Journal of Applied Analysis and Computation
Volume 8, Number 4, August 2018, 1186–1193

Website:http://jaac-online.com/
DOI:10.11948/2018.1186

ASYMPTOTIC DYNAMICS FOR REACTION DIFFUSION EQUATIONS IN UNBOUNDED DOMAIN*

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Abstract In this paper we study the asymptotic dynamics for reaction diffusion equation defined in $\mathbb{R}^n$. We will prove that the equation possesses a fixed point when the nonlinearity satisfies some restrictive conditions and then we show that the fixed point is an exponential attractor.

Keywords Reaction diffusion equation, semigroup, exponential attractor.

MSC(2010) 35K57, 35B40, 35B41.

1. Introduction

We are interested in the long time behavior of the following semilinear reaction diffusion equation:

$$\frac{\partial u}{\partial t} - \triangle u + f(u) + \lambda u = g(x) \ \text{ in } \ \mathbb{R}^+ \times \mathbb{R}^n \tag{1.1}$$

with initial data

$$u(0,x) = u_0(x) \ \text{ in } \ \mathbb{R}^n. \tag{1.2}$$

We assume that the nonlinearity $f(s)$ satisfies the following conditions:

$$f(0) = 0, (f(s_1) - f(s_2))(s_1 - s_2) \geq -C_0|s_1 - s_2|^2, \text{ for any } s_1, s_2 \in \mathbb{R}, \tag{1.3}$$

$$\alpha_1|s|^p - k_1|s|^2 \leq f(s)s \leq \alpha_2|s|^p + k_2|s|^2, \ 2 < p < \infty, \tag{1.4}$$

where $C_0, \alpha_1, \alpha_2, k_1, k_2$ are positive constants, $\lambda > k_1$.

It is known that the asymptotic dynamics of this problem has been studied extensively, especially for the case of bounded domains; see, e.g. [1,5,8,11,13] and the references therein, where many results associated with this problem concentrate on the existence of global attractors. In general, the existence of attractors depends on some kind of compactness. For bounded domains, the compactness is obtained by a priori estimates and compactness of Sobolev embeddings. For unbounded domains, it becomes a difficult question to get the compactness. To avoid this difficulty, asymptotically compact method was introduced in [10, 12], and the existence of global attractor in $L^p(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ was proved for Eq.(1.1). However, there

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*The authors were supported by National Natural Science Foundation of China (11761044, 71701084) and the key constructive discipline of Lanzhou City University(LZCU-ZDJSXK-201706).

are few articles study the existence of exponential attractors when the domain is unbounded. The likely reason is that the existence of exponential attractors techniques in bounded domain can not be used directly in the case. For this reason, we study the existence of exponential attractors for problem (1.1).

As far as we know, there are no results on exponential attractors for Eq.(1.1). In this paper, we study the existence of exponential attractors for Eq.(1.1). It is worth mentioning that Sobolev embedding is not compact in unbounded domains. Those methods([2–4,6,7,9]) which can be used to prove the existence of exponential attractors in bounded domains can not be used to deal with the problem (1.1).

As far as our problems are concerned, the nonlinearity $f(s)$ is a function with polynomial growth of arbitrary order, in many cases, not only $\lambda > k_1$, but also $\lambda > C_0$, so we impose an additional condition that

$$\lambda > \max\{k_1, C_0\}, \tag{1.5}$$

and study the asymptotic dynamics under conditions (1.3)–(1.5).

This paper is organized as follows. In the next section, we recall some basic concepts about attractors and exponential attractors. In Section 3, we prove that the dynamical system generated by (1.1) exists an exponential attractor $\mathcal{M} = \{\theta(x)\}$ in $L^r(\mathbb{R}^n)(2 \leq r < 2p-2)$ and $H^1(\mathbb{R}^n)$, respectively.

For convenience, here let $|\cdot|$ be the norm of $L^2(\mathbb{R}^n)$ or absolute value, $|\cdot|_p$ be the norm of $L^p(\mathbb{R}^n)(p > 2)$, C_i denote constants which may vary from line to line.

