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QUALITATIVE ANALYSIS OF STOCHASTIC RATIO-DEPENDENT PREDATOR-PREY SYSTEMS *

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Abstract In this paper, two stochastic ratio-dependent predator-prey systems are considered. One is just with white noise, and the other one is taken into both white noise and color noise account. Sufficient criteria for extinction and persistence in time average are established. The critical value between persistence and extinction is obtained. Moreover, we show that there is stationary distribution for the stochastic system with regime-switching. Finally, examples and simulations are carried on to verify these results.

Keywords Stochastic ratio-dependent predator-prey system, persistence in time average, extinction, stationary distribution.

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1. Introduction

The dynamic relationship between predators and their preys has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [13]. The functional response is the important component depicting the predator-prey relationship [15]. Considering predators have to search for food (and therefore have to share or compete for food), Arditi etc [1,2,17] proposed a ratio-dependent function. And so the ratio-dependent predator-prey system takes the form of

$$\begin{cases} \dot{x}(t) = x(t)\left(a - bx(t) - \frac{cy(t)}{my(t)+x(t)}\right), \\ \dot{y}(t) = y(t)\left(-d + \frac{fx(t)}{my(t)+x(t)}\right). \end{cases} \tag{1.1}$$

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Here, $x(t)$ and $y(t)$ represent population densities of the prey and the predator at time t, respectively; parameters a, b, c, d, f, m are positive constants. The prey growth is of logistic type with growth rate a and carrying capacity rate a/b in the absence of predation. c, m, f, d stand for the prey capturing rate, half capturing saturation constant, conversion rate and the predator death rate, respectively.

The classical predator-prey theory relies on the functional response on prey density, while the ratio-dependent predator-prey theory is based on the assumption that a functional response depends on the ratio of prey to predator abundance, which solves the problem of the paradoxes of enrichment [1, 3, 16]. Lots of authors studied the dynamics of system (1.1). Hsu etc [18] and Kuang etc [24] showed that the ratio-dependent model (1.1) is capable of producing richer and more reasonable dynamics. Berezovskaya etc [4], Xiao etc [35], Tang etc [32] and Li etc [25] showed that there exist numerous kinds of topological structures in a neighborhood of the origin, which is a degenerate equilibrium.

As most of ecosystems are exposed within the open environment, the randomly fluctuating environmental forces are not ignored. Considering the continuous fluctuations in the environment (e.g. variation in intensity of sunlight, temperature, water level, etc.), parameters involved in models are not absolute constants, but they always fluctuate around some average value. Recently, a lot of authors introduced environmental noise into predator-prey models, such as [5–7, 22]. Especially, we [22] investigated the dynamics of the following stochastic ratio-dependent predator prey system

$$\begin{cases} dx(t) = x(t)\left(a - bx(t) - \dfrac{cy(t)}{my(t)+x(t)}\right)dt + \alpha x(t)dB_1(t), \\ dy(t) = y(t)\left(-d + \dfrac{fx(t)}{my(t)+x(t)}\right)dt + \beta y(t)dB_2(t), \end{cases} \tag{1.2}$$

where $B_1(t), B_2(t)$ are mutually independent Brownian motions, α, β represent intensities of white noises.

Due to seasonal effects of weather, food supply, mating habits, hunting or harvesting seasons, etc, Cushing [8] pointed out that it is necessary and important to consider models with periodic ecological parameters or perturbations which might be quite naturally exposed. Thus, the assumption of periodicity of parameters is a way of incorporating the periodicity of the environment. Lots of authors considered the behavior of non-autonomous predator-prey models (see [9, 10, 26, 34, 37] for example). Fan etc [14] assumed that all of parameters in system (1.1) are not constants but are T-periodic functions, and investigated its dynamics. They obtained sufficient conditions for existence, uniqueness and stability of a positive periodic solution. In detail,

- If $\hat{f} > \check{d}, \hat{m}\hat{a} > \check{c}$, then there is a periodic solution of the deterministic system, and it is persistent;
- If $\check{f} > \hat{d}$ or $\dfrac{\hat{c}}{\check{m}} > \check{a} + \check{d}$, then the deterministic system is not persistent;
- If $\dfrac{\hat{c}}{\check{m}} > \check{a} + \check{d}$, then $\lim\limits_{t\to\infty}(x(t), y(t)) = (0, 0)$.

Therefore, this paper incorporates the varying property of parameters and stochastic fluctuation of environment into the model, and considers the following non-autonomous stochastic ratio-dependent predator-prey system

$$\begin{cases} dx(t) = x(t)\left(a(t) - b(t)x(t) - \dfrac{c(t)y(t)}{m(t)y(t)+x(t)}\right)dt + \alpha(t)x(t)dB_1(t), \\ dy(t) = y(t)\left(-d(t) + \dfrac{f(t)x(t)}{m(t)y(t)+x(t)}\right)dt + \beta(t)y(t)dB_2(t), \end{cases} \tag{1.3}$$

where $B_1(t), B_2(t)$ are mutually independent Brownian motions, $\alpha(t), \beta(t)$ represent intensities of white noises, which are continuous functions in time t with T-periodicity, and other parameters are all T-periodic functions with the same meaning as in system (1.1). Recently, non-autonomous stochastic population models are considered [19, 21, 33, 36, 38]. As far as we know, the stochastic ratio-dependent predator-prey system with periodic coefficients is not considered. The purpose of the present paper is to investigate the dynamics of system (1.3).

Apart from white noise, variability of the environment regimes (such as temperature, rainfall, humidity, wind etc.) may have an important impact on the dynamics of population. The effects of environment regimes in memoryless conditions to population are called color noise and can be illustrated as a Markovian switching between two or more regimes of environment. Hence, the traditional stochastic epidemic models cannot describe this phenomena. There are lots of works have been done on the population dynamics with regime-switching [11,12,27–29]. In this paper, we also take the regime-switching into account, and we get the following system:

$$\begin{cases} dx(t) = x(t)\left(a(\xi(t)) - b(\xi(t))x(t) - \dfrac{c(\xi(t))y(t)}{m(\xi(t))y(t)+x(t)}\right)dt + \alpha(\xi(t))x(t)dB_1(t), \\ dy(t) = y(t)\left(-d(\xi(t)) + \dfrac{f(\xi(t))x(t)}{m(\xi(t))y(t)+x(t)}\right)dt + \beta(\xi(t))y(t)dB_2(t), \end{cases} \tag{1.4}$$

where $\xi(t)$ is a continuous time Markov chain with a finite state space $\mathcal{M} = \{1, 2, \cdots, N\}, 1 \leq N < \infty$. Lv etc [29] discussed the dynamics of this system. They showed that it is persistent in mean under some conditions, and it is extinction when white noise is stronger. But they did not obtain the threshold value for the persistence and extinction of the population. We fill this gap in this paper.

The main focus of this article is to discuss how white noise and color noise affect the population dynamics. Due to the degeneration of the zero equilibrium, there is some difficult to investigate the dynamics around the origin. The rest of this paper is organized as follows. In Sec. 2, we recall some results for stochastic dynamical systems and introduce some notations. Sec. 3 discusses the dynamics of system (1.3). The existence and uniqueness of a positive solution is shown, and conditions for persistence and extinction of system (1.3) are given. Sec. 4 discusses the dynamics of system (1.4). Apart from showing the persistence and extinction of the population, we mainly show that there is a stationary distribution of system (1.4). Finally, in Sec. 5, some examples and simulations are given to illustrate obtained results.

2. Preliminary

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e.

it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $B_1(t)$ and $B_2(t)$ denote the independent standard Brownian motions defined on this probability space, and $\xi(t)$ is independent of $B_i(t), i = 1, 2$.

Suppose the generator $\Gamma = (\gamma_{ij})_{N\times N}$ of the Markov chain is given by

$$P\{\xi(t+\Delta) = j|\xi(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), \text{if } \ i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta), \text{if } \ i = j, \end{cases}$$

where $\Delta > 0, \gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, while $\gamma_{ii} = -\sum_{i\neq j}\gamma_{ij}$. Assume further that Markov chain $\xi(t)$ is irreducible and has a unique stationary distribution $\pi = \{\pi_1, \pi_2, \cdots, \pi_N\}$ which can be determined by equation

$$\pi\Gamma = 0 \tag{2.1}$$

subject to

$$\sum_{l=1}^{N} \pi_l = 1, \quad \text{and} \quad \pi_l > 0, \forall\ l \in \mathcal{M}.$$

Let $(X(t), \xi(t))$ be the diffusion process described by the following equation:

$$dX(t) = b(X(t), \xi(t))dt + \sigma(X(t), \xi(t))dB(t), X(0) = x_0, \xi(0) = \xi, \tag{2.2}$$

where $b(\cdot,\cdot) : \mathbb{R}^n\times\mathcal{M} \to \mathbb{R}^n, \sigma(\cdot,\cdot) : \mathbb{R}^n\times\mathcal{M} \to \mathbb{R}^{n\times n}$, and $D(x, l) = \sigma(x, l)\sigma^\top(x, l) = (d_{ij}(x, l))$. For each $l \in \mathcal{M}$, let $V(\cdot, l)$ be any twice continuously differentiable function, the operator L can be defined by

$$LV(x,l) = \sum_{i=1}^{n} b_i(x,l)\frac{\partial V(x,l)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} d_{ij}(x,l)\frac{\partial^2 V(x,l)}{\partial x_i \partial x_j} + \sum_{j=1}^{N}\gamma_{lj}V(x,j).$$

Now, we recall some results on the stationary distribution for stochastic differential equations under regime switching. For more details, readers can refer to [31,39].

