Journal of Applied Analysis and Computation
Volume 8, Number 3, June 2018, 1011–1020
Website:http://jaac-online.com/
DOI:10.11948/2018.1011

OSCILLATORY BEHAVIOR OF A FRACTIONAL PARTIAL DIFFERENTIAL EQUATION*

Jiangfeng Wang[1,2], Fanwei Meng[1,†]

Abstract In this paper, a fractional partial differential equation subject to the Robin boundary condition is considered. Based on the properties of Riemann-Liouville fractional derivative and a generalized Riccati technique, we obtained sufficient conditions for oscillation of the solutions of such equation. Examples are given to illustrate the main results.

Keywords Oscillation, partial differential equation, fractional derivative.

MSC(2010) 35B05, 35R11, 34K37.

1. Introduction

Fractional differential equations are generalizations of classical differential equations of integer order and have gained increasing attention due to their various applications in various fields of engineering, chemical physics, electrical networks, control theory of dynamical systems, industrial robotics, economics and so on. The research of fractional differential equations and their applications have received more and more attention very recently, see the monographs [7, 14, 17].

Recently, the oscillation behavior of solutions for partial differential equation has been developed rapidly and some results are established. However, to the best of our knowledge very little is known about the oscillatory behavior of fractional partial differential equations up to now. Some articles about oscillation theory of partial differential equations and fractional partial differential equations have been published, such as [1–6, 8–13, 15, 16, 18–21].

A few paper studied the oscillation of fractional partial differential equations which involve the Riemann-Liouville fractional derivative. Prakash et al. [15] considered the oscillation of the fractional differential equation

$$\frac{\partial}{\partial t}\left(r(t)D^{\alpha}_{+,t}u(x,t)\right)+q(t)f\left(\int_0^t (t-v)^{-\alpha}u(x,v)dv\right)=a(t)\Delta u(x,t), \qquad (1.1)$$

†the corresponding author. Email address:fwmeng@mail.qfnu.edu.cn(F. Meng)
1School of Mathematical Science, Qufu Normal University, No.57 Jingxuanxi Road, Qufu 273165, Shandong, China
2Department of Mathematics, Jining University, No.1 Xingtan Road, Qufu, 273155, Shandong, China
*The authors were supported by Natural Science Foundation of China (Grants 11671227), the Natural Science Foundation of Shandong Province (Grant ZR2014AL007, ZR2015PA005, ZR2017BF021) and the Science and Technology Project of High Schools of Shandong Province (Grant J14LJ09).

where $(x,t) \in G = \Omega \times \mathbb{R}_+$.

Harikrishnan et al. [6] established the oscillation of the fractional differential equation of the form

$$D^{\alpha}_{+,t}\left(r(t)D^{\alpha}_{+,t}u(x,t)\right) + q(x,t)f\left(u(x,t)\right) = a(t)\Delta u(x,t) + g(x,t), (x,t) \in G. \quad (1.2)$$

Li [11] studied the forced oscillation of fractional partial differential equations with the damping term of the form

$$\frac{\partial}{\partial t}\left(D^{\alpha}_{+,t}u(x,t)\right) + p(x,t)D^{\alpha}_{+,t}u(x,t) = a(t)\Delta u(x,t) - q(x,t)u(x,t) + f(x,t), (x,t) \in G. \quad (1.3)$$

However, to the best of our knowledge, very little is known regarding the oscillatory behavior of fractional differential equations. To develop the qualitative properties of fractional partial differential equation, it is of great interest to study the oscillatory behavior of fractional partial differential equation. In this paper, we establish several oscillation criteria for fractional partial differential equation by applying a generalized Riccati transformation technique and by using the properties of the Riemann-Liouville fractional derivative. These results are considered essentially new. Examples are given to illustrate the main results.

