Journal of Applied Analysis and Computation
Volume 8, Number 6, December 2018, 1707–1726
Website:http://jaac.ijournal.cn
DOI:10.11948/2018.1707

ASYMPTOTIC BEHAVIOR OF NABLA HALF ORDER *H*-DIFFERENCE EQUATIONS*

Baoguo Jia[1], Feifei Du[1], Lynn Erbe[2] and Allan Peterson[2,†]

Abstract In this paper we study the half order nabla fractional difference equation ${}_{\rho(a)}\nabla_h^{0.5}x(t) = cx(t),\ t \in (h\mathbb{N})_{a+h}$, where ${}_{\rho(a)}\nabla_h^{0.5}x(t)$ denotes the Riemann-Liouville nabla half order h-difference of $x(t)$. We will establish the asymptotic behavior of the solutions of this equation satisfying $x(a) = A > 0$ for various values of the constant c.

Keywords Laplace transform, Mittag-Leffler function, Riemann-Liouville fractional h-difference, oscillation.

MSC(2010) 39A12, 39A70.

1. Introduction

Discrete fractional calculus has generated interest in recent years. Some of the work has employed the forward or delta difference. We refer the readers to [3, 14, 15, 17] for example, and more recently [5, 6, 11, 16, 20]. Probably more work has been developed for the backward or nabla difference and we refer the readers [1, 9, 10, 13]. There has been some work to develop relations between the forward and backward fractional operators, Δ^ν and ∇^ν [4] and fractional calculus on time scales [7, 8].

From [9], we know that if $c \in (0,1)\cup(1,\sqrt{2})$, then the trivial solution of the half order nabla fractional difference equation (NFDE)

$$ {}_0\nabla^{0.5}x(t) = cx(t), \quad c \neq 1, \quad t \in \mathbb{N}_2 \tag{1.1}$$

is unstable. If $c \in (-\infty, 0]\cup(\sqrt{2}, +\infty)$, then the trivial solution of (1.1) is asymptotic stable. However, when $c = \sqrt{2}$, the asymptotic stability of the trivial solution of (1.1) is still an open problem in [9]. In the book [12, Chapter 6] (Proposition 4.4, Page 140), the authors (J. Čermák and T. Kisela) generalized the corresponding results to h-difference equations and solved that open problem.

In this paper, we will further investigate the asymptotic behavior of the fractional initial value problem

$$ {}_{\rho(a)}\nabla_h^{0.5}x(t) = cx(t),\ x(a) = A > 0,\ ch^{0.5} \neq 1, \quad t \in (h\mathbb{N})_{a+h}, \tag{1.2}$$

†the corresponding author. Email address: apeterson1@math.unl.edu (A. Peterson)
[1]School of Mathematics, Sun Yat-sen University, 510275 Guangzhou, China
[2]Department of Mathematics, University of Nebraska-Lincoln, 68588-0130 Lincoln, USA
*This work is supported by the National Natural Science Foundation of China (No.11271380), Guangdong Province Key Laboratory of Computational Science, and the International Program for Ph.D. Candidates, Sun Yat-sen University.

and obtain the new results that as a special case ($\nu = 0.5, h = 1$), qualitative properties of (1.1) are different when $c \in (0,1)$ and $c \in (1, \sqrt{2})$ (see detail, Remark 1.1), qualitative properties of (1.1) are different when $c \in (-\infty, 0]$ and $c \in (\sqrt{2}, +\infty)$(see detail, Remark 1.2).

It is worth mentioning that in the case $c = \sqrt{2}$, we also solved that open problem, i.e. solutions of (1.1) are stable (and oscillatory), however, our techniques are different from their methods.

To the best of author's observation, however, few papers ([2, 18]) have been published in literature regarding the oscillation of solutions for fractional difference equations. A primary purpose of this paper is to establish one explicit oscillation criterion for a type of linear fractional difference equations involving the Riemann-Liouville's operator. Two numerical examples are provided to demonstrate the effectiveness of the main theorems.

Consider the solutions of initial value problem for the NFDE

$$_{\rho(a)}\nabla_h^{\nu} x(t) = cx(t),\ x(a) = A > 0,\ ch^{\nu} \neq 1,\ t \in (h\mathbb{N})_{a+h}, \tag{1.3}$$

where $(h\mathbb{N})_{a+h} := \{a+h, a+2h, \cdots\}$ and $_{\rho(a)}\nabla_h^{\nu} x(t)$ denotes Riemann-Liouville nabla h-difference of $x(t)$ on sets $(h\mathbb{N})_a := \{a, a+h, a+2h, \cdots\}$. In this paper, we will discuss the asymptotic behavior of the solutions of (1.3). The following Theorem is obtained.

Theorem A. *Assume $c > 0$, $ch^{0.5} \neq 1$. Then the unique solution of the fractional initial value problem* (1.2) *satisfies*
(i) When $0 < c < \frac{1}{h^{0.5}}$,

$$\lim_{n\to\infty} x(a+nh) = +\infty.$$

(ii) When $\frac{1}{h^{0.5}} < c < \left(\frac{2}{h}\right)^{0.5}$,

$$\limsup_{n\to\infty} x(a+nh) = +\infty, \quad \liminf_{n\to\infty} x(a+nh) = -\infty.$$

(iii) When $c = \left(\frac{2}{h}\right)^{0.5}$,

$$\limsup_{n\to\infty} x(a+nh) = 2\sqrt{2}(\sqrt{2}-1)A, \quad \liminf_{n\to\infty} x(a+nh) = -2\sqrt{2}(\sqrt{2}-1)A.$$

(iv) When $c > \left(\frac{2}{h}\right)^{0.5}$,

$$\lim_{n\to\infty} x(a+nh) = 0.$$

In the following two remarks we compare our results to the known results in the literature.

Remark 1.1. From the reference [12, Chapter 6], we know when $0 < c < (\frac{1}{h})^{0.5}$ or $(\frac{1}{h})^{0.5} < c < (\frac{2}{h})^{0.5}$,

$$\lim_{n\to\infty} |x(a+nh)| = \infty.$$

However, qualitative properties of (1.2) are different when $0 < c < (\frac{1}{h})^{0.5}$ and $(\frac{1}{h})^{0.5} < c < (\frac{2}{h})^{0.5}$. From [19] (for $h = 1$, but the technique is valid for $h > 0$ and $h \neq 1$), we know when $0 < c < (\frac{1}{h})^{0.5}$, the solution $x(t)$ of (1.2) is positive and tends to infinity. But from this paper, we know when $(\frac{1}{h})^{0.5} < c < (\frac{2}{h})^{0.5}$, the solution $x(t)$ of (1.2) is oscillatory and tends to infinity.

Remark 1.2. From the reference [12, Chapter 6], we know when $c \leq 0$ or $c > (\frac{2}{h})^{0.5}$,

$$\lim_{n\to\infty} x(a+nh) = 0.$$

However, qualitative properties of (1.2) are different when $c \leq 0$ and $c > (\frac{2}{h})^{0.5}$. From [19] (for $h = 1$, but the technique is valid for $h > 0$ and $h \neq 1$), we know when $c \leq 0$, the solution of (1.2) is positive and tends to zero. But from this paper, we know when $c > (\frac{2}{h})^{0.5}$, for large t, the solution $x(t)$ of (1.2) is negative and is increasing to zero.

