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# THREE POSITIVE SOLUTIONS FOR BOUNDARY VALUE PROBLEM FOR DIFFERENTIAL EQUATION WITH RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE*

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**Abstract** In this paper, by using the Avery-Peterson fixed point theorem, we establish the existence result of at least three positive solutions of boundary value problem of nonlinear differential equation with Riemann-Liouville's fractional order derivative. An example illustrating our main result is given. Our results complements and extends previous work in the area of boundary value problems of nonlinear fractional differential equations.

**Keywords** Fractional differential equation, fixed point, positive solution, cone, Avery-Peterson fixed point theorem.

**MSC(2010)** 26A33, 34B15.

## 1. Introduction

Due to the development of the theory of fractional calculus and its applications, such as in the fields of physics, Bode's analysis of feedback amplifiers, aerodynamics and polymer rheology etc, many works on the basic theory of fractional calculus and fractional order differential equations have been established [10, 19–23, 27]. Recently, there have been many papers dealing with the solutions or positive solutions of boundary value problems for nonlinear fractional differential equations(FBVPs) with various boundary conditions [1, 5, 9, 13, 16, 17, 26, 32, 34, 36–39] and nonlocal boundary conditions [2, 3, 6, 24, 29, 31, 33] and references along this line.

In a recent paper [11], Moustafa El-Shahed considered a class of fractional boundary value problem of the form

$$D^{\alpha}_{0+}u(t) + \lambda a(t)f(u(t)),\ t \in (0,\ 1),$$

$$u(0) = u'(0) = u'(1) = 0$$

where $\lambda$ is a positive parameter, $a(t) : (0, 1) \to [0, \infty)$ is continuous with $\int_0^1 a(t)dt > 0$ and $f : [0,\ +\infty) \to f : [0,\ +\infty)$ is continuous. Here $D^{\alpha}_{0+}$ was the Riemann-Liouville's fractional derivative of order $\alpha$. By using the fixed point theorem, the

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author established the existence and nonexistence of positive solutions for this nonlinear fractional boundary value problem.

However, in this work, the existence of positive solutions for FBVPs were established under the assumption that the derivative of the unknown function was not involved in the nonlinear term. To our best of knowledge, few papers can be found in the literature for positive solutions of FBVPs where the derivative of the unknown function is involved in the nonlinear term. The purpose of this paper is to fill this gap.

In this paper, we will consider positive solutions for the following FBVPs

$$D_{0+}^{\alpha}u(t) + f(t,\ u(t),\ D_{0+}^{\beta}u(t)),\ t \in (0,\ 1), \tag{1.1}$$

$$u(0) = u'(0) = u'(1) = 0 \tag{1.2}$$

where $2 < \alpha \leq 3,\ 0 < \beta < 1$ and $f : C([0,1] \times R^{+} \times R \to R^{+}$.

Boundary value problems for differential equations of integer order where the derivative of the unknown function is involved in the nonlinear term have been studied by extensively, see Guo and Ge [14, 15], Avery and Peterson [4], Bai and Ge [7, 8], Yang, Liu and Jia [35] etc. In [4], Avery and Peterson gave a new triple fixed point theorem, which can be regarded as an extension of Leggett-Williams fixed point theorem. Recently, this fixed point theorem has been used as a classical method for seeking the positive solutions for BVPs of nonlinear differential equations where the lower order derivatives of unknown function is involved in the nonlinear term, see [4, 12, 18, 25, 28, 31, 35].

However, this classical method can not be used directly to investigate the existence of positive solutions of boundary value problems of nonlinear differential equations of fractional order. The main reason is that we cannot derive the concavity or convexity of the function $u(t)$ by the sign of its fractional order derivative. In this paper, by obtaining some new inequalities of the unknown function and defining a special cone, we overcome the difficulties bringing by the lack of the concavity or convexity of the the unknown function . By an application of Avery-Peterson fixed point theorem, the existence of at least three positive solutions of problem (1.1-1.2) is established. An example is given to illustrate the main results of this paper. Our results extend and complements some previous works in the area of boundary value problems of nonlinear fractional differential equations, such as Moustafa El-Shahed [29].