In this paper, we consider the oscillatory properties of solutions to the fractional partial differential equations of the form

$$\begin{aligned} &D^{\alpha}_{+,t}\left(r(t)D^{\alpha}_{+,t}u(x,t)\right) + p(t)D^{\alpha}_{+,t}u(x,t) + q(x,t)f\left(\int_0^t (t-v)^{-\alpha}u(x,v)dv\right) \\ =&a(t)\Delta u(x,t), \quad (x,t) \in G = \Omega \times \mathbb{R}_+, \end{aligned} \quad (1.4)$$

with the Robin boundary condition

$$\frac{\partial u(x,t)}{\partial N} + g(x,t)u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \quad (1.5)$$

where $\alpha \in (0,1)$ is a constant, $D^{\alpha}_{+,t}$ is the Riemann-Liouville fractional derivative of order α of u with respect t, Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary $\partial\Omega$, Δ is the Laplacian operator and N is the unit exterior normal vector to $\partial\Omega$, and $g(x,t)$ is a nonnegative continuous function on $\partial\Omega \times \mathbb{R}_+$.

Throughout this paper, we assume that:

(A_1) $r(t) \in C^1(\mathbb{R}_+, \mathbb{R}_+), p(t) \in C(\mathbb{R}_+, R), a(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$;

(A_2) $q(x,t) \in C(\bar{G}, \mathbb{R}_+)$ and $\min_{x\in\Omega} q(x,t) = q(t)$

(A_3) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(x)/x > \mu$ for certain constant $\mu > 0$ and for all $x \neq 0$.

By a solution of (1.4) we mean a nontrivial function $u(x,t) \in C^{1+\alpha}(\bar{\Omega} \times [0,\infty))$ such that $\int_0^t (t-v)^{-\alpha}u(x,v)dv \in C^1(\bar{G};\mathbb{R}), D^{\alpha}_{+,t}u(x,t) \in C^1(\bar{G};\mathbb{R})$ and satisfies (1.4) on $\bar{G}$.

A solution $u(x,t)$ of (1.4) is called oscillatory in G if it is neither eventually positive nor eventually negative. Otherwise, it is called non-oscillatory. Equation (1.4) is said to be oscillatory if all its solutions are oscillatory.

2. Preliminaries and lemmas

In this section, we recall several definitions of fractional integral and fractional derivative, which will be used in the following proof. There are several kinds of definitions of fractional integral and fractional derivatives [7]. In this article, we use Riemann-Liouville definition. For convenience, throughout the rest of this article, we denote

$$v(t) = \frac{1}{|\Omega|}\int_{\Omega} u(x,t)dx, \qquad |\Omega| = \int_{\Omega} dx. \tag{2.1}$$

Definition 2.1 ([7]). The Riemann-Liouville fractional integral $I_+^{\alpha}y$ of order $\alpha \in R^+$ is defined by

$$(I_+^{\alpha}y)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-v)^{\alpha-1}y(v)dv, \quad t > 0. \tag{2.2}$$

Here $\Gamma(\alpha)$ is the gamma function defined by $\Gamma(\alpha) = \int_0^{+\infty} s^{\alpha-1}e^{-s}ds$ for $\alpha > 0$. This integral is called left-sided fractional integral.

Definition 2.2 ([7]). The Riemann-Liouville fractional partial derivative of order $0 < \alpha < 1$ of a function $u(x,t)$ is defined by

$$(D_{+,t}^{\alpha}u)(x,t) = \frac{\partial}{\partial t}\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-v)^{-\alpha}u(x,v)dv, \quad t > 0, \tag{2.3}$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$, where Γ is the gamma function.

Definition 2.3 ([7]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y : \mathbb{R}_+ \to \mathbb{R}$ on the half-axis $\mathbb{R}_+$ is given by

$$(D_+^{\alpha}y)(t) := \frac{\mathrm{d}^{[\alpha]}}{\mathrm{d}x^{[\alpha]}}(I_+^{[\alpha]-\alpha}y)(t) = \frac{1}{\Gamma([\alpha]-\alpha)}\frac{\mathrm{d}^{[\alpha]}}{\mathrm{d}x^{[\alpha]}}\int_0^t (t-v)^{[\alpha]-\alpha-1}y(v)dv, \quad t > 0, \tag{2.4}$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$, where $[\alpha]$ is the ceiling function of α.

