

GLOBAL ASYMPTOTIC STABILITY OF A GENERALIZED SIRS EPIDEMIC MODEL WITH TRANSFER FROM INFECTIOUS TO SUSCEPTIBLE*

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Abstract In this paper, we propose a generalized SIRS epidemic model with varying total population size caused by the death rate due to the disease and transfer from infectious to susceptible, where the incidence rate employed includes a wide range of monotonic and concave incidence rates. Applying the geometric approach developed by Smith, Li and Muldowey, we prove that the endemic equilibrium is globally asymptotically stable provided that the rate of loss of immunity δ is in a critical interval $[\eta, \bar{\delta})$ when the basic reproduction number R_0 is greater than unity.

Keywords Generalized SIRS epidemic model, varying total population size, transfer from infectious to susceptible, global asymptotic stability, geometric approach.

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1. Introduction

Mathematical models of infectious diseases have been proven to be very important in better understanding of epidemiological patterns and disease control in human populations. The classical SIR model is the Kermack and McKendrick model, which the total population is divided into three compartments Susceptible (S), Infected (I), and Recovery (R). In most communicable diseases such as cholera, pertussis, influenza and malaria, it has been observed that recovered individuals can return to the susceptible period (temporary immunity). This observation has been described by SIRS and SEIRS models (see [9–11, 13, 15, 19–21] and the references therein).

Many researchers were devoted to generalizing the nonlinear incidence rates or considering different factors to probe into the preserved dynamics in epidemic models. By direct Lyapunov method, Korobeinikov [9] studied the global stability for the SIR and SIRS models with the more general incidence rates $f(S, I)$. Moreover, the author observed that the same Lyapunov function could be applied to examine the global property for the SIR with the varying total population size caused by the death rate due to the disease. Similar conclusions for an SEIR model were reached

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in work [10]. Most works (see [9, 10, 20]) assumed that the incidence rates $f(S, I)$ are monotonic and concave with respect to I to achieve threshold dynamics of epidemic models. Recently, Li etc [13] proposed an SIRS model with the incidence $Sf(I)$ (also see [11]) and transfer from infectious to susceptible. Tang etc [19] studied the SIRS model with the incidence $\beta f(S)g(I)$ and vaccination in susceptible. Singh etc [17] developed an effective SEIRS model by taking into consideration the human immunity. Chen etc [2] considered the stability and attraction for the endemic equilibrium of an SEIQ epidemic model with transfer from infectious to susceptible.

Motivated by the above discussions, we propose the following SIRS epidemic model with varying total population size and transfer from infectious to susceptible

$$\begin{cases} \dot{S} = \Lambda - \mu S - f(S, I) + \gamma_1 I + \delta R, \\ \dot{I} = f(S, I) - (\mu + \gamma_1 + \gamma_2 + \alpha)I, \\ \dot{R} = \gamma_2 I - (\mu + \delta)R, \end{cases} \quad (1.1)$$

with the initial conditions

$$S(0) \geq 0, \quad I(0) \geq 0, \quad R(0) \geq 0, \quad (1.2)$$

where S is the number of susceptible individuals, I is the number of infectious individuals, R is the number of recovered individuals, Λ is the recruitment rate of susceptible individuals (we assume that all recruitment are susceptible), μ is the natural death rate, γ_1 is the transfer rate from the infected class to the susceptible class, γ_2 is the transfer rate from the infected class to the recovered class, α is the disease-induced death rate, δ is the immunity loss rate. Assume that all the parameters $\Lambda, \mu, \alpha, \gamma_1, \gamma_2$ and δ are positive constants with $\Lambda > \mu$.

In our proposition, $f(S, I)$ is a continuously differentiable function on \mathbb{R}_+^2 satisfying $f(0, I) = f(S, 0) = 0$ for $S, I \geq 0$ and the following hypotheses [5]:

- (H_1) f is a strictly monotonically increasing function of $S \geq 0$, for any fixed $I > 0$, and f is a strictly monotonically increasing function of $I > 0$, for any fixed $S \geq 0$;
- (H_2) $\Phi(S, I) = \frac{f(S, I)}{I}$ is a bounded and monotonically decreasing function of $I > 0$, for any fixed $S \geq 0$, and $\phi(S) = \lim_{I \rightarrow 0^+} \Phi(S, I)$ is continuous on $S \geq 0$.