2. Preliminaries

For any real number z, we define the exponential function, $E_z(t,a)$, to be the unique solution of the initial value problem (IVP)

$$\nabla_h y(t) = zy(t), \quad t \in (h\mathbb{N})_{a+h}, \quad y(a) = 1.$$

It is easy to see that $y(t) = (1-zh)^{-1}y(t-h)$ and it follows that

$$E_z(t,a) = (1-zh)^{-\frac{t-a}{h}}.$$

Also if we define the box difference by

$$\boxminus z = -\frac{z}{1-zh},$$

then

$$E_{\boxminus z}(t,a) = (1-zh)^{\frac{t-a}{h}}.$$

Consequently, for any function $x : (h\mathbb{N})_{a+h} \to \mathbb{R}$, its nabla h-Laplace transform has the form

$$\begin{aligned}\mathcal{L}_a\{x\}(z) &= \int_a^\infty E_{\boxminus z}(\rho(t),a)x(t)\nabla_h t = \int_a^\infty (1-zh)^{\frac{t-h-a}{h}} x(t)\nabla_h t \\ &= h\sum_{k=1}^{\infty}(1-zh)^{k-1}x(a+kh).\end{aligned}$$

Lemma 2.1. *Assume $x : (h\mathbb{N})_{a+h} \to \mathbb{R}$. Then*

$$\mathcal{L}_a\{x\}(z) = h\sum_{k=1}^{\infty}(1-zh)^{k-1}x(a+kh), \tag{2.1}$$

for those values of z such that this infinite series converges.

Remark 2.1.

$$\mathcal{L}_a\{1\}(z) = h\sum_{k=1}^{\infty}(1-zh)^{k-1} = \frac{h}{1-(1-zh)} = \frac{1}{z},$$

for $|1-zh| < 1$, which is a standard formula.

Definition 2.1. Let $\nu \neq -1, -2, -3, \cdots$. Then we difine the ν-th order nabla fractional h-Taylor monomial, $\hat{H}_\nu(t,a)$, by

$$\hat{H}_\nu(t,a) = \frac{(t-a)_h^{\overline{\nu}}}{\Gamma(\nu+1)} = h^\nu \frac{\Gamma(\frac{t-a}{h}+\nu)}{\Gamma(\nu+1)\Gamma(\frac{t-a}{h})},$$

where $t \in (h\mathbb{N})_a$.

Definition 2.2 (Nabla Fractional Sum)**.** Let $x : (h\mathbb{N})_{a+h} \to \mathbb{R}$ and $\nu > 0$ be given. The ν-th order h-sum with staring point a is given by

$$_a\nabla_h^{-\nu} x(t) = \int_a^t \hat{H}_{\nu-1}(t,\rho(\tau))x(\tau)\nabla_h\tau = \frac{1}{\Gamma(\nu)}\int_a^t (t-\rho(\tau))_h^{\overline{\nu-1}} x(\tau)\nabla_h\tau, \quad (2.2)$$

for $t \in (h\mathbb{N})_a$, where by convention $_a\nabla_h^{-\nu}x(a) = 0$ and $\rho(\tau) = \tau - h$, the h-rising factorial function is defined as

$$t_h^{\overline{\nu}} = h^\nu \frac{\Gamma(\frac{t}{h}+\nu)}{\Gamma(\frac{t}{h})}, \quad t, \nu \in \mathbb{R}.$$

Definition 2.3 (Riemann-Liouville fractional difference)**.** For $x(t)$ defined on $(h\mathbb{N})_a$ and $m-1 < \nu < m$, where m denotes a positive integer, $m = \lceil \nu \rceil$, where $\lceil \cdot \rceil$ is the ceiling of function. The ν-th Riemann-Liouville nabla fractional difference is defined as

$$_a\nabla_h^\nu x(t) = \nabla_h^m \, {_a\nabla_h^{-(m-\nu)}} x(t), \quad (2.3)$$

for $t \in (h\mathbb{N})_{a+mh}$.

The proof of the following lemma is different from [16, Theorem 3.74].

Lemma 2.2.

$$\mathcal{L}_a\{\hat{H}_\nu(\cdot,a)\}(z) = \frac{1}{z^{\nu+1}},$$

for $|zh-1| < 1$.

Proof.

$$\begin{aligned}
\mathcal{L}_a\{\hat{H}_\nu(\cdot,a)\}(z) &= h\sum_{k=1}^{\infty}(1-zh)^{k-1}\hat{H}_\nu(a+kh,a) \\
&= h\sum_{k=1}^{\infty}(1-zh)^{k-1}h^\nu\frac{\Gamma(k+\nu)}{\Gamma(\nu+1)\Gamma(k)} \\
&= h^{1+\nu}\sum_{k=1}^{\infty}(1-zh)^{k-1}\frac{(k+\nu-1)(k+\nu-2)\cdots(\nu+1)}{(k-1)!} \\
&= h^{1+\nu}\sum_{k=0}^{\infty}(1-zh)^{k}\frac{(k-1+\nu+1)(k-2+\nu+1)\cdots(\nu+1)}{k!} \\
&= h^{1+\nu}[1-(1-zh)]^{-(\nu+1)} = \frac{1}{z^{\nu+1}},
\end{aligned}$$

for $|1-zh| < 1$, where we use

$$[1-(1-sh)]^{-(\nu+1)} = \sum_{k=0}^{\infty}\frac{[-(\nu+1)][-(\nu+1)-1]\cdots[-(\nu+1)-k+1]}{k!}[-(1-zh)]^k$$

$$= \sum_{k=0}^{\infty} \frac{(\nu+1)(\nu+2)\cdots(\nu+k)}{k!}(1-zh)^k.$$

□

Definition 2.4. For $f, g : (h\mathbb{N})_{a+h} \to \mathbb{R}$, we define the nabla convolution product of f and g by

$$(f * g)(t) := \int_a^t f(t-\rho(\tau)+a)g(\tau)\nabla_h\tau = h\sum_{k=1}^{n} f((n-k+1)h+a)g(a+kh),$$

where $t = a + nh,\ n = 1, 2, \cdots$.

Lemma 2.3 (Convolution Theorem)**.** *Assume $f, g : (h\mathbb{N})_{a+h} \to \mathbb{R}$ and their nabla h-Laplace transforms converge for $|1 - zh| < r$ for some $r > 0$. Then*

$$\mathcal{L}_a\{f * g\}(z) = \mathcal{L}_a\{f\}(z) \cdot \mathcal{L}_a\{g\}(z),$$

for $|1 - zh| < r$.

Lemma 2.4. *Assume $\nu \in \mathbb{R} \setminus \{0, -1, -2, \cdots\}$ and $f : (h\mathbb{N})_{a+h} \to \mathbb{R}$. Then*

$${}_a\nabla_h^{-\nu} f(t) = (\hat{H}_{\nu-1}(\cdot, a) * f)(t),$$

for $t \in (h\mathbb{N})_{a+h}$.

Proof.

$$\begin{aligned}(\hat{H}_{\nu-1}(\cdot, a) * f)(t) &= \int_a^t \hat{H}_{\nu-1}(t-\rho(\tau)+a, a)f(\tau)\nabla_h\tau \\ &= \int_a^t \hat{H}_{\nu-1}(t, \rho(\tau))f(\tau)\nabla_h\tau = {}_a\nabla_h^{-\nu} f(t).\end{aligned}$$

□

Lemma 2.5. *Assume the nabla h-Laplace transform of $f : (h\mathbb{N})_{a+h} \to \mathbb{R}$ converges for $|1 - zh| < r$ for some $r > 0$. Then*

$$\mathcal{L}_a\{\nabla_h f\}(z) = z\mathcal{L}_a\{f\}(z) - f(a),$$

for $|1 - zh| < r$.

