractional integral and fractional derivatives [7]. In this article, we use Riemann-Liouville definition. For convenience, throughout the rest of this article, we denote

$$v(t) = \frac{1}{|\Omega|} \int_{\Omega} u(x, t) dx, \quad |\Omega| = \int_{\Omega} dx. \tag{2.1}$$

Definition 2.1 ([7]). The Riemann-Liouville fractional integral $I_+^\alpha y$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$(I_+^\alpha y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - v)^{\alpha-1} y(v) dv, \quad t > 0. \tag{2.2}$$

Here $\Gamma(\alpha)$ is the gamma function defined by $\Gamma(\alpha) = \int_0^{+\infty} s^{\alpha-1} e^{-s} ds$ for $\alpha > 0$. This integral is called left-sided fractional integral.

Definition 2.2 ([7]). The Riemann-Liouville fractional partial derivative of order $0 < \alpha < 1$ of a function $u(x, t)$ is defined by

$$(D_{+,t}^\alpha u)(x, t) = \frac{\partial}{\partial t} \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - v)^{-\alpha} u(x, v) dv, \quad t > 0, \tag{2.3}$$

provided the right hand side is pointwise defined on \mathbb{R}_+ , where Γ is the gamma function.

Definition 2.3 ([7]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y : \mathbb{R}_+ \rightarrow \mathbb{R}$ on the half-axis \mathbb{R}_+ is given by

$$(D_+^\alpha y)(t) := \frac{d^{[\alpha]}}{dx^{[\alpha]}} (I_+^{[\alpha]-\alpha} y)(t) = \frac{1}{\Gamma([\alpha] - \alpha)} \frac{d^{[\alpha]}}{dx^{[\alpha]}} \int_0^t (t - v)^{[\alpha]-\alpha-1} y(v) dv, \quad t > 0, \tag{2.4}$$

provided the right hand side is pointwise defined on \mathbb{R}_+ , where $[\alpha]$ is the ceiling function of α .

Lemma 2.1 (Lemma 2.4, [2]). *Let y be a solution of (1.1) and*

$$F(t) := \int_0^t (t - v)^{-\alpha} y(v) dv \quad \text{for } \alpha \in (0, 1) \quad \text{and } t > 0. \tag{2.5}$$

Then

$$F'(t) = \Gamma(1 - \alpha) (D_+^\alpha y)(t). \tag{2.6}$$

3. Main result

We define the following functions that will be used in the proof of our results, suppose that there exists a function $\varphi \in C^1[[t_0, \infty), (0, \infty)]$, let $\xi(t) = r(t)\varphi'(t) - p(t)\varphi(t), \eta(t) = \frac{1}{r(t)\varphi(t)}$. Also we recall a class function defined on $D = \{(t, s) : t \geq s \geq t_0\}$.

A function $H \in C(D, \mathbb{R})$ is said to belong to the class \wp if

- (i) $H(t, t) = 0$ for $t \geq t_0$ and $H(t, s) > 0$ when $t \neq s$;
- (ii) $H(t, s)$ has partial derivatives on D such that $\frac{\partial H}{\partial t}(t, s) = h_1(t, s)\sqrt{H(t, s)}, \frac{\partial H}{\partial s}(t, s) = -h_2(t, s)\sqrt{H(t, s)}$ for some $h_1, h_2 \in L^1_{loc}(D, \mathbb{R})$.

Theorem 3.1. *Let conditions (A₁)-(A₃) hold, suppose that there exists a function $\varphi \in C^1[[t_0, \infty), (0, \infty)]$. If for every $T \geq t_0$, there exists an interval $(a, b) \subset [T, \infty)$ and there exists $c \in (a, b)$, $H \in \varphi$, such that*

$$\begin{aligned} & \frac{1}{H(c, a)} \int_a^c \left[\mu H(s, a) \varphi(s) q(s) - \frac{1}{4\Gamma(1-\alpha)\eta(t)} \Phi_1^2(s, a) \right] ds \\ & + \frac{1}{H(b, c)} \int_c^b \left[\mu H(b, s) \varphi(s) q(s) - \frac{1}{4\Gamma(1-\alpha)\eta(t)} \Phi_2^2(b, s) \right] ds > 0 \end{aligned} \quad (3.1)$$

where

$$\begin{aligned} \Phi_1(s, a) &= h_1(s, a) + \xi(s)\eta(s)\sqrt{H(s, a)}, \\ \Phi_2(b, s) &= h_2(b, s) - \xi(s)\eta(s)\sqrt{H(b, s)}. \end{aligned} \quad (3.2)$$