Lemma 2.1 (Lemma 2.4, [2]). *Let y be a solution of* (1.1) *and*

$$F(t) := \int_0^t (t-v)^{-\alpha}y(v)dv \quad \textit{for} \quad \alpha \in (0,1) \quad \textit{and} \quad t > 0. \tag{2.5}$$

Then

$$F'(t) = \Gamma(1-\alpha)(D_+^{\alpha}y)(t). \tag{2.6}$$

3. Main result

We define the following functions that will be used in the proof of our results, suppose that there exists a function $\varphi \in C^1\left[[t_0,\infty),(0,\infty)\right]$, let $\xi(t) = r(t)\varphi'(t) - p(t)\varphi(t)$, $\eta(t) = \frac{1}{r(t)\varphi(t)}$. Also we recall a class function defined on $D = \{(t,s) : t \geq s \geq t_0\}$. A function $H \in C\left(D,\mathbb{R}\right)$ is said to belong to the class $\wp$ if

(i) $H(t,t) = 0$ for $t \geq t_0$ and $H(t,s) > 0$ when $t \neq s$;

(ii) $H(t,s)$ has partial derivatives on D such that $\frac{\partial H}{\partial t}(t,s) = h_1(t,s)\sqrt{H(t,s)}$, $\frac{\partial H}{\partial s}(t,s) = -h_2(t,s)\sqrt{H(t,s)}$ for some $h_1, h_2 \in L^1_{loc}(D,\mathbb{R})$.

Theorem 3.1. *Let conditions (A_1)-(A_3) hold, suppose that there exists a function $\varphi \in C^1\left[[t_0,\infty),(0,\infty)\right]$. If for every $T \geq t_0$, there exists an interval $(a,b) \subset [T,\infty)$ and there exists $c \in (a,b), H \in \wp$,such that*

$$\begin{aligned}&\frac{1}{H(c,a)}\int_a^c\left[\mu H(s,a)\varphi(s)q(s)-\frac{1}{4\Gamma(1-\alpha)\eta(t)}\Phi_1^2(s,a)\right]ds\\&+\frac{1}{H(b,c)}\int_c^b\left[\mu H(b,s)\varphi(s)q(s)-\frac{1}{4\Gamma(1-\alpha)\eta(t)}\Phi_2^2(b,s)\right]ds>0\end{aligned}\tag{3.1}$$

where

$$\begin{aligned}\Phi_1(s,a)&=h_1(s,a)+\xi(s)\eta(s)\sqrt{H(s,a)},\\\Phi_2(b,s)&=h_2(b,s)-\xi(s)\eta(s)\sqrt{H(b,s)}.\end{aligned}\tag{3.2}$$

Then every solution $u(x,t)$ of (1.4) is oscillatory in G.

Proof. Suppose to the contrary that u is a non-oscillatory solution of (1.4). Without loss of generality, we can assume that there exists $u(x,t)>0$ in $G\times[t_0,\infty)$ for some $t_0>0$. Integrating (1.4) with respect x over the domain Ω, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[r(t)(\int_\Omega(D_{+,t}^\alpha u)(x,t)dx\right]+\int_\Omega p(t)D_{+,t}^\alpha u(x,t)dx\\&+\int_\Omega q(x,t)f(\int_0^t(t-v)^{-\alpha}u(x,v)dv)dx\\=&a(t)\int_\Omega\Delta u(x,t)dx.\end{aligned}\tag{3.3}$$

Using Green's formula, it is obvious that

$$\int_\Omega\Delta u(x,t)dx=\int_{\partial\Omega}\frac{\partial u(x,t)}{\partial N}ds=-\int_{\partial\Omega}g(x,t)u(x,t)ds\leq 0,\quad t\geq t_1.\tag{3.4}$$

where ds is surface element on $\partial\Omega$. By using Jensen's inequality and (A_2) , we can obtain

$$\begin{aligned}&\int_\Omega q(x,t)f\left(\int_0^t(t-\mu)^{-\alpha}u(x,\mu)d\mu\right)dx\\\geq&q(t)f\left[\int_\Omega\left(\int_0^t(t-\mu)^{-\alpha}u(x,\mu)d\mu\right)dx\right]\\\geq&q(t)f\left[\int_0^t(t-\mu)^{-\alpha}\left(\int_\Omega u(x,\mu)dx\right)d\mu\right]\\\geq&q(t)\int_\Omega dxf\left[\int_\Omega\left(\int_0^t(t-\mu)^{-\alpha}u(x,\mu)d\mu\right)dx(\int_\Omega dx)^{-1}\right]\\=&q(t)\int_\Omega dxf\left[\int_0^t(t-\mu)^{-\alpha}(\int_\Omega u(x,mu)dx)(\int_\Omega dx)^{-1}d\mu\right]\\=&q(t)\left|\Omega\right|f\left(G(t)\right).\end{aligned}\tag{3.5}$$