It is easy to see that the function $\Phi(S, I) = \frac{f(S, I)}{I}$ is a monotonically decreasing function of $I > 0$ for any fixed $S \geq 0$ iff $\frac{f(S, I)}{I} - \frac{\partial f(S, I)}{\partial I} \geq 0$. As for the function $f(S, I)$, in some cases where it is monotonically increases with respect to S, I and concave with respect to I (i.e., $\frac{\partial^2 f(S, I)}{\partial I^2} \leq 0$), the hypotheses (H_1)-(H_2) naturally hold. Therefore, the function $f(S, I)$ employed in this paper includes the following forms: the bilinear incidence βSI introduced in [2, 17], the standard incidence $\frac{\beta SI}{S+I+R}$ advanced in [15, 21] and the saturated incidence $\frac{\beta SI}{1+\alpha_1 S+\alpha_2 I}$ proposed in [8] where α_1, α_2 are positive constants. However, hypotheses (H_1)-(H_2) cannot be applied in the following cases, for examples, the incidence rates $\beta S^p I^q$ and $\frac{\beta S I^q}{1+\alpha_1 I^q}$ if $p > 0, 0 < q < 1$. When $0 < q < 1$, one has $\phi(S) = \lim_{I \rightarrow 0^+} \Phi(S, I) = \lim_{I \rightarrow 0^+} \frac{\partial f(S, I)}{\partial I} = +\infty$. It implies that the basic reproduction number may not be well defined if (H_2) is not satisfied.

The global stability of equilibria for SIRS models or other differential equations has been extensively studied by applying Lyapunov functions(see [1, 9–11, 13, 19,

21] and the references therein). In this paper, we investigate the global stability of the endemic equilibrium only for a reasonably small δ applying the geometric approach developed by Smith [18], Li and Muldowney [12]. We will prove that the endemic equilibrium is globally asymptotically stable provided that the rate of loss of immunity δ is less than a critical value $\bar{\delta}$ when the basic reproduction number R_0 is greater than unity.

The rest of the paper is organized as follows. In "Basic properties of the generalized SIRS Model" section, the basic properties of the generalized SIRS model (1.1) are investigated. We calculate the basic reproduction number and give the existence and uniqueness of endemic equilibrium. Furthermore, the local stability of equilibria and the uniform persistence of model (1.1) are carefully addressed. In "Global asymptotic stability of the endemic equilibrium" section, the global asymptotic stability of the endemic equilibrium is discussed. A brief discussion is presented to conclude this paper.

2. Basic properties of the generalized SIRS model

In this section, we state the basic properties of model (1.1) with disease-induced mortality. The varying total population size N satisfies the equation $N = S + I + R$. From model (1.1), we have

$$\dot{N} = \dot{S} + \dot{I} + \dot{R} = \Lambda - \mu(S + I + R) - \alpha I \leq \Lambda - \mu N,$$

which implies that the biologically feasible region

$$\Omega = \{(S, I, R) \in \mathbb{R}_+^3 : S + I + R \leq \frac{\Lambda}{\mu}\}$$

for model (1.1) is bounded and positively invariant. Obviously, every solution of model (1.1) with the initial condition (1.2) exists globally and is nonnegative.

In the following, we calculate the basic reproduction number of model (1.1) using the method of the next generation matrix developed by van den Driessche and Watmough [4].

