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EXISTENCE THEOREMS AND HYERS-ULAM STABILITY FOR A CLASS OF HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS WITH *P*-LAPLACIAN OPERATOR*

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Abstract In this paper, we prove necessary conditions for existence and uniqueness of solution (EUS) as well Hyers-Ulam stability for a class of hybrid fractional differential equations (HFDEs) with p-Laplacian operator. For these aims, we take help from topological degree theory and Leray Schauder-type fixed point theorem. An example is provided to illustrate the results.

Keywords Hybrid fractional differential equations, Hyers-Ulam stability, Caputo's fractional derivative, existence and uniqueness, topological degree theory.

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1. Introduction

Mathematical models by FDEs have attracted the attention of scientists due to useful and realistic in memory problems as compared to the models of integer order differential equations. In last two decades, scientists have shown a great contribution by applying FDEs to day life problems in various scientific fields like; viscoelastic theory, image processing, biology, hydrodynamics, signals, fluid dynamics, control theory, computer networking and many others [1, 13, 23, 25, 32].

The exploration of different aspects of FDEs is very popular among the scientists. We highlight some recent and important contributions of scientists for the investigation of EUS of FDEs of different classes of FDEs. For example, Khan *et al.* [24] considered the existence of solution and error estimation for a coupled system of differential-integral equations via uper and lower solution method. Baleanu *et al.* [6] proved existence of solution for a nonlinear FDE on partially ordered Banach spaces and provided applications. Mahmudov and Unul [29] studied a FDE of order $\epsilon \in (2, 3]$ with integral conditions, an impulsive FDE [31] and FDE with

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p-Laplacian operator [30], for the existence of solutions. Hu *et al.* [19] studied a coupled system of fractional differential equations with nonlinear p-Laplacian operator at resonance. One can study global existence of FDE [9], EUS for the superlinear FDE in [10] and eventually large and small solutions of FDEs in [11] and FDEs with nonlocal boundary conditions [37].

The applications of hybrid fractional differential equations (HFDEs) in different scientific fields like; fractals theory, plasma physics, economics, metallurgy, electromagnetic theory, signal and image processing, biology, control theory ecology and many more, have greatly attracted the attention of researchers. Recently, some authors have investigated different aspects of FDEs including; existence and uniqueness of solutions (EUS) and Hyers-Ulam stability for FDEs by different mathematical techniques. The Hyers-Ulam stability we mean that a FDE has a very close exact solution to the approximate solution of the differential equation and the error is very small which can be estimated. Dhage and Lakshmikantham [15] investigated the EUS to the ordinary hybrid differential equation of first order with perturbation of first type

$$\frac{d}{dt}\Big(\frac{u(t)}{f(t,u(t))}\Big) = h(t,u(t)), \quad u(t_0) = u_0 \in \mathbb{R}, \tag{1.1}$$

where $f \in C([t_0, t_0 + a] \times \mathbb{R}, \mathbb{R} - 0)$, $a \in \mathbb{R}^+$, $[t_0, t_0 + a]$ is a bounded interval, $f(t, u(t))$ is continuous and $h(t, u(t))$ is Caratheodory class of functions. Dhage and Jadhav [14] studied the EUS of the ordinary hybrid differential equation of first order with perturbation of second type

$$\frac{d}{dt}\Big(u(t) - f(t,u(t))\Big) = h(t,u(t)), \quad u(t_0) = u_0 \in \mathbb{R}. \tag{1.2}$$

Herzallah and Baleanu [18] considered the EUS for the following first type and second type hybrid FDEs

$$\begin{cases} D^{\epsilon}\Big(\dfrac{u(t)}{f(t,u(t))}\Big) = h(t,u(t)), \quad u(t_0) = u_0 \in \mathbb{R}, \\ D^{\epsilon}\Big(u(t) - f(t,u(t))\Big) = h(t,u(t)), \quad u(t_0) = u_0 \in \mathbb{R}, \end{cases} \tag{1.3}$$

for $t \in [0, T]$, D^{ϵ} is Caputo fractional derivative of order $0 < \epsilon < 1$. In literature one can see the contributions of scientists by considering more general problems of HFDEs than (1.1)-(1.3) for the EUS, we refer the readers to [8, 12, 18, 35].

Recently, FDEs with p-Laplacian operator have been considered by a large number of scientists. For example, Li [28] considered existence of positive solutions for the following FDE

$$\begin{aligned} &\mathcal{D}^{\beta}(\phi_p({}^{c}\mathcal{D}u(t))) = -f(t,u(t)), \\ &\phi_p({}^{c}\mathcal{D}u(0)) = (\phi_p(\mathcal{D}u(0)))' = \phi_p({}^{c}\mathcal{D}u(1)), \\ &u''(0) = u'(1) = 0,\ au(0) + bu'(0) = \int_0^1 g(t)u(t)dt, \end{aligned}$$

where $\epsilon, \beta \in (2, 3]$, $\epsilon + \beta \in (5, 6]$. ${}^{c}\mathcal{D}$ is Caputo fractional derivative while $\mathcal{D}$ is Riemann-Liouville fractional derivative.

Inspired from the above contribution of the scientists and the work in the references, we use topological degree method to study EUS and Hyers-Ulam stability of a class of nonlinear HFDE with p-Laplacian operator of the type

$$\begin{cases} \mathcal{D}^{\beta}\left[\phi_p[\mathcal{D}^{\epsilon}(u(t)-\psi_2(t,u(t)))]\right]+\psi_1(t,u(t))=0, \\ \left(\phi_p[\mathcal{D}^{\epsilon}(u(t)-\psi_2(t,u(t))]\right)^{(i)}|_{t=0}=0,\ \text{for } i=0,2,3,\dots,n-1, \\ \left(\phi_p[\mathcal{D}^{\epsilon}(u(t)-\psi_2(t,u(t))]\right)'|_{t=\eta}=0,\ u^{(j)}(0)=0,\ \text{for } j=2,3,\dots,n-1, \\ u(0)=\dfrac{1}{\Gamma(\epsilon)}\displaystyle\int_0^a (a-s)^{\epsilon-1}\psi_2(u(s))ds, \quad u'(\eta)=\psi_2'(u(\eta)), \end{cases} \tag{1.4}$$