## 2. Preliminaries

**Definition 2.1.** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u(t)$ is given by

$$I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds.$$

**Definition 2.2.** The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $u(t)$ is given by

$$D_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^{n}\int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}}ds$$

where $n = [\alpha] + 1$. As examples, for $\lambda > -1$, we have

$$D_{0+}^{\alpha} u^{\lambda} = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda-\alpha)} u^{\lambda-\alpha}.$$

**Lemma 2.1.** *Let $\alpha > 0$. The fractional differential equation $D_{0+}^{\alpha} u(t) = 0$ has solution*

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + ... + C_n t^{\alpha-n}$$

*for some $C_i \in R,\ i = 1,\ 2,\ \cdots,\ n$.*

**Lemma 2.2.** *Assume that $u(t)$ with a fractional derivative of order $\alpha > 0$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + ... + C_n t^{\alpha-n},$$

*for some $C_i \in R,\ i = 1,\ 2,\ \cdots,\ n$.*

**Lemma 2.3.** *Let $\alpha > 0,\ \beta > 0$ and $f \in L_p(0,\ 1)(1 \le p \le \infty)$, then for almost everywhere on [0, 1], we have*

$$I^{\alpha} I^{\beta} f(t) = I^{\alpha+\beta} f(t).$$

**Definition 2.3.** Let $E$ be a real Banach space. A nonempty convex closed set $P$ is called a cone provided that:
(1) $au \in P$, for all $u \in P,\ a \ge 0$;
(2) $u, -u \in P$ implies $u = 0$.

**Definition 2.4.** The map $\alpha$ is said to be a nonnegative continuous convex functional on the cone $P$ of a Banach space $E$ provided that $\alpha : P \to [0, +\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \le t\alpha(x) + (1-t)\alpha(y),\ x,\ y \in P,\ t \in [0,1].$$

**Definition 2.5.** The map $\beta$ is said to be a nonnegative continuous concave functional on the cone $P$ of a Banach space $E$ provided that $\beta : P \to [0, +\infty)$ is continuous and

$$\beta(tx + (1-t)y) \ge t\beta(x) + (1-t)\beta(y),\ x,\ y \in P,\ t \in [0,1].$$

Let $\gamma, \theta$ be nonnegative continuous convex functionals on $P$, $\alpha$ be a nonnegative continuous concave functional on $P$ and $\psi$ be a nonnegative continuous functional on P. Then for positive numbers $a,\ b,\ c$ and $d$, we define the following convex sets:

$$P(\gamma,\ d) = \{x \in P | \gamma(x) < d\},$$
$$P(\gamma,\ \alpha,\ b,\ d) = \{x \in P | b \le \alpha(x),\ \gamma(x) \le d\},$$
$$P(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d) = \{x \in P | b \le \alpha(x),\ \theta(x) \le c,\ \gamma(x) \le d\},$$

and a closed set

$$R(\gamma,\ \psi,\ a,\ d) = \{x \in P | a \le \psi(x),\ \gamma(x) \le d\}.$$

**Lemma 2.4.** *Let $P$ be a cone in Banach space E. Let $\gamma,\ \theta$ be nonnegative continuous convex functionals on $P$, $\alpha$ be a nonnegative continuous concave functional on $P$, and $\psi$ be a nonnegative continuous functional on $P$ satisfying*

$$\psi(\lambda x) \le \lambda \psi(x), \quad \textit{for} \ \ 0 \le \lambda \le 1, \tag{2.1}$$

$$\alpha(x) \leq \psi(x),\ \|x\| \leq l\gamma(x)\ \textit{for}\ \ x \in \overline{P(\gamma,\ d)}, \tag{2.2}$$

*Suppose* $T : \overline{P(\gamma,\ d)} \to \overline{P(\gamma,\ d)}$ *is completely continuous and there exist positive numbers* $a,\ b,\ c$ *with* $a < b$ *such that*

($S_1$) $\{x \in P(\gamma, \theta, \alpha, b, c, d) | \alpha(x) > b\} \neq \emptyset$ *and* $\alpha(Tx) > b$ *for* $x \in P(\gamma, \theta, \alpha, b, c, d)$*;*

($S_2$) $\alpha(Tx) > b$ *for* $x \in P(\gamma,\ \alpha,\ b,\ d)$ *with* $\theta(Tx) > c$*;*

($S_3$) $0 \notin R(\gamma,\ \psi,\ a,\ d)$ *and* $\psi(Tx) < a$ *for* $x \in R(\gamma,\ \psi,\ a,\ d)$ *with* $\psi(x) = a$.