Set $X = (R, I)^T$. Then it follows from model (1.1) that $\frac{dX}{dt} = \mathbb{F}_1(X) - \mathbb{V}_1(X)$, where

$$\mathbb{F}_1(X) = \begin{pmatrix} 0 \\ f(S, I) \end{pmatrix}, \quad \mathbb{V}_1(X) = \begin{pmatrix} (\mu + \delta)R - \gamma_2 I \\ (\mu + \gamma_1 + \gamma_2 + \alpha)I \end{pmatrix}.$$

The Jacobian matrices of $\mathbb{F}_1(X)$ and $\mathbb{V}_1(X)$ at the point $(0, 0)$ with $S = \frac{\Lambda}{\mu}$ are, respectively,

$$F_1 = \begin{pmatrix} 0 & 0 \\ \frac{\partial f(\frac{\Lambda}{\mu}, 0)}{\partial I} & 0 \end{pmatrix}, \quad V_1 = \begin{pmatrix} -\gamma_2 & \mu + \delta \\ \mu + \gamma_1 + \gamma_2 + \alpha & 0 \end{pmatrix}.$$

The inverse of V_1 is

$$V_1^{-1} = \begin{pmatrix} 0 & \frac{1}{\mu + \gamma_1 + \gamma_2 + \alpha} \\ \frac{1}{\mu + \delta} & \frac{\gamma_2}{(\mu + \delta)(\mu + \gamma_1 + \gamma_2 + \alpha)} \end{pmatrix},$$

thus the next generation matrix is

$$F_1 V_1^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\mu + \gamma_1 + \gamma_2 + \alpha} \cdot \frac{\partial f(\frac{\Lambda}{\mu}, 0)}{\partial I} \end{pmatrix}.$$

We obtain the basic reproduction number of model (1.1) as follows:

$$R_0 = \rho(F_1 V_1^{-1}) = \frac{1}{\mu + \gamma_1 + \gamma_2 + \alpha} \cdot \frac{\partial f(\frac{\Lambda}{\mu}, 0)}{\partial I}.$$

If we choose $\frac{dX}{dt} = \mathbb{F}_2(X) - \mathbb{V}_2(X)$ with $F(S, I) = f(S, I) - \gamma_1 I$, where

$$\mathbb{F}_2(X) = \begin{pmatrix} 0 \\ F(S, I) \end{pmatrix}, \quad \mathbb{V}_2(X) = \begin{pmatrix} (\mu + \delta)R - \gamma_2 I \\ (\mu + \gamma_2 + \alpha)I \end{pmatrix}.$$

The Jacobian matrices of $\mathbb{F}_2(X)$ and $\mathbb{V}_2(X)$ at the point $(0, 0)$ with $S = \frac{\Lambda}{\mu}$ are, respectively,

$$F_2 = \begin{pmatrix} 0 & 0 \\ \frac{\partial F(\frac{\Lambda}{\mu}, 0)}{\partial I} & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} -\gamma_2 & \mu + \delta \\ \mu + \gamma_2 + \alpha & 0 \end{pmatrix}.$$

The inverse of V_2 is

$$V_2^{-1} = \begin{pmatrix} 0 & \frac{1}{\mu + \gamma_2 + \alpha} \\ \frac{1}{\mu + \delta} & \frac{\gamma_2}{(\mu + \delta)(\mu + \gamma_2 + \alpha)} \end{pmatrix},$$

thus the next generation matrix is

$$F_2 V_2^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\mu + \gamma_2 + \alpha} \cdot \frac{\partial F(\frac{\Lambda}{\mu}, 0)}{\partial I} \end{pmatrix}.$$

We obtain the basic reproduction number of model (1.1) as follows:

$$R'_0 = \rho(F_2 V_2^{-1}) = \frac{1}{\mu + \gamma_2 + \alpha} \cdot \left(\frac{\partial f(\frac{\Lambda}{\mu}, 0)}{\partial I} - \gamma_1 \right).$$

We note that the basic reproduction number R_0 is equivalent to R'_0 , that is, $R_0 = 1$ is equivalent to $R'_0 = 1$, $R_0 < 1$ is equivalent to $R'_0 < 1$ and $R_0 > 1$ is equivalent to $R'_0 > 1$. In this paper, we use the basic reproduction number R_0 .