Combining (3.3)-(3.5) and using definitions, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[r(t)D_+^\alpha v(t)\right]+p(t))D_+^\alpha v(t)+q(t)f(G(t))\leq 0,\tag{3.6}$$

where

$$v(t) = \frac{\int_\Omega u(x,t)dx}{|\Omega|},$$

$$G(t) = \int_0^t (t-\xi)^{-\alpha} v(\xi) d\xi.$$

Define the function $w(t)$ by

$$w(t) = \varphi(t)\frac{r(t)D_{+,t}^{\alpha}v(t)}{G(t)}, \quad for \quad t \geq t_1. \tag{3.7}$$

Then we have $w(t) > 0$ for $t \geq t_1$. Differentiating (3.7) for $t \geq t_1$, we have

$$\begin{aligned}
&w'(t)\\
=&\frac{\varphi'(t)}{\varphi(t)}w(t) + \varphi(t)\frac{(r(t)D_+^\alpha v(t))'}{G(t)} - G'(t)\frac{\varphi(t)r(t)D_+^\alpha v(t)}{G^2(t)}\\
\leq&\frac{\varphi'(t)}{\varphi(t)}w(t) + \frac{\varphi(t)\left[-p(t)D_+^\alpha v(t) - q(t)f(G(t))\right]}{G(t)}\\
&-\Gamma(1-\alpha)D_+^\alpha v(t)\frac{\varphi(t)r(t)D_+^\alpha v(t)}{G^2(t)}\\
=&\frac{\varphi'(t)}{\varphi(t)}w(t) - \frac{p(t)}{r(t)}w(t) - \varphi(t)q(t)\frac{f(G(t))}{G(t)} - \frac{\Gamma(1-\alpha)}{\varphi(t)r(t)}w^2(t)\\
\leq& -\mu\varphi(t)q(t) + (\frac{\varphi'(t)}{\varphi(t)} - \frac{p(t)}{r(t)})w(t) - \frac{\Gamma(1-\alpha)}{\varphi(t)r(t)}w^2(t)\\
\leq& -\mu\varphi(t)q(t) + \xi(t)\eta(t)w(t) - \Gamma(1-\alpha)\eta(t)w^2(t).
\end{aligned} \tag{3.8}$$

Multiplying (3.8) by $H(s,t)$ and integrating with respect to s from t to c for $t \in (a, c]$, we have

$$\begin{aligned}
\int_t^c \mu H(s,t)\varphi(s)q(s)ds \leq& -\int_t^c H(s,t)w'(s)ds\\
&+\int_t^c H(s,t)\xi(s)\eta(s))w(s)ds\\
&-\int_t^c H(s,t)\Gamma(1-\alpha)\eta(s)w^2(s)ds.
\end{aligned} \tag{3.9}$$

In view of (i) and (ii), we see that

$$\int_t^c H(s,t)w'(s)ds = H(c,t)w(c) - \int_t^c h_1(s,t)\sqrt{H(s,t)}w(s)ds. \tag{3.10}$$