Lemma 2.1 ([31]). *If the following conditions are satisfied:*
(i) $\gamma_{ij} > 0$ *for any* $i \neq j$;
(ii) for each $l \in \mathcal{M}, D(x, l) = (d_{ij}(x, l))$ *is symmetric and satisfies*

$$\lambda|\zeta|^2 \leq \langle D(x,l)\zeta, \zeta\rangle \leq \lambda^{-1}|\zeta|^2 \quad \textit{for all} \ \ \zeta \in \mathbb{R}^n,$$

with some constant $\lambda \in (0, 1]$ *for all* $x \in \mathbb{R}^n$;
(iii) there exists a nonempty open set $\mathcal{D}$ *with compact closure, satisfying that, for each* $l \in \mathcal{M}$, *there is a nonnegative function* $V(\cdot, l) : \mathcal{D}^c \to \bar{\mathbb{R}}_+$ *such that* $V(\cdot, l)$ *is twice continuously differential and that for some* $\alpha > 0$,

$$LV(x,l) \leq -\alpha, (x,l) \in \mathcal{D}^c \times \mathcal{M},$$

then $(X(t), \xi(t))$ *of system* (2.2) *is positive recurrent and ergodic. That is to say, there exists a unique stationary distribution* $\mu(\cdot,\cdot)$ *such that for any Borel measurable function* $f(\cdot,\cdot) : \mathbb{R}^n \times \mathcal{M} \to \mathbb{R}$ *satisfying*

$$\sum_{l=1}^{N}\int_{\mathbb{R}^n} |f(x,l)|\mu(dx,l) < \infty,$$

we have

$$P\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t f(X(s),\xi(s))ds = \sum_{l=1}^{N}\int_{\mathbb{R}^n} f(x,l)\mu(dx,l)\right) = 1.$$

At last of this section, we introduce some notations used in this paper.

If $f(t)$ is a continuous T-periodic function defined on $[0,+\infty)$, then define

$$\check{f} = \sup_{0\le t<+\infty} f(t),\quad \hat{f} = \inf_{0\le t<+\infty} f(t),\quad \langle f\rangle = \frac{1}{T}\int_0^T f(s)ds.$$

While if $f(l)$ is a function defined on $\mathcal{M}$, then define

$$\check{f} = \max_{l\in\mathcal{M}} f(l),\quad \hat{f} = \min_{l\in\mathcal{M}} f(l).$$

Let $\mathbb{R}^2_+ = \{(x,y)\in\mathbb{R}^2 : x>0, y>0\}$. Define

$$\lambda_1 = \sum_{\kappa=1}^{N}\pi_\kappa\left(a(\kappa) - \frac{c(\kappa)}{m(\kappa)} - \frac{\alpha^2(\kappa)}{2}\right),\quad \lambda_2 = \sum_{\kappa=1}^{N}\pi_\kappa\left(f(\kappa) - d(\kappa) - \frac{\beta^2(\kappa)}{2}\right).$$

3. The Dynamics of System (1.3)

To investigate the long time behavior of system (1.3), the existence of a positive solution should be discussed first. From results in [22], it is easy to get the following conclusions.

Theorem 3.1. *There is a unique positive local solution* $(x(t),y(t))$ *for* $t\in[0,\tau_e)$ *a.s. of system* (1.3) *for any initial value* $(x(0),y(0))\in\mathbb{R}^2_+$.

Let $\Phi(t),\phi(t),\Psi(t),\psi(t)$ be solutions of the following equations

$$\begin{aligned}
d\Phi(t) &= \Phi(t)\left(a(t) - b(t)\Phi(t)\right)dt + \alpha(t)\Phi(t)dB_1(t),\\
d\phi(t) &= \phi(t)\left(a(t) - \frac{c(t)}{m(t)} - b(t)\phi(t)\right)dt + \alpha(t)\phi(t)dB_1(t),\\
d\Psi(t) &= \Psi(t)\left(-d(t) + \frac{f(t)\Phi(t)}{m(t)\Psi(t)}\right)dt + \beta(t)\Psi(t)dB_2(t),\\
d\psi(t) &= \psi(t)\left(-d(t) + f(t) - \frac{f(t)m(t)}{\phi(t)}\psi(t)\right)dt + \beta(t)\psi(t)dB_2(t),
\end{aligned}$$

with initial values $\Phi(0)=\phi(0)=x(0)$ and $\Psi(0)=\psi(0)=y(0)$, respectively, where coefficients are all continuous T-periodic functions. For $t\in[0,\tau_e)$, using the comparison theorem of stochastic differential equations, gives

$$\begin{aligned}
\phi(t) &\le x(t) \le \Phi(t),\\
\psi(t) &\le y(t) \le \Psi(t) \quad \text{a.s.}
\end{aligned}$$

Besides, expressions of $\Phi(t),\phi(t),\Psi(t),\psi(t)$ are [23, 30]

$$\Phi(t) = \frac{e^{\int_0^t\left(a(s)-\frac{\alpha^2(s)}{2}\right)ds+\alpha(s)dB_1(s)}}{\frac{1}{x(0)} + \int_0^t b(s)e^{\int_0^s\left(a(r)-\frac{\alpha^2(r)}{2}\right)dr+\alpha(r)dB_1(r)dr}ds},$$

$$\phi(t) = \frac{e^{\int_0^t \left(a(s) - \frac{c(s)}{m(s)} - \frac{\alpha^2(s)}{2}\right) ds + \alpha(s) dB_1(s)}}{\frac{1}{x(0)} + \int_0^t b(s) e^{\int_0^s \left(a(r) - \frac{c(r)}{m(r)} - \frac{\alpha^2(r)}{2}\right) dr + \alpha(r) dB_1(r) dr} ds},$$

$$\Psi(t) = e^{\int_0^t \left(-d(s) - \frac{\beta^2(s)}{2}\right) ds + \beta(s) dB_2(s)}$$

$$\left(y(0) + \int_0^t \frac{f(s)\Phi(s)}{m(s)} e^{\int_0^s \left(d(r) + \frac{\beta^2(r)}{2}\right) dr - \beta(r) dB_2(r)} ds\right),$$

$$\psi(t) = \frac{e^{\int_0^t \left(f(s) - d(s) - \frac{\beta^2(s)}{2}\right) ds + \beta(s) dB_2(s)}}{\frac{1}{y(0)} + \int_0^t \frac{f(s)m(s)}{\phi(s)} e^{\int_0^s \left(f(r) - d(r) - \frac{\beta^2(r)}{2}\right) dr + \beta(r) dB_2(r) dr} ds}.$$

It is clear that all these solutions are well defined for all $t \in [0, \tau_e)$ a.s. and arbitrarily large magnitude of τ_e, which in turn implies that $\tau_e = \infty$. Thus a positive solution is global existence.

Theorem 3.2. *There is a unique positive solution $(x(t), y(t))$ for $t \geq 0$ a.s. of system (1.3) for any initial value $(x(0), y(0)) \in \mathbb{R}_+^2$. Furthermore, for all $t \geq 0$,*

$$\phi(t) \leq x(t) \leq \Phi(t),$$
$$\psi(t) \leq y(t) \leq \Psi(t) \quad a.s.$$

It is shown that the solution $(x(t), y(t))$ is between $(\phi(t), \psi(t))$ and $(\Phi(t), \Psi(t))$, respectively. By Lemma A.1 of [22], one can conclude the following results.

Lemma 3.1. *If $\langle a - \frac{\alpha^2}{2}\rangle > 0$, then*

$$\lim_{t\to\infty} \frac{\log \Phi(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t b(s)\Phi(s)ds = \langle a - \frac{\alpha^2}{2}\rangle \quad a.s.$$

Lemma 3.2. *If $\langle a - \frac{c}{m} - \frac{\alpha^2}{2}\rangle > 0$, then*

$$\lim_{t\to\infty} \frac{\log \phi(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t b(s)\phi(s)ds = \langle a - \frac{c}{m} - \frac{\alpha^2}{2}\rangle \quad a.s.$$

By the similar reasoning as in section 3.1 of [22], one can get the following results.

Lemma 3.3. *If $\langle a - \frac{\alpha^2}{2}\rangle > 0$, then*

$$\limsup_{t\to\infty} \frac{\log \Psi(t)}{t} \leq 0 \quad a.s.$$

Lemma 3.4. *If $\langle a - \frac{c}{m} - \frac{\alpha^2}{2}\rangle > 0$ and $\langle f - d - \frac{\beta^2}{2}\rangle > 0$, then*

$$\liminf_{t\to\infty} \frac{\log \psi(t)}{t} \geq 0 \quad a.s.$$

As the ratio-dependent function depends on the ratio of prey to predator, it is necessary and interesting to consider the long time behavior of $\frac{x(t)}{y(t)}$ or $\frac{y(t)}{x(t)}$. For convenience, set $u(t) = \frac{x(t)}{y(t)}$, $v(t) = \frac{y(t)}{x(t)}$.