where ψ_1, ψ_2, are continuous functions, $\psi_2^{(k)}(t,u(t))|_{t=0}=0$, for $k=0,1,2,\dots,n-1$. $n-1<\epsilon,\beta\leq n$, n is a positive integer greater than or equal to 3, $0<a,b<1$, $\psi_1,\psi_2\in\mathcal{L}[0,1]$ and $\mathcal{D}^{\epsilon},\mathcal{D}^{\beta}$ stand for Caputo fractional derivative, $\phi_p(r)=|r|^{p-2}r$ is p-Laplacian operator where $1/p+1/q=1$, ϕ_q denotes inverse of p-Laplacian operator. Our suggested problem is more general and complicated than the problems considered earlier and mentioned above. To the best of our knowledge, the topological degree theory has not been widely used for the study of EUS for HFDEs with IBCs having orders in $(n-1,n]$ for $n\geq 3$ involving the nonlinear p-Laplacian operator. In most of the previously studied cases the authors would need to the assumption of compactness of the operators which would restrict the impact of the problem and mathematical method at large. In this paper, following the recent contributions of Khan *et al.* [26], we investigate three important aspects of the HFDE with nonlinear p-Laplacian operator (1.4) including existence of solution, uniqueness of solution and Hyers-Ulam stability of the suggested problem. For these objectives, we are going to convert the problem (1.4) to an integral equation by the help of Green functions. After this, we will prove results for existence and uniqueness by topological degree method. By the use of this technique, we do not need to the assumption of the compactness of the operator. Then after, Hyers-Ulam stability will be investigated. In literature, we could not find any published work on the Hyers-Ulam stability of HFDEs with nonlinear p-Laplacian operator and integral boundary condition. Therefore, this work may get the attention of researchers to the study of Hyers-Ulam stability as well many other types of stability for more complex problems. We also suggest the readers that the problem (1.4) has potentials to be studied for further aims including multiplicity results.

2. Axillary results

Definition 2.1. The fractional integral of order $\epsilon>0$ of a function $f:(0,+\infty)\to\mathbb{R}$ is given by

$$\mathcal{I}^{\epsilon}\psi(t)=\frac{1}{\Gamma(\epsilon)}\int_0^t (t-s)^{\epsilon-1}\psi(s)\,ds,$$

provided that the integral on right side is point wise defined on the interval $(0,+\infty)$, where

$$\Gamma(\epsilon)=\int_0^{+\infty} e^{-s}s^{\epsilon-1}ds.$$

Definition 2.2. The Caputo fractional derivative of order $\epsilon>0$, for a continuous function $\psi(t):(0,+\infty)\to\mathbb{R}$ is defined by

$$\mathcal{D}^{\epsilon}\psi(t)=\frac{1}{\Gamma(k-\epsilon)}\int_0^t (t-s)^{k-\epsilon-1}\psi^{(k)}(s)ds,$$

for $k = [\epsilon] + 1$, where $[\epsilon]$ is used for the integer part of ϵ, provided that the integral on right side is point wise defined on $(0, +\infty)$.

For the proof of the following lemma we refer the readers to [13, 27, 32]

Lemma 2.1 ([27]). *Let $\mathcal{R}(\epsilon) > 0$ and $n \in N$, if $\psi(t) \in AC^n[a, b]$ or $\psi(t) \in C^n[a, b]$, then*

$$\mathcal{I}^{\epsilon}_{a+}\mathcal{D}^{\epsilon}_{a+}\psi(t) = \psi(t) + \sum_{k=0}^{n-1} \frac{\psi^{(k)}(a)}{k!}(t-a)^k. \tag{2.1}$$

Proof. Let $\epsilon \notin N$. If $\psi(t) \in AC^n[a, b](\psi(t) \in C^n[a, b])$, then

$$\mathcal{D}^{\epsilon}_{a+}\psi(t) = \frac{1}{\Gamma(n-\epsilon)} \int_0^t \frac{y^{(n)}(s)ds}{(t-s)^{\epsilon-n+1}} = I^{n-\epsilon}_{a^+} D^n \psi(t). \tag{2.2}$$

We further have

$$I^{\epsilon}_{a+} D^{\epsilon}_{a+}\psi(t) = I^{\epsilon}_{a+} I^{n-\epsilon}_{a^+} D^n \psi(t) = I^n_{a+} D^n_{a+}\psi(t). \tag{2.3}$$

This leads to the proof of (2.1). □

In our case without loss of generality, we consider $a = 0$.

Consider the space of real and continuous functions $\mathcal{V} = C([0, 1], \mathbb{R})$ with topological norm $\|\upsilon\| = \sup\{|\upsilon(t)| : 0 \leq t \leq 1\}$ for $\upsilon \in \mathcal{V}$. $\mathcal{S}$ represents the class of all bounded mappings in $\mathcal{V}$.

Definition 2.3. The mapping $\xi : \mathcal{S} \to (0, \infty)$ for Kuratowski measure of non-compactness is defined as:

$$\xi(\wp) = \inf\{d > 0 : \wp \quad \text{the finite cover for sets of diameter} \leq d\},$$

where $\wp \in \mathcal{S}$.

Definition 2.4. Let $\mathcal{F} : \vartheta \to \mathcal{V}$ be a bounded and continuous mapping $\vartheta \subset \mathcal{V}$. Then $\mathcal{F}$ is a ξ-Lipschitz, where $\zeta \geq 0$ such that

$$\xi(\mathcal{F}(\wp)) \leq \zeta\xi(\wp) \qquad \forall \text{ bounded } \wp \subset \vartheta.$$

Then $\mathcal{F}$ is called strict ξ-contraction if $\zeta < 1$.

Definition 2.5 ([4, 26]). The function $\mathcal{F}$ is ξ-condensing if

$$\xi(\mathcal{F}(\wp)) < \xi(\wp) \qquad \forall \text{ bounded } \wp \subset \vartheta \quad \text{such that} \quad \xi(\wp) > 0.$$

Therefore $\xi(\mathcal{F}(\wp)) \geq \xi(\wp)$ yields that $\xi(\wp) = 0$.

Further, we have $\mathcal{F} : \vartheta \to \mathcal{V}$ is Lipschitz for $\zeta > 0$ such that

$$\|\mathcal{F}(\upsilon) - \mathcal{F}(\bar{\upsilon})\| \leq \zeta\|\upsilon - \bar{\upsilon}\| \quad \text{for all } \upsilon, \bar{\upsilon} \in \vartheta.$$

If $\zeta < 1$, then $\mathcal{F}$ is called strict contraction.

Proposition 2.1 ([4, 26]). *The mapping $\mathcal{F}$ is ξ-Lipschitz with constant $\zeta = 0$ if and only if $\mathcal{F} : \vartheta \to \mathcal{V}$ is compact.*

Proposition 2.2 ([4, 26]). *The $\mathcal{F}$ operator is ξ-Lipschitz with constant ζ if and only if $\mathcal{F} : \vartheta \to \mathcal{V}$ is Lipschitz with constant ζ.*

Theorem 2.1 ([21]). *Let $\mathcal{F} : \mathcal{V} \to \mathcal{V}$ be an ξ-condensing and*

$$\mathcal{H} = \{z \in \mathcal{V} : \textit{there exist} \quad 0 \leq \lambda \leq 1 \quad \textit{such that} \quad z = \lambda \mathcal{T} z\}.$$

If $\mathcal{H}$ is a bounded set in $\mathcal{V}$, i.e., there exists $a > 0$ such that $\mathcal{H} \subset \wp_a(0)$, then

$$\deg(I - \lambda \mathcal{G}, \wp_r(0), 0) = 1 \quad \textit{for all} \quad \lambda \in [0, 1].$$

Consequently, $\mathcal{F}$ has at least one fixed point and the set of fixed points of $\mathcal{F}$ lies in $\wp_a(0)$.