Proof.

$$\begin{aligned}\mathcal{L}_a\{\nabla_h f\}(z) &= h\sum_{k=1}^{\infty}(1-zh)^{k-1}\nabla_h f(a+kh) \\ &= \sum_{k=1}^{\infty}(1-zh)^{k-1}[f(a+kh) - f(a+kh-h)] \\ &= \frac{1}{h}\mathcal{L}_a\{f\}(z) - \sum_{k=1}^{\infty}(1-zh)^{k-1}f(a+kh-h) \\ &= \frac{1}{h}\mathcal{L}_a\{f\}(z) - \sum_{k=0}^{\infty}(1-zh)^{k}f(a+kh)\end{aligned}$$

$$= \frac{1}{h}\mathcal{L}_a\{f\}(z) - \frac{1-zh}{h}\mathcal{L}_a\{f\}(z) - f(a)$$
$$= z\mathcal{L}_a\{f\}(z) - f(a).$$

□

Lemma 2.6. *Assume the nabla h-Laplace transform of* $f : (h\mathbb{N})_{a+h} \to \mathbb{R}$ *converges for* $|1-zh| < r$ *for some* $r > 0$. *Then*

$$\mathcal{L}_{a+h}\{f\}(z) = \frac{\mathcal{L}_a\{f\}(z)}{1-zh} - \frac{hf(a+h)}{1-zh},$$

for $|1-zh| < r$.

Proof.

$$\mathcal{L}_{a+h}\{f\}(z) = h\sum_{k=1}^{\infty}(1-zh)^{k-1}f(a+h+kh) = h\sum_{j=2}^{\infty}(1-zh)^{j-2}f(a+jh)$$
$$= h\sum_{j=1}^{\infty}(1-zh)^{j-2}f(a+jh) - \frac{hf(a+h)}{1-zh}$$
$$= \frac{\mathcal{L}_a\{f\}(z)}{1-zh} - \frac{hf(a+h)}{1-zh}.$$

This completes the proof.

□

Lemma 2.7. *Assume* $\nu > 0$ *and the nabla h-Laplace transform of* $f : (h\mathbb{N})_{a+h} \to \mathbb{R}$ *converges for* $|1-zh| < r$ *for some* $r > 0$. *Then*

$$\mathcal{L}_a\{{}_a\nabla_h^{-\nu}f\}(z) = \frac{1}{z^\nu}\mathcal{L}_a\{f\}(z)$$

for $|1-zh| < \min\{1, r\}$.

Lemma 2.8. *Given* $f : (h\mathbb{N})_{a+h} \to \mathbb{R}$ *and* $0 < \nu < 1$. *Then we have*

$$\mathcal{L}_a\{{}_a\nabla_h^{\nu}f\}(z) = z^\nu\mathcal{L}_a\{f\}(z).$$

Proof. Using Lemma 2.5 and Lemma 2.7 we have that

$$\mathcal{L}_a\{{}_a\nabla_h^{\nu}f\}(z) = \mathcal{L}_a\{\nabla_h{}_a\nabla_h^{-(1-\nu)}f\}(z) = z\mathcal{L}_a\{{}_a\nabla_h^{-(1-\nu)}f\}(z) - {}_a\nabla_h^{-(1-\nu)}f(a)$$
$$= \frac{z}{z^{1-\nu}}\mathcal{L}_a\{f\}(z) = z^\nu\mathcal{L}_a\{f\}(z),$$

where we use ${}_a\nabla_h^{-(1-\nu)}f(a) = 0$ (Definition 2.2).

□

Lemma 2.9. *Given* $f : (h\mathbb{N})_{a+h} \to \mathbb{R}$ *and* $0 < \nu < 1$. *Then we have*

$$\mathcal{L}_{a+h}\{{}_a\nabla_h^{\nu}f\}(z) = z^\nu\mathcal{L}_{a+h}\{f\}(z) + \frac{h(z^\nu - h^{-\nu})}{1-zh}f(a+h).$$

Proof. From Lemma 2.5, we have

$$\begin{aligned}\mathcal{L}_{a+h}\{{}_a\nabla_h^{\nu} f\}(z) &= \mathcal{L}_{a+h}\{\nabla_h {}_a\nabla_h^{-(1-\nu)} f\}(z)\\ &= z\mathcal{L}_{a+h}\{{}_a\nabla_h^{-(1-\nu)} f\}(z) - {}_a\nabla_h^{-(1-\nu)} f(a+h).\end{aligned}$$

Using Lemma 2.6, we get that

$$\mathcal{L}_{a+h}\{{}_a\nabla_h^{-(1-\nu)} f\}(z) = \frac{\mathcal{L}_a\{{}_a\nabla_h^{-(1-\nu)} f\}(z)}{1-zh} - \frac{h_a\nabla_h^{-(1-\nu)} f(a+h)}{1-zh}.$$

So

$$\begin{aligned}\mathcal{L}_{a+h}\{{}_a\nabla_h^{\nu} f\}(z) &= \frac{z\mathcal{L}_a\{{}_a\nabla_h^{-(1-\nu)} f\}(z)}{1-zh} - \frac{{}_a\nabla_h^{-(1-\nu)} f(a+h)}{1-zh}\\ &= \frac{z\mathcal{L}_a\{f\}(z)}{z^{1-\nu}(1-zh)} - \frac{h^{1-\mu} f(a+h)}{1-zh}\\ &= \frac{z^{\nu}\mathcal{L}_a\{f\}}{1-zh} - \frac{h^{1-\nu} f(a+h)}{1-zh},\end{aligned}$$

where we use

$$\begin{aligned}{}_a\nabla_h^{-(1-\nu)} f(a+h) &= \int_a^{a+h} \hat{H}_{-\nu}(a+h,\rho(\tau)) f(\tau)\nabla_h\tau\\ &= h\hat{H}_{-\nu}(a+h,a) f(a+h)\\ &= h^{1-\nu} f(a+h).\end{aligned}$$

Applying Lemma 2.6 again, we obtain

$$\mathcal{L}_{a+h}\{{}_a\nabla_h^{\nu} f\}(s) = z^{\nu}\mathcal{L}_{a+h}\{f\}(z) + \frac{h(z^{\nu} - h^{-\nu})}{1-zh} f(a+h).$$

This completes the proof. □

3. Behavior of Solutions of ν-th Order Riemann-Liouville Fractional Difference Equations

Consider the ν order nabla fractional initial value problems

$$_{\rho(a)}\nabla_h^{\nu} x(t) = cx(t),\ x(a) = A,\ ch^{\nu} \neq 1,\ t \in (h\mathbb{N})_{a+h}. \tag{3.1}$$

A solution $x(t)$ of (3.1) is said to be oscillatory if for every integer $N > 0$, there exists $t \geq N$ such that $x(t)x(t+h) \leq 0$; otherwise it is called nonoscillatory. An equation is said to be oscillatory if all of its solution are oscillatory.

Definition 3.1. A function $f : (h\mathbb{N})_a \to \mathbb{R}$ is said to be exponential of order $r > 0$ if there exist a constant $M > 0$ and a $r > 1$ such that

$$|f(t)| \leq Mr^t$$

for $t \in (h\mathbb{N})_a$.

Lemma 3.1. *Assume $0<\nu<1$ and $1-h^\nu c\neq 0$. If $f:(h\mathbb{N})_a\to\mathbb{R}$ is exponential bounded, then each solution of the fractional initial value problem*

$$_{\rho(a)}\nabla_h^\nu x(t)=cx(t)+f(t),\quad t\in(h\mathbb{N})_{a+h},\quad x(a)=A\in\mathbb{R}$$

is exponential bounded and hence its h-Laplace transform exists.