Using (3.10) in (3.9) leads to

$$\begin{aligned}
&\int_t^c \mu H(s,t)\varphi(s)q(s)ds\\
\leq& -H(c,t)w(c)\\
&-\int_t^c \left[\Gamma(1-\alpha)\eta(s)H(s,t)w^2(s) - \left(h_1(s,t)\sqrt{H(s,t)} + \xi(s)\eta(s)H(s,t)w(s)\right)\right] ds
\end{aligned}$$

$$
\begin{aligned}
=& - H(c,t)w(c) - \int_t^c \left(\sqrt{\Gamma(1-\alpha)\eta(s)H(s,t)}w(s) - \frac{1}{2\sqrt{\Gamma(1-\alpha)\eta(s)}}\Phi_1(s,t) \right)^2 ds \\
&+ \int_t^c \frac{1}{4\Gamma(1-\alpha)\eta(s))}\Phi_1^2(s,t)ds \\
\le& - H(c,t)w(c) + \int_t^c \frac{1}{4\Gamma(1-\alpha)\eta(s)}\Phi_1^2(s,t)ds. \qquad (3.11)
\end{aligned}
$$

Similarly, if (3.8) is multiplied by $H(t,s)$ and then integrated from c to t for $t \in [c,b)$, then we gets

$$
\begin{aligned}
&\int_c^t \mu H(s,t)\varphi(s)q(s)ds \\
\le& H(t,c)w(c) \\
&- \int_c^t \left[\Gamma(1-\alpha)\eta(s)H(s,t)w^2(s) + \left(h_2(t,s)\sqrt{H(s,t)} - \xi(s)\eta(s)H(s,t)w(s) \right) \right] ds \\
=& H(t,c)w(c) - \int_c^t \left(\sqrt{\Gamma(1-\alpha)\eta(s)H(s,t)}w(s) + \frac{1}{2\sqrt{\Gamma(1-\alpha)\eta(s)}}\Phi_2(s) \right)^2 ds \\
&+ \int_c^t \frac{1}{4\Gamma(1-\alpha)\eta(s)}\Phi_2^2(t,s)ds \\
\le& H(t,c)w(c) + \int_c^t \frac{1}{4\Gamma(1-\alpha)\eta(s)}\Phi_2^2(t,s)ds. \qquad (3.12)
\end{aligned}
$$

Letting $t \to a^+$ in (3.11) and $t \to b^-$ in (3.12) and adding the resulting inequalities we have

$$
\begin{aligned}
&\frac{1}{H(c,a)} \int_a^c \left[\mu H(s,a)\varphi(s)q(s) - \frac{1}{4\Gamma(1-\alpha)\eta(s)}\Phi_1^2(s,a) \right] ds \\
&+ \frac{1}{H(b,c)} \int_c^b \left[\mu H(b,s)\varphi(s)q(s) - \frac{1}{4\Gamma(1-\alpha)\eta(s)}\Phi_2^2(b,s) \right] ds \le 0 \qquad (3.13)
\end{aligned}
$$

which contradicts the assumption (3.1). The proof is complete.

□

Theorem 3.2. *Let conditions* (A_1)-(A_3) *hold, suppose that there exists a function* $\varphi \in C^1[[t_0,\infty),(0,\infty)]$, *and there exists* $H \in \wp$ *such that*

$$
\limsup_{t\to\infty} \int_l^t \left[\mu H(s,l)\varphi(s)q(s) - \frac{1}{4\Gamma(1-\alpha)\eta(s)}\Phi_1^2(s,l) \right] ds > 0, \qquad (3.14)
$$

and

$$
\limsup_{t\to\infty} \int_l^t \left[\mu H(t,s)\varphi(s)q(s) - \frac{1}{4\Gamma(1-\alpha)\eta(s)}\Phi_2^2(t,s) \right] ds > 0, \qquad (3.15)
$$

hold for every $l \in [t_0,\infty)$, $t_1 > t_0$, *where* Φ_1, Φ_2 *are the same in Theorem 3.1, then every solution of* (1.4) *is oscillatory.*

Proof. Suppose to the contrary that u is a non-oscillatory solution of (1.4). Without loss of generality, we can assume that there exists $u(x,t) > 0$ in $G \times [t_0,\infty)$ for some $t_2 \geq t_1$. Set $l = a \geq t_2$ in (3.14). Clearly, we see from (3.14) that there exists $c > a$ such that

$$\int_a^c \left[\mu H(s,a)\varphi(s)q(s) - \frac{1}{4\Gamma(1-\alpha)\eta(s)}\Phi_1^2(s,a)\right] ds > 0. \tag{3.16}$$

Similarly setting $l = c \geq t_2$ in (3.15), it follows that there exists $b > c$ such that

$$\int_c^b \left[\mu H(b,s)\varphi(s)q(s) - \frac{1}{4\Gamma(1-\alpha)\eta(s)}\Phi_2^2(b,s)\right] ds > 0. \tag{3.17}$$

From (3.16) and (3.17) we see that (3.11) is satisfied. Therefore, in view of Theorem 3.1, we may conclude that every solution of (1.4) is oscillatory.