The next lemma has an important role in this paper.

Lemma 2.2 ([26, 34]). *Let ϕ_p be a p-Laplacian operator.*
(1) If $1 < p \leq 2, \ell_1\ell_2 > 0$ and $|\ell_1|, |\ell_2| \geq \rho > 0$, then

$$|\phi_p(\ell_1) - \phi_p(\ell_2)| \leq (p-1)\rho^{p-2}|\ell_1 - \ell_2|.$$

(2) If $p > 2$, and $|\ell_1|, |\ell_2| \leq \rho^*$, then

$$|\phi_p(\ell_1) - \phi_p(\ell_2)| \leq (p-1){\rho^*}^{p-2}|\ell_1 - \ell_2|.$$

Recently, different sorts of stabilities for differential equations have been considered by many scientists. For example, Urs [36] considered Hyers-Ulam stability for the following coupled system

$$\begin{aligned} &u''(t) - \vartheta_1(t, \upsilon(t)) = \vartheta_2(t, z(t)), \quad \upsilon''(t) - \vartheta_1(t, z(t)) = \vartheta_2(t, \upsilon(t)), \\ &\upsilon(t)|_{t=0} = \upsilon(t)_{t=T}, \quad z(t)|_{t=0} = z(t)_{t=T}. \end{aligned}$$

Gâvrută *et al.* [16] proved Hyers-Ulam stability for the following differential equation of second order

$$u'' + \beta(x)u = 0, \tag{2.4}$$

with conditions

$$u(a) = u(b) = 0, \tag{2.5}$$

where $u \in C^2[a, b]$, $\beta(x) \in C[a, b]$, $-\infty < a < b < +\infty$. They gave the following definition for the Hyers-Ulam stability of (2.4).

Definition 2.6 ([16]). The equation (2.4) has Hyers-Ulam stability with boundary conditions (2.5), if there exists a positive constant $\mathcal{D}$ satisfying:
For every $\epsilon > 0$, $y \in C^2[a, b]$, if

$$|u'' + \beta(t)u| \leq \epsilon,$$

and $u(a) = 0 = u(b)$, then there exists some $u^* \in C^2[a, b]$ satisfying

$$u^{*\prime\prime} + \beta(t)u^* = 0,$$

and $u^*(a) = 0 = u^*(b)$, such that $|u(t) - u^*(t)| < \mathcal{D}\epsilon$.

Zada *et al.* [38] recently studied Hyers-Ulam stability and Hyers-Ulam-Rassias stability for the following differential equation

$$\begin{cases} u^{(n)}(t) = h(t, \{u^{(i)}(t)\}_{i=0}^{n-1}, \{u^{(i)}(t-\mu)\}_{i=0}^{n-1}), & t \in [t_0, T], \\ u^{(i)} = g^{(i)}, i = 0, 1, \ldots, n-1, & t \in [t_0 - \mu, t_0], \end{cases} \tag{2.6}$$

where $\mu > 0$, $t_0 < T$ and function $g : [t_0 - \mu, t_0] \to \mathbb{R}$ is $n - 1$ times continuously differentiable. They introduced the following definition for the Hyers-Ulam stability.

Definition 2.7 ([38]). The problem (2.6) is said to be Hyers-Ulam stable on $[t_0-\mu, T]$ if and only if for every function $u \in \mathcal{D}^n[t_0-\mu, T]$ and $\delta \geq 0$ such that

$$\begin{cases} \left|u^{(n)}(t) - h(t, \{u^{(i)}(t)\}_{i=0}^{n-1}, \{u^{(i)}(t-\mu)\}_{i=0}^{n-1})\right| \leq \delta, & t \in [t_0, T], \\ \left|u^{(i)} - g^{(i)}\right| \leq \delta, i = 0, 1, \dots, n-1, & t \in [t_0-\mu, t_0], \end{cases}$$

there is a function $u_0 \in \mathcal{D}^n[t_0-\mu, T]$ with

$$\begin{cases} u_0^{(n)}(t) = h(t, \{u_0^{(i)}(t)\}_{i=0}^{n-1}, \{u_0^{(i)}(t-\mu)\}_{i=0}^{n-1}), & t \in [t_0, T], \\ u_0^{(i)} = g^{(i)}, i = 0, 1, \dots, n-1, & t \in [t_0-\mu, t_0], \end{cases}$$

and

$$|u(t) - u_0(t)| < C(\delta),$$

where $C(\delta) \to 0$ as $\delta \to 0$.

3. Main results

Theorem 3.1. *Let $\psi_1 \in C[0,1]$ be an integrable function satisfying (1.4). Then the solution of*

$$\begin{cases} \mathcal{D}^\beta\left[\phi_p[\mathcal{D}^\epsilon(u(t) - \psi_2(t, u(t)))]\right] + \psi_1(t, u(t)) = 0, \\ \left(\phi_p[\mathcal{D}^\epsilon(u(t) - \psi_2(t, u(t))]\right)^{(i)}|_{t=0} = 0, \ \textit{for } i = 0, 2, 3, \dots, n-1, \\ \left(\phi_p[\mathcal{D}^\epsilon(u(t) - \psi_2(t, u(t))]\right)'|_{t=\eta} = 0, \ u^{(j)}(0) = 0, \ \textit{for } j = 2, 3, \dots, n-1, \\ u(0) = \frac{1}{\Gamma(\epsilon)}\int_0^a (a-s)^{\epsilon-1}\psi_2(u(s))ds, \quad u'(\eta) = \psi_2'(u(\eta)), \end{cases} \tag{3.1}$$

is

$$\begin{aligned} u(x) = &\int_0^1 \mathcal{H}^\epsilon(t,s)\phi_q\Big(\int_0^1 \mathcal{H}^\beta(s,\vartheta)\psi_1(\vartheta, u(\vartheta))d\vartheta\Big)ds \\ &+ \frac{1}{\Gamma(\epsilon-1)}\int_0^a (a-s)^{\epsilon-1}\psi_2(u(s))ds + \psi_2(t, u(t)), \end{aligned}$$

where $\mathcal{H}^\epsilon(t,s)$, $\mathcal{H}^\beta(t,s)$ are Green functions defined by

$$\mathcal{H}^\epsilon(t,s) = \begin{cases} \frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)} - \frac{t(\delta-s)^{\epsilon-2}}{\Gamma(\epsilon-1)}, & s \leq t \leq \delta, \\ \frac{-t(\delta-s)^{\epsilon-1}}{\Gamma(\epsilon-1)}, & t \leq s \leq \delta, \\ \frac{-(t-s)^{\epsilon-1}}{\Gamma(\epsilon)}, & \delta \leq s \leq t, \end{cases} \tag{3.2}$$

$$\mathcal{H}^\beta(t,s) = \begin{cases} \frac{-(t-s)^{\beta-1}}{\Gamma(\beta)} + \frac{t(\eta-s)^{\beta-2}}{\Gamma(\beta-1)}, & s \leq t \leq \eta, \\ \frac{t(\eta-s)^{\beta-2}}{\Gamma(\beta-1)}, & t \leq s \leq \eta, \\ \frac{-(t-s)^{\beta-1}}{\Gamma(\beta)} & \eta \leq s \leq t. \end{cases} \tag{3.3}$$