Proof. Using Leibniz formula, it is easy to get that

$$\begin{aligned}{}_{\rho(a)}\nabla_h^\nu x(t)&=\nabla_h {}_{\rho(a)}\nabla_h^{-(1-\nu)}x(t)=\nabla_h\int_{\rho(a)}^t\hat{H}_{-\nu}(t,\rho(s))x(s)\nabla_h s\\&=\int_{\rho(a)}^t\hat{H}_{-\nu-1}(t,\rho(s))x(s)\nabla_h s.\end{aligned}$$

Let $t=a+kh$, $k\geq 1$. We have

$$\begin{aligned}&h[\hat{H}_{-\nu-1}(a+kh,a-h)x(a)+\hat{H}_{-\nu-1}(a+kh,a)x(a+h)+\cdots\\&\quad+\hat{H}_{-\nu-1}(a+kh,a+(k-1)h)x(a+kh)]\\=&cx(a+kh)+f(a+kh).\end{aligned}$$

That is

$$\begin{aligned}&h^{-\nu}\Big[x(a+kh)-\nu x(a+kh-h)-\frac{\nu(-\nu+1)}{2!}x(a+kh-2h)-\cdots\\&-\frac{\nu(-\nu+1)\cdots(-\nu+k-1)}{k!}x(a)\Big]\\=&cx(a+kh)+f(a+kh).\end{aligned}$$

That is

$$\begin{aligned}&[1-h^\nu c]x(a+kh)\\=&\nu x(a+kh-h)+\frac{\nu(-\nu+1)}{2!}x(a+kh-2h)+\cdots\\&+\frac{\nu(-\nu+1)\cdots(-\nu+k-1)}{k!}x(a)+h^\nu f(a+kh).\end{aligned}$$

Since $f(a+kh)$ is exponentially bounded, there is an $M>0$ and a $r>1$ such that

$$|f(a+kh)|\leq Mr^{a+kh},$$

for $k\geq 1$. Taking large numbers R,B with $R>r$ and $BR^a>|A|$ and

$$B>\frac{2Mh^\nu}{|1-h^\nu c|},\quad |1-h^\nu c||R^h-1|>2.$$

We now prove by induction that

$$|x(a+kh)|\leq BR^{a+kh},\tag{3.2}$$

for $k=1,2,\cdots$. It is easy to see that the inequality (3.2) is true for $k=0$. Now assume that $k_0\geq 1$ and that the inequality (3.2) is true for $1\leq k\leq k_0-1$. Using

$$\Big|\frac{\nu(-\nu+1)\cdots(-\nu+i-1)}{i!}\Big|\leq 1,$$

for $i = 1, 2, \cdots, k$, we have

$$|x(a+k_0h)| \leq \frac{B}{|1-h^\nu c|}[R^{a+k_0h-h} + R^{a+k_0h-2h} + \cdots + R^a] + \frac{h^\nu M r^{a+k_0h}}{|1-h^\nu c|}$$
$$= \frac{BR^a}{|1-h^\nu c|}\frac{R^{k_0h}-1}{R^h-1} + \frac{h^\nu M r^{a+k_0h}}{|1-h^\nu c|}$$
$$\leq \frac{BR^{a+k_0h}}{|1-h^\nu c||R^h-1|} + \frac{BR^{a+k_0h}}{2} \leq BR^{a+k_0h},$$

which completes the induction. In the following, we will show that

$$\mathcal{L}_a\{x\}(z) = h\sum_{k=1}^{\infty}(1-zh)^{k-1}x(a+kh)$$

converges for $\left|1-zh\right| < \frac{1}{R^h}$. To see this, consider

$$h\sum_{k=1}^{\infty}\left|(1-zh)^{k-1}x(a+kh)\right| = h\sum_{k=1}^{\infty}\left|(1-zh)^{k-1}\right|\left|x(a+kh)\right|$$
$$\leq h\sum_{k=1}^{\infty}\left|(1-zh)^{k-1}\right|BR^{a+kh}$$
$$= hBR^{a+h}\sum_{k=1}^{\infty}\left(\left|1-zh\right|R^h\right)^{k-1},$$

which converges since $\left|1-zh\right|R^h < 1$. It follows that $\mathcal{L}_a\{x\}(z)$ converges absolutely for $\left|1-zh\right| < \frac{1}{R^h}$.

□

Theorem 3.1. *Assume* $0 < \nu < 1$. *Then the unique solution of the fractional initial value problem*

$$_{\rho(a)}\nabla_h^\nu x(t) = cx(t), \quad t \in (h\mathbb{N})_{a+h}, \quad x(a) = A \in \mathbb{R} \tag{3.3}$$

satisfies

$$\mathcal{L}_{\rho(a)}\{x\}(z) = \frac{hA(h^{-\nu}-c)}{z^\nu - c}. \tag{3.4}$$

Proof. We begin by taking the h-Laplace transform based at a of both sides of the fractional equation (3.3) to get that

$$\mathcal{L}_a\{_{\rho(a)}\nabla_h^\nu x\}(z) = c\mathcal{L}_a\{x\}(z).$$

From Lemma 2.9, and using the initial condition, we have that

$$z^\nu\mathcal{L}_a\{x\}(z) + \frac{h(z^\nu - h^{-\nu})}{1-zh}A = c\mathcal{L}_a\{x\}(z).$$

From Lemma 2.6, we get that

$$(z^\nu - c)\Big[\frac{\mathcal{L}_{\rho(a)}\{x\}(z)}{1-zh} - \frac{hA}{1-zh}\Big] + \frac{h(z^\nu - h^{-\nu})}{1-zh}A = 0.$$

So

$$\mathcal{L}_{\rho(a)}\{x\}(z) = \frac{hA(h^{-\nu}-c)}{z^\nu - c}.$$

□

Remark 3.1. When we take the h-Laplace transform of both sides of the fractional initial value problem (3.3), base point must be a, can't be $\rho(a)$, since the equation ${}_{\rho(a)}\nabla_h^{\nu}x(t) = cx(t)$ is defined on $(h\mathbb{N})_{a+h}$.

Remark 3.2. For $\nu = 0.5$, in Section 4, we get the inverse h-Laplace transform of $\frac{1}{z^{0.5}-c}$, For $\nu \neq 0.5$ and $c > 0$, it is not easy to get the inverse h-Laplace transform of $\frac{1}{z^{\nu}-c}$. So, in order to obtain behavior of solutions of ν-th ($\nu \neq 0.5$) order Riemann-Liouville fractional difference equations, a new technique should be developed.

4. Asymptotic Behavior of Half Order Riemann-Liouville Fractional Difference Equations

For $\nu = 0.5$, we obtain an asymptotic expansion of the solution $x(t)$ of (1.3). This enables us to obtain various properties of the solution $x(t)$ of (1.3). For example, we get a monotonicity result and sign condition for $x(t)$ for large t when $c \in (\sqrt{\frac{2}{h}}, +\infty)$ and the asymptotic estimate of $x(t)$ when $c \in (0, \sqrt{\frac{1}{h}})$.

Definition 4.1. For $|c| < h^{-\nu}$, $0 < \nu < 1$, we define the discrete Mittage-Leffler function by

$$E^h_{c,\nu,\nu-1}(t,\rho(a)) = \sum_{k=0}^{\infty} c^k \hat{H}_{\nu k+\nu-1}(t,\rho(a)), \quad t \in (h\mathbb{N})_a.$$

Remark 4.1. It is easy to see the above series is convergent for $|c| < h^{-\nu}$.