Theorem 3.3. *Let $(x(t), y(t))$ be a solution of system (1.3) with any initial value $(x(0), y(0)) \in \mathbb{R}_+^2$. If*

$$f(t) \geq \frac{c(t)}{m(t)} \quad or \quad \frac{\alpha^2(t)}{2} \geq a(t) + d(t) + \frac{\beta^2(t)}{2} - f(t) \quad for \quad all \quad t \geq 0$$

and

$$\langle a + d - \frac{c}{m} - \frac{\alpha^2}{2} + \frac{\beta^2}{2} \rangle < 0$$

then

$$\lim_{t\to\infty} \frac{x(t)}{y(t)} = 0, \quad \lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = 0 \quad a.s.$$

Proof. Applying Itô's formula to the second equation of system (1.3), yields

$$\begin{aligned} d\frac{1}{y(t)} &= -\frac{1}{y^2(t)}dy(t) + \frac{1}{y^3(t)}(dy(t))^2 \\ &= \frac{1}{y(t)}\left(d(t) + \beta^2(t) - \frac{f(t)x(t)}{m(t)y(t) + x(t)}\right)dt - \frac{\beta(t)}{y(t)}dB_2(t). \end{aligned}$$

Then

$$\begin{aligned} du(t) &= \frac{1}{y(t)}dx(t) + x(t)d\frac{1}{y(t)} + dx(t)d\frac{1}{y(t)} \\ &= \frac{x(t)}{y(t)}\left(a(t) + d(t) + \beta^2(t) - b(t)x(t) - \frac{f(t)x(t) + c(t)y(t)}{m(t)y(t) + x(t)}\right)dt \\ &\quad + \frac{x(t)}{y(t)}(\alpha(t)dB_1(t) - \beta(t)dB_2(t)) \\ &\leq u(t)\left(a(t) + d(t) + \beta^2(t) - \frac{f(t)u(t) + c(t)}{m(t) + u(t)}\right)dt \\ &\quad + u(t)(\alpha(t)dB_1(t) - \beta(t)dB_2(t)), \end{aligned}$$

and

$$\begin{aligned} d\log u(t) &\leq \left(a(t) + d(t) + \frac{\beta^2(t)}{2} - \frac{\alpha^2(t)}{2} - \frac{f(t)u(t) + c(t)}{m(t) + u(t)}\right)dt \\ &\quad + \alpha(t)dB_1(t) - \beta(t)dB_2(t) \\ &= \Bigg[\frac{\left(a(t) + d(t) + \frac{\beta^2(t)}{2} - \frac{\alpha^2(t)}{2} - f(t)\right)u(t)}{m(t) + u(t)} \\ &\quad \frac{\left(a(t) + d(t) + \frac{\beta^2(t)}{2} - \frac{\alpha^2(t)}{2}\right)m(t) - c(t)}{m(t) + u(t)}\Bigg]dt \\ &\quad + \alpha(t)dB_1(t) - \beta(t)dB_2(t). \end{aligned}$$

If $f(t) \geq \frac{c(t)}{m(t)}$ or $\frac{\alpha^2(t)}{2} \geq a(t) + d(t) + \frac{\beta^2(t)}{2} - f(t)$ for all $t \geq 0$, then

$$d\log u(t) \leq \left(a(t) + d(t) - \frac{c(t)}{m(t)} - \frac{\alpha^2(t)}{2} + \frac{\beta^2(t)}{2}\right)dt + \alpha(t)dB_1(t) - \beta(t)dB_2(t),$$

which implies that

$$\frac{\log u(t)-\log u(0)}{t} \le \frac{1}{t}\int_0^t \left(a(s)+d(s)-\frac{c(s)}{m(s)}-\frac{\alpha^2(s)}{2}+\frac{\beta^2(s)}{2}\right)ds + \frac{M_1(t)}{t}-\frac{M_2(t)}{t}, \tag{3.1}$$

where $M_1(t)=\int_0^t \alpha(s)dB_1(s), M_2(t)=\int_0^t \beta(s)dB_2(s)$. They are local martingales whose quadratic variations are $\langle M_1, M_1\rangle_t = \int_0^t \alpha^2(s)ds \le \check{\alpha}t$ and $\langle M_2, M_2\rangle_t = \int_0^t \beta^2(s)ds \le \check{\beta}t$, respectively. Then according to the strong law of large numbers for martingales (see e.g. [20]), one can derive that

$$\lim_{t\to\infty}\frac{M_i(t)}{t}=0 \ \text{ a.s.,} \quad i=1,2. \tag{3.2}$$

Taking the superior limit of (3.1) and applying (3.2), yields

$$\limsup_{t\to\infty}\frac{\log u(t)}{t} \le \langle a+d-\frac{c}{m}-\frac{\alpha^2}{2}+\frac{\beta^2}{2}\rangle < 0.$$

Therefore,

$$\lim_{t\to\infty} u(t)=0 \ \text{ a.s.}$$

which implies that for a arbitrary $0<\epsilon<1$, there exists a $\tau_1>0$ and a set $\tilde{\Omega}_1$ satisfying $P\{\tilde{\Omega}_1\}\ge 1-\epsilon$, for $t\ge\tau_1$ and $\omega\in\tilde{\Omega}_1$, $u(t)\le\epsilon$. In this situation, it is easy to have

$$\begin{aligned} dx(t) &= x(t)\left(a(t)-\frac{c(t)}{m(t)}-b(t)x(t)+\frac{c(t)u(t)}{m^2(t)+m(t)u(t)}\right)dt+\alpha(t)x(t)dB_1(t)\\ &\le x(t)\left(a(t)-\frac{c(t)}{m(t)}-b(t)x(t)+\frac{c(t)u(t)}{m^2(t)}\right)dt+\alpha(t)x(t)dB_1(t)\\ &\le x(t)\left(a(t)-\frac{c(t)}{m(t)}+\frac{c(t)}{m^2(t)}\epsilon\right)dt+\alpha(t)x(t)dB_1(t), \end{aligned}$$

and

$$\begin{aligned} dy(t) &= y(t)\left(-d(t)+\frac{f(t)u(t)}{m(t)+u(t)}\right)dt+\beta(t)y(t)dB_2(t)\\ &\le y(t)\left(-d(t)+\frac{f(t)}{m(t)}u(t)\right)dt+\beta(t)y(t)dB_2(t)\\ &\le y(t)\left(-d(t)+\frac{f(t)}{m(t)}\epsilon\right)dt+\beta(t)y(t)dB_2(t). \end{aligned}$$

Then

$$\begin{aligned} \frac{\log x(t)-\log x(\tau_1)}{t} &\le \frac{1}{t}\int_{\tau_1}^t\left(a(s)-\frac{c(s)}{m(s)}-\frac{\alpha^2(s)}{2}+\frac{c(s)}{m^2(s)}\epsilon\right)ds+\frac{M_1(t)-M_1(\tau_1)}{t},\\ \frac{\log y(t)-\log y(\tau_1)}{t} &\le \frac{1}{t}\int_{\tau_1}^t\left(-d(s)-\frac{\beta^2(s)}{2}+\frac{f(s)}{m(s)}\epsilon\right)ds+\frac{M_2(t)-M_2(\tau_1)}{t}. \end{aligned}$$

Taking (3.2) together implies

$$\limsup_{t\to\infty} \frac{\log x(t)}{t} \leq \frac{1}{T}\int_0^T \left(a(s) - \frac{c(s)}{m(s)} - \frac{\alpha^2(s)}{2} + \frac{c(s)}{m^2(s)}\epsilon\right) ds,$$
$$\limsup_{t\to\infty} \frac{\log y(t)}{t} \leq \frac{1}{T}\int_0^T \left(-d(s) - \frac{\beta^2(s)}{2} - \frac{\alpha^2(s)}{2} + \frac{f(s)}{m(s)}\epsilon\right) ds.$$

Then letting $\epsilon \to 0$, gives

$$\limsup_{t\to\infty} \frac{\log x(t)}{t} \leq \frac{1}{T}\int_0^T \left(a(s) - \frac{c(s)}{m(s)} - \frac{\alpha^2(s)}{2}\right) ds = \langle a - \frac{c}{m} - \frac{\alpha^2}{2}\rangle < 0 \quad \text{a.s.}$$
$$\limsup_{t\to\infty} \frac{\log y(t)}{t} \leq \frac{1}{T}\int_0^T \left(-d(s) - \frac{\beta^2(s)}{2}\right) ds = -\langle d + \frac{\beta^2}{2}\rangle < 0 \quad \text{a.s.}$$

Therefore

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = 0 \quad \text{a.s.}$$

□

Remark 3.1. Theorem 3.3 shows two situations which will make both the prey and the predator dying out. In the first case, $f(t) \geq \frac{c(t)}{m(t)}$ is needed. In fact, $f(t) = l(t)c(t)$, where $l(t)$ represents the efficiency at which consumed prey is converted into predator births. So $f(t) \geq \frac{c(t)}{m(t)}$ equals to $l(t) \geq \frac{1}{m(t)}$, that is to say that the conversion efficient is not less than $\frac{1}{m(t)}$ at time t. While, in the other one, white noise $\dot{B}_1(t)$ is so large that $\frac{\alpha^2(t)}{2} \geq a(t) + d(t) + \frac{\beta^2}{2} - f(t)$. From simulations, Examples 5.2, 5.3 also illustrate that whether the prey population is persistent or both the prey and the predator are persistent of the corresponding deterministic system, the large white noise can always cause the population to die out.

Besides, in general, when the prey dies out, the predator will always tend to zero, because there is no food, which is verified by the dynamics of many prey-predator systems. But for the ratio-dependent prey-predator system, it is difficulty to prove this phenomena. In the proof of Theorem 3.3, the extinction of the prey $y(t)$ is obtained by showing that $\frac{x(t)}{y(t)}$ will tend to zero.