It is easy to verify that model (1.1) always has a disease-free equilibrium $E_0(\frac{\Lambda}{\mu}, 0, 0)$. Next, we show the existence and uniqueness of the endemic equilibrium.

Theorem 2.1. *If $R_0 > 1$, model (1.1) also has a unique endemic equilibrium $E^*(S^*, I^*, R^*)$.*

Proof. If (S, I, R) is an equilibrium of model (1.1), then we have the following system

$$\begin{cases} \Lambda - \mu S - f(S, I) + \gamma_1 I + \delta R = 0, \\ f(S, I) - (\mu + \gamma_1 + \gamma_2 + \alpha)I = 0, \\ \gamma_2 I - (\mu + \delta)R = 0. \end{cases}$$

If $I \neq 0$, we have

$$\begin{cases} S = \frac{\Lambda(\mu+\delta) - [(\mu+\alpha)(\mu+\delta) + \gamma_2\mu]I}{\mu(\mu+\delta)}, \\ R = \frac{\gamma_2 I}{\mu+\delta}, \\ f(S, I) = (\mu + \gamma_1 + \gamma_2 + \alpha)I. \end{cases}$$

We consider the following function

$$h(I) := \Phi \left(\frac{\Lambda(\mu + \delta) - [(\mu + \alpha)(\mu + \delta) + \gamma_2\mu]I}{\mu(\mu + \delta)}, I \right) - (\mu + \gamma_1 + \gamma_2 + \alpha).$$

By (H_1) and (H_2) , we know that h is a continuous and strictly monotone decreasing function of $I > 0$, and for $R_0 > 1$,

$$\lim_{I \rightarrow 0^+} h(I) = (\mu + \gamma_1 + \gamma_2 + \alpha)(R_0 - 1) > 0,$$

and

$$h \left(\frac{\Lambda(\mu + \delta)}{(\mu + \alpha)(\mu + \delta) + \gamma_2\mu} \right) < 0.$$

Hence, if $R_0 > 1$, there exists a unique endemic equilibrium $E^*(S^*, I^*, R^*)$. \square

We will obtain the local stability of E_0 and E^* by the Routh-Hurwitz criterion.

Theorem 2.2. *If $R_0 < 1$, then the disease-free equilibrium E_0 is locally asymptotically stable; If $R_0 > 1$, then the disease-free equilibrium E_0 is unstable.*

Proof. The Jacobian matrix of model (1.1) at E_0 is

$$J_0 = \begin{pmatrix} -\mu & -\frac{\partial f(\frac{\Lambda}{\mu}, 0)}{\partial I} + \gamma_1 & \delta \\ 0 & \frac{\partial f(\frac{\Lambda}{\mu}, 0)}{\partial I} - (\mu + \gamma_1 + \gamma_2 + \alpha) & 0 \\ 0 & \gamma_2 & -(\mu + \delta) \end{pmatrix}.$$

The characteristic equation of J_0 is given by

$$(\lambda + \mu)(\lambda + \mu + \gamma_1 + \gamma_2 + \alpha - \frac{\partial f(\frac{\Lambda}{\mu}, 0)}{\partial I})(\lambda + \mu + \delta) = 0,$$

where

$$\lambda + \mu + \gamma_1 + \gamma_2 + \alpha - \frac{\partial f(\frac{\Lambda}{\mu}, 0)}{\partial I} = \lambda + (\mu + \gamma_1 + \gamma_2 + \alpha)(1 - R_0).$$

If $R_0 < 1$, all the roots of above equation have negative real parts which implies that E_0 is locally asymptotically stable. If $R_0 > 1$, the above equation has a positive root which implies that E_0 is unstable. \square

Theorem 2.3. *If $R_0 > 1$, then the endemic equilibrium E^* is locally asymptotically stable.*

Proof. The Jacobian matrix of model (1.1) at E^* is

$$J^* = \begin{pmatrix} -\mu - L & M - (\mu + \gamma_2 + \alpha) & \delta \\ L & -M & 0 \\ 0 & \gamma_2 & -(\mu + \delta) \end{pmatrix},$$

where $L = \frac{\partial f(S^*, I^*)}{\partial S}$ and $M = \mu + \gamma_1 + \gamma_2 + \alpha - \frac{\partial f(S^*, I^*)}{\partial I}$.