*Then* $T$ *has at least three fixed points* $x_1,\ x_2,\ x_3 \in \overline{P(\gamma,\ d)}$ *such that:*

$$\gamma(x_i) \leq d,\ i = 1,\ 2,\ 3;\ b < \alpha(x_1);\ a < \psi(x_2),\ \alpha(x_2) < b;\ \psi(x_3) < a. \tag{2.3}$$

# 3. Main results

**Lemma 3.1.** *Given* $y(t) \in C[0,1]$*. Then the following FBVPs*

$$D^{\alpha}_{0+}u(t) + y(t) = 0,\ t \in (0,\ 1), \tag{3.1}$$

$$u(0) = u'(0) = u'(1) = 0, \tag{3.2}$$

*is equivalent to operator equation*

$$u(t) = \int_0^1 G(t,\ s)y(s)ds,$$

*where*

$$G(t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} & 0 \leq t \leq s \leq 1 \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} - \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} & 0 \leq s \leq t \leq 1 \end{cases}$$

**Lemma 3.2.** *Let* $G(t,s)$ *be given as in the statement of Lemma 3.1. Then we find that*

(1) $G(t,s)$ *is a continuous and nonnegative function on the unit square* $[0,1]\times[0,1]$*;*

(2) $t^{\alpha-1}G(1,\ s) \leq G(t,s) \leq G(1,\ s) = \dfrac{(1-s)^{\alpha-2} - (1-s)^{\alpha-1}}{\Gamma(\alpha)},\ t,\ s \in [0,1]$.

**Lemma 3.3.** *Assume that* $y(t) > 0$ *and* $u(t)$ *is a solution of problem (3.1-3.2). Then*

$$\max_{0\leq t\leq 1}|u(t)| \leq \gamma_1 \max_{0\leq t\leq 1}|D^{\beta}_{0+}u(t)|,$$

*where* $\gamma_1 = \dfrac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\dfrac{1}{1-\beta} > 0$.

**Proof.** Considering Lemma 2.1-2.3 and the boundary conditions (3.2), we have

$$\begin{aligned} u(t) &= \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}y(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^1 [(1-s)^{\alpha-2} - (1-s)^{\alpha-1}]y(s)ds. \end{aligned}$$

Then, from the definition of the fractional derivative of Riemann-Liouvile, we have

$$D_{0+}^{\beta}u(t) = -\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}y(s)ds + \frac{t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-2}y(s)ds.$$

Thus

$$\max_{0\le t\le 1} |D_{0+}^{\beta}u(t)| \ge |D_{0+}^{\beta}u(1)| = \frac{1}{\Gamma(\alpha-\beta)}\int_0^1 [(1-s)^{\alpha-2} - (1-s)^{\alpha-\beta-1}]y(s)ds.$$

Define the function

$$h(s) = \frac{(1-s)^{\alpha-2} - (1-s)^{\alpha-1}}{(1-s)^{\alpha-2} - (1-s)^{\alpha-\beta-1}},\ s \in (0,\ 1).$$

The fact that $h(s)$ is decreasing on $(0,\ 1)$ and

$$\lim_{s\to 0} h(s) = \lim_{s\to 0}\frac{(1-s)^{\alpha-2} - (1-s)^{\alpha-1}}{(1-s)^{\alpha-2} - (1-s)^{\alpha-\beta-1}} = \frac{1}{1-\beta}$$

ensures that

$$h(s) < \frac{1}{1-\beta},\ 0 < s < 1.$$

Then

$$\begin{aligned}\max_{0\le t\le 1} u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^1 [(1-s)^{\alpha-2} - (1-s)^{\alpha-1}]y(s)ds \\ &\le \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\frac{1}{\Gamma(\alpha-\beta)}\frac{1}{1-\beta}\int_0^1 [(1-s)^{\alpha-2} - (1-s)^{\alpha-\beta-1}]y(s)ds \\ &\le \gamma_1 \max_{0\le t\le 1} |D_{0+}^{\beta}u(t)|.\end{aligned}$$