Theorem 3.4. *Let* $(x(t), y(t))$ *be a solution of system* (1.3) *with any initial value* $(x(0), y(0)) \in \mathbb{R}^2_+$. *If*

$$f(t) \geq \frac{c(t)}{m(t)} \quad \text{or} \quad \frac{\beta^2(t)}{2} \geq \frac{c(t)}{m(t)} - d(t) \quad \text{for} \quad \text{all} \quad t \geq 0$$

and

$$\langle a - \frac{\alpha^2}{2}\rangle > 0, \quad \langle f - d - \frac{\beta^2}{2}\rangle < 0$$

then

$$\lim_{t\to\infty} v(t) = 0, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t b(s)x(s)ds = \langle a - \frac{\alpha^2}{2}\rangle, \quad \lim_{t\to\infty} y(t) = 0 \quad a.s.$$

Proof. Also applying Itô's formula, one can get

$$\begin{aligned} dv(t) = d\frac{1}{u(t)} &= -v^2(t)du(t) + v^3(t)(du(t))^2 \\ &= v(t)\left(-a(t) - d(t) + \alpha^2(t) + b(t)x(t) + \frac{c(t)v(t) + f(t)}{m(t)v(t) + 1}\right)dt \\ &\quad - v(t)(\alpha(t)dB_1(t) - \beta(t)dB_2(t)) \end{aligned}$$

and

$$\begin{aligned} d\log v(t) &= \left(-a(t) - d(t) + \frac{\alpha^2(t)}{2} - \frac{\beta^2(t)}{2} + b(t)x(t) + \frac{c(t)v(t) + f(t)}{m(t)v(t) + 1}\right)dt \\ &\quad - \alpha(t)dB_1(t) + \beta(t)dB_2(t) \\ &= \Bigg(-a(t) + \frac{\alpha^2(t)}{2} + b(t)x(t) \\ &\quad + \frac{(c(t) - d(t)m(t) - \frac{\beta^2(t)}{2}m(t))v(t) + f(t) - d(t) - \frac{\beta^2(t)}{2}}{m(t)v(t) + 1}\Bigg)dt \\ &\quad - \alpha(t)dB_1(t) + \beta(t)dB_2(t) \\ &\le \left(-a(t) + \frac{\alpha^2(t)}{2} + f(t) - d(t) - \frac{\beta^2(t)}{2} + b(t)x(t)\right)dt \\ &\quad - \alpha(t)dB_1(t) + \beta(t)dB_2(t), \end{aligned}$$

where the last inequality is base on the condition $f(t) \ge \frac{c(t)}{m(t)}$ or $\frac{\beta^2(t)}{2} \ge \frac{c(t)}{m(t)} - d(t)$ for all $t \ge 0$. Then

$$\begin{aligned} \frac{\log v(t) - \log v(0)}{t} &\le \frac{1}{t}\int_0^t \left(-a(s) + \frac{\alpha^2(s)}{2} + f(s) - d(s) - \frac{\beta^2(s)}{2}\right)ds \\ &\quad + \frac{1}{t}\int_0^t b(x)x(s)ds - \frac{M_1(t)}{t} + \frac{M_2(t)}{t}, \end{aligned} \tag{3.3}$$

where $M_1(t)$ and $M_2(t)$ are the same as in the proof of Theorem 3.3. On the other hand, using results of Theorem 3.2 and Lemma 3.1 together, yields

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t b(s)x(s)ds \le \frac{1}{T}\int_0^T \left(a(s) - \frac{\alpha^2(s)}{2}\right)ds = \langle a - \frac{\alpha^2}{2}\rangle \quad \text{a.s.} \tag{3.4}$$

if $\langle a - \frac{\alpha^2}{2}\rangle > 0$. Taking the superior limit in (3.3) and using (3.2), (3.4), one can conclude that

$$\limsup_{t\to\infty} \frac{\log v(t)}{t} \le \frac{1}{T}\int_0^T \left(f(s) - d(s) - \frac{\beta^2(s)}{2}\right)ds = \langle f - d - \frac{\beta^2}{2}\rangle < 0 \quad \text{a.s.}$$

That is to say, for any $0 < \epsilon < 1$, there is a $\tau_2 > 0$ and a set $\tilde{\Omega}_2$ such that $P\{\tilde{\Omega}\} \ge 1 - \epsilon$, for $t \ge \tau_2, \omega \in \tilde{\Omega}$, $v(t) \le \epsilon$ a.s. Now referring to the first equation in system (1.3) again, it is true that

$$dx(t) = x(t)\left(a(t) - b(t)x(t) - \frac{c(t)v(t)}{m(t)v(t) + 1}\right)dt + \alpha(t)x(t)dB_1(t)$$

$$\geq x(t)\left(a(t)-\check{c}\epsilon-b(t)x(t)\right)dt+\alpha(t)x(t)dB_1(t)$$

for $t \geq \tau_2$ and $\omega \in \tilde{\Omega}_2$. Together with Lemma 3.1, if $\langle a-\check{c}\epsilon-\frac{\alpha^2}{2}\rangle > 0$, then

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t b(s)x(s) \geq \frac{1}{T}\int_0^T\left(a(s)-\frac{\alpha^2(s)}{2}\right)ds-\check{c}\epsilon \quad \text{a.s.}$$

Letting $\epsilon \to 0$, yields

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t b(s)x(s) \geq \frac{1}{T}\int_0^T\left(a(s)-\frac{\alpha^2(s)}{2}\right)ds \quad \text{a.s.}$$

which together with (3.4) implies

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t b(s)x(s) = \frac{1}{T}\int_0^T\left(a(s)-\frac{\alpha^2(s)}{2}\right)ds = \langle a-\frac{\alpha^2}{2}\rangle \quad \text{a.s.}$$

In addition, note that

$$\begin{aligned}\frac{\log y(t)-\log y(0)}{t} &= \frac{1}{t}\int_0^t\left(f(s)-d(s)-\frac{\beta^2(s)}{2}\right)ds-\frac{1}{t}\int_0^t\frac{f(s)m(s)y(s)}{m(s)y(s)+x(s)}ds\\ &\quad+\frac{1}{t}\int_0^t\beta(s)dB_2(s)\\ &\leq \frac{1}{t}\int_0^t\left(f(s)-d(s)-\frac{\beta^2(s)}{2}\right)ds+\frac{1}{t}\int_0^t\beta(s)dB_2(s),\end{aligned}$$

then

$$\lim_{t\to\infty}\frac{\log y(t)}{t} \leq \frac{1}{T}\int_0^T\left(f(s)-d(s)-\frac{\beta^2(s)}{2}\right)ds = \langle f-d-\frac{\beta^2}{2}\rangle < 0 \quad \text{a.s.} \tag{3.5}$$

where (3.2) is used. Hence $\lim_{t\to\infty} y(t) = 0$ a.s. □

Remark 3.2. Theorem 3.4 also gives two cases in which the phenomena will happen. One is the stochastic system has the similar dynamics as the deterministic system. The other is the large white noise $\dot{B}_2(t)$ makes the system non-persistent, even if the deterministic system is persistent. Also see Examples 5.4, 5.5.

Theorem 3.5. *Let $(x(t), y(t))$ be a solution of system (1.3) with any initial value $(x(0), y(0)) \in \mathbb{R}^2_+$. If*

$$\langle a-\frac{c}{m}-\frac{\alpha^2}{2}\rangle > 0, \quad \langle f-d-\frac{\beta^2}{2}\rangle > 0$$

then

$$\lim_{t\to\infty}\frac{\log x(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{\log y(t)}{t} = 0 \quad a.s.$$

and

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\int_0^t\left(b(s)x(s)+\frac{c(s)y(s)}{m(s)y(s)+x(s)}\right)ds = \langle a-\frac{\alpha^2}{2}\rangle \quad a.s.\\ &\lim_{t\to\infty}\frac{1}{t}\int_0^t\frac{f(s)y(s)}{m(s)y(s)+x(s)}ds = \langle d+\frac{\beta^2}{2}\rangle \quad a.s.\end{aligned}$$

Proof. When $\langle a - \frac{c}{m} - \frac{\alpha^2}{2} \rangle > 0$, $\langle f - d - \frac{\beta^2}{2} \rangle > 0$, then from Lemma 3.1-Lemma 3.4, it is easy to get

$$\lim_{t\to\infty} \frac{\log x(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{\log y(t)}{t} = 0 \quad a.s. \tag{3.6}$$

Applying Itô's formula to the first equation of system (1.3), yields

$$d\log x(t) = \left(a(t) - \frac{\alpha^2(t)}{2} - b(t)x(t) - \frac{c(t)y(t)}{m(t)y(t)+x(t)}\right)dt + \alpha(t)dB_1(t),$$
$$d\log y(t) = \left(-d(t) - \frac{\beta^2(t)}{2} + \frac{f(t)x(t)}{m(t)y(t)+x(t)}\right)dt + \beta(t)dB_2(t),$$

which together with (3.6) tells us that the result is true. □

Remark 3.3. The asymptotic behavior of system (1.3) shows that $\langle a - \frac{c}{m} - \frac{\alpha^2}{2} \rangle$, $\langle f - d - \frac{\beta^2}{2} \rangle$ is the critical value between persistence and extinction of the population in system (1.3).

4. The Dynamics of System (1.4)

In this section, we discuss the dynamics of system (1.4). Lv etc [29] pointed out there is a unique positive solution of system (1.4). As the arguments in the previous section, we can obtain the results about the persistence and extinction of the population.