Remark 4.2. Similar to the proof of [19], we get that $E^h_{c,\nu,\nu-1}(t,\rho(a))$ is the unique solution of the IVP

$$\begin{aligned} {}_{\rho(a)}\nabla_h^{\nu}x(t) &= cx(t), \quad t \in (h\mathbb{N})_{a+h}, \\ x(a) &= \frac{h^{\nu-1}}{1-ch^{\nu}}. \end{aligned}$$

Consider the half order nabla fractional initial value problem

$$ {}_{\rho(a)}\nabla_h^{0.5}x(t) = cx(t),\ t \in (h\mathbb{N})_{a+h},\ c \neq \sqrt{\frac{1}{h}}, \tag{4.1}$$

$$ x(a) = A > 0. \tag{4.2}$$

Theorem 4.1. *Assume* $0 \le c \le \sqrt{\frac{2}{h}}$. *Then the unique solution* $x(t)$ *of the fractional initial value problem* (4.1)-(4.2) *satisfies*

$$x(a+mh) = A(c-h^{-0.5})h(hc^2-1)^{-1}\frac{(-1)^m h^{-0.5}}{(hc^2-1)^m}\Big[2h^{0.5}c - R_m[(hc^2-1)]\Big],$$

where $\lim_{m\to\infty} R_m[(hc^2-1)] = 0$.

Proof. Let $1 - zh = y$, $\nu = 0.5$. Then

$$\frac{1}{c-z^{\nu}} = \frac{1}{c-\sqrt{z}} = \frac{c+\sqrt{z}}{c^2-z} = \frac{c+\sqrt{\frac{1-y}{h}}}{c^2-\frac{1-y}{h}} \tag{4.3}$$

$$
\begin{aligned}
&=[c+h^{-0.5}(1-y)^{\frac{1}{2}}]\times h(hc^2-1)^{-1}\Big[1+\frac{y}{hc^2-1}\Big]^{-1}\\
&=h(hc^2-1)^{-1}\times\Big[c+h^{-0.5}\Big(1-0.5y+\frac{0.5(0.5-1)}{2!}y^2\\
&\quad+\cdots+(-1)^m\frac{0.5(0.5-1)\cdots(0.5-m+1)}{m!}y^m+\cdots\Big)\Big]\\
&\quad\times\Big[1-\frac{y}{hc^2-1}+\frac{y^2}{(hc^2-1)^2}+\cdots+(-1)^m\frac{y^m}{(hc^2-1)^m}+\cdots\Big]\\
&=h(hc^2-1)^{-1}\sum_{m=0}^{\infty}c_m y^m,
\end{aligned}
$$

where

$$
\begin{aligned}
a_i &= \begin{cases}(-1)^i\frac{0.5(0.5-1)\cdots(0.5-i+1)}{i!}h^{-0.5}, & i\geq 1,\\ c+h^{-0.5}, & i=0,\end{cases}\\
b_i &=(-1)^i\frac{1}{(hc^2-1)^i},\\
c_m &=a_0b_m+a_1b_{m-1}+\cdots+a_mb_0\\
&=\frac{0.5(0.5-1)\cdots(0.5-m+1)(-1)^m h^{-0.5}}{m!}\\
&\quad+\frac{-1}{hc^2-1}\frac{0.5(0.5-1)\cdots(0.5-m+2)(-1)^{m-1}h^{-0.5}}{(m-1)!}\\
&\quad+\cdots+(-0.5)\frac{(-1)^{m-1}h^{-0.5}}{(hc^2-1)^{m-1}}+(c+h^{-0.5})\frac{(-1)^m}{(hc^2-1)^m}\\
&=(-1)^m\Big[\frac{0.5(0.5-1)\cdots(0.5-m+1)h^{-0.5}}{m!}+\frac{0.5(0.5-1)\cdots(0.5-m+2)h^{-0.5}}{(m-1)!(hc^2-1)}\\
&\quad+\cdots+\frac{0.5h^{-0.5}}{(hc^2-1)^{m-1}}+\frac{(c+h^{-0.5})}{(hc^2-1)^m}\Big]\\
&=\frac{(-1)^m h^{-0.5}}{(hc^2-1)^m}\Big[\frac{0.5(0.5-1)\cdots(0.5-m+1)}{m!}(hc^2-1)^m\\
&\quad+\frac{0.5(0.5-1)\cdots(0.5-m+2)}{(m-1)!}(hc^2-1)^{m-1}\\
&\quad+\cdots+0.5(hc^2-1)+h^{0.5}(c+h^{-0.5})\Big].
\end{aligned}
$$

Note that for $|hc^2-1|\leq 1$, we have

$$
\begin{aligned}
[1+(hc^2-1)]^{0.5} =&1+0.5(hc^2-1)+\frac{0.5(0.5-1)}{2!}(hc^2-1)^2\\
&+\cdots+\frac{0.5(0.5-1)\cdots(0.5-m+1)}{m!}(hc^2-1)^m+R_m[(hc^2-1)],
\end{aligned}
\tag{4.4}
$$

where $\lim_{m\to\infty}R_m[(hc^2-1)]=0$. Therefore using (4.4) and $c>0$, we get that

$$
c_m =\frac{(-1)^m h^{-0.5}}{(hc^2-1)^m}\Big[h^{0.5}|c|-R_m[(hc^2-1)]-1+h^{0.5}(c+h^{-0.5})\Big] \tag{4.5}
$$

$$=\frac{(-1)^m h^{-0.5}}{(hc^2-1)^m}\Big[2h^{0.5}c - R_m[(hc^2-1)]\Big].$$

From (2.1), we get that

$$\mathcal{L}_{\rho(a)}\{x\}(z) = h\sum_{m=0}^{\infty}(1-zh)^m x(a+mh). \tag{4.6}$$

From (3.4) and (4.3), we get that

$$\mathcal{L}_{\rho(a)}\{x\}(z) = hA(c-h^{-0.5})h(hc^2-1)^{-1}\sum_{m=0}^{\infty}c_m(1-zh)^m. \tag{4.7}$$

From (4.6) and (4.7), we get

$$x(a+mh) = A(c-h^{-0.5})h(hc^2-1)^{-1}\frac{(-1)^m h^{-0.5}}{(hc^2-1)^m}\Big[2h^{0.5}c - R_m[(hc^2-1)]\Big]. \tag{4.8}$$

This completes the proof. □

Corollary 4.1. *Assume* $\sqrt{\frac{1}{h}} < c < \sqrt{\frac{2}{h}}$*. Then the solution* $x(t)$ *of the fractional initial value problem* (4.1)-(4.2) *satisfies*

$$\limsup_{m\to\infty} x(a+mh) = +\infty$$

and

$$\liminf_{m\to\infty} x(a+mh) = -\infty.$$

Proof. When $\sqrt{\frac{1}{h}} < c < \sqrt{\frac{2}{h}}$, we have $0 < hc^2 - 1 < 1$. From (4.8) we get the desired results. □

Corollary 4.2. *Assume* $\sqrt{\frac{1}{h}} < c < \sqrt{\frac{2}{h}}$*. Then every solution* $x(t)$ *of half order fractional initial value problem* (4.1)-(4.2) *is unstable and oscillatory.*

Corollary 4.3. *Assume* $1 < c < \sqrt{2}$*. Then every solution* $x(t)$ *of half order fractional initial value problem*

$$\begin{aligned} {}_{\rho(a)}\nabla^{0.5}x(t) &= cx(t), \quad t\in\mathbb{N}_{a+1},\ c\neq 1 \\ x(a) &= A > 0 \end{aligned} \tag{4.9}$$

is unstable and oscillatory.

From (4.5), we can get an asymptotic estimate of the half order Mitagg-Leffler function $E^h_{c,0.5,-0.5}(t,\rho(a))$.

Corollary 4.4. *Assume that* $0 < c < \sqrt{\frac{1}{h}}$*. Then every solution* $x(t)$ *of half order fractional initial value problem* (4.1)-(4.2) *tends to infinity. Furthermore,*

$$E^h_{c,0.5,-0.5}(a+mh,\rho(a)) = \frac{h^{-0.5}}{(1-hc^2)^{m+1}}\Big[2h^{0.5}c - R_m[(hc^2-1)]\Big],$$

where $\lim_{m\to\infty} R_m[(hc^2-1)] = 0$.