2. Preliminaries

Let X be a complete metric space. A one-parameter family of mappings $S(t) : X \to X$ is called a semigroup provided that

(i) S(0)=I;

(ii) $S(t+s) = S(t)S(s)$ for all $t, s \geq 0$.

The pair $(S(t), X)$ is usually referred to as a dynamical system.

Definition 2.1. A set B is called a bounded absorbing set to $(S(t), X)$, if for any bounded set $D \subset X$, there exists $T = T(D)$ such that $S(t)D \subset B$ for all $t \geq T$.

Definition 2.2. A set $\mathcal{A} \subset X$ is called a global attractor for $(S(t), X)$ if

(i) $\mathcal{A}$ is compact in X,

(ii) $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$,

(iii) for any $B \subset X$ that is bounded, $dist(S(t)B, \mathcal{A}) \to 0$ as $t \to \infty$, where $dist(B, \mathcal{A}) = \sup_{b \in B} \inf_{a \in \mathcal{A}} \| b - a \|_X$.

Definition 2.3. Let $n(\mathcal{M}, \varepsilon)$, $\varepsilon > 0$, denote the minimum number of ball of X of radius ε which is necessary to cover $\mathcal{M}$. The fractal dimension of $\mathcal{M}$, which is also called the capacity of $\mathcal{M}$, is the number

$$\dim_f \mathcal{M} = \overline{\lim_{\varepsilon \to 0^+}} \frac{\ln n(\mathcal{M}, \varepsilon)}{\ln \frac{1}{\varepsilon}}.$$

Definition 2.4. A set E is called positively invariant w.r.t $S(t)$ if for all $t \geq 0$, $S(t)E \subset E$.

Definition 2.5. A set $\mathcal{M}$ is called an exponential attractor for $S(t)$, if

(i) the set $\mathcal{M}$ is compact in X and has finite fractal dimension,

(ii) the set $\mathcal{M}$ is positively invariant, i.e., $S(t)\mathcal{M} \subset \mathcal{M}$,

(iii) the set $\mathcal{M}$ is an exponentially attracting set for the semigroup $S(t)$. i.e., there exists a constant $l > 0$ such that, for any bounded set $B \subset X$, there exists a constant $k(B) > 0$ such that

$$dist(S(t)B, \mathcal{M}) \leq k(B)e^{-lt}.$$

3. Existence of exponential attractor

Our main purpose is to prove the existence of exponential attractor in $(L^2(\mathbb{R}^n), L^r(\mathbb{R}^n))$ $(2 \leq r < 2p-2)$ and $(L^2(\mathbb{R}^n), H^1(\mathbb{R}^n))$. For that matter, we shall need some results in [10].

Lemma 3.1 ([10]). *Assume that $g(x) \in L^2(\mathbb{R}^n)$ and (1.3), (1.4) hold, $S(t)$ be the semigroup associated with (1.1). Then $S(t)$ has a $(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$,$(L^2(\mathbb{R}^n), L^p(\mathbb{R}^n))$ and $(L^2(\mathbb{R}^n), H^1(\mathbb{R}^n))$-bounded absorbing set, that is, there is a positive constant $\rho > 0$ such that for any bounded subset $B \subset L^2(\mathbb{R}^n)$, there exists $T = T(B)$ such that* $|S(t)u_0| + |S(t)u_0|_p + |\nabla S(t)u_0| \leq \rho$ *for any* $t \geq T$ *and* $u_0 \in B$.

Theorem 3.1 ([10]). *Assume that $g(x) \in L^2(\mathbb{R}^n)$ and (1.3), (1.4) hold, $S(t)$ be the semigroup associated with (1.1). Then $S(t)$ has a $(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$,$(L^2(\mathbb{R}^n), L^p(\mathbb{R}^n))$-global attractor $\mathcal{A}$, which is a nonempty, compact, invariant set in $L^2(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$, and attracts every bounded subset of $L^2(\mathbb{R}^n)$.*

In the following, we will give our main results.