Theorem 4.1. *Let* $(x(t), y(t), \xi(t))$ *be a solution of system* (1.4) *with any initial value* $(x(0), y(0), \xi(0)) \in \mathbb{R}^2_+ \times \mathcal{M}$. *If*

$$f(l) \geq \frac{c(l)}{m(l)} \quad or \quad \frac{\alpha^2(l)}{2} \geq a(l) + d(l) + \frac{\beta^2(l)}{2} - f(l) \quad for \;\; each \;\; l \in \mathcal{M}$$

and

$$\sum_{\kappa=1}^{N} \pi_\kappa \left(a(\kappa) + d(\kappa) - \frac{c(\kappa)}{m(\kappa)} - \frac{\alpha^2(\kappa)}{2} + \frac{\beta^2(\kappa)}{2}\right) < 0,$$

then

$$\lim_{t\to\infty} \frac{x(t)}{y(t)} = 0, \quad \lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = 0 \quad a.s.$$

Theorem 4.2. *Let* $(x(t), y(t), \xi(t))$ *be a solution of system* (1.4) *with any initial value* $(x(0), y(0), \xi(0)) \in \mathbb{R}^2_+ \times \mathcal{M}$. *If*

$$f(l) \geq \frac{c(l)}{m(l)} \quad or \quad \frac{\beta^2(l)}{2} \geq \frac{c(l)}{m(l)} - d(l) \quad for \;\; each \;\; l \in \mathcal{M}$$

and

$$\sum_{\kappa=1}^{N} \pi_\kappa \left(a(\kappa) - \frac{\alpha^2(\kappa)}{2}\right) > 0, \quad \sum_{\kappa=1}^{N} \pi_\kappa \left(f(\kappa) - d(\kappa) - \frac{\beta^2(\kappa)}{2}\right) < 0$$

then

$$\lim_{t\to\infty} v(t) = 0, \lim_{t\to\infty} \frac{1}{t}\int_0^t b(\xi(s))x(s)ds = \sum_{\kappa=1}^{N} \pi_\kappa \left(a(\kappa) - \frac{\alpha^2(\kappa)}{2}\right), \lim_{t\to\infty} y(t) = 0 \ a.s.$$

Theorem 4.3. *Let $(x(t), y(t), \xi(t))$ be a solution of system (1.4) with any initial value $(x(0), y(0), \xi(0)) \in \mathbb{R}_+^2 \times \mathcal{M}$. If*

$$\sum_{\kappa=1}^{N} \pi_\kappa \left(a(\kappa) - \frac{c(\kappa)}{m(\kappa)} - \frac{\alpha^2(\kappa)}{2}\right) > 0, \quad \sum_{\kappa=1}^{N} \pi_\kappa \left(f(\kappa) - d(\kappa) - \frac{\beta^2(\kappa)}{2}\right) > 0$$

then

$$\lim_{t\to\infty} \frac{\log x(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{\log y(t)}{t} = 0 \quad a.s.$$

and

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \left(b(\xi(s))x(s) + \frac{c(\xi(s))y(s)}{m(\xi(s))y(s) + x(s)}\right) ds = \sum_{\kappa=1}^{N} \pi_\kappa \left(a(\kappa) - \frac{\alpha^2(\kappa)}{2}\right) \quad a.s.$$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{f(\xi(s))y(s)}{m(\xi(s))y(s) + x(s)} ds = \sum_{\kappa=1}^{N} \pi_\kappa \left(d(\kappa) + \frac{\beta^2(\kappa)}{2}\right) \quad a.s.$$

In the remain of this section, we mainly investigate the existence of a stationary distribution of system (1.4).

Theorem 4.4. *Assume that $\gamma_{ij} > 0$ for any $i \neq j$, and $\lambda_1 > 0, \lambda_2 > 0$. Then for any initial value $(x(0), y(0), \xi(0)) \in \mathbb{R}_+^2 \times \mathcal{M}$, the solution $(x(t), y(t), \xi(t))$ of system (1.4) admits a unique ergodic stationary distribution.*

Proof. It is easy to see if all conditions in Lemma 2.1 are satisfied, then system system (1.4) is positive recurrent. Obviously, condition (i) in Lemma 2.1 is true. By using the same method as those in [31], we obtain that condition (ii) holds. Now we mainly verify condition (iii).

Let $(x(t), y(t), \xi(t))$ be a solution of system (1.4) with the initial value $(x(0), y(0), \xi(0)) \in \mathbb{R}_+^2 \times \mathcal{M}$. By Itô's formula, yields

$$d\frac{1}{x(t)} = -\frac{1}{x(t)}\left(a(l) - \alpha^2(l) - b(l)x(t) - \frac{c(l)y(t)}{m(l)y(t) + x(t)}\right) + \frac{\alpha(l)}{x(t)} dB_1(t),$$

and

$$\begin{aligned} dx^{-\theta}(t) &= \theta\left(\frac{1}{x(t)}\right)^{\theta-1} d\frac{1}{x(t)} + \frac{\theta(\theta-1)}{2}\left(\frac{1}{x(t)}\right)^{\theta-2}\left(d\frac{1}{x(t)}\right)^2 \\ &= -\theta x^{-\theta}(t)\left(a(l) - \frac{\theta+1}{2}\alpha^2(l) - b(l)x(t) - \frac{c(l)y(t)}{m(l)y(t)+x(t)}\right) dt \\ &\quad + \theta\alpha(l)x^{-\theta}(t)dB_1(t) \\ &\leq -\theta x^{-\theta}(t)\left(a(l) - \frac{\theta+1}{2}\alpha^2(l) - b(l)x(t) - \frac{c(l)}{m(l)}\right) dt + \theta\alpha(l)x^{-\theta}(t)dB_1(t) \end{aligned}$$

$$:= -\theta x^{-\theta}(t)\left(Q_1(l) - \frac{\theta}{2}\alpha^2(l) - b(l)x(t)\right)dt + \theta\alpha(l)x^{-\theta}(t)dB_1(t).$$

Define $V_1(x,l) = \frac{1}{\theta}e^{\theta\varsigma_1(l)}x^{-\theta}$, where $0 < \theta < 1$ determined later, and $\varsigma_1(l) = (\varsigma_1(1), \varsigma_1(2), \cdots, \varsigma_1(N))^\top$ satisfying the following Poisson system

$$\Gamma\varsigma_1 = -\sum_{\kappa=1}^{N}\pi_\kappa Q_1(\kappa) + Q_1(l).$$

Then

$$\begin{aligned} LV_1 &\le -e^{\theta\varsigma_1(l)}x^{-\theta}(t)\left(\sum_{\kappa=1}^{N}\pi_\kappa Q_1(\kappa) - \frac{\theta}{2}\alpha^2(l) - b(l)x(t)\right) \\ &\le -e^{\theta\varsigma_1(l)}x^{-\theta}(t)\left(\lambda_1 - \frac{\theta}{2}\check{\alpha}^2 - \check{b}x(t)\right) \\ &\le e^{\theta\varsigma_1(l)}x^{-\theta}(t)\left(\check{b}x(t) - \frac{\lambda_1}{2}\right) \end{aligned}$$

provided $0 < \theta < \min\{1, \frac{\lambda_1}{\check{\alpha}^2}\}$. Besides,

$$\begin{aligned} d(\check{f}x(t) + \hat{c}y(t)) = &\left(\check{f}x(t)(a(l) - b(l)x(t)) - \hat{c}d(l)y(t) + \frac{\hat{c}f(l) - \check{f}c(l)}{m(l)y(t) + x(t)}x(t)y(t)\right)dt \\ &+ \check{f}\alpha(l)x(t)dB_1(t) + \hat{c}\beta(l)y(t)dB_2(t), \end{aligned}$$

and so

$$\begin{aligned} L(\check{f}x(t) + \hat{c}y(t)) &\le \check{f}x(t)(a(l) - b(l)x(t)) - \hat{c}d(l)y(t) \\ &\le -\frac{\check{f}b(l)}{2}x^2(t) + \frac{\check{f}a^2(l)}{2b(l)} - \hat{c}d(l)y(t) \\ &\le -\frac{\check{f}\hat{b}}{2}x^2(t) + \frac{\check{f}\check{a}^2}{2\hat{b}} - \hat{c}\hat{d}y(t). \end{aligned}$$

Besides, note that

$$\begin{aligned} d(-\log y(t)) &= \left(d(l) + \frac{\beta^2(l)}{2} - f(l) + \frac{f(l)m(l)y(t)}{m(l)y(t) + x(t)}\right)dt - \beta(l)dB_2(t) \\ &\le \left(d(l) + \frac{\beta^2(l)}{2} - f(l) + \frac{\check{f}\check{m}y(t)}{\hat{m}y(t) + x(t)}\right)dt - \beta(l)dB_2(t) \\ &:= \left(-Q_2(l) + \frac{\check{f}\check{m}y(t)}{\hat{m}y(t) + x(t)}\right)dt - \beta(l)dB_2(t). \end{aligned}$$

Let $\varsigma_2(l) = (\varsigma_2(1), \varsigma_2(2), \cdots, \varsigma_2(N))^\top$ satisfying the following Poisson system

$$\Gamma\varsigma_2 = -\sum_{\kappa=1}^{N}\pi_\kappa Q_2(\kappa) + Q_2(l).$$

Thus

$$L(-\log y + \varsigma_2) \le -\sum_{\kappa=1}^{N}\pi_\kappa Q_2(\kappa) + \frac{\check{f}\check{m}y(t)}{\hat{m}y(t) + x(t)}$$

$$= -\lambda_2 + \frac{\check{f}\check{m}y(t)}{\hat{m}y(t) + x(t)}.$$

Define a C^2-function $\widetilde{V} : \mathbb{R}_+^2 \times \mathcal{M} \to \mathbb{R}$ by

$$\widetilde{V}(x, y, \xi(t)) = \frac{1}{\theta} e^{-\theta\varsigma_1(l)} x^{-\theta} + \check{f}x + \hat{c}y + M(-\log y + \varsigma_2(l)),$$

where $M > 0$ to be determined later. Note that for each $l \in \mathcal{M}$, the function $\tilde{V}(\cdot, l)$ is not only continuous, but also tends to $+\infty$ as (x, y) approaches the boundary of $\mathbb{R}_+^2$ and as $\|(x, y)\| \to \infty$, where $\|\cdot\|$ denotes the Euclidean norm of a point in $\mathbb{R}_+^2$. Thus it must be lower bounded and achieve this lower bound at a point (x_0, y_0, l) in the interior of $\mathbb{R}_+^2$. Therefore $V(x, y) = \tilde{V}(x, y) - \min_{l\in\mathcal{M}} \tilde{V}(x_0, y_0, l)$ is a nonnegative C^2 function: $\mathbb{R}_+^2 \to \bar{\mathbb{R}}_+$. Then

$$\begin{aligned} LV \le{}& e^{\theta\varsigma_1(l)} x^{-\theta}(t) \left(\check{b}x(t) - \frac{\lambda_1}{2} \right) - \frac{\check{f}\hat{b}}{2} x^2(t) + \frac{\check{f}\check{a}^2}{2\hat{b}} - \hat{c}\hat{d}y(t) \\ &+ M \left(-\lambda_2 + \frac{\check{f}\check{m}y(t)}{\hat{m}y(t) + x(t)} \right). \end{aligned}$$