When $h = 1$, denote

$$E^h_{c,\mu,\mu-1}(t,\rho(a)) := E_{c,\mu,\mu-1}(t,\rho(a)) = \sum_{k=0}^{\infty} c^k H_{\mu k+\mu-1}(t,\rho(a)), \quad t \in \mathbb{N}_a,$$

where $H_{\mu k+\mu-1}(t,\rho(a)) = \frac{\Gamma(t-\rho(a)+\mu k+\mu-1)}{\Gamma(\mu k+\mu)\Gamma(t-\rho(a))}$. (See [16, Definition 3.98]).

Corollary 4.5. *Assume that $0 < c < 1$. Then every solution $x(t)$ of half order fractional initial value problem (4.9) tends to infinity. Furthermore,*

$$E_{c,0.5,-0.5}(a+m,\rho(a)) = \frac{1}{(1-c^2)^{m+1}}\Big[2c - R_m[(c^2-1)]\Big],$$

where $\lim_{m\to\infty} R_m[(c^2-1)] = 0$.

Theorem 4.2. *Assume $c = \sqrt{\frac{2}{h}}$. Then the unique solution $x(t)$ of the fractional initial value problem (4.1)-(4.2) satisfies*

$$x(a+mh) = A(\sqrt{2}-1)(-1)^m\Big[2\sqrt{2} - R_m[1]\Big], \tag{4.10}$$

where $\lim_{m\to\infty} R_m[1] = 0$.

Proof. From (4.8), when $c = \sqrt{\frac{2}{h}}$, we get the desired result. □

Corollary 4.6. *Assume $c = \sqrt{\frac{2}{h}}$. Then the unique solution $x(t)$ of the fractional initial value problem (4.1)-(4.2) satisfies*

$$\limsup_{t\to\infty} x(t) = 2\sqrt{2}(\sqrt{2}-1)A$$

and

$$\liminf_{t\to\infty} x(t) = -2\sqrt{2}(\sqrt{2}-1)A.$$

Proof. From (4.10), we get the desired results.

□

Remark 4.3. From (4.8) we know that if $c = 0$, then the solution $x(t)$ of half order fractional initial value problem (4.1)-(4.2) is asymptotically stable.

So, in this paper, we provide a new approach which is different from the one in [9] and [19] to prove the asymptotic stability of fractional difference equation (1.1) when $c = 0$.

In the following, we prove $x(t)$ tends to zero by use of (4.5) and Stolz-Cesáro theorem.

Lemma 4.1 (Stolz-Cesáro theorem)**.** *Let $\{A_n\}_{n\geq 1}$ and $\{B_n\}_{n\geq 1}$ be two sequences of real number. If B_n is positive, strictly increasing and unbounded and the following limit exists:*

$$\lim_{n\to\infty} \frac{A_{n+1}-A_n}{B_{n+1}-B_n} = l,$$

then

$$\lim_{n\to\infty} \frac{A_n}{B_n} = l.$$

Theorem 4.3. *Assume $c > \sqrt{\frac{2}{h}}$. Then the solution $x(t)$ of the fractional initial value problem (4.1)-(4.2) satisfies*

$$\lim_{t\to\infty} x(t) = 0.$$

Furthermore, for large t, $x(t) < 0$ and $x(t)$ is increasing.

Proof. Let

$$A_m = \sum_{i=1}^{m} \binom{0.5}{i}(hc^2-1)^i + h^{0.5}(c+h^{-0.5}),$$

$$B_m = (hc^2-1)^m,$$

then

$$c_m = (-1)^m h^{-0.5}\frac{A_m}{B_m}.$$

For $c > \sqrt{\frac{2}{h}}$, B_m is positive, strictly increasing and unbounded.

$$\begin{aligned}
&\lim_{m\to\infty} \frac{A_{m+1}-A_m}{B_{m+1}-B_m}\\
=&\lim_{m\to\infty} \frac{\binom{0.5}{m+1}(hc^2-1)^{m+1}}{(hc^2-1)^{m+1}-(hc^2-1)^m}\\
=&\lim_{m\to\infty} \frac{\Gamma(1.5)}{\Gamma(m+2)\Gamma(0.5-m)}\cdot\frac{hc^2-1}{hc^2-2}\\
=&\lim_{m\to\infty} \frac{\Gamma(1.5)\Gamma(0.5+m)}{\Gamma(m+2)\Gamma(0.5-m)\Gamma(0.5+m)}\cdot\frac{hc^2-1}{hc^2-2}\\
=&\lim_{m\to\infty} \frac{\Gamma(1.5)\Gamma(0.5+m)}{\Gamma(m+2)}\cdot\frac{\sin(\pi(0.5+m))}{\pi}\cdot\frac{hc^2-1}{hc^2-2}\\
=&\lim_{m\to\infty} \frac{\Gamma(0.5+m)}{\Gamma(m+2)(m+2)^{-1.5}}\cdot\frac{\Gamma(1.5)}{(m+2)^{1.5}}\cdot\frac{\sin(\pi(0.5+m))}{\pi}\cdot\frac{hc^2-1}{hc^2-2}\\
=&0,
\end{aligned}$$

where we used

$$\lim_{n\to\infty}\frac{\Gamma(n+\alpha)}{\Gamma(n)n^{\alpha}} = 1,\ \alpha\in\mathbb{C} \text{ (Stirling formula)},$$

$$\left|\frac{\sin(\pi(0.5+m))}{\pi}\right| \le \frac{1}{\pi},$$

and

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)},\ z\notin\mathbb{Z}.$$

By the Stolz-Cesáro Theorem (Lemma 4.1), we know that

$$\lim_{m\to\infty}\frac{A_m}{B_m} = 0.$$

So, we have

$$\lim_{m\to\infty} c_m = 0. \tag{4.11}$$

From (4.8), we have

$$x(a+mh) = A(c-h^{-0.5})h(hc^2-1)^{-1}c_m. \tag{4.12}$$

So from (4.11) and (4.12) we have

$$\lim_{m\to\infty} x(a+mh) = 0.$$

Next we further characterize asymptotic behavior of $x(t)$. Note that

$$\begin{aligned}
&(-1)^m\Big[\frac{0.5(0.5-1)\cdots(0.5-m+1)}{m!}(hc^2-1)^m \\
&\quad+\frac{0.5(0.5-1)\cdots(0.5-m+2)}{(m-1)!}(hc^2-1)^{m-1}\Big] \\
=&(-1)^m\Big[\frac{(-1)^{m-1}0.5(-0.5+1)\cdots(-0.5+m-1)}{m!}(hc^2-1)^m \\
&\quad+\frac{(-1)^{m-2}0.5(-0.5+1)\cdots(-0.5+m-2)}{(m-1)!}(hc^2-1)^{m-1}\Big] \\
=&\frac{0.5(-0.5+1)\cdots(-0.5+m-2)(hc^2-1)^{m-1}}{(m-1)!}\Big[\frac{0.5-m+1}{m}(hc^2-1)+1\Big] \\
=&\frac{0.5(-0.5+1)\cdots(-0.5+m-2)}{(m-2)!(m-2)^{-0.5}}\cdot\frac{(hc^2-1)^{m-1}(m-2)^{-0.5}}{m-1} \\
&\quad\cdot\Big[\frac{0.5-m+1}{m}(hc^2-1)+1\Big] \\
\to&-\infty,
\end{aligned}$$

where we use

$$\begin{aligned}
\frac{0.5(-0.5+1)\cdots(-0.5+m-2)}{(m-2)!(m-2)^{-0.5}} &\to \frac{-1}{\Gamma(-0.5)}, \\
\frac{(hc^2-1)^{m-1}(m-2)^{-0.5}}{m-1} &\to +\infty, \\
\frac{0.5-m+1}{m}(hc^2-1)+1 &\to -hc^2+2<0,
\end{aligned}$$

as $m\to+\infty$. Therefore

$$c_m < 0,$$

for large m. By virtue of (4.12), we know $x(t) < 0$ for large t.