Let the Banach space $E = \{u(t) \in C[0,\ 1],\ D_{0+}^{\beta}u(t) \in C[0,\ 1]\}$ be endowed with the norm

$$\|u\| = \max\{\max_{0\le t\le 1} |u(t)|,\ \max_{0\le t\le 1} |D_{0+}^{\beta}u(t)|\},\ u \in E.$$

Let $\gamma_0 = (\frac{1}{3})^{\alpha-1}$. We define the cone $P \subset E$ by

$$P = \{u \in E \mid u(t) \ge 0, \min_{1/3\le t\le 2/3} u(t) \ge \gamma_0 \max_{0\le t\le 1} u(t), \max_{0\le t\le 1} u(t) \le \gamma_1 \max_{0\le t\le 1} |D_{0+}^{\beta}u(t)|\}.$$

□

**Lemma 3.4.** *Let* $T:\ P \to E$ *be the operator defined by*

$$Tu(t) := \int_0^1 G(t,\ s)f(s,\ u(s),\ D_{0+}^{\beta}u(s))ds.$$

*Then* $T : P \to P$ *is completely continuous.*

**Proof.** The operator $T$ is nonnegative and continuous obviously. Let $\Omega \subset K$ be bounded. Then there exist a positive constant $R_1 > 0$ such that $\|u\| \le R_1,\ u \in \Omega$. Denote

$$R = \max_{0\le t\le 1,\ u\in\Omega} |f(t,\ u(t),\ D^{\beta}_{0+}u(t)| + 1.$$

Then for $u \in \Omega$, we have

$$\begin{aligned}
|(Tu)(t)| &\le \int_0^1 G(t,\ s)|f(s,\ u(s),\ D^{\beta}_{0+}u(s))|ds \\
&\le \int_0^1 G(1,\ s)|f(s,\ u(s),\ D^{\beta}_{0+}u(s))|ds \\
&\le \frac{R}{(\alpha-1)\Gamma(\alpha+1)}, \\
|D^{\beta}_{0+}(Tu)(t)| &= |\frac{t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-2} f(s,\ u,\ D^{\beta}_{0+}u)ds \\
&\quad - \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1} f(s,\ u,\ D^{\beta}_{0+}u)ds| \\
&\le \frac{R}{(\alpha-\beta)\Gamma(\alpha-\beta)} + \frac{R}{\Gamma(\alpha-\beta)(\alpha-1)}.
\end{aligned}$$

Hence $T(\Omega)$ is bounded. On the other hand, for $u \in \Omega,\ t_1,\ t_2 \in [0,\ 1]$, one has

$$\begin{aligned}
|Tu(t_2) - Tu(t_1)| =& |\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} f(s,\ u(s),\ D^{\beta}_{0+}u(s))ds \times (t_2^{\alpha-1} - t_1^{\alpha-1}) \\
&+ \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,\ u(s),\ D^{\beta}_{0+}u(s))ds \\
&- \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,\ u(s),\ D^{\beta}_{0+}u(s))ds| \\
\le& \frac{R}{\Gamma(\alpha)(\alpha-1)} \times |t_2^{\alpha-1} - t_1^{\alpha-1}| + \frac{R}{\Gamma(\alpha+1)} \times |t_2^{\alpha} - t_1^{\alpha}|,
\end{aligned}$$

$$\begin{aligned}
&|D^{\beta}_{0+}(Tu)(t_2) - D^{\beta}_{0+}(Tu)(t_1)| \\
=& |\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-2} f(s,\ u(s),\ D^{\beta}_{0+}u(s))ds \times |t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1}| \\
&+ \frac{1}{\Gamma(\alpha-\beta)}(\int_0^{t_1} (t_1-s)^{\alpha-\beta-1} f(s,\ u(s),\ D^{\beta}_{0+}u(s))ds \\
&- \int_0^{t_2} (t_2-s)^{\alpha-\beta-1} f(s,\ u(s),\ D^{\beta}_{0+}u(s))ds)| \\
\le& \frac{R}{\Gamma(\alpha-\beta)(\alpha-1)} \times |t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1}| + \frac{R}{\Gamma(\alpha-\beta)(\alpha-\beta)} \times |t_2^{\alpha-\beta} - t_1^{\alpha-\beta}|.
\end{aligned}$$