Now choose $M > 0$ so large that $\sup_{x\in(0,+\infty), l\in\mathcal{M}} \check{b}e^{\theta\varsigma_1(l)} x^{1-\theta} - \frac{\check{f}\hat{b}}{4} x^2 + \frac{\check{f}\check{a}^2}{2\hat{b}} - M\lambda_2 \le -2$. For arbitrary $0 < \epsilon_1, \epsilon_2 < 1$, define a bounded open domain $D_\epsilon \subset \mathbb{R}_+^2$ as follows:

$$\mathcal{D} = \{(x, y) \in R_+^2 : \epsilon_1 \le x \le 1/\epsilon_1, \epsilon_2 \le y \le 1/\epsilon_2\}.$$

The remainder of the proof only needs to verify that LV is negative in $\mathbb{R}_+^2 \setminus \mathcal{D}$, where $\mathbb{R}_+^2 \setminus \mathcal{D}$ is the following sets:

$$\mathcal{D}_1^c = \{(x, y) \in \mathbb{R}_+^2 : 0 < x < \epsilon_1\}, \quad \mathcal{D}_2^c = \{(x, y) \in \mathbb{R}_+^2 : \epsilon_1 \le x \le 1/\epsilon_1, 0 < y < \epsilon_2\},$$
$$\mathcal{D}_3^c = \{(x, y) \in \mathbb{R}_+^2 : x > 1/\epsilon_1\}, \qquad \mathcal{D}_4^c = \{(x, y) \in \mathbb{R}_+^2 : \epsilon_1 \le x \le 1/\epsilon_1, y > 1/\epsilon_2\}.$$

Case (i): For $(x, y, l) \in \mathcal{D}_1^c \times \mathcal{M}$,

$$\begin{aligned} LV \le{}& e^{\theta\varsigma_1(l)} x^{-\theta}(t) \left(\check{b}x(t) - \frac{\lambda_1}{2} \right) - \frac{\check{f}\hat{b}}{2} x^2(t) + \frac{\check{f}\check{a}^2}{2\hat{b}} - \hat{c}\hat{d}y(t) \\ &+ M \left(-\lambda_2 + \frac{\check{f}\check{m}y(t)}{\hat{m}y(t) + x(t)} \right) \\ \le{}& -2 - \frac{\lambda_1 e^{\theta \min_{l\in\mathcal{M}} \varsigma_1(l)}}{2\epsilon_1^\theta} + M\frac{\check{f}\check{m}}{\hat{m}}. \end{aligned}$$

Choose

$$0 < \epsilon_1 \le \left(\frac{\lambda_1 \hat{m} e^{\theta \min_{l\in\mathcal{M}} \varsigma_1(l)}}{2M\check{f}\check{m}} \right)^{1/\theta} \le \left(\frac{\lambda_1 e^{\theta \min_{l\in\mathcal{M}} \varsigma_1(l)}}{2M\check{f}} \right)^{1/\theta},$$

such that

$$\frac{\lambda_1 e^{\theta \min_{l\in\mathcal{M}} \varsigma_1(l)}}{2\epsilon_1^\theta} \ge M\frac{\check{f}\check{m}}{\hat{m}}.$$

Then

$$LV \le -1.$$

Case (ii): For $(x, y, l) \in \mathcal{D}_2^c \times \mathcal{M}$,

$$\begin{aligned}
LV &\le e^{\theta\varsigma_1(l)} x^{-\theta}(t)\left(\check{b}x(t) - \frac{\lambda_1}{2}\right) - \frac{\check{f}\hat{b}}{2}x^2(t) + \frac{\check{f}\check{a}^2}{2\hat{b}} - \hat{c}\hat{d}y(t) \\
&\quad + M\left(-\lambda_2 + \frac{\check{f}\check{m}y(t)}{\hat{m}y(t) + x(t)}\right) \\
&\le -2 + M\frac{\check{f}\check{m}\epsilon_2}{\hat{m}\epsilon_2 + \epsilon_1} \\
&\le -2 + M\check{f}\check{m}\epsilon_1
\end{aligned}$$

provided $\epsilon_2 = \epsilon_1^2$. It is clear that in order to make

$$LV \le -1,$$

it only needs to satisfy

$$0 < \epsilon_1 < \frac{1}{M\check{f}\check{m}}.$$

Case (iii): For $(x, y, l) \in \mathcal{D}_3^c \times \mathcal{M}$, when

$$\epsilon_1 \le \left(\frac{\hat{b}\hat{m}}{4M\check{m}}\right)^{1/2} \le \left(\frac{\hat{b}}{4M}\right)^{1/2},$$

it can conclude that

$$\begin{aligned}
LV &\le e^{\theta\varsigma_1(l)} x^{-\theta}(t)\left(\check{b}x(t) - \frac{\lambda_1}{2}\right) - \frac{\check{f}\hat{b}}{2}x^2(t) + \frac{\check{f}\check{a}^2}{2\hat{b}} - \hat{c}\hat{d}y(t) \\
&\quad + M\left(-\lambda_2 + \frac{\check{f}\check{m}y(t)}{\hat{m}y(t) + x(t)}\right) \\
&\le -2 - \frac{\check{f}\hat{b}}{4\epsilon_1^2} + M\frac{\check{f}\check{m}}{\hat{m}} \\
&\le -1.
\end{aligned}$$

Case (iv): For $(x, y, l) \in \mathcal{D}_4^c \times \mathcal{M}$,

$$\begin{aligned}
LV &\le e^{\theta\varsigma_1(l)} x^{-\theta}(t)\left(\check{b}x(t) - \frac{\lambda_1}{2}\right) - \frac{\check{f}\hat{b}}{2}x^2(t) + \frac{\check{f}\check{a}^2}{2\hat{b}} - \hat{c}\hat{d}y(t) \\
&\quad + M\left(-\lambda_2 + \frac{\check{f}\check{m}y(t)}{\hat{m}y(t) + x(t)}\right) \\
&\le -2 - \frac{\hat{c}\hat{d}}{\epsilon_2} + M\frac{\check{f}\check{m}}{\hat{m}}.
\end{aligned}$$

If $\epsilon_2 \le \frac{\hat{c}\hat{d}\hat{m}}{M\check{f}\check{m}} \le \frac{\hat{c}\hat{d}}{M\check{f}}$ such that $-\frac{\hat{c}\hat{d}}{\epsilon_2} + M\frac{\check{f}\check{m}}{\hat{m}} \le 0$, then

$$LV \le -1.$$

Hence, taking these four cases together, if $\epsilon_2 = \epsilon_1^2$ and

$$0 < \epsilon_1 \le \min\left\{ \left(\frac{\lambda_1 e^{\theta \min\limits_{l\in\mathcal{M}} \varsigma_1(l)}}{2M\check{f}} \right)^{1/\theta}, \frac{1}{M\check{f}\check{m}}, \left(\frac{\hat{b}}{4M}\right)^{1/2}, \left(\frac{\hat{c}\hat{d}}{M\check{f}}\right)^{1/2} \right\},$$

then

$$LV \le -1$$

is always true.

Therefore, there is a unique ergodic stationary distribution of system (1.4) according to Lemma 2.1. □

5. Examples and simulations

In this section, examples and simulations are given to illustrate previous findings.

Example 5.1. Assume that parameters of system (1.3) are given by

$$a(t) = 1.1 + 0.4\cos t, b(t) = 0.5 + 0.4\cos t, c(t) = 2.7 + 0.1\sin t, d(t) = 0.8 + 0.2\sin t,$$
$$f(t) = 2.8 + 0.1\sin t, m(t) = 1, \alpha(t) = 0.1 + 0.08\sin t, \beta(t) = 0.2 + 0.1\cos t,$$

and the initial value $(x(0), y(0)) = (1.5, 2)$. Obviously, $\frac{\hat{c}}{\check{m}} = 2.6 > \check{a} + \check{d} = 2.5$, then the corresponding deterministic system tends to origin. While for the stochastic system (1.3), $f(t) \ge \frac{c(t)}{m(t)}$ and

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} \left(a(t) + d(t) - \frac{c(t)}{m(t)} - \frac{\alpha^2(t)}{2} + \frac{\beta^2(t)}{2} \right) dt \\ =& \frac{1}{2\pi}\int_0^{2\pi} \left(-\frac{7841}{10^4} + \frac{21}{50}\cos t + \frac{92}{10^3}\sin t + \frac{41}{10^4}\cos 2t \right) dt < 0. \end{aligned}$$

Hence, according to the result of Theorem 3.3, one can get $\lim\limits_{t\to\infty} x(t) = 0, \lim\limits_{t\to\infty} y(t) = 0$ a.s. See Figure 1.