Proof. Applying operator $\mathcal{I}^\beta$ on (3.1) and using Lemma 2.1, we get the following equivalent integral form

$$\phi_p[\mathcal{D}^\epsilon(u(t) - \psi_2(t, u(t)))] = -\mathcal{I}^\beta\psi_1(t, u(t)) + c_1 + c_2 t + c_3 t^2 + \ldots + c_n t^{n-1}. \tag{3.4}$$

By $\Big(\phi_p(\mathcal{D}^\epsilon(u(t)-\psi_2(t,u(t))\Big)^{(i)}|_{t=0}=0$, for $i=0,2,3,\ldots,n-1$, we get $c_1=c_3=c_4=\ldots=c_n=0$. And $\Big(\phi_p(\mathcal{D}^\epsilon(u(t)-\psi_2(t,u(t)))\Big)'|_{t=\eta}=0$, implies

$$c_2=\mathcal{I}_\eta^{\beta-1}\psi_1(t,u(t))=\frac{1}{\Gamma(\beta-1)}\int_0^\eta(\eta-s)^{\beta-2}\psi_1(s,u(s))ds.$$

From the values of c_i for $i=1,2,3,\ldots,n$, and (3.4), we have

$$\begin{aligned}\phi_p\Big(\mathcal{D}^\epsilon(u(t)-\psi_2(t,u(t))\Big)&=-\mathcal{I}^\beta\psi_1(t,u(t))+t\mathcal{I}^{\beta-1}\psi_1(t,u(t))|_{t=\eta}\\&=-\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\psi_1(s,u(s))ds+t\int_0^\eta\frac{(\eta-s)^{\beta-2}}{\Gamma(\beta-1)}\psi_1(s,u(s))ds\\&=\int_0^1\mathcal{G}^\beta(t,s)\psi_1(s,u(s))ds,\end{aligned}\tag{3.5}$$

where $\mathcal{H}^\beta(t,s)$ is a Green's function given in (3.3). From (3.5), we further have

$$\mathcal{D}^\epsilon(u(t)-\psi_2(t,u(t)))=\phi_q\Big(\int_0^1\mathcal{H}^\beta(t,s)\psi_1(s,u(s))ds\Big).\tag{3.6}$$

Applying fractional integral operator $\mathcal{I}^\epsilon$ on (3.6) and using Lemma 2.2 again, we have

$$\begin{aligned}u(t)=\psi_2(t,u(t))+\mathcal{I}^\epsilon\Big(\phi_q(\int_0^1\mathcal{H}^\beta(t,s)\psi_1(s,u(s))ds)\Big)+k_1+k_2t\\+k_3t^2+\ldots+k_nt^{n-1}.\end{aligned}\tag{3.7}$$

Using conditions $u^{(j)}(0)=0$ for $j=2,3,\ldots,n-1$ in (3.7), we obtain $k_3=k_4=\ldots=k_n=0$. From condition $u'(\delta)=\big(\psi_2(t,u(t))\big)'|_{t=\delta}$, we have $k_2=-\mathcal{I}^{\epsilon-1}\Big(\phi_q(\int_0^1\mathcal{H}^\beta(t,s)\psi_1(s,u(s))ds)\Big)|_{t=\delta}$.

From condition $u(0)=\frac{1}{\Gamma(\epsilon)}\int_0^a(a-s)^{\epsilon-1}\psi_2(u(s))ds$ we get

$$k_1=\frac{1}{\Gamma(\epsilon)}\int_0^a(a-s)^{\epsilon-1}\psi_2(u(s))ds.$$

Now putting the values of k_i for $i=1,2,3$ in (3.7), we have

$$\begin{aligned}u(t)=\psi_2(t,u(t))&+\mathcal{I}^\epsilon\Big(\phi_q(\int_0^1\mathcal{H}^\beta(t,s)\psi_1(s,u(s))ds)\Big)\\&-t\mathcal{I}^\epsilon\Big(\phi_q(\int_0^1\mathcal{H}^\beta(t,s)\psi_1(s,u(s))ds)\Big)|_{t=\delta}+\mathcal{I}_a^\epsilon\psi_2(u(t))\\=&\Big(\int_0^t\frac{(t-s)^{\epsilon-1}}{\Gamma(\epsilon)}-t\int_0^\delta\frac{(\delta-s)^{\epsilon-2}}{\Gamma(\epsilon-1)}\Big)\phi_q\Big(\int_0^1\mathcal{H}^\beta(s,\vartheta)\psi_1(\vartheta,u(\vartheta))d\vartheta ds\Big)\\&+\frac{1}{\Gamma(\epsilon)}\int_0^a(a-s)^{\epsilon-1}\psi_2(u(s))ds+\psi_2(t,u(t))\\=&\int_0^1\mathcal{H}^\epsilon(t,s)\phi_q\Big(\int_0^1\mathcal{H}^\beta(s,\vartheta)\psi_1(\vartheta,u(\vartheta))d\vartheta\Big)ds+\int_0^a\frac{(a-s)^{\epsilon-1}}{\Gamma(\epsilon)}\psi_2(u(s))ds\\&+\psi_2(t,u(t)),\end{aligned}$$

where $\mathcal{H}^{\epsilon}(t,s)$, $\mathcal{H}^{\beta}(t,s)$ are Green functions defined by (3.2), (3.3), respectively. $\square$

By Theorem 3.1, our problem (1.4) is equivalent to the following integral equation

$$u(t) = \int_0^1 \mathcal{H}^{\epsilon}(t,s)\phi_q\big(\int_0^1 \mathcal{H}^{\beta}(s,\vartheta)\psi_1(\vartheta,u(\vartheta))d\vartheta\big)ds + \int_0^a \frac{(a-s)^{\epsilon-1}}{\Gamma(\epsilon)}\psi_2(u(s))ds + \psi_2(t,u(t)).$$