The characteristic equation of J^* is given by

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0,$$

where

$$\begin{aligned} a &= M + L + 2\mu + \delta, \\ b &= (2\mu + \delta)M + (2\mu + \delta + \gamma_2 + \alpha)L + \mu(\mu + \delta), \\ c &= \mu(\mu + \delta)M + (\mu + \delta)(\mu + \alpha)L. \end{aligned}$$

Firstly, from hypothesis (H_1) , we have $L = \frac{\partial f(S^*, I^*)}{\partial S} > 0$. Secondly, from Theorem 2.1 and hypothesis (H_2) , we find that

$$M = \mu + \gamma_1 + \gamma_2 + \alpha - \frac{\partial f(S^*, I^*)}{\partial I} = \frac{f(S^*, I^*)}{I^*} - \frac{\partial f(S^*, I^*)}{\partial I} \geq 0.$$

Hence, $a > 0, b > 0, c > 0$ and

$$\begin{aligned} ab - c &= (M + L + 2\mu + \delta)[(2\mu + \delta)M + (2\mu + \delta + \gamma_2 + \alpha)L + \mu(\mu + \delta)] \\ &\quad - [\mu(\mu + \delta)M + (\mu + \delta)(\mu + \alpha)L] \\ &> \mu(2\mu + \delta)(\mu + \delta) > 0. \end{aligned}$$

So, by the Routh-Hurwitz criterion, we obtain that the endemic equilibrium E^* is locally asymptotically stable for $R_0 > 1$. This concludes the proof of Theorem 2.3. \square

In order to obtain sufficient conditions that E^* is globally asymptotically stable for $R_0 > 1$, we prove the uniform persistence of model (1.1). Let Ω_1 denote the interior of Ω and $\partial\Omega$ denote the boundary of Ω . We can prove the following results.

Theorem 2.4. *If $R_0 > 1$, then model (1.1) is uniformly persistent, which means that there exists a positive constant C such that every solution (S, I, R) of model (1.1) with the initial data $(S(0), I(0), R(0)) \in \Omega_1$ satisfies*

$$\liminf_{t \rightarrow \infty} S(t) \geq C, \quad \liminf_{t \rightarrow \infty} I(t) \geq C, \quad \liminf_{t \rightarrow \infty} R(t) \geq C,$$

where C is independent of initial data in Ω_1 .

Proof. From Theorem 2.1 we get for $R_0 > 1$, there exists a unique endemic equilibrium E^* . From Theorem 2.2 we know that $R_0 > 1$ implies that the disease-free equilibrium E_0 is unstable. By Theorem 4.3 in [6], the instability of E_0 , together with $E_0 \in \partial\Omega$, imply the uniform persistence of the state variables of model (1.1). Therefore, there exists a positive constant C such that every solution (S, I, R) of model (1.1) with the initial data $(S(0), I(0), R(0)) \in \Omega_1$ satisfies

$$\liminf_{t \rightarrow \infty} S(t) \geq C, \quad \liminf_{t \rightarrow \infty} I(t) \geq C, \quad \liminf_{t \rightarrow \infty} R(t) \geq C,$$

where C is independent of initial data in Ω_1 . \square

The uniform persistence, together with boundedness of Ω , are equivalent to the existence of a compact set in the interior of Ω which is absorbing for model (1.1)(see [7]). So, we have

Theorem 2.5. *If $R_0 > 1$, then there exists a compact absorbing set $K \subset \Omega_1$.*

3. Global asymptotic stability of the endemic equilibrium

From Theorem 2.1 and Theorem 2.3, we find that for $R_0 > 1$, the unique endemic equilibrium E^* exists and is locally asymptotically stable. In this section we discuss the possible global asymptotical stability of E^* using the geometric approach developed by Smith [18], Li and Muldowney [12].