Then every solution $u(x, t)$ of (1.4) is oscillatory in G .

Proof. Suppose to the contrary that u is a non-oscillatory solution of (1.4). Without loss of generality, we can assume that there exists $u(x, t) > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$. Integrating (1.4) with respect x over the domain Ω , we obtain

$$\begin{aligned} & \frac{d}{dt} \left[r(t) \left(\int_{\Omega} (D_{+,t}^{\alpha} u)(x, t) dx \right) + \int_{\Omega} p(t) D_{+,t}^{\alpha} u(x, t) dx \right. \\ & \left. + \int_{\Omega} q(x, t) f \left(\int_0^t (t-v)^{-\alpha} u(x, v) dv \right) dx \right. \\ & \left. = a(t) \int_{\Omega} \Delta u(x, t) dx. \right. \end{aligned} \quad (3.3)$$

Using Green's formula, it is obvious that

$$\int_{\Omega} \Delta u(x, t) dx = \int_{\partial\Omega} \frac{\partial u(x, t)}{\partial N} ds = - \int_{\partial\Omega} g(x, t) u(x, t) ds \leq 0, \quad t \geq t_1. \quad (3.4)$$

where ds is surface element on $\partial\Omega$. By using Jensen's inequality and (A₂), we can obtain

$$\begin{aligned} & \int_{\Omega} q(x, t) f \left(\int_0^t (t-\mu)^{-\alpha} u(x, \mu) d\mu \right) dx \\ & \geq q(t) f \left[\int_{\Omega} \left(\int_0^t (t-\mu)^{-\alpha} u(x, \mu) d\mu \right) dx \right] \\ & \geq q(t) f \left[\int_0^t (t-\mu)^{-\alpha} \left(\int_{\Omega} u(x, \mu) dx \right) d\mu \right] \\ & \geq q(t) \int_{\Omega} dx f \left[\int_{\Omega} \left(\int_0^t (t-\mu)^{-\alpha} u(x, \mu) d\mu \right) dx \left(\int_{\Omega} dx \right)^{-1} \right] \\ & = q(t) \int_{\Omega} dx f \left[\int_0^t (t-\mu)^{-\alpha} \left(\int_{\Omega} u(x, \mu) dx \right) \left(\int_{\Omega} dx \right)^{-1} d\mu \right] \\ & = q(t) |\Omega| f(G(t)). \end{aligned} \quad (3.5)$$

Combining (3.3)-(3.5) and using definitions, we get

$$\frac{d}{dt} [r(t)D_{+}^{\alpha}v(t)] + p(t)D_{+}^{\alpha}v(t) + q(t)f(G(t)) \leq 0, \quad (3.6)$$

where

$$v(t) = \frac{\int_{\Omega} u(x, t) dx}{|\Omega|},$$

$$G(t) = \int_0^t (t - \xi)^{-\alpha} v(\xi) d\xi.$$

Define the function $w(t)$ by

$$w(t) = \varphi(t) \frac{r(t) D_{+,t}^{\alpha} v(t)}{G(t)}, \quad \text{for } t \geq t_1. \tag{3.7}$$