Define $\mathcal{F}_i^* : \mathcal{V} \to \mathcal{V}$ for $(i=1,2)$ by

$$\mathcal{F}_1^* u(t) = \int_0^1 \mathcal{H}^{\epsilon}(t,s)\phi_q\big(\int_0^1 \mathcal{H}^{\beta}(s,\vartheta)\psi_1(\vartheta,u(\vartheta))d\vartheta\big)ds, \tag{3.8}$$

$$\mathcal{F}_2^* u(t) = \psi_2(t,u(t)) + \frac{1}{\Gamma(\epsilon)}\int_0^a (a-s)^{\epsilon-1}\psi_2(s,u(s))ds. \tag{3.9}$$

By Theorem 3.1, solution of our problem (1.4) is a fixed point $u(t)$ of the operator $\mathcal{F}$ defined by

$$\mathcal{F}(u) = \mathcal{F}_1^*(u) + \mathcal{F}_2^*(u) = u.$$

To proceed further, we introduce the following assumptions:

$(\mathcal{Q}_1)$ With $a_1, \mathbb{M}^*_{\psi_1} > 0$, $k_1 \in [0,1]$, function ψ_1 satisfies

$$|\psi_1(t,u(t))| \le \phi_p\big(a_1|u(t)|^{k_1} + \mathbb{M}^*_{\psi_1}\big).$$

$(\mathcal{Q}_2)$ There exists a real valued constant λ_{ψ_1} such that for all $u,v \in \mathcal{V}$,

$$|\psi_1(t,u) - \psi_1(t,v)| \le \lambda_{\psi_1}|u(t)-v(t)|.$$

$(\mathcal{Q}_3)$ With $a_2, \mathbb{M}^*_{\psi_2} > 0$, $k_1 \in [0,1]$, function ψ_2 satisfies

$$|\psi_2(t,u)| \le a_2|u|^{k_1} + \mathbb{M}^*_{\psi_2}.$$

$(\mathcal{Q}_4)$ There exists a real valued constant λ_{ψ_2} such that $\forall\ u,v \in \mathcal{V}$,

$$|\psi_2(t,u) - \psi_2(t,v)| \le \lambda_{\psi_2}|u(t)-v(t)|.$$

For simplicity in calculations, we define the following terms:
$\Delta_1 = \big(\frac{1}{\Gamma(\epsilon+1)} + \frac{\delta^{\epsilon-1}}{\Gamma(\epsilon)}\big)\big(\frac{1}{\Gamma(\beta+1)} + \frac{\eta^{\beta-1}}{\Gamma(\beta)}\big)^{q-1}$, $\Delta_{\psi_1} = (p-1)\rho_1^{p-2}\lambda_{\psi_1}\big(\frac{1}{\Gamma(\epsilon+1)} + \frac{\delta^{\epsilon-1}}{\Gamma(\epsilon)}\big)\big(\frac{1}{\Gamma(\beta+1)} + \frac{\eta^{\epsilon-1}}{\Gamma(\beta)}\big)$, $\mathcal{M}^* = \Delta(\mathbb{M}_{\psi_1} + \mathbb{M}_{\psi_2})\Omega$, $\Omega = (a_1+a_2)(\Delta_1 + 1 + \frac{a^{\epsilon}}{\Gamma(\epsilon+1)})$.

Theorem 3.2. *With assumptions $(\mathcal{Q}_1)$, $(\mathcal{Q}_3)$, the operator $\mathcal{F} : \mathcal{V}^* \to \mathcal{V}$ is continuous and satisfies*

$$\mathcal{F}(u(t)) \le \Omega\|u\|^k + \mathcal{M}^*,$$

for each $u \in \wp_r \subset \mathcal{V}$.

Proof. Let us consider a bounded set $\wp_r = \{u \in \mathcal{V} : \|u\| \le r\}$ with sequence $\{u_n\}$ converging to u in $\wp_r$. To show that $\|\mathcal{F}^*(u_n) - \mathcal{F}^*(u)\| \to 0$ as $n \to \infty$, let us consider

$$
\begin{aligned}
|\mathcal{F}_1^* u_n(t) - \mathcal{F}_1^* u(t)| &= \Big| \int_0^1 \mathcal{H}^\epsilon(t,s)\phi_q\big(\int_0^1 \mathcal{H}^\beta(s,\vartheta)\psi_1(\vartheta,u_n(\vartheta))d\vartheta\big)ds \\
&\quad - \int_0^1 \mathcal{H}^\epsilon(t,s)\phi_q\big(\int_0^1 \mathcal{H}^\beta(s,\vartheta)\psi_1(\vartheta,u(\vartheta))d\vartheta ds\big)\Big| \quad (3.10)\\
&\le \int_0^1 \big|\mathcal{H}^\epsilon(t,s)\big|\Big|\phi_q\big(\int_0^1 \mathcal{H}^\beta(s,\vartheta)\psi_1(\vartheta,u_n(\vartheta))d\vartheta\big)ds \\
&\quad -\phi_q\big(\int_0^1 \mathcal{H}^\beta(s,\vartheta)\psi_1(\vartheta,u(\vartheta))d\vartheta\big)\Big|ds.
\end{aligned}
$$

By the estimate (3.10) and continuity of the function ψ_1, we have $|\mathcal{F}_1^* u_n(t) - \mathcal{F}_1^* u(t)| \to 0$ as $n \to +\infty$. This proves that $\mathcal{F}_1^*$ is continuous. Now, for the continuity of $\mathcal{F}_2^*$, let us consider

$$
\begin{aligned}
\big|\mathcal{F}_2^* u_n(t) - \mathcal{F}_2^* u(t)\big| &= \big|\psi_2(t,u_n(t)) + \int_0^a \frac{(a-s)^{\epsilon-1}}{\Gamma(\epsilon)}\psi_2(s,u_n(s))ds \\
&\quad -\Big(\psi_2(t,u(t)) + \int_0^a \frac{(a-s)^{\epsilon-1}}{\Gamma(\epsilon)}\psi_2(s,u(s))ds\Big)\big| \quad (3.11)\\
&\le \int_0^a \frac{(a-s)^{\epsilon-1}}{\Gamma(\epsilon)}\big|\psi_2(s,u_n(s))ds - \psi_2(s,u(s))\big|ds \\
&\quad +\big|\psi(t,u_n(t)) - \psi_2(t,u(t))\big|.
\end{aligned}
$$

With the help of (3.11) and continuity of the function $\psi_2(t,u(t)) : ([0,1]\times\mathbb{R}) \to \mathbb{R}$, we have $\big|\mathcal{F}_2^* u_n(t) - \mathcal{F}_2^* u(t)\big| \to 0$, as $n \to +\infty$. This implies $\mathcal{F}_2^*$ is continuous. Consequently, from (3.10), (3.11) we have $\mathcal{F} = \mathcal{F}_1^*(u(t)) + \mathcal{F}_2^*(u(t))$ is continuous.