We rewrite model (1.1) as the following autonomous dynamical system

$$\dot{x} = g(x), \quad (3.1)$$

where $g : \Omega \rightarrow \mathbb{R}_+^3$ and $g \in C^1(\Omega)$, $x = (S, I, R)^T$.

From Section 2, we have the following conditions hold:

- (A₁) Ω_1 is simply connected;
- (A₂) there is a compact absorbing set $K \subset \Omega_1$;
- (A₃) E^* is the only equilibrium of the system (3.1) in Ω_1 .

Let A be a matrix-valued function that is C^1 on Ω_1 and set $B = A_g A^{-1} + A J^{[2]} A^{-1}$, where the matrix A_g is $(a_{ij}(x))_g = (\nabla a_{ij}(x) \cdot g(x))$, and $J^{[2]}$ is the second additive compound matrix of $J = \frac{\partial g}{\partial x}$ (see [16]). The Lozinskiĭ measure $\bar{\mu}(B)$ of B with respect to the norm $|\cdot|$ in \mathbb{R}^3 is defined as (see [3])

$$\bar{\mu}(B) = \lim_{h \rightarrow 0^+} \frac{|I + hB| - 1}{h}.$$

Now, we present an important lemma and prove the main result as follows.

Lemma 3.1 (Theorem 3.5, [12]). *Assume that (3.1) satisfies (A₁), (A₂) and (A₃). Then E^* is globally asymptotically stable in Ω_1 if the quantity \bar{q}_2 satisfies*

$$\bar{q}_2 = \limsup_{t \rightarrow \infty} \sup_{x_0 \in K} \frac{1}{t} \int_0^t \bar{\mu}(B(x(s, x_0))) ds < 0, \quad (3.2)$$

where $x(t, x_0)$ is the solution of (3.1) with the initial condition $x(0, x_0) = x_0 \in K$.

Theorem 3.1. *Assume that $R_0 > 1$. Then there exist η and $\bar{\delta} > 0$ such that the unique endemic equilibrium E^* is globally asymptotically stable in Ω_1 when δ is in the critical interval $[\eta, \bar{\delta})$.*

Proof. The Jacobian matrix associated with a solution $x(t, x_0)$ of (3.1) is given by

$$J = \begin{pmatrix} -\mu - \frac{\partial f(S, I)}{\partial S} - \frac{\partial f(S, I)}{\partial I} + \gamma_1 & \delta & 0 \\ \frac{\partial f(S, I)}{\partial S} & \frac{\partial f(S, I)}{\partial I} - m & 0 \\ 0 & \gamma_2 & -(\mu + \delta) \end{pmatrix},$$

where $m = \mu + \gamma_1 + \gamma_2 + \alpha$. The second additive compound matrix $J^{[2]}$ of J is given by

$$J^{[2]} = \begin{pmatrix} \frac{\partial f(S, I)}{\partial I} - \frac{\partial f(S, I)}{\partial S} - (m + \mu) & 0 & -\delta \\ \gamma_2 & -\frac{\partial f(S, I)}{\partial S} - (2\mu + \delta) & -\frac{\partial f(S, I)}{\partial I} + \gamma_1 \\ 0 & \frac{\partial f(S, I)}{\partial S} & \frac{\partial f(S, I)}{\partial I} - (m + \mu + \delta) \end{pmatrix}.$$

Choosing a proper function $A = \text{diag}\{1, \frac{I}{R}, \frac{I}{R}\}$, then

$$\begin{aligned} A^{-1} &= \text{diag}\{1, \frac{R}{I}, \frac{R}{I}\}, \\ A_g &= \text{diag}\{0, \frac{\dot{I}}{R} - \frac{I\dot{R}}{R^2}, \frac{\dot{I}}{R} - \frac{I\dot{R}}{R^2}\}, \\ A_g A^{-1} &= \text{diag}\{0, \frac{\dot{I}}{I} - \frac{\dot{R}}{R}, \frac{\dot{I}}{I} - \frac{\dot{R}}{R}\}. \end{aligned}$$