Theorem 3.2. *Assume that $g(x) \in L^2(\mathbb{R}^n)$ and (1.3)-(1.5) hold, $S(t)$ be the semigroup associated with (1.1). Then there is only one element in $\mathcal{A}$ of the global attractor of $S(t)$.*

We only prove that $|u(x) - v(x)|^2 < \varepsilon$ in $L^2(\mathbb{R}^n)$, for any $\varepsilon > 0$, $u(x)$,$v(x) \in \mathcal{A}$.

Proof. By the definition of attractor and Lemma 3.1, we get

$$S(t)\mathcal{A} = \mathcal{A}, \text{ for any } t \in \mathbb{R}^+, \text{ and } |\theta(x)| \leq \rho \text{ for any } \theta(x) \in \mathcal{A}.$$

For any $u(x), v(x) \in \mathcal{A}$, $\forall t \geq 0$, there exist $u_0(x), v_0(x) \in \mathcal{A}$, such that $u(x) = S(t)u_0(x)$, $v(x) = S(t)v_0(x)$, $S(t)u_0(x)$ and $S(t)v_0(x)$ to be solutions associated with equation (1.1) with initial data $u_0(x)$,$v_0(x) \in \mathcal{A}$.

For any $\varepsilon > 0$, there exists $T > 0$, $\forall t \geq T$, we have

$$4\rho^2 e^{-2(\lambda - C_0)t} < \varepsilon.$$

For any $t \geq T$, we set $u_1(t) = S(t)u_0(x)$, $v_1(t) = S(t)v_0(x)$, $w(t) = u_1(t) - v_1(t)$, by (1.1), we get

$$w_t - \Delta w + \lambda w + f(u_1) - f(v_1) = 0. \tag{3.1}$$

Taking inner product of (3.1) with $w(t)$, we have

$$\frac{1}{2}\frac{d}{dt}|w|^2 + |\nabla w|^2 + \lambda |w|^2 + (f(u_1) - f(u_2), w) = 0.$$

By (1.3), we find

$$\frac{d}{dt}|w|^2 + 2|\nabla w|^2 + 2(\lambda - C_0)|w|^2 \le 0. \tag{3.2}$$

Applying the Gronwall lemma to (3.2), we get

$$|w(t)|^2 \le e^{-2(\lambda - C_0)t}|u_0(x) - v_0(x)|^2, \tag{3.3}$$

that is to say

$$|u(x) - v(x)|^2 \le 4\rho^2 e^{-2(\lambda - C_0)t} < \varepsilon,$$

we get

$$u(x) = v(x)$$

which says that there exists an element $\theta(x)$ in $\mathcal{A}$, which is a fixed point of semigroup $S(t)$, i.e., $\mathcal{A} = \{\theta(x)\}$ and $S(t)\theta(x) = \theta(x)$, for any $t \ge 0$. □

Theorem 3.3. *Assume that $g(x) \in L^2(\mathbb{R}^n)$ and (1.3)-(1.5) hold, $S(t)$ be the semigroup associated with (1.1). Then the global attractor $\mathcal{A}$ of $S(t)$ is an exponential attractor of $S(t)$ in $L^2(\mathbb{R}^n)$.*

Proof. By Theorem 3.2, we know that $S(t)\theta(x) = \theta(x)$. Obviously $\mathcal{A}$ is a compact set, $\dim_f \mathcal{A} = 0$. We only prove that $\theta(x)$ exponentially attracts bounded set B in $L^2(\mathbb{R}^n)$.