Now, for the inequality (3.10), by (3.8) and assumption $(\mathcal{Q}_1)$, we have

$$
\begin{aligned}
|\mathcal{F}_1^* u(t)| &= \Big| \int_0^1 \mathcal{H}^\epsilon(t,s)\phi_q\big(\int_0^1 \mathcal{H}^\beta(s,\vartheta)\psi_1(\vartheta,u(\vartheta))d\vartheta\big)ds\Big| \\
&= \int_0^1 |\mathcal{H}^\epsilon(t,s)|\phi_q\big(\int_0^1 |\mathcal{H}^\beta(s,\vartheta)||\psi_1(\vartheta,u(\vartheta))|d\vartheta\big)ds \qquad (3.12)\\
&\le \int_0^1 |\mathcal{H}^\epsilon(t,s)|\phi_q\big(\int_0^1 |\mathcal{H}^\beta(s,\vartheta)|\phi_p\big(a_1\|u\|^{k_1} + \mathbb{M}_{\psi_1}^*\big)d\vartheta\big)ds \\
&\le \big(\frac{1}{\Gamma(\epsilon+1)} + \frac{\delta^{\epsilon-1}}{\Gamma(\epsilon)}\big)\big(\frac{1}{\Gamma(\beta+1)} + \frac{\eta^{\beta-1}}{\Gamma(\beta)}\big)^{q-1}\Big(a_1\|u\|^{k_1} + \mathbb{M}_{\psi_1}^*\Big) \\
&= \Delta_1\Big(a_1\|u\| + \mathbb{M}_{\psi_1}^*\Big).
\end{aligned}
$$

From (3.9) and assumption $\mathcal{Q}_3$, we get

$$
\begin{aligned}
\big|\mathcal{F}_2^* u(t)\big| &= \big|\psi_2(t,u(t)) + \frac{1}{\Gamma(\epsilon)}\int_0^a (a-s)^{\epsilon-1}\psi_2(s,u(s))ds\big| \\
&\le |\psi_2(s,u(s))\big| + \frac{1}{\Gamma(\epsilon)}\int_0^a (a-s)^{\epsilon-1}\big|\psi_2(s,u(s))\big|ds \qquad (3.13)\\
&= \big(1 + \frac{a^\epsilon}{\Gamma(\epsilon+1)}\big)\big(a_2\|u\| + \mathbb{M}_{\psi_2}^*\big).
\end{aligned}
$$

In view of the estimates (3.12) and (3.13), we can obtain

$$\begin{aligned}\big|\mathcal{F}(u(t))\big| &\le \Delta_1\Big(a_1\|u\|^{k_1}+\mathbb{M}^*_{\psi_1}\Big)+\big(1+\frac{a^\epsilon}{\Gamma(\epsilon+1)}\big)\big(a_2\|u\|^{k_1}+\mathbb{M}^*_{\psi_2}\big)\\ &\le \Omega\|u\|^k+\mathcal{M}^*_2.\end{aligned}$$

This completes the proof. □

Theorem 3.3. *With assumption* $(\mathcal{Q}_1)$, *the operator* $\mathcal{F}^*_1:\mathcal{V}^*\to\mathcal{V}^*$ *is compact and* ξ*-Lipschitz with constant zero.*

Proof. Theorem 3.2 implies that the operator $\mathcal{F}^*_1:\mathcal{V}\to\omega$ is bounded. Next, let $\mathcal{Y}\subset\wp_r\subset\omega^*$. Then, by assumption $(\mathcal{Q}_1)$, Lemma 3.1, equation (3.8), for any $t_1,t_2\in[0,1]$, we have

$$\begin{aligned}&\big|\mathcal{F}^*_1u(t_1)-\mathcal{F}^*_1u(t_2)\big|\\ &=\Big|\int_0^1\mathcal{H}^\epsilon(t_1,s)\phi_q\big(\int_0^1\mathcal{H}^\beta(s,\vartheta)\psi_1(\vartheta,u(\vartheta))d\vartheta\big)ds\\ &\quad-\int_0^1\mathcal{H}^\epsilon(t_2,s)\phi_q\big(\int_0^1\mathcal{H}^\beta(s,\vartheta)\psi_1(\vartheta,u(\vartheta))d\vartheta\big)ds\Big|\\ &\le\int_0^1\big|\mathcal{H}^\epsilon(t_1,s)-\mathcal{H}^\epsilon(t_2,s)\big|\phi_q\big(\int_0^1\big|\mathcal{H}^\beta(s,\vartheta)\big|\phi_p\big(a_1\|u\|^{k_1}+\mathbb{M}^*_{\psi_1}\big)d\vartheta\big)ds\\ &\le\Big(\frac{|t_1^\epsilon-t_2^\epsilon|}{\Gamma(\epsilon+1)}+\frac{|t_1-t_2|\delta^{\epsilon-1}}{\Gamma(\epsilon)}\Big)\Big(\frac{\eta^{\beta-1}}{\Gamma(\beta)}+\frac{1}{\Gamma(\beta+1)}\Big)^{q-1}(a_1\|u\|^{k_1}+\mathbb{M}^*_{\psi_1}).\end{aligned}\tag{3.14}$$

As $t_1\to t_2$ the right hand side of (3.14) approaches to zero. Thus $\mathcal{F}^*_1$ is an equicontinuous operator on $\mathcal{Y}$. By Arzela-Ascoli theorem, $\mathcal{F}^*_1(\mathcal{H})$ is compact. Hence $\mathcal{H}$ is ξ-Lipschitz with constant zero. □

Theorem 3.4. *With assumptions* $\mathcal{Q}_1$*-*$\mathcal{Q}_4$ *and* $\Omega<1$, *the FDE with* p*-Laplacian operator* (1.4) *has a solution and the set containing solutions of the problem* (1.4) *is bounded in* $\mathcal{V}^*$.