Thus,

$$\|Tu(t_2) - Tu(t_1)\| \to 0, \text{ for } t_1 \to t_2.$$

By means of the Arzela-Ascoli theorem, $T$ is completely continuous. Furthermore, for $u \in P$, we have

$$\begin{aligned}
\min_{\frac{1}{3}\le t\le \frac{2}{3}} Tu(t) &= \min_{\frac{1}{3}\le t\le \frac{2}{3}} \int_0^1 G(t,\ s)f(s,\ u(s),\ D_{0+}^{\beta}u(s))ds \\
&\ge (\frac{1}{3})^{\alpha-1}\int_0^1 G(1,\ s)f(s,\ u(s),\ D_{0+}^{\beta}u(s))ds \\
&= \gamma_0 \max_{0\le t\le 1} Tu(t), \\
\max_{0\le t\le 1} Tu(t) &= \frac{1}{\Gamma(\alpha)}\int_0^1 [(1-s)^{\alpha-2}-(1-s)^{\alpha-1}]f(s,\ u(s),\ D_{0+}^{\beta}u(s))ds \\
&\le \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\frac{1}{\Gamma(\alpha-\beta)}\frac{1}{1-\beta}\int_0^1[(1-s)^{\alpha-2}-(1-s)^{\alpha-\beta-1}]f(s,u(s),D_{0+}^{\beta}u(s))ds \\
&\le \gamma_1 \max_{0\le t\le 1}|D_{0+}^{\beta}Tu(t)|.
\end{aligned}$$

Thus, $T: P \to P$ is completely continuous.

Let the nonnegative continuous concave functional $\alpha$, the nonnegative continuous convex functionals $\gamma,\ \theta$ and the nonnegative continuous functional $\psi$ be defined on the cone by

$$\gamma(u) = \max_{0\le t\le 1}|D_{0+}^{\beta}u(t)|,\ \theta(u) = \psi(u) = \max_{0\le t\le 1}|u(t)|,\ \alpha(u) = \min_{\frac{1}{3}\le t\le \frac{2}{3}}|u(t)|.$$

By Lemmas 3.2 and 3.3, the functionals defined above satisfy

$$\gamma_0\theta(u) \le \alpha(u) \le \theta(u) = \psi(u),\ \|u\| \le \gamma_2\gamma(u),\ u \in P,$$

where $\gamma_2 = \max\{\gamma_1,\ 1\}$. Therefore condition (2.1, 2.2) of Lemma 2.4 are satisfied. Assume that there exist constants $0 < a,\ b,\ d,\ c = b/\gamma_0$ with

$$3^{\alpha-1}(2\alpha-\beta-1)\Gamma(\alpha+1)b < \Gamma(\alpha-\beta+1)d$$

such that

$(A_1)$ $f(t,\ u,\ v) \le \frac{\Gamma(\alpha-\beta)(\alpha-1)(\alpha-\beta)}{2\alpha-\beta-1}d, (t,\ u,\ v) \in [0,1]\times[0,\ \gamma_2 d]\times[-d,\ d]$;

$(A_2)$ $f(t,\ u,\ v) > \frac{(\alpha-1)\Gamma(\alpha+1)}{\gamma_0}b, (t,\ u,\ v) \in [1/3,\ 2/3]\times[b,\ b/\gamma_0]\times[-d,d]$;

$(A_3)$ $f(t,\ u,\ v) < (\alpha-1)\Gamma(\alpha+1)a, (t,\ u,\ v) \in [0,\ 1]\times[0,\ a]\times[-d,\ d]$.