Example 5.2. Assume that parameters of system (1.3) are given by

$$a(t) = 1.2 + 0.5\cos t, b(t) = 0.5 + 0.4\cos t, c(t) = 0.5 + 0.1\sin t, d(t) = 0.9 + 0.3\sin t,$$
$$f(t) = 1.4 + 0.1\sin t, m(t) = 1, \alpha(t) = 1.85 + 0.1\sin t, \beta(t) = 0.2 + 0.1\cos t$$

with the same initial value as in Example 5.1. Note that $\hat{f} = 1.3 > \check{d} = 1.2$ and $\hat{m}\hat{a} = 0.7 > \check{c} = 0.6$, then there is a periodic solution of the corresponding deterministic system. While the large white noise $\dot{B}_1(t)$ makes

$$\begin{aligned} a(t) + d(t) + \frac{\beta^2(t)}{2} - f(t) &= \frac{289}{4\times 10^2} + \frac{13\cos t}{25} + \frac{\sin t}{5} + \frac{\cos 2t}{4\times 10^2} \\ &\le \frac{25200}{2\times 10^4} + \frac{37\sin t}{2\times 10^2} \\ &\le \frac{34275}{2\times 10^4} + \frac{37\sin t}{2\times 10^2} - \frac{5\cos 2t}{2\times 10^3} \le \frac{\alpha^2(t)}{2} \end{aligned}$$

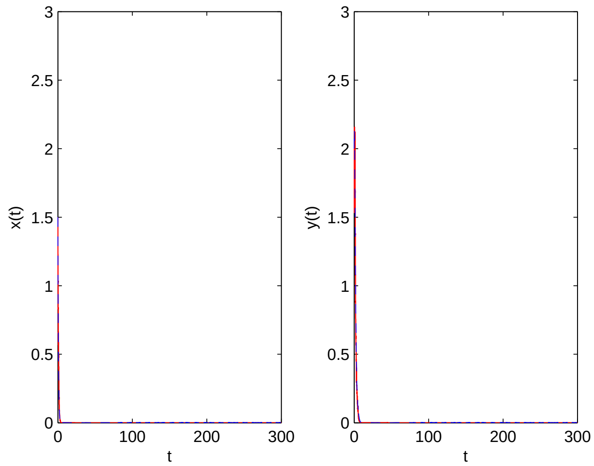

Figure 1. Simulation of paths of $(x(t), y(t))$ of system (1.3) (red solid line) and the corresponding deterministic system (blue dotted line), respectively. Population in both systems will die out.

for all $t \geq 0$, and

$$\frac{1}{2\pi}\int_0^{2\pi}\left(a(t)+d(t)-\frac{c(t)}{m(t)}-\frac{\alpha^2(t)}{2}+\frac{\beta^2(t)}{2}\right)dt$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\left(-\frac{1825}{2\times 10^4}+\frac{52\cos t}{10^2}+\frac{3\sin t}{2\times 10^2}+\frac{\cos 2t}{2\times 10^2}\right)dt<0,$$

then from Theorem 3.3, it shows that $\lim_{t\to\infty} x(t)=0$, $\lim_{t\to\infty} y(t)=0$ a.s. See Figure 2.

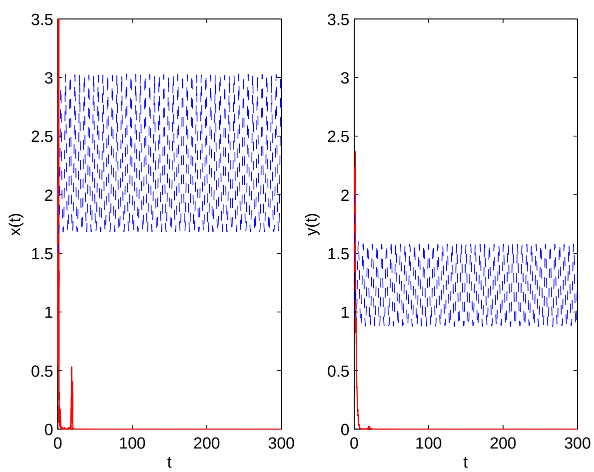

Figure 2. Simulation of paths of $(x(t), y(t))$ of system (1.3) (red solid line) and the corresponding deterministic system (blue dotted line), respectively. In this situation, there is a periodic solution of the corresponding deterministic system, while the large white noise makes both the prey and the predator die out a.s.

Example 5.3. Parameters and the initial value of system (1.3) are the same as in Example 5.2 except

$$c(t) = 0.9 + 0.1\sin t, d(t) = 1.7 + 0.1\sin t, \alpha(t) = 2 + 0.5\cos t, \beta(t) = 0.2 + 0.1\cos t.$$

In this situation,

$$\begin{aligned} a(t) + d(t) + \frac{\beta^2(t)}{2} - f(t) &= \frac{30445}{2\times 10^4} + \frac{104\cos t}{2\times 10^2} + \frac{\cos 2t}{4\times 10^2} \\ &\le \frac{33}{16} + \cos t + \frac{\cos 2t}{16} = \frac{\alpha^2(t)}{2} \end{aligned}$$

for all $t \ge 0$, and

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi}\left(a(t) + d(t) - \frac{c(t)}{m(t)} - \frac{\alpha^2(t)}{2} + \frac{\beta^2(t)}{2}\right)dt \\ =& \frac{1}{2\pi}\int_0^{2\pi}\left(-\frac{4}{10^2} - \frac{48\cos t}{10^2} - \frac{6\cos 2t}{10^2}\right)dt < 0, \end{aligned}$$

then the large white noise $\dot{B}_1(t)$ also makes conditions in Theorem 3.3 are satisfied, and so $\lim_{t\to\infty} x(t) = 0, \lim_{t\to\infty} y(t) = 0$ a.s. While, $\check{f} = 1.5 < \hat{d} = 1.6$, then the corresponding deterministic system is not persistent. See Figure 3.

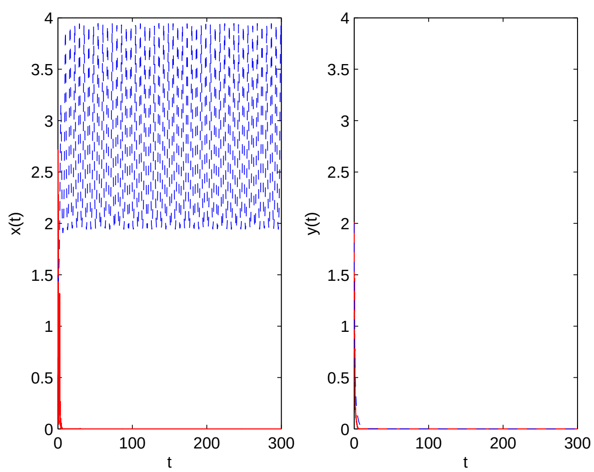

Figure 3. Simulation of paths of $(x(t), y(t))$, $\frac{1}{t}\int_0^t b(s)x(s)ds$ of system (1.3) (red solid line) and the corresponding deterministic system (blue dotted line), respectively. In both of two systems, the prey is persistent, but the predator is not, and $b(t)x(t)$ is stable in time average of system (1.3).

Example 5.4. Choose

$$c(t) = 0.1 + 0.1\sin t, f(t) = 0.4 + 0.1\sin t, \alpha(t) = 0.1 + 0.08\sin t.$$

Other parameters and the initial value have the same values as in Example 5.2. For the corresponding deterministic system, $\check{f} = 0.5 < \hat{d} = 0.6$, then it is not persistent.

For the stochastic system (1.3), it is clear that $f(t) \geq \frac{c(t)}{m(t)}$ for all $t \geq 0$, and

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi}\left(f(t)-d(t)-\frac{\beta^2(t)}{2}\right)dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(-\frac{5}{10}-\frac{2\sin t}{10}-\frac{(0.2+0.1\cos t)^2}{2}\right)dt < 0. \end{aligned}$$

Besides,

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi}\left(a(t)-\frac{\alpha^2(t)}{2}\right)dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{11934}{10^4}-\frac{8\sin t}{10^3}+\frac{16\cos 2t}{10^4}\right)dt \\ &= \frac{11934}{10^4} > 0. \end{aligned}$$

Theorem 3.4 shows that system (1.3) is also not persistent. That is

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t b(t)x(s)ds = \frac{11934}{10^4}, \quad \lim_{t\to\infty} y(t) = 0 \ \text{ a.s.}$$

Figure 4 also illustrates this.

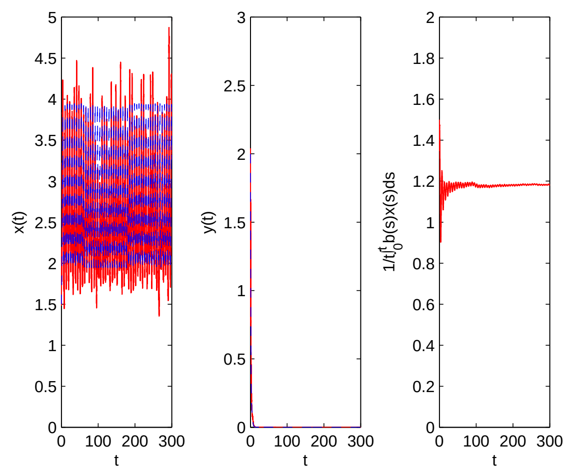

Figure 4. Simulation of paths of $(x(t), y(t))$, $\frac{1}{t}\int_0^t b(s)x(s)ds$ of system (1.3) (red solid line) and the corresponding deterministic system (blue dotted line), respectively. In both of two systems, the prey is persistent, but the predator is not, and $b(t)x(t)$ is stable in time average of system (1.3).