Proof. For existence of solution of the FDE with nonlinear p-Laplacian operator (1.4), we take help from Theorem 2.1. Let us consider the set

$$\mathcal{S}=\{u\in\mathcal{V}^*:\ \text{there exist }\lambda\in[0,1],\text{ such that }u=\lambda\mathcal{F}(u)\},$$

to show that $\mathcal{S}$ is bounded. For this we assume a contrary path. Assume that $u\in\mathcal{S}$, such that $\|u\|=\mathcal{K}\to\infty$. But from Theorem 3.2, we obtain

$$\begin{aligned}\|u\|=\|\lambda\mathcal{F}(u)\|\le\|\mathcal{F}(u)\|&\le\|\mathcal{F}^*_1(u)\|+\|\mathcal{F}^*_2(u)\|\\ &\le\mathcal{M}^*_2+\Omega\|u\|^k.\end{aligned}\tag{3.15}$$

Since $\mathcal{K}=\|u\|$, (3.15) implies that

$$\begin{aligned}\|u\|&\le\mathcal{M}^*_2+\Omega\|u\|^k,\\ 1&\le\Omega\frac{\|u\|^k}{\|u\|}+\frac{\mathcal{M}^*_2}{\|u\|},\\ 1&\le\Omega\frac{1}{\mathcal{K}^{1-k}}+\frac{\mathcal{M}^*_2}{\mathcal{K}}\to0,\ \text{as }\mathcal{K}\to\infty.\end{aligned}$$

This is a contradiction. Ultimately, $\|u\| < \infty$ which implies the set $\mathcal{S}$ is bounded and by Theorem 2.1, the operator $\mathcal{F}$ has a fixed point which is a solution of our problem (1.4). Consequently, $\mathcal{S}$ which is containing the solutions of (1.4) is a bounded subset of $\mathcal{V}^*$. □

Theorem 3.5. *Let assumptions* $(\mathcal{Q}_1)$, $(\mathcal{Q}_4)$ *hold. Then the FDE with nonlinear p-Laplacian* (1.4) *has a unique solution provided that* $\Delta_{\psi_1} + \lambda_{\psi_2}\big(1 + \frac{a^\epsilon}{\Gamma(\epsilon+1)}\big) < 1$.

Proof. From (3.8), assumptions $(\mathcal{Q}_1)$ and Lemma 2.2, for any t_1, $t_2 \in [0,1]$, we have

$$\begin{aligned}
|\mathcal{F}_1^* u(t) - \mathcal{F}_1^* \bar{u}(t)| &= \Big| \int_0^1 \mathcal{H}^\epsilon(t,s)\phi_q\big(\int_0^1 \mathcal{H}^\beta(s,\vartheta)\psi_1(\vartheta,u(\vartheta))d\vartheta\big) \\
&\quad - \int_0^1 \mathcal{H}^\epsilon(t,s)\phi_q\big(\int_0^1 \mathcal{H}^\beta(s,\vartheta)\psi_1(\vartheta,\bar{u}(\vartheta))d\vartheta\big)\Big| ds \\
&= \int_0^1 \Big|\mathcal{H}^\epsilon(t,s)\Big|\Big|\phi_q\big(\int_0^1 \mathcal{G}^\beta(s,\vartheta)\psi_1(\vartheta,u(\vartheta))d\vartheta ds \\
&\quad -\phi_q\big(\int_0^1 \mathcal{G}^\beta(s,\vartheta)\psi_1(\vartheta,\bar{u}(\vartheta))d\vartheta\big)ds\Big| \qquad (3.16) \\
&= (q-1)\rho^{q-2}\int_0^1 \big|\mathcal{H}^\epsilon(t,s)\big| \int_0^1 \big|\mathcal{H}^\beta(s,\vartheta)\big|\big|\psi_1(\vartheta,u(\vartheta)) \\
&\quad -\psi_1(\vartheta,\bar{u}(\vartheta))\big|d\vartheta ds \\
&\le (q-1)\rho^{q-2}\lambda_{\psi_1}\big(\frac{1}{\Gamma(\epsilon+1)} + \frac{\delta^{\epsilon-1}}{\Gamma(\epsilon)}\big)\big(\frac{1}{\Gamma(\beta+1)} + \frac{\eta^{\epsilon-1}}{\Gamma(\beta)}\big) \\
&\quad \Big(\big|u(t)-\bar{u}(t)\big|\Big) \\
&= \Delta_{\psi_1}\Big(\big\|u-\bar{u}\big\|\Big).
\end{aligned}$$

From (3.9), assumptions $(\mathcal{Q}_4)$ and Lemma 2.2, for any t_1, $t_2 \in [0,1]$, we get

$$\begin{aligned}
\big|\mathcal{F}_2^* u(t) - \mathcal{F}_2^* \bar{u}(t)\big| &= \big|\psi_2(t,u(t))\frac{1}{\Gamma(\epsilon)}\int_0^a (a-s)^{\epsilon-1}\psi_2(s,u(s))ds \\
&\quad -\big(\psi_2(t,\bar{u}(t)) + \frac{1}{\Gamma(\epsilon)}\int_0^a (a-s)^{\epsilon-1}\psi_2(s,\bar{u}(s))ds\big)\big| \qquad (3.17) \\
&\le \frac{1}{\Gamma(\epsilon)}\int_0^a (a-s)^{\epsilon-1}\big|\psi_2(s,u(s)) - \psi_2(s,\bar{u}(s))\big|ds \\
&\quad +\big|\psi_2(t,u(t)) - \psi_2(t,\bar{u}(t))\big| \\
&\le \lambda_{\psi_2}\Big(1 + \frac{a^\epsilon}{\Gamma(\epsilon+1)}\Big)\big\|u(t)-\bar{u}(t)\big\|.
\end{aligned}$$

From (3.16) and (3.17), we can obtain

$$\begin{aligned}
\big|\mathcal{F}(u(t)) - \mathcal{F}(\bar{u}(t))\big| &\le \Delta_{\psi_1}\Big(\big|u(t)-\bar{u}(t)\big|\Big) + \lambda_{\psi_2}\Big(1+\frac{a^\epsilon}{\Gamma(\epsilon+1)}\Big)\big\|u-\bar{u}(t)\big\| \\
&\le \Big(\Delta_{\psi_1} + \lambda_{\psi_2}\big(1+\frac{a^\epsilon}{\Gamma(\epsilon+1)}\big)\Big)\big\|u(t)-\bar{u}(t)\big\|,
\end{aligned}$$

with $\Delta_{\psi_1} + \lambda_{\psi_2}\big(1+\frac{a^\epsilon}{\Gamma(\epsilon+1)}\big) < 1$. The Banach's contraction principle implies that $\mathcal{F}$

has a unique fixed point. Thus, the FDE with nonlinear p-Laplacian operator (1.4) has a unique solution. □

4. Hyers-Ulam stability

Here we present Hyers-Ulam stability for the FDE with nonlinear p-Laplacian operator (1.4). In view of Definition 2.7 and the work given in [4, 22, 38] we give the following definition.