Therefore,

$$AJ^{[2]}A^{-1} = \begin{pmatrix} \frac{\partial f(S,I)}{\partial I} - \frac{\partial f(S,I)}{\partial S} - (m + \mu) & 0 & -\delta \frac{R}{I} \\ \gamma_2 \frac{I}{R} & -\frac{\partial f(S,I)}{\partial S} - (2\mu + \delta) & -\frac{\partial f(S,I)}{\partial I} + \gamma_1 \\ 0 & \frac{\partial f(S,I)}{\partial S} & \frac{\partial f(S,I)}{\partial I} - (m + \mu + \delta) \end{pmatrix}.$$

Let

$$B = A_g A^{-1} + AJ^{[2]}A^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where

$$\begin{aligned} B_{11} &= \frac{\partial f(S,I)}{\partial I} - \frac{\partial f(S,I)}{\partial S} - (m + \mu), \\ B_{12} &= (0, -\delta \frac{R}{I}), \quad B_{21} = (\gamma_2 \frac{I}{R}, 0)^T, \\ B_{22} &= \begin{pmatrix} \frac{\dot{I}}{I} - \frac{\dot{R}}{R} - \frac{\partial f(S,I)}{\partial S} - (2\mu + \delta) & -\frac{\partial f(S,I)}{\partial I} + \gamma_1 \\ \frac{\partial f(S,I)}{\partial S} & \frac{\dot{I}}{I} - \frac{\dot{R}}{R} + \frac{\partial f(S,I)}{\partial I} - (m + \mu + \delta) \end{pmatrix}. \end{aligned}$$

Consider the norm in \mathbb{R}^3 as $|(u, v, w)| = \max\{|u|, |v| + |w|\}$, where (u, v, w) is the vector in \mathbb{R}^3 . Using the method of estimating $\bar{\mu}$ in [14], we have

$$\bar{\mu}(B) \leq \sup\{g_1, g_2\}, \quad (3.3)$$

where

$$g_1 = B_{11} + |B_{12}|, \quad g_2 = \bar{\mu}_1(B_{22}) + |B_{21}|,$$

and $|B_{12}|, |B_{21}|$ are matrix norms with respect to the l_1 vector norm and $\bar{\mu}_1(B_{22})$ is the Lozinskiĭ measure of the matrix B_{22} with respect to the l^1 norm in \mathbb{R}^2 . Then we get

$$\begin{aligned} B_{11} &= \frac{\partial f(S,I)}{\partial I} - \frac{\partial f(S,I)}{\partial S} - (m + \mu), \\ |B_{12}| &= \delta \frac{R}{I}, \quad |B_{21}| = \gamma_2 \frac{I}{R}, \\ \bar{\mu}_1(B_{22}) &= \max\{\frac{\dot{I}}{I} - \frac{\dot{R}}{R} - (2\mu + \delta), \frac{\dot{I}}{I} - \frac{\dot{R}}{R} + \frac{\partial f(S,I)}{\partial I} - (m + \mu + \delta) + |-\frac{\partial f(S,I)}{\partial I} + \gamma_1|\} \\ &= \frac{\dot{I}}{I} - \frac{\dot{R}}{R} - (2\mu + \delta) + \max\{0, 2\sigma_1 - 2\gamma_1 - \gamma_2 - \alpha\}, \end{aligned}$$

where $\sigma_1 = \max\{\frac{\partial f(S,I)}{\partial I}, C \leq S, I \leq \frac{\Lambda}{\mu}\}$. Therefore, we have

$$g_1 = \frac{\partial f(S,I)}{\partial I} - \frac{\partial f(S,I)}{\partial S} - (m + \mu) + \delta \frac{R}{I}, \quad (3.4)$$

$$g_2 = \frac{\dot{I}}{I} - \frac{\dot{R}}{R} - 2\mu - \delta + \gamma_2 \frac{I}{R} + \max\{0, 2\sigma_1 - 2\gamma_1 - \gamma_2 - \alpha\}. \quad (3.5)$$