$$\begin{aligned}
c_m - c_{m-1} =& h^{-0.5}\binom{0.5}{m}(-1)^m + h^{-0.5}\binom{0.5}{m-1}\frac{-1}{hc^2-1}(-1)^{m-1} \\
&+\cdots+h^{-0.5}\binom{0.5}{2}\Big(\frac{-1}{hc^2-1}\Big)^{m-2}(-1)^2 \\
&+h^{-0.5}\binom{0.5}{1}\Big(\frac{-1}{hc^2-1}\Big)^{m-1}(-1)^1+(c+h^{-0.5})\Big(\frac{-1}{hc^2-1}\Big)^m
\end{aligned}$$

$$
\begin{aligned}
&-\left[h^{-0.5}\binom{0.5}{m-1}(-1)^{m-1}+h^{-0.5}\binom{0.5}{m-2}\frac{-1}{hc^2-1}(-1)^{m-2}\right.\\
&\left.+(c+h^{-0.5})\left(\frac{-1}{hc^2-1}\right)^{m-1}\right]\\
=&h^{-0.5}\binom{1.5}{m}(-1)^m+h^{-0.5}\binom{1.5}{m-1}\frac{-1}{hc^2-1}(-1)^{m-1}\\
&+\cdots+h^{-0.5}\binom{1.5}{2}\left(\frac{-1}{hc^2-1}\right)^{m-2}(-1)^2\\
&+h^{-0.5}\binom{0.5}{1}\left(\frac{-1}{hc^2-1}\right)^{m-1}(-1)^1+\left(\frac{-1}{hc^2-1}\right)^m hc^2(c+h^{-0.5})\\
=&h^{-0.5}\left(\frac{-1}{hc^2-1}\right)^m\left[\binom{1.5}{m}(hc^2-1)^m+\binom{1.5}{m-1}(hc^2-1)^{m-1}\right.\\
&\left.+\cdots+\binom{1.5}{2}(hc^2-1)^2+0.5(hc^2-1)+(ch^{0.5}+1)hc^2\right]
\end{aligned}
$$

Because

$$
\begin{aligned}
&(-1)^m\left[\binom{1.5}{m}(hc^2-1)^m+\binom{1.5}{m-1}(hc^2-1)^{m-1}\right]\\
=&(-1)^m\binom{1.5}{m-1}(hc^2-1)^m\left[\frac{2.5-m}{m}+\frac{1}{hc^2-1}\right]\\
=&(-1)^{m-1}\binom{1.5}{m-1}(hc^2-1)^m\left[\frac{-2.5+m}{m}-\frac{1}{hc^2-1}\right]\\
=&\binom{m-3.5}{m-1}(hc^2-1)^m\left[\frac{-2.5+m}{m}-\frac{1}{hc^2-1}\right]\\
=&\frac{\Gamma(m-2.5)}{\Gamma(m)\Gamma(-1.5)}(hc^2-1)^m\left[\frac{-2.5+m}{m}-\frac{1}{hc^2-1}\right]\\
=&\frac{\Gamma(m-2.5)}{\Gamma(m)m^{-2.5}}\frac{(hc^2-1)^m}{m^{2.5}\Gamma(-1.5)}\left[\frac{-2.5+m}{m}-\frac{1}{hc^2-1}\right]\to+\infty,
\end{aligned}
$$

where we use $hc^2-1>1$,

$$\lim_{m\to\infty}\frac{\Gamma(m-2.5)}{\Gamma(m)m^{-2.5}}=1,$$

$$\lim_{m\to\infty}\frac{(hc^2-1)^m}{m^{2.5}\Gamma(-1.5)}=+\infty,$$

and

$$\lim_{m\to\infty}\left[\frac{-2.5+m}{m}-\frac{1}{hc^2-1}\right]=1-\frac{1}{hc^2-1}>0.$$

Similar to the proof $c_m<0$ for large m, we can get

$$c_m-c_{m-1}>0.$$

For large m, from (4.12), we know

$$x(a+mh) - x(a+(m-1)h) > 0,$$

for large m. i.e. $x(t)$ is increasing for large t. □

Let $t = a + (k-1)h$, $k \geq 2$, from Lemma 3.1, it is easy to get the recursion formula for the equation (4.1)

$$x(a+(k-1)h) = \frac{1}{ch^{\nu}-1}\left[\sum_{i=1}^{k-1}\binom{k-i-\mu-1}{k-i}x(a+(i-1)h)\right], \ k \geq 2. \quad (4.13)$$

From formula (4.13), we know $x(t)$ depends on initial value $x(a)$ and is independent of staring point $\rho(a)$.

As a result, by virtue of [9, 19], Corollary 4.3, Corollary 4.5, Theorem 4.2, and Theorem 4.3 we can summarize the results as following: A solution $x(t)$ of half order fractional initial value problem (4.9)

(1) tends to zero if $c \leq 0$.
(2) tends to infinity if $0 < c < 1$.
(3) is oscillatory unstable if $1 < c < \sqrt{2}$.
i.e.

$$\limsup_{t\to\infty} x(t) = +\infty$$

and

$$\liminf_{t\to\infty} x(t) = -\infty.$$

(4) is oscillatory stable if $c = \sqrt{2}$. i.e.

$$\limsup_{t\to\infty} x(t) = 2\sqrt{2}(\sqrt{2}-1)A$$

and

$$\liminf_{t\to\infty} x(t) = -2\sqrt{2}(\sqrt{2}-1)A.$$

(5) tends to zero if $c > \sqrt{2}$.

5. Examples

In this section we give two examples concerning Remark 4.2, Corollary 4.2, Corollary 4.3, and Theorem 4.2. We use the recursion formula (4.13) to enable us to plot the solutions of the initial value problems in our two examples. Figure 3, Table 1, Figure 6, and Table 2 are concerned with Theorem 4.3.

Example 5.1. Consider the initial value problem in the form

$$\begin{aligned} {}_0\nabla^{0.5}x(t) &= cx(t), \quad t \in \mathbb{N}_2, \quad c \neq 1, \\ x(1) &= 1. \end{aligned} \quad (5.1)$$

We plot the solution $x(t)$ in Figures 1, 2, 3. Note that $x(t)$ of fractional initial value problem (5.1)
(1) is unstable and oscillatory if $c = 1.4$.
(2) is oscillatory stable if $c = \sqrt{2}$.
(3) tends to zero if $c = 1.5$.

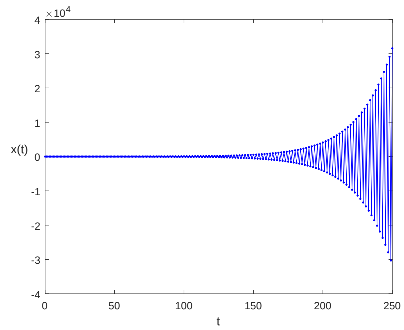

Figure 1. $c = 1.4,\ a = h = 1$.

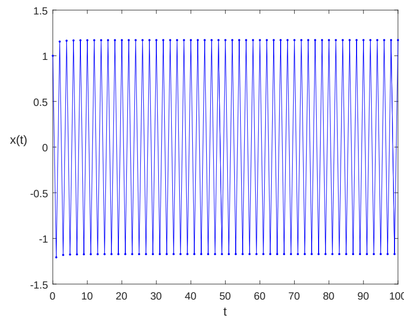

Figure 2. $c = \sqrt{2},\ a = h = 1$.