Then we have $w(t) > 0$ for $t \geq t_1$. Differentiating (3.7) for $t \geq t_1$, we have

$$\begin{aligned} & w'(t) \\ &= \frac{\varphi'(t)}{\varphi(t)} w(t) + \varphi(t) \frac{(r(t) D_{+,t}^{\alpha} v(t))'}{G(t)} - G'(t) \frac{\varphi(t) r(t) D_{+,t}^{\alpha} v(t)}{G^2(t)} \\ &\leq \frac{\varphi'(t)}{\varphi(t)} w(t) + \frac{\varphi(t) [-p(t) D_{+,t}^{\alpha} v(t) - q(t) f(G(t))]}{G(t)} \\ &\quad - \Gamma(1 - \alpha) D_{+,t}^{\alpha} v(t) \frac{\varphi(t) r(t) D_{+,t}^{\alpha} v(t)}{G^2(t)} \\ &= \frac{\varphi'(t)}{\varphi(t)} w(t) - \frac{p(t)}{r(t)} w(t) - \varphi(t) q(t) \frac{f(G(t))}{G(t)} - \frac{\Gamma(1 - \alpha)}{\varphi(t) r(t)} w^2(t) \\ &\leq -\mu \varphi(t) q(t) + \left(\frac{\varphi'(t)}{\varphi(t)} - \frac{p(t)}{r(t)} \right) w(t) - \frac{\Gamma(1 - \alpha)}{\varphi(t) r(t)} w^2(t) \\ &\leq -\mu \varphi(t) q(t) + \xi(t) \eta(t) w(t) - \Gamma(1 - \alpha) \eta(t) w^2(t). \end{aligned} \tag{3.8}$$

Multiplying (3.8) by $H(s, t)$ and integrating with respect to s from t to c for $t \in (a, c]$, we have

$$\begin{aligned} \int_t^c \mu H(s, t) \varphi(s) q(s) ds &\leq - \int_t^c H(s, t) w'(s) ds \\ &\quad + \int_t^c H(s, t) \xi(s) \eta(s) w(s) ds \\ &\quad - \int_t^c H(s, t) \Gamma(1 - \alpha) \eta(s) w^2(s) ds. \end{aligned} \tag{3.9}$$

In view of (i) and (ii), we see that

$$\int_t^c H(s, t) w'(s) ds = H(c, t) w(c) - \int_t^c h_1(s, t) \sqrt{H(s, t)} w(s) ds. \tag{3.10}$$

Using (3.10) in (3.9) leads to

$$\begin{aligned} & \int_t^c \mu H(s, t) \varphi(s) q(s) ds \\ &\leq -H(c, t) w(c) \\ &\quad - \int_t^c \left[\Gamma(1 - \alpha) \eta(s) H(s, t) w^2(s) - \left(h_1(s, t) \sqrt{H(s, t)} + \xi(s) \eta(s) H(s, t) w(s) \right) \right] ds \end{aligned}$$

$$\begin{aligned}
&= -H(c, t)w(c) - \int_t^c \left(\sqrt{\Gamma(1-\alpha)\eta(s)H(s, t)}w(s) - \frac{1}{2\sqrt{\Gamma(1-\alpha)\eta(s)}}\Phi_1(s, t) \right)^2 ds \\
&\quad + \int_t^c \frac{1}{4\Gamma(1-\alpha)\eta(s)}\Phi_1^2(s, t)ds \\
&\leq -H(c, t)w(c) + \int_t^c \frac{1}{4\Gamma(1-\alpha)\eta(s)}\Phi_1^2(s, t)ds. \tag{3.11}
\end{aligned}$$

Similarly, if (3.8) is multiplied by $H(t, s)$ and then integrated from c to t for $t \in [c, b)$, then we get

$$\begin{aligned}
&\int_c^t \mu H(s, t)\varphi(s)q(s)ds \\
&\leq H(t, c)w(c) \\
&\quad - \int_c^t \left[\Gamma(1-\alpha)\eta(s)H(s, t)w^2(s) + \left(h_2(t, s)\sqrt{H(s, t)} - \xi(s)\eta(s)H(s, t)w(s) \right) \right] ds \\
&= H(t, c)w(c) - \int_c^t \left(\sqrt{\Gamma(1-\alpha)\eta(s)H(s, t)}w(s) + \frac{1}{2\sqrt{\Gamma(1-\alpha)\eta(s)}}\Phi_2(s) \right)^2 ds \\
&\quad + \int_c^t \frac{1}{4\Gamma(1-\alpha)\eta(s)}\Phi_2^2(t, s)ds \\
&\leq H(t, c)w(c) + \int_c^t \frac{1}{4\Gamma(1-\alpha)\eta(s)}\Phi_2^2(t, s)ds. \tag{3.12}
\end{aligned}$$