□

If we choose $H(t,s) = (t-s)^{\lambda}, t \geq s \geq t_0$, where $\lambda > 1$ is a constant. Then, we obtain the following useful oscillation criterion.

Corollary 3.1. *Let conditions (A_1)-(A_3) hold, suppose that there exists a function $\varphi \in C^1\big[[t_0,\infty),(0,\infty)\big]$, such that the following two inequalities hold:*

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_l^t (s-l)^{\lambda} \left\{\mu\varphi(s)q(s) - \frac{1}{4\Gamma(1-\alpha)\eta(s)}\left(\frac{\lambda}{(s-l)} + \xi(s)\eta(s)\right)^2\right\} ds > 0 \tag{3.18}$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_l^t (t-s)^{\lambda} \left\{\mu\varphi(s)q(s) - \frac{1}{4\Gamma(1-\alpha)\eta(s)}\left(\frac{\lambda}{(t-s)} - \xi(s)\eta(s)\right)^2\right\} ds > 0 \tag{3.19}$$

for each $l \geq t_0, \lambda > 1$, then Eq.(1.4) is oscillatory.

More generally, one may consider $H(t,s) = [R(t) - R(s)]^{\lambda}$, where λ is constant and $R(t) = \int_{t_1}^t \frac{1}{r(s)}ds$ and $\lim_{t\to\infty} R(t) = \infty$. If we choose $\varphi(t) = 1$, by Theorem 3.2, we have the following oscillatory criterion.

Theorem 3.3. *Let conditions (A_1)-(A_3) hold, $p(t) \geq 0$ and $q(t) \geq 0$ for all $t \in [t_0,\infty)$ and $\lim_{t\to\infty} R(t) = \infty$. Then every solution of Eq.(1.3) is oscillatory provided for each $l \geq t_0$, and for some $\lambda > 1$, the following two inequalities hold:*

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \Bigg\{&\left(\mu q(s) - \frac{p^2(s)}{4\Gamma(1-\alpha)r(s)}\right)\Big[R(s) - R(l)\Big]^{\lambda} \\ &+ \frac{\lambda p(s)}{2\Gamma(1-\alpha)r(s)}\Big[R(s) - R(l)\Big]^{\lambda-1}\Bigg\} ds > \frac{\lambda^2}{4\Gamma(1-\alpha)(\lambda-1)}\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \Bigg\{&\left(\mu q(s) - \frac{p^2(s)}{4\Gamma(1-\alpha)r(s)}\right)\Big[R(t) - R(s)\Big]^{\lambda} \\ &+ \frac{\lambda p(s)}{2\Gamma(1-\alpha)r(s)}\Big[R(t) - R(s)\Big]^{\lambda-1}\Bigg\} ds > \frac{\lambda^2}{4\Gamma(1-\alpha)(\lambda-1)}.\end{aligned} \tag{3.21}$$

Proof. Since $H(t,s) = [R(t) - R(s)]^{\lambda}$, it is easy to see that

$$h_1(t,s) = \lambda \left[R(t) - R(s)\right]^{\frac{\lambda-2}{2}} \frac{1}{r(t)},$$

and

$$h_2(t,s) = \lambda \left[R(t) - R(s)\right]^{\frac{\lambda-2}{2}} \frac{1}{r(s)}.$$

Noting that

$$\int_l^t r(s) h_1^2(s,l) ds = \int_l^t r(s)\lambda^2 \left[R(s) - R(l)\right]^{\lambda-2} \frac{1}{r^2(s)} ds = \frac{\lambda^2}{\lambda - 1} \left[R(t) - R(l)\right]^{\lambda-1}$$

and

$$\int_l^t r(s) h_2^2(s,l) ds = \int_l^t r(s)\lambda^2 \left[R(t) - R(s)\right]^{\lambda-2} \frac{1}{r^2(s)} ds = \frac{\lambda^2}{\lambda - 1} \left[R(t) - R(l)\right]^{\lambda-1}.$$