□

**Theorem 3.1.** *Under assumptions $(A_1)$–$(A_3)$, the boundary value problem (1.1, 1.2) has at least three positive solutions $u_1(t),\ u_2(t),\ u_3(t)$ satisfying*

$$\begin{gathered}
\max_{0\le t\le 1}|D_{0+}^{\beta}u(t)| \le d,\ i = 1,\ 2,\ 3; \\
b < \min_{1/3\le t\le 2/3}|u_1(t)|;\ a < \max_{0\le t\le 1}|u_2(t)|,\ \min_{1/3\le t\le 2/3}|u_2(t)| < b; \\
\max_{0\le t\le 1}|u_3(t)| \le a.
\end{gathered}$$

**Proof.** Problem (1.1, 1.2) has a solution $u = u(t)$ if and only if $u$ solves the operator equation

$$u(t) = \int_0^1 G(t,s)f(s,\ u(s),\ D_{0+}^{\beta}u(s))ds = (Tu)(t).$$

For $u \in \overline{P(\gamma,\ d)}$, we have $\gamma(u) = \max_{0\le t\le 1} |D_{0+}^{\beta} u(t)| < d$. From assumption $(A_1)$, we obtain

$$f(t,\ u(t),\ D_{0+}^{\beta} u(t)) \le \frac{\Gamma(\alpha-\beta)(\alpha-1)(\alpha-\beta)}{2\alpha-\beta-1} d.$$

Thus

$$\begin{aligned}\gamma(Tu) &= \max_{0\le t\le 1} |D_{0+}^{\beta}(Tu)| \\ &= |\frac{t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-2} f(s,\ u(s),\ D_{0+}^{\beta} u(s))ds \\ &\quad - \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1} f(s,\ u(s),\ D_{0+}^{\beta} u(s))ds| \\ &\le [\frac{1}{(\alpha-\beta)\Gamma(\alpha-\beta)} + \frac{1}{\Gamma(\alpha-\beta)(\alpha-1)}] \times \frac{\Gamma(\alpha-\beta)(\alpha-1)(\alpha-\beta)}{2\alpha-\beta-1} d = d.\end{aligned}$$

Hence, $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$.

The fact that the constant function $u(t) = c = \frac{b}{\gamma_0} \in P(\gamma,\theta,\alpha,b,c,d)$ and $\alpha\left(\frac{b}{\gamma_0}\right) > b$ implies that $\{u \in P(\gamma,\theta,\alpha,b,c,d|\alpha(u) > b)\} \ne \emptyset$.

For $u \in P(\gamma,\theta,\alpha,b,c,d)$, we have $b \le u(t) \le \frac{b}{\gamma_0}$ and $|D_{0+}^{\beta} u(t)| < d,\ 0 \le t \le 1$. From assumption $(A_2)$,

$$f(t,\ u(t),\ D_{0+}^{\beta} u(t)) > \frac{(\alpha-1)\Gamma(\alpha+1)}{\gamma_0} b.$$

Thus

$$\begin{aligned}\alpha(Tu) &= \min_{\frac{1}{3}\le t\le \frac{2}{3}} \int_0^1 G(t,\ s) f(s,\ u(s),\ D_{0+}^{\beta} u(s))ds \\ &\ge \gamma_0 \int_0^1 G(1,\ s) f(s,\ u(s),\ D_{0+}^{\beta} u(s))ds \\ &\ge \gamma_0 \int_0^1 G(1,\ s)ds \times \frac{(\alpha-1)\Gamma(\alpha+1)}{\gamma_0} b = b,\end{aligned}$$

which means $\alpha(Tu) > b,\ \forall u \in P(\gamma,\theta,\alpha,b,\frac{b}{\gamma_0},d)$. These ensure that condition $(S1)$ of Lemma 2.4 is satisfied. Also, for all $u \in P(\gamma,\alpha,b,d)$ with $\theta(Tu) > \frac{b}{\gamma_0}$,

$$\alpha(Tu) \ge \gamma_0\theta(Tu) > \gamma_0 \times c = \gamma_0 \times \frac{b}{\gamma_0} = b.$$

Thus, condition $(S_2)$ of Lemma 2.4 holds. Finally we show that $(S_3)$ also holds. We see that $\psi(0) = 0 < a$ and $0 \notin R(\gamma,\psi,a,d)$. Suppose that $u \in R(\gamma,\psi,a,d)$ with $\psi(x) = a$. Then by assumption $(A_3)$,

$$\begin{aligned}\psi(Tu) = \max_{0\le t\le 1} |(Tu)(t)| &= \int_0^1 G(t,\ s) f(s,\ u(s),\ D_{0+}^{\beta} u(s))ds \\ &\le \int_0^1 G(1,\ s)ds \times (\alpha-1)\Gamma(\alpha+1)a \\ &= a.\end{aligned}$$