Letting $t \rightarrow a^+$ in (3.11) and $t \rightarrow b^-$ in (3.12) and adding the resulting inequalities we have

$$\begin{aligned}
&\frac{1}{H(c, a)} \int_a^c \left[\mu H(s, a)\varphi(s)q(s) - \frac{1}{4\Gamma(1-\alpha)\eta(s)}\Phi_1^2(s, a) \right] ds \\
&+ \frac{1}{H(b, c)} \int_c^b \left[\mu H(b, s)\varphi(s)q(s) - \frac{1}{4\Gamma(1-\alpha)\eta(s)}\Phi_2^2(b, s) \right] ds \leq 0 \tag{3.13}
\end{aligned}$$

which contradicts the assumption (3.1). The proof is complete. \square

Theorem 3.2. *Let conditions (A_1) - (A_3) hold, suppose that there exists a function $\varphi \in C^1[[t_0, \infty), (0, \infty)]$, and there exists $H \in \wp$ such that*

$$\limsup_{t \rightarrow \infty} \int_l^t \left[\mu H(s, l)\varphi(s)q(s) - \frac{1}{4\Gamma(1-\alpha)\eta(s)}\Phi_1^2(s, l) \right] ds > 0, \tag{3.14}$$

and

$$\limsup_{t \rightarrow \infty} \int_l^t \left[\mu H(t, s)\varphi(s)q(s) - \frac{1}{4\Gamma(1-\alpha)\eta(s)}\Phi_2^2(t, s) \right] ds > 0, \tag{3.15}$$

hold for every $l \in [t_0, \infty)$, $t_1 > t_0$, where Φ_1, Φ_2 are the same in Theorem 3.1, then every solution of (1.4) is oscillatory.

Proof. Suppose to the contrary that u is a non-oscillatory solution of (1.4). Without loss of generality, we can assume that there exists $u(x, t) > 0$ in $G \times [t_0, \infty)$ for

some $t_2 \geq t_1$. Set $l = a \geq t_2$ in (3.14). Clearly, we see from (3.14) that there exists $c > a$ such that

$$\int_a^c \left[\mu H(s, a) \varphi(s) q(s) - \frac{1}{4\Gamma(1 - \alpha)\eta(s)} \Phi_1^2(s, a) \right] ds > 0. \tag{3.16}$$

Similarly setting $l = c \geq t_2$ in (3.15), it follows that there exists $b > c$ such that

$$\int_c^b \left[\mu H(b, s) \varphi(s) q(s) - \frac{1}{4\Gamma(1 - \alpha)\eta(s)} \Phi_2^2(b, s) \right] ds > 0. \tag{3.17}$$

From (3.16) and (3.17) we see that (3.11) is satisfied. Therefore, in view of Theorem 3.1, we may conclude that every solution of (1.4) is oscillatory. □

If we choose $H(t, s) = (t - s)^\lambda, t \geq s \geq t_0$, where $\lambda > 1$ is a constant. Then, we obtain the following useful oscillation criterion.