For any $u_0(x) \in B$, we set $u(t) = S(t)u_0$, $v(t) = S(t)\theta(x)$, using the same proof as (3.3), we have

$$|u(t) - v(t)|^2 \le e^{-2(\lambda - C_0)t}|u_0(x) - \theta(x)|^2,$$

hence

$$|S(t)u_0(x) - \theta(x)|^2 \le e^{-2(\lambda - C_0)t}|u_0(x) - \theta(x)|^2,$$

which implies $\theta(x)$ is an exponential attractor of $S(t)$ in $L^2(\mathbb{R}^n)$. □

Imposing another condition that $g(x) \in L^2(\mathbb{R}^n) \cap L^{\frac{3p-4}{p-1}}(\mathbb{R}^n)$, we prove that $\theta(x)$ is an exponential attractor in $L^r(\mathbb{R}^n)(2 < r < 2p-2)$.

Lemma 3.2. *Assume that $g(x) \in L^2(\mathbb{R}^n) \cap L^{\frac{3p-4}{p-1}}(\mathbb{R}^n)$ and (1.3), (1.4) hold, $S(t)$ be the semigroup associated with (1.1). Then the semigroup $S(t)$ exists bounded absorbing set in $L^{2p-2}(\mathbb{R}^n)$.*

Proof. Multiplying Eq.(1.1) by $|u|^{p-2}u$, we get

$$\frac{1}{p}\frac{d}{dt}|u|_p^p - (\Delta u, |u|^{p-2}u) + \lambda|u|_p^p + (f(u), |u|^{p-2}u) = (g(x), |u|^{p-2}u). \tag{3.4}$$

By (1.4) and Young's inequality, we have

$$(f(u), |u|^{p-2}u) \ge \alpha_1|u|_{2p-2}^{2p-2} - k_1|u|_p^p, \tag{3.5}$$

$$|(g(x), |u|^{p-2}u)| \le \frac{\alpha_1}{2}|u|_{2p-2}^{2p-2} + \frac{1}{2\alpha_1}|g(x)|^2. \tag{3.6}$$

And

$$-(\Delta u, |u|^{p-2}u) = (\nabla u, |u|^{p-2}\nabla u) + (\nabla u, u\nabla|u|^{p-2}) = (p-1)(|u|^{p-2}\nabla u, \nabla u) \ge 0. \tag{3.7}$$

Using (3.5)-(3.7) in (3.4), we find that

$$\frac{d}{dt}|u|_p^p + (\lambda - k_1)p|u|_p^p + \frac{\alpha_1 p}{2}|u|_{2p-2}^{2p-2} \le \frac{p}{2\alpha_1}|g(x)|^2. \tag{3.8}$$

By Lemma 3.1, for any bounded subset $B \subset L^2(\mathbb{R}^n)$, there exits $T = T(B) > 0$, such that

$$|S(t)u_0(x)|_p \le \rho, \quad \forall t \ge T.$$

Integrating the inequality (3.8) from t to $t+1$, we find that

$$\int_t^{t+1} |u(s)|_{2p-2}^{2p-2} ds \le C_1, \quad \forall t \ge T(B). \tag{3.9}$$

Multiplying Eq.(1.1) by $|u|^{2p-4}u$, we have

$$\frac{1}{2p-2}\frac{d}{dt}|u|_{2p-2}^{2p-2} - (\Delta u, |u|^{2p-4}u) + \lambda|u|_{2p-2}^{2p-2} + (f(u), |u|^{2p-4}u) = (g(x), |u|^{2p-4}u). \tag{3.10}$$

By Young's inequality, we get

$$|(g(x), |u|^{2p-4}u)| \le \frac{\alpha_1}{2}|u|_{3p-4}^{3p-4} + C_2|g(x)|_{\frac{3p-4}{p-1}}^{\frac{3p-4}{p-1}}.$$

Using the same proof as in (3.8), we obatain

$$\frac{d}{dt}|u|_{2p-2}^{2p-2} + (\lambda - k_1)(2p-2)|u|_{2p-2}^{2p-2} \le C_3. \tag{3.11}$$