Thus, all conditions of Lemma 2.4 are satisfied. Hence problem (1.1, 1.2) has at least three positive solutions $u_1(t),\ u_2(t),\ u_3(t)$ satisfying

$$\max_{0\le t\le 1}|D^{\beta}_{0+}u(t)|\le d,\ i=1,\ 2,\ 3;$$

$$b<\min_{1/3\le t\le 2/3}|u_1(t)|; a<\max_{0\le t\le 1}|u_2(t)|,\ \min_{1/3\le t\le 2/3}|u_2(t)|<b;$$

$$\max_{0\le t\le 1}|u_3(t)|\le a.$$

□

## 4. Example

Consider the nonlinear FBVP

$$D^{\alpha}_{0+}u(t)+f(t,\ u(t),\ D^{\beta}_{0+}u(t))=0,\ t\in(0,\ 1), \tag{4.1}$$

$$u(0)=u'(0)=u'(1)=0, \tag{4.2}$$

where $\alpha=2.7,\ \beta=0.6, n=3$ and

$$f(t,u,v)=\begin{cases}\frac{1}{20}e^t+5u^4+\frac{1}{100}\sin\left(\frac{v}{10000}\right), & 0\le u\le 6,\\ \frac{1}{20}e^t+6480+\frac{1}{100}\sin\left(\frac{v}{10000}\right), & u>6.\end{cases}$$

Choose $a=1,\ b=3,\ d=10000$. By a simple computation, we obtain that

$$\gamma_0=\left(\frac{1}{3}\right)^{1.7},\ \gamma_1=\frac{5}{2}\frac{\Gamma(2.1)}{\Gamma(2.7)}\approx 1.6937,\ \gamma_2=\max\{\gamma_1,\ 1\}=\gamma_1,$$

$$(\alpha-1)\Gamma(\alpha+1)a\approx 7.0901,\ \frac{(\alpha-1)\Gamma(\alpha+1)}{\gamma_0}b\approx 137.6830,$$

$$\frac{\Gamma(\alpha-\beta)(\alpha-1)(\alpha-\beta)}{2\alpha-\beta-1}d\approx 9831.$$

We can check that the nonlinear term $f(t,\ u,\ v)$ satisfies

(1) $f(t,u,v)<\frac{\Gamma(\alpha-\beta)(\alpha-1)(\alpha-\beta)}{2\alpha-\beta-1}d, (t,u,v)\in[0,1]\times[0,16937]\times[-10000,10000];$

(2) $f(t,\ u,\ v)>\frac{(\alpha-1)\Gamma(\alpha+1)}{\gamma_0}b,\ (t,\ u,\ v)\in[\frac{1}{2},\ 1]\times[3,\ 3\times 3^{1.7}]\times[-10000,\ 10000];$

(3) $f(t,\ u,\ v)<(\alpha-1)\Gamma(\alpha+1)a,\ (t,\ u,\ v)\in[0,\ 1]\times[0,\ 1]\times[-10000,\ 10000].$

Then all assumptions of Theorem 3.1 are satisfied. Thus problems (4.1–4.2) has at least three positive solutions $u_1(t),\ u_2(t),\ u_3(t)$ satisfying

$$\max_{0\le t\le 1}|D^{\beta}_{0+}u(t)|\le 10000,\ i=1,\ 2,\ 3;$$

$$3<\min_{1/2\le t\le 1}|u_1(t)|,\ 1<\max_{0\le t\le 1}|u_2(t)|,$$

$$\min_{1/2\le t\le 1}|u_2(t)|<3, \max_{0\le t\le 1}|u_3(t)|\le 1.$$

**Remark.** We see that the fractional derivative of the function $u(t)$ of order $\beta$ is involved in the nonlinear term of problems (4.1–4.2). The early results for positive solutions of FBVPs, to author's best knowledge, are not applicable to this problem. Our results complements some previous works in the area of FBVPS, such as Moustafa El-Shahed [29].

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