Corollary 3.1. *Let conditions (A₁)-(A₃) hold, suppose that there exists a function $\varphi \in C^1[[t_0, \infty), (0, \infty)]$, such that the following two inequalities hold:*

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{\lambda-1}} \int_l^t (s-l)^\lambda \left\{ \mu \varphi(s) q(s) - \frac{1}{4\Gamma(1 - \alpha)\eta(s)} \left(\frac{\lambda}{(s-l)} + \xi(s)\eta(s) \right)^2 \right\} ds > 0 \tag{3.18}$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{\lambda-1}} \int_l^t (t-s)^\lambda \left\{ \mu \varphi(s) q(s) - \frac{1}{4\Gamma(1 - \alpha)\eta(s)} \left(\frac{\lambda}{(t-s)} - \xi(s)\eta(s) \right)^2 \right\} ds > 0 \tag{3.19}$$

for each $l \geq t_0, \lambda > 1$, then Eq.(1.4) is oscillatory.

More generally, one may consider $H(t, s) = [R(t) - R(s)]^\lambda$, where λ is constant and $R(t) = \int_{t_1}^t \frac{1}{r(s)} ds$ and $\lim_{t \rightarrow \infty} R(t) = \infty$. If we choose $\varphi(t) = 1$, by Theorem 3.2, we have the following oscillatory criterion.

Theorem 3.3. *Let conditions (A₁)-(A₃) hold, $p(t) \geq 0$ and $q(t) \geq 0$ for all $t \in [t_0, \infty)$ and $\lim_{t \rightarrow \infty} R(t) = \infty$. Then every solution of Eq.(1.3) is oscillatory provided for each $l \geq t_0$, and for some $\lambda > 1$, the following two inequalities hold:*

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \left\{ \left(\mu q(s) - \frac{p^2(s)}{4\Gamma(1 - \alpha)r(s)} \right) [R(s) - R(l)]^\lambda \right. \\ \left. + \frac{\lambda p(s)}{2\Gamma(1 - \alpha)r(s)} [R(s) - R(l)]^{\lambda-1} \right\} ds > \frac{\lambda^2}{4\Gamma(1 - \alpha)(\lambda - 1)} \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \left\{ \left(\mu q(s) - \frac{p^2(s)}{4\Gamma(1 - \alpha)r(s)} \right) [R(t) - R(s)]^\lambda \right. \\ \left. + \frac{\lambda p(s)}{2\Gamma(1 - \alpha)r(s)} [R(t) - R(s)]^{\lambda-1} \right\} ds > \frac{\lambda^2}{4\Gamma(1 - \alpha)(\lambda - 1)}. \end{aligned} \tag{3.21}$$

Proof. Since $H(t, s) = [R(t) - R(s)]^\lambda$, it is easy to see that

$$h_1(t, s) = \lambda [R(t) - R(s)]^{\frac{\lambda-2}{2}} \frac{1}{r(t)},$$

and

$$h_2(t, s) = \lambda [R(t) - R(s)]^{\frac{\lambda-2}{2}} \frac{1}{r(s)}.$$

Noting that

$$\int_l^t r(s) h_1^2(s, l) ds = \int_l^t r(s) \lambda^2 [R(s) - R(l)]^{\lambda-2} \frac{1}{r^2(s)} ds = \frac{\lambda^2}{\lambda-1} [R(t) - R(l)]^{\lambda-1}$$

and

$$\int_l^t r(s) h_2^2(s, l) ds = \int_l^t r(s) \lambda^2 [R(t) - R(s)]^{\lambda-2} \frac{1}{r^2(s)} ds = \frac{\lambda^2}{\lambda-1} [R(t) - R(l)]^{\lambda-1}.$$

In view of $\lim_{t \rightarrow \infty} R(t) = \infty$, it follows that

$$\lim_{t \rightarrow \infty} \frac{1}{4\Gamma(1-\alpha)R^{\lambda-1}(t)} \int_l^t r(s) h_1^2(s, l) ds = \frac{\lambda^2}{4\Gamma(1-\alpha)(\lambda-1)} \quad (3.22)$$

and

$$\lim_{t \rightarrow \infty} \frac{1}{4\Gamma(1-\alpha)R^{\lambda-1}(t)} \int_l^t r(s) h_2^2(s, l) ds = \frac{\lambda^2}{4\Gamma(1-\alpha)(\lambda-1)}. \quad (3.23)$$