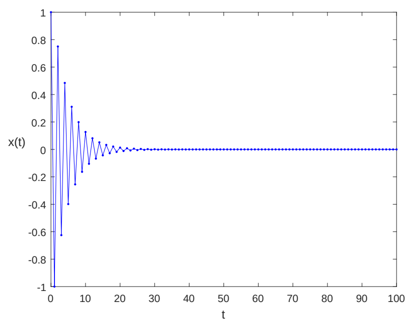

Figure 3. $c = 1.5,\ a = h = 1$.

Table 1. Table of $x(t)$ for $c = 1.5$ in Example 5.1.

t	$x(t)$	t	$x(t)$	t	$x(t)$
70	-0.0001091500	78	-0.0000924634	86	-0.0000797224
71	-0.0001063871	79	-0.0000906218	87	-0.0000783268
72	-0.0001045051	80	-0.0000889699	88	-0.0000769932
73	-0.0001020604	81	-0.0000872639	89	-0.0000756796
74	-0.0001002011	82	-0.0000856946	90	-0.0000744169
75	-0.0000980024	83	-0.0000841054	91	-0.0000731777
76	-0.0000961981	84	-0.0000826179	92	-0.0000719815
77	-0.0000941955	85	-0.0000811311	93	-0.0000708104

Example 5.2. Consider the initial value problem in the form

$$\begin{aligned} {}_{\rho(0)}\nabla_4^{0.5}x(t) &= cx(t), \quad t \in (4\mathbb{N})_4, \quad c \neq \frac{1}{2}, \\ x(0) &= 1. \end{aligned} \tag{5.2}$$

We plot the solution $x(t)$ in Figures 4, 5, 6. Note that if $x(t)$ is a solution of the fractional initial value problem (5.2), then $x(t)$

(1) is oscillatory unstable if $c = 0.7$.

(2) is oscillatory stable if $c = \sqrt{\frac{1}{2}}$.

(3) tends to zero if $c = 0.8$.

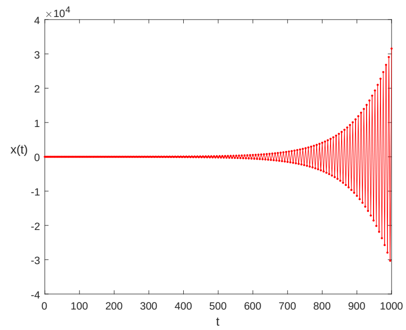

Figure 4. $c = 0.7,\ a = 0,\ h = 4.$

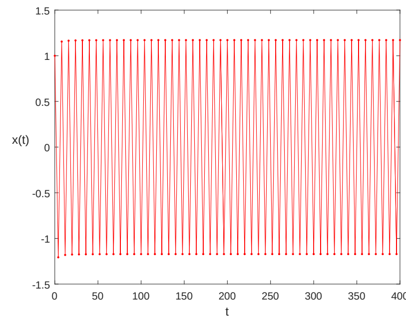

Figure 5. $c = \sqrt{\frac{1}{2}},\ a = 0,\ h = 4.$

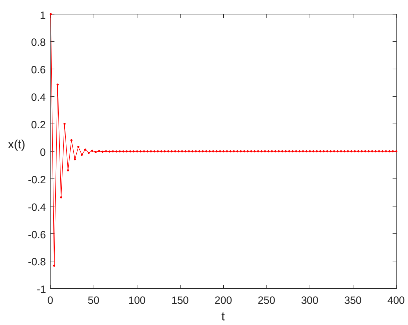

Figure 6. $c = 0.8,\ a = 0,\ h = 4.$

Table 2. Table of $x(t)$ for $c = 0.8$ in Example 5.2.

t	$x(t)$	t	$x(t)$	t	$x(t)$
120	-0.000397414	152	-0.000280561	184	-0.000210912
124	-0.000381588	156	-0.000269971	188	-0.000204241
128	-0.000361905	160	-0.000259891	192	-0.000197910
132	-0.000346954	164	-0.000250513	196	-0.000191902
136	-0.000330996	168	-0.000241628	200	-0.000186190
140	-0.000317508	172	-0.000233293	204	-0.000180757
144	-0.000304084	176	-0.000225402	208	-0.000175583
148	-0.000292110	180	-0.000217962	212	-0.000170651

References

[1] G. Anastassiou, *Foundations of nabla fractional calculus on time scales and inequalities*, Comput. Appl. Math., 2010, 59, 3750–3762.

[2] J. O. Alzabut and T. Abdeljawad, *Sufficient conditions for the oscillation of nonlinear fractional difference equations*, J. Fract. Calc. Appl., 2014, 5(1), 177–187.

[3] F. M. Atici and P. W. Eloe, *Initial value problems in discrete fractional calculus*, P. Amer. Math. Soc., 2009, 137, 981–989.

[4] F. M. Atici and P. W. Eloe, *Discrete fractional calculus with the nabla operator*, Electron. J. Qual. Theo., 2009, 3, 1–12.

[5] Z. Bai and R. Xu, *The asymptotic behavior of solutions for a class of nonlinear fractional difference equations with damping term*, Discrete. Dyn. Nat. Soc., 2018. DOI: org/10.1155/2018/5232147.

[6] D. Baleanu, G. Wu, Y. Bai and F. Chen, *Stability analysis of Caputo-like discrete fractional systems*, Commun. Nonlinear. Sci., 2017, 48, 520–530.

[7] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[8] M. Bohner and A. Peterson, *Dynamic Equations on time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[9] J. Čermák, T. Kisela and L. Nechvátal, *Stability and asymptotic properties of a linear fractional difference equation*, Adv. Differ. Equ., 2012. DOI: org/10.1186/1687-1847-2012-122.

[10] J. Čermák, T. Kisela and L. Nechvátal, *Stability regions for linear fractional differential systems and their discretizations*, Appl. Math. Comput., 2013, 219, 7012–7022.

[11] J. Čermák, I. Győri and L. Nechvátal, *On explicit stability conditions for a linear fractional difference system*, Fract. Calc. Appl. Anal., 2015, 18(3), 651–672.

[12] R. A. Z. Daou and X. Moreau, *Fractional Calculus: Theory*, Nova Science Publishers, New York, 2015.

[13] F. Du, L. Erbe, B. Jia and A. Peterson, *Two asymptotic results of solutions for nabla fractional (q, h)-difference equations*, Turk. J. Math., 2018. DOI: 10.3906/mat-1802-49.

[14] R. A. C. Ferreira, *A discrete fractional Gronwall inequality*, Proc. Amer. Math. Soc., 2012, 140, 1605–1612.

[15] R. A. C. Ferreira and D. F. M. Torres, *Fractional h-difference equations arising from the calculus of variations*, Appl. Anal. Discr. Math., 2011, 5, 110–121.

[16] C. Goodrich and A. Peterson, *Discrete Fractional Calculus*, Springer, New York, 2015.

[17] W. Kelley and A. Peterson, *Difference Equations: An Introduction With Applications*, Second Edition, Harcourt/Academic Press, New York, 2001.

[18] W. Li and W. Sheng, *Sufficient conditions for oscillation of a nonlinear fractional nabla difference system*, Springer Plus, 2016. DOI: org/10.1186/s40064-016-2820-2.

[19] B. Jia, L. Erbe and A. Peterson, *Comparison theorems and asymptotic behavior of solutions of discrete fractional equations*, Electron. J. Qual. Theo., 2015, 89, 1–18.

[20] R. Xu and Y. Zhang, *Generalized Gronwall fractional summation inequalities and their applications*, J. Inequal. Appl., 2015. DOI:10.1186/s13660-015-0763-8