In view of $\lim_{t\to\infty} R(t) = \infty$, it follows that

$$\lim_{t\to\infty} \frac{1}{4\Gamma(1-\alpha) R^{\lambda-1}(t)} \int_l^t r(s) h_1^2(s,l) ds = \frac{\lambda^2}{4\Gamma(1-\alpha)(\lambda-1)} \tag{3.22}$$

and

$$\lim_{t\to\infty} \frac{1}{4\Gamma(1-\alpha) R^{\lambda-1}(t)} \int_l^t r(s) h_2^2(s,l) ds = \frac{\lambda^2}{4\Gamma(1-\alpha)(\lambda-1)}. \tag{3.23}$$

From (3.20) and (3.22), we have that

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \left\{ H(s,l)\mu q(s) - \frac{r(s)}{4\Gamma(1-\alpha)} \left[h_1(s,l) - \frac{p(s)}{r(s)} \sqrt{H(s,l)} \right]^2 \right\} ds \\
=&\limsup_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \left\{ \left[R(s) - R(l) \right]^{\lambda} \mu q(s) + \frac{\lambda p(s)}{2\Gamma(1-\alpha) r(s)} \left[R(s) - R(l) \right]^{\lambda-1} \right. \\
&\left. - \frac{p^2(s)}{4\Gamma(1-\alpha) r(s)} \left[R(s) - R(l) \right]^{\lambda} \right\} ds - \mathop{limsup}_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \frac{r(s)}{4\Gamma(1-\alpha)} h_1^2(s,l) ds \\
=&\limsup_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \left\{ \left(\mu q(s) - \frac{p^2(s)}{4\Gamma(1-\alpha) r(s)} \right) \left[R(s) - R(l) \right]^{\lambda} \right. \\
&\left. + \frac{\lambda p(s)}{2\Gamma(1-\alpha) r(s)} \left[R(s) - R(l) \right]^{\lambda-1} \right\} ds - \frac{\lambda^2}{4\Gamma(1-\alpha)(\lambda-1)} > 0,
\end{aligned} \tag{3.24}$$

i.e., (3.14) holds. Similarly, (3.21) implies that (3.15) holds. By Theorem 3.2, every solution of (1.4) is oscillatory. The proof is complete. □

4. Example

Example 4.1. Consider the fractional partial differential equations

$$\begin{aligned}
&D_{+,t}^{\alpha} \left(D_{+,t}^{\alpha} u(x,t) \right) - \frac{1}{t} D_{+,t}^{\alpha} u(x,t) + \frac{e^x}{t^2} f \left(\int_0^t (t-\xi)^{-\alpha} u(x,v) dv \right) \\
=&\frac{e^t}{8} \Delta u(x,t), \quad (x,t) \in (0,\pi) \times (0,\infty)
\end{aligned} \tag{4.1}$$

with the Robin boundary condition

$$u_x(0,t) = u_x(\pi,t) = 0, \tag{4.2}$$

where $\alpha \in (0,1), p(t) = -\frac{1}{t}, q(t) = \underset{x\in\Omega}{min} q(x,t) = \underset{x\in(0,\pi)}{min} \frac{e^x}{t^2} = \frac{1}{t^2}, r(t) = 1, a(t) = \frac{e^t}{8}, f(u) = u$. Set $t_0 \geq 0$ and $\mu = 1$. Thus all the conditions of the theorem (3.1) hold. Therefore every solution of (4.1) is oscillatory.