Rewriting the second and the third equations in (1.1), we obtain

$$\frac{\dot{I}}{I} = \frac{f(S,I)}{I} - m, \quad (3.6)$$

$$\frac{\dot{R}}{R} = \gamma_2 \frac{I}{R} - (\mu + \delta). \quad (3.7)$$

Substituting (3.6) and (3.7) into (3.4) and (3.5), respectively, we have

$$g_1 = \frac{\dot{I}}{I} - \frac{f(S,I)}{I} + \frac{\partial f(S,I)}{\partial I} - \frac{\partial f(S,I)}{\partial S} - \mu + \delta \frac{R}{I},$$

$$g_2 = \frac{\dot{I}}{I} - \mu + \max\{0, 2\sigma_1 - 2\gamma_1 - \gamma_2 - \alpha\}.$$

By the hypothesis (H2), we get

$$\begin{aligned} \bar{\mu}(B) &\leq \frac{\dot{I}}{I} - \mu + \max\{\delta \frac{R}{I} - \frac{\partial f(S,I)}{\partial S}, 0, 2\sigma_1 - 2\gamma_1 - \gamma_2 - \alpha\} \\ &\leq \frac{\dot{I}}{I} - \mu + \max\{\frac{\delta \Lambda}{C\mu} - \sigma_2, 0, 2\sigma_1 - 2\gamma_1 - \gamma_2 - \alpha\}, \end{aligned}$$

where $\sigma_2 = \min\{\frac{\partial f(S,I)}{\partial S}, C \leq S, I \leq \frac{\Lambda}{\mu}\}$.

Set

$$\sigma = \max\{\frac{\delta \Lambda}{C\mu} - \sigma_2, 0, 2\sigma_1 - 2\gamma_1 - \gamma_2 - \alpha\}.$$

If $\mu > \sigma$, then we have

$$\begin{aligned} \frac{1}{t} \int_0^t \bar{\mu}(B) ds &\leq \frac{1}{t} \int_0^{\bar{t}} \bar{\mu}(B) ds + \frac{1}{t} \ln \frac{I(t)}{I(\bar{t})} - (\mu - \sigma) \frac{t - \bar{t}}{t} \\ &= \frac{1}{t} \left(\int_0^{\bar{t}} \bar{\mu}(B) ds + \ln \frac{I(t)}{I(\bar{t})} + (\mu - \sigma) \bar{t} \right) - (\mu - \sigma), \end{aligned}$$

for all $x_0 = (S(0), I(0), R(0))^T \in K$, which implies $\bar{q}_2 < 0$.

Let

$$\eta = \frac{C\mu}{\Lambda} (2\sigma_1 - 2\gamma_1 - \gamma_2 - \alpha + \sigma_2), \quad \bar{\delta} = \frac{C\mu}{\Lambda} (\mu + \sigma_2).$$

Then $\eta \leq \delta < \bar{\delta}$ implies $\mu > \sigma$. Therefore, if $\eta \leq \delta < \bar{\delta}$, we have $\bar{q}_2 < 0$. From Lemma 3.1 we obtain that the unique endemic equilibrium E^* is globally asymptotically stable in Ω_1 . \square

4. Discussion

In this paper, we propose a generalized SIRS epidemic model. The incidence function $f(S, I)$ employed in this paper can be applied generally for a wide class of incidence functions such as the bilinear incidence, the standard incidence and the saturated incidence.

We calculate the basic reproduction number R_0 by using the method of the next generation matrix and give some basic properties of the generalized SIRS model. We investigate the global stability of the endemic equilibrium by applying the geometric approach. When R_0 is greater than unity, there exists a unique endemic equilibrium, which is globally asymptotically stable provided that the rate of loss of immunity δ is in a critical interval $[\eta, \bar{\delta})$.

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