Definition 4.1. The integral equation (3.8) is Hyers-Ulam stable if there exists positive constant $\mathcal{D}^*$ satisfying:
For every $\lambda > 0$, if

$$\begin{aligned}\Big|u(t) - \int_0^1 \mathcal{H}^\epsilon(t,s)\phi_q\big(\int_0^1 \mathcal{H}^\beta(s,\vartheta)\psi_1(\vartheta,u(\vartheta))d\vartheta\big)ds \\ - \frac{1}{\Gamma(\epsilon)}\int_0^a (a-s)^{\epsilon-1}\psi_2(s,u(s))ds - \psi_2(t,u(t))\Big| \leq \lambda,\end{aligned}$$

then there exists a $u^*(t)$ satisfying

$$\begin{aligned}u^*(t) = \int_0^1 \mathcal{H}^\epsilon(t,s)\phi_q\big(\int_0^1 \mathcal{H}^\beta(s,\vartheta)\psi_1(\vartheta,u^*(\vartheta))d\vartheta\big)ds \\ +\frac{1}{\Gamma(\epsilon)}\int_0^a (a-s)^{\epsilon-1}\psi_2(s,u^*(s))ds + \psi_2(t,u^*(t)),\end{aligned} \tag{4.1}$$

such that

$$|u(t) - u^*(t)| \leq \mathcal{D}^*\lambda.$$

Theorem 4.1. *With the assumptions $(\mathcal{Q}_1)$ and $(\mathcal{Q}_2)$, the FDE with nonlinear p-Laplacian operator (1.4) is Hyers-Ulam stable.*

Proof. In view of Theorem 3.5 and Definition 4.1, let $u(t)$ be the real solution of (3.8) and $u^*(t)$ be an approximation satisfying (4.1). Then, we get

$$\begin{aligned}&\big|u(t) - u^*(t)\big| \\ &= \Big|\int_0^1 \mathcal{H}^\epsilon(t,s)\phi_q\big(\int_0^1 \mathcal{H}^\beta(s,\vartheta)\psi_1(\vartheta,u(\vartheta))d\vartheta\big)ds \\ &\quad + \int_0^a \frac{(a-s)^{\epsilon-1}\psi_2(s,v(s))ds}{\Gamma(\epsilon)} + \psi_2(t,u(t)) \\ &\quad - \int_0^1 \mathcal{H}^\epsilon(t,s)\phi_q\big(\int_0^1 \mathcal{H}^\beta(s,\vartheta)\psi_1(\vartheta,u^*(\vartheta))d\vartheta ds\big) \\ &\quad - \int_0^a \frac{(a-s)^{\epsilon-1}\psi_2(s,u^*(s))ds}{\Gamma(\epsilon)} - \psi_2(t,u^*(t))\Big| \\ &\leq (q-1)\rho^{q-2}\Big(\int_0^1 |\mathcal{H}^\epsilon(t,s)|\int_0^1 |\mathcal{H}^\beta(s,\vartheta)||\psi_1(\vartheta,u(\vartheta)) - \psi_1(\vartheta,u^*(\vartheta))|d\vartheta ds\Big) \\ &\quad +\frac{1}{\Gamma(\epsilon)}\int_0^a (a-s)^{\epsilon-1}|\psi_2(s,u(s)) - \psi_2(s,u^*(s))|ds + |\psi_2(s,u(s)) - \psi_2(s,u^*(s))|\end{aligned}$$

$$\leq (q-1)\rho^{q-2}\lambda_{\psi_1}\Big(\frac{1}{\Gamma(\epsilon+1)}+\frac{\delta^{\epsilon-1}}{\Gamma(\epsilon)}\Big)\Big(\frac{1}{\Gamma(\beta+1)}+\frac{\eta^{\beta-1}}{\Gamma(\beta)}\Big)\|u-u^*\|$$
$$+\frac{\lambda_{\psi_2}a^{\epsilon}}{\Gamma(\epsilon+1)}\|u-u^*\|+\lambda_{\psi_2}\|u-u^*\|$$
$$=(\Delta_{\psi_1}+\lambda_{\psi_2}+\frac{\lambda_{\psi_2}a^{\epsilon}}{\Gamma(\epsilon+1)})\|u-u^*\|, \tag{4.2}$$

where $\mathcal{D}^* = \Delta_{\psi_1}+\lambda_{\psi_2}+\frac{\lambda_{\psi_2}a^{\epsilon}}{\Gamma(\epsilon+1)}$. Hence, in view of the estimate (4.2), we can conclude that the integral equation (3.8) is Hyers-Ulam stable. Consequently, the FDE with nonlinear p-Laplacian operator (1.4) is Hyers-Ulam stable. □

5. Illustrative example

Here, we present an application of our theorems which were proved in Section 2 and Section 3.

Example 5.1. Consider the following HFDE with p-Laplacian operator for $n=3$

$$\begin{aligned}
&\mathcal{D}^{\frac{8}{3}}\big(\phi_5(\mathcal{D}^{\frac{7}{3}}(u(t)-\psi_2(t,u(t))))\big)+\psi_1(t,v(t))=0,\\
&\big(\phi_5(\mathcal{D}^{\frac{7}{3}}(u(t)-\psi_2(t,u(t))))\big)|_0^{(i)}=0, \text{ for } i=0,2,\\
&\big(\phi_5(\mathcal{D}^{\frac{7}{3}}(u(t)-\psi_2(t,u(t))))\big)'|_{0.5}=0,\\
&u(0)=\frac{1}{\Gamma(\frac{7}{3})}\int_0^a (a-s)^{\frac{7}{3}-1}\psi_2(s,u(s))ds,\\
&u''(0)=0,\ u'(0.5)=\psi_2'(u(0.5)),
\end{aligned} \tag{5.1}$$

where $t\in[0,1]$, $\eta=\delta=a=b=0.5$, $p=5, \epsilon=7/3, \beta=8/3$. $\psi_1(t,u(t))=\frac{-24t}{17}+\frac{1}{15}\sin(u(t))$, $\psi_2(t,u(t))=t^3(\frac{30}{18}+\frac{1}{15}\cos(u))$, $\lambda_{\psi_1}=\lambda_{\psi_2}=\frac{1}{15}$. By simple calculations, we have $\max\{\Delta_{\psi_1}+\lambda_{\psi_2}\big(1+\frac{a^{\epsilon}}{\Gamma(\epsilon+1)}\big),\Omega\}<1$. By Theorem 3.5, we can conclude that (5.1) has a unique solution. With similar fashion, the satisfaction of the conditions of Theorem 4.1 can be checked easily, and consequently the problem (5.1) is Hyers-Ulam stable.

6. Conclusion

In this paper, three important aspects of the FDE with nonlinear p-Laplacian operator (1.4) have been considered. They are existence of solution, uniqueness of solution and Hyers-Ulam stability. For these aims, we converted the problem (1.4) to an integral equation by the help of Green function. After this, we have proved results for existence and uniqueness by topological degree method. Then, Hyers-Ulam stability was investigated. The problem (1.4) has the potentials to be studied for further aims. For future, we also have a plan to investigate its multiplicity results. The references of the paper are helpful for the readers to related concepts.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions

All the authors have equal contributions in this article.

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