Example 4.2. Consider the fractional partial differential equations

$$\begin{aligned} &D^{\frac{1}{2}}_{+,t}\left(D^{\frac{1}{2}}_{+,t}u(x,t)\right) - D^{\frac{1}{2}}_{+,t}u(x,t) + (x^2+\frac{1}{t})f\left(\int_0^t (t-\xi)^{-\frac{1}{2}}u(x,v)dv\right) \\ =&3e^{-t}\Delta u(x,t), \quad (x,t)\in(0,\pi)\times(0,\infty) \end{aligned} \tag{4.3}$$

with the Robin boundary condition

$$u_x(0,t) = u_x(\pi,t) = 0, \tag{4.4}$$

where $\alpha \in (0,1), p(t) = -1, q(x,t) = (x^2+\frac{1}{t}), q(t) = \underset{x\in\Omega}{min} q(x,t) = \underset{x\in(0,\pi)}{min}(x^2+\frac{1}{t}) = \frac{1}{t}, r(t) = 1, a(t) = 3e^{-t}, f(u) = u$. Set $t_0 \geq 0$ and $\mu = 1$. Thus all the conditions of the theorem (3.1) hold. Therefore every solution of (4.3) is oscillatory.

References

[1] D. Chen, *Oscillatory behavior of a class of fractional differential equations with damping*, U.P.B. Sci. Bull., Series A, 2013, 75(1),107–118.

[2] D. X. Chen, P. X. Qu and Y. H. Lan,*Forced oscillation of certain fractional differential equations*, Adv. Differ. Equ., 2013, 2013(1), 125.

[3] Q. Feng and F. Meng, *Oscillation of solutions to nonlinear forced fractional differential equations*, Electron. J. Differ. Eq., 2013, 2013(169), 1471–1484.

[4] S. R. Grace, R. P. Agarwal, P. J. Y. Wong, et al, *On the oscillation of fractional differential equations.* Fract. Calc. Appl. Anal., 2012, 15(2), 222–231.

[5] Z. Han, Y. Zhao, S. Ying, et al, *Oscillation Theorem for a Kind of Fractional Differential Equations*, Journal of Binzhou University, 2013, 2013(9–10), 216–219.

[6] S. Harikrishnan, P. Prakash and J. J. Nieto, *Forced oscillation of solutions of a nonlinear fractional partial differential equation*, Appl. Math. Comput., 2015(254), 14–19.

[7] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[8] F. Lu and F. Meng, *Oscillation theorems for superlinear second-order damped differential equations*, Appl. Math. Comput., 2007, 189(1), 796–804.

[9] W. N. Li, *Oscillation of solutions for certain fractional partial differential equations*, Advances in Difference Equations, (2016) 2016: 16.

[10] W. N. Li, *On the forced oscillation of certain fractional partial differential equations*, Appl. Math. Lett., 50(2015), 5–9.

[11] W. N. Li, *Forced oscillation criteria for a class of fractional partial differential equations with damping term*, Mathematical Problems in Engineering, vol. 2015, Article ID 410904, 2015.

[12] W. N. Li and W. Sheng, *Oscillation properties for solutions of a kind of partial fractional differential equations with damping term*, J. Nonlinear Sci. Appl., 9(2016), 1600–1608.

[13] F. Meng, *An oscillation theorem for second order superlinear differential equations*, Indian J. Pure Ap. Mat., 1996, 27(7).

[14] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[15] P. Prakash, S. Harikrishnan, J. J. Nieto, et al, *Oscillation of a time fractional partial differential equation*, Electron. J. Qua. Theo., 2014, 15(11), 1–10.

[16] P. Prakash, S. Harikrishnan and M. Benchohra, *Oscillation of certain nonlinear fractional partial differential equation with damping term*, Appl. Math. Lett., 2015(43), 72–79.

[17] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Elsevier, Amsterdam, 1993.

[18] E. Tunç and H. Avci, *Interval oscillation criteria for second order nonlinear differential equations with nonlinear damping// Seminar on nonlinear partial differential equations* , Springer-Verlag, 1984, 357–365.

[19] J. Wang, F. Meng and S. Liu, *Integral average method for oscillation of second order partial differential equations with delays*, Appl. Math. Comput., 2007, 187(2), 815–823.

[20] J. Wang, F. Meng and S. Liu, *Interval oscillation criteria for second order partial differential equations with delays*, J. Comput. Appl. Math., 2008, 212(2), 397–405.

[21] B. Zheng, *Oscillation for a class of nonlinear fractional differential equations with damping term*, J.Adv.Math.Stud., 2013(9–10), 216–219.