From (3.20) and (3.22), we have that

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \left\{ H(s, l) \mu q(s) - \frac{r(s)}{4\Gamma(1-\alpha)} \left[h_1(s, l) - \frac{p(s)}{r(s)} \sqrt{H(s, l)} \right]^2 \right\} ds \\ &= \limsup_{t \rightarrow \infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \left\{ [R(s) - R(l)]^\lambda \mu q(s) + \frac{\lambda p(s)}{2\Gamma(1-\alpha)r(s)} [R(s) - R(l)]^{\lambda-1} \right. \\ & \quad \left. - \frac{p^2(s)}{4\Gamma(1-\alpha)r(s)} [R(s) - R(l)]^\lambda \right\} ds - \limsup_{t \rightarrow \infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \frac{r(s)}{4\Gamma(1-\alpha)} h_1^2(s, l) ds \\ &= \limsup_{t \rightarrow \infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \left\{ \left(\mu q(s) - \frac{p^2(s)}{4\Gamma(1-\alpha)r(s)} \right) [R(s) - R(l)]^\lambda \right. \\ & \quad \left. + \frac{\lambda p(s)}{2\Gamma(1-\alpha)r(s)} [R(s) - R(l)]^{\lambda-1} \right\} ds - \frac{\lambda^2}{4\Gamma(1-\alpha)(\lambda-1)} > 0, \end{aligned} \quad (3.24)$$

i.e., (3.14) holds. Similarly, (3.21) implies that (3.15) holds. By Theorem 3.2, every solution of (1.4) is oscillatory. The proof is complete. \square

4. Example

Example 4.1. Consider the fractional partial differential equations

$$\begin{aligned} & D_{+,t}^\alpha (D_{+,t}^\alpha u(x, t)) - \frac{1}{t} D_{+,t}^\alpha u(x, t) + \frac{e^x}{t^2} f \left(\int_0^t (t-\xi)^{-\alpha} u(x, v) dv \right) \\ &= \frac{e^t}{8} \Delta u(x, t), \quad (x, t) \in (0, \pi) \times (0, \infty) \end{aligned} \quad (4.1)$$

with the Robin boundary condition

$$u_x(0, t) = u_x(\pi, t) = 0, \quad (4.2)$$

where $\alpha \in (0, 1)$, $p(t) = -\frac{1}{t}$, $q(t) = \min_{x \in \Omega} q(x, t) = \min_{x \in (0, \pi)} \frac{e^x}{t^2} = \frac{1}{t^2}$, $r(t) = 1$, $a(t) = \frac{e^t}{8}$, $f(u) = u$. Set $t_0 \geq 0$ and $\mu = 1$. Thus all the conditions of the theorem (3.1) hold. Therefore every solution of (4.1) is oscillatory.

Example 4.2. Consider the fractional partial differential equations

$$\begin{aligned} & D_{+,t}^{\frac{1}{2}} \left(D_{+,t}^{\frac{1}{2}} u(x, t) \right) - D_{+,t}^{\frac{1}{2}} u(x, t) + \left(x^2 + \frac{1}{t} \right) f \left(\int_0^t (t - \xi)^{-\frac{1}{2}} u(x, v) dv \right) \\ & = 3e^{-t} \Delta u(x, t), \quad (x, t) \in (0, \pi) \times (0, \infty) \end{aligned} \quad (4.3)$$

with the Robin boundary condition

$$u_x(0, t) = u_x(\pi, t) = 0, \quad (4.4)$$

where $\alpha \in (0, 1)$, $p(t) = -1$, $q(x, t) = (x^2 + \frac{1}{t})$, $q(t) = \min_{x \in \Omega} q(x, t) = \min_{x \in (0, \pi)} (x^2 + \frac{1}{t}) = \frac{1}{t}$, $r(t) = 1$, $a(t) = 3e^{-t}$, $f(u) = u$. Set $t_0 \geq 0$ and $\mu = 1$. Thus all the conditions of the theorem (3.1) hold. Therefore every solution of (4.3) is oscillatory.

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