Example 5.5. In this example, parameters and the initial value have the same values as in Example 5.2 except

$$\begin{aligned} &d(t) = 0.2 + 0.1\sin t, f(t) = 0.45 + 0.1\sin t, \\ &\alpha(t) = 0.1 + 0.08\sin t, \beta(t) = 1.1 + 0.2\cos t. \end{aligned}$$

Obviously, $\hat{f} = 0.35 > \check{d} = 0.3, \hat{m}\hat{a} = 0.7 > \check{c} = 0.6$, and so the corresponding deterministic system has a periodic solution. While, white noise $\dot{B}_2(t)$ is so large that

$$\frac{\beta^2(t)}{2} = \frac{(11 + 2\cos t)^2}{2 \times 10^2} \geq \frac{81}{2 \times 10^2} \geq \frac{c(t)}{m(t)} - d(t) = \frac{3}{10}$$

for all $t \geq 0$, and

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} \left(f(t) - d(t) - \frac{\beta^2(t)}{2} \right) dt \\ = &\frac{1}{2\pi} \int_0^{2\pi} \left(-\frac{73}{2 \times 10^2} - \frac{22\cos t + \cos 2t}{10^2} \right) dt < 0. \end{aligned}$$

In addition,

$$\frac{1}{2\pi} \int_0^{2\pi} \left(a(t) - \frac{\alpha^2(t)}{2} \right) dt > 0$$

is also satisfied. Therefore

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t b(t)x(s)ds = \frac{11934}{10^4}, \quad \lim_{t\to\infty} y(t) = 0 \ \text{ a.s.}$$

which as Theorem 3.4 said. See Figure 5.

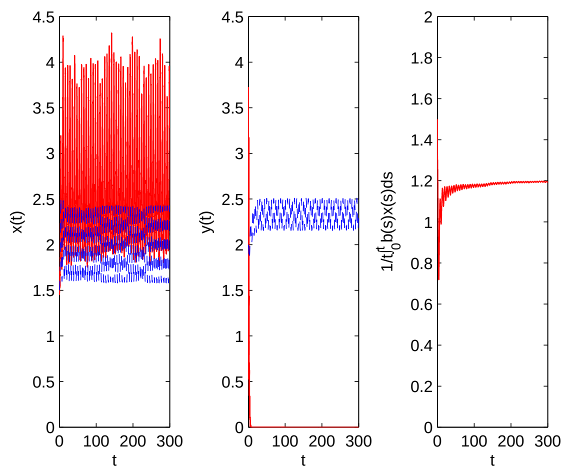

Figure 5. Simulation of paths of $x(t), y(t), \frac{1}{t}\int_0^t b(s)x(s)ds$ of system (1.3) (red solid line) and the corresponding deterministic system (blue dotted line), respectively. There is a periodic solution of the corresponding deterministic system, but the large white noise makes the predator extinction and the prey be stable in time average.

Example 5.6. Assume that parameters of system (1.3) are given by

$$\begin{aligned} &a(t) = 1.2 + 0.5\cos(t), b(t) = 0.5 + 0.4\cos(t), c(t) = 0.4 + 0.2\sin(t), \\ &d(t) = 0.4 + 0.3\sin(t), f(t) = 1.1 + 0.3\sin(t), m(t) = 1, \end{aligned}$$

$$\alpha(t) = 0.5 + 0.3\sin(t), \beta(t) = 0.2 + 0.1\cos(t),$$

and the same initial value as in Example 5.1. It is easy to compute that

$$\frac{1}{2\pi}\int_0^{2\pi}\left(a(t) - \frac{c(t)}{m(t)} - \frac{\alpha^2(t)}{2}\right)dt$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{261}{400} + \frac{1}{2}\cos t - \frac{7}{20}\sin t + \frac{9}{400}\cos 2t\right)dt > 0$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\left(f(t) - d(t) - \frac{\beta^2(t)}{2}\right)dt$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{271}{400} - \frac{1}{50}\cos t - \frac{1}{400}\cos 2t\right)dt > 0,$$

and so according to Theorem 3.5, system (1.3) is persistent. While, for the corresponding deterministic system, $\hat{f} = 0.8 > \check{d} = 0.7$ and $\hat{m}\hat{a} = 0.7 > \check{c} = 0.6$, there is also a periodic solution. Fig. 6 shows that the path of system (1.3) is around the periodic solution of the deterministic system after a long time (see the fist line in Figure 6).

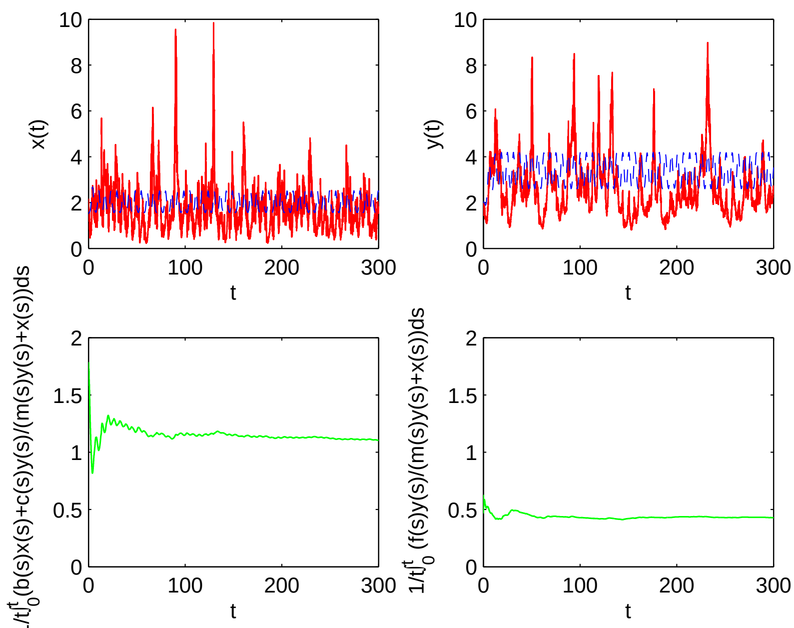

Figure 6. Simulation of paths of $(x(t), y(t))$ of system (1.3) (red solid line) and the corresponding deterministic system (blue dotted line), respectively, and the time average of the solution of system (1.3) (green solid line).

Example 5.7. Consider (1.4) with Markov chain $\xi(t)$ taking values in $\mathcal{M} = \{1, 2, 3\}$,

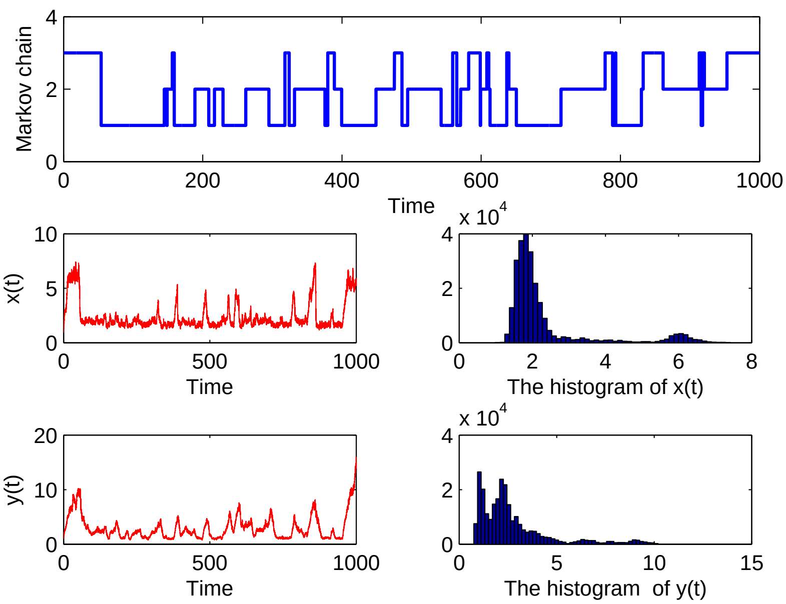

Figure 7. Simulation of paths of $x(t), y(t)$ (see the fist column in the second and third lines), and their histogram (see the second column in the second and third lines) of system (1.4) with the Markov chain (see the picture in the first line).

which is regarded as equations

$$\begin{cases} dx(t) = x(t)\left(a(l) - b(l)x(t) - \dfrac{c(l)y(t)}{m(l)y(t)+x(t)}\right)dt + \alpha(l)x(t)dB_1(t), \\ dy(t) = y(t)\left(-d(l) + \dfrac{f(l)x(t)}{m(l)y(t)+x(t)}\right)dt + \beta(l)y(t)dB_2(t), \end{cases} \quad l = 1, 2, 3,$$

where

$$\begin{aligned} &a(1) = 1.2, a(2) = 1.7, a(3) = 0.7, b(1) = 0.5, b(2) = 0.9, b(3) = 0.1, \\ &c(1) = 0.4, c(2) = 0.6, c(3) = 0.2, d(1) = 0.4, d(2) = 0.7, d(3) = 0.1, \\ &f(1) = 1.1, f(2) = 1.4, f(3) = 0.8, m(1) = 1.2, m(2) = 1.5, m(3) = 1, \\ &\alpha(1) = 0.1, \alpha(2) = 0.12, \alpha(3) = 0.08, \beta(1) = 0.09, \beta(2) = 0.1, \beta(3) = 0.06 \end{aligned}$$

switching according to the movement of the Markov chain $\xi(t)$. Let the generator $\Gamma = (\gamma_{ij})_{3\times 3}$ of the Markov chain $\xi(t)$ be

$$\Gamma = \begin{pmatrix} -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \end{pmatrix}.$$

By solving equation (2.1) we obtain the probability distribution is

$$\pi = \left(\frac{3}{7}, \frac{2}{7}, \frac{2}{7}\right).$$

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$$\lambda_1 = \sum_{\kappa=1}^{3} \pi_\kappa \left(a(\kappa) - \frac{c(\kappa)}{m(\kappa)} - \frac{\alpha^2(\kappa)}{2} \right) \approx 0.88 > 0,$$

$$\lambda_2 = \sum_{\kappa=1}^{3} \pi_\kappa \left(f(\kappa) - d(\kappa) - \frac{\beta^2(\kappa)}{2} \right) \approx 0.67 > 0.$$

Therefore, according to Theorem 4.4, system (1.4) admits a unique stationary distribution. See Figure 7.

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