By (3.9) and Gronwall lemma, we get

$$|u|_{2p-2}^{2p-2} \le C_4, \quad \forall t \ge T(B). \tag{3.12}$$

□

Theorem 3.4. *Assume that $g(x) \in L^2(\mathbb{R}^n) \cap L^{\frac{3p-4}{p-1}}(\mathbb{R}^n)$ and (1.3)-(1.5) hold. Then the semigroup $S(t)$ exists an exponential attractor $\{\theta(x)\}$ in $L^r(\mathbb{R}^n)(2 < r < 2p-2)$.*

Proof. For any bounded set $B \subset L^2(\mathbb{R}^n)$, $\forall u_0(x) \in B$, set $u(t) = S(t)u_0(x)$, $v(t) = S(t)\theta(x)$. By Hölder's inequality, we obtain

$$\begin{aligned}|u(t) - v(t)|_r^r &= \int_{\mathbb{R}^n} |u-v|^{\frac{(r-2)(p-1)}{p-2}} |u-v|^{\frac{2p-2-r}{p-2}} dx \\ &\le (\int_{\mathbb{R}^n} |u-v|^{2p-2} dx)^{\frac{r-2}{2(p-2)}} (\int_{\mathbb{R}^n} |u-v|^2 dx)^{\frac{2p-2-r}{2(p-2)}} \\ &= |u(t) - v(t)|_{2p-2}^{\frac{(r-2)(p-1)}{p-2}} |u(t) - v(t)|^{\frac{2p-2-r}{p-2}},\end{aligned}$$

by (3.12) and (3.3), we get

$$|u(t) - \theta(x)|_r^r \le C_5 e^{-(\lambda - C_0)\frac{2p-2-r}{p-2}t} |u_0(x) - \theta(x)|^{\frac{2p-2-r}{2(p-2)}},$$

which means that $\theta(x)$ exponentially attracts bounded set in the norm of $L^r(\mathbb{R}^n)$. □

Hereafter, we will prove that the existence of exponential attractor in $H^1(\mathbb{R}^n)$. For this purpose, we will give a priori estimates about u_t endowed with L^2-norm.

Lemma 3.3. *Assume $f(s)$ is a $\mathcal{C}^1$ function and (1.3)-(1.5) hold, $S(t)$ be the semigroup associated with Eq.(1.1). Then for any bounded subset B in $L^2(\mathbb{R}^n)$, there exists a positive constant $T = T_B > 0$ such that*

$$|u_t(s)|^2 \leq M \quad \text{for any } u_0 \in B \text{ and } s \geq T,$$

where $u_t(s) = \frac{d}{dt}(S(t)u_0)|_{t=s}$ and M is a positive constant which is independent of B.

Proof. By (1.3) and (1.4), we know that

$$f'(s) \geq -C_0, \quad |f(s)| \leq C_6(|s|^{p-1} + |s|). \tag{3.13}$$

Taking inner product of (1.1) with u_t, we get

$$|u_t|^2 + \frac{1}{2}\frac{d}{dt}|\nabla u|^2 + \frac{\lambda}{2}\frac{d}{dt}|u|^2 + (f(u), u_t) = (g(x), u_t). \tag{3.14}$$

Using Young's inequality and (3.13), we have

$$|(f(u), u_t)| \leq \int_{\mathbb{R}^n} |f(u)|^2 dx + \frac{1}{4}|u_t|^2 \leq C_7(|u|_{2p-2}^{2p-2} + |u|^2) + \frac{1}{4}|u_t|^2, \tag{3.15}$$

and

$$|(g(x), u_t)|^2 \leq |g(x)|^2 + \frac{1}{4}|u_t|^2. \tag{3.16}$$

It follows from (3.13)-(3.16), we get

$$|u_t|^2 + \frac{d}{dt}(|\nabla u|^2 + \lambda|u|^2) \leq C_8(|g(x)|^2 + |u|_{2p-2}^{2p-2} + |u|^2).$$

Integrating the above inequality from t to $t+1$, we have

$$\int_t^{t+1} |u_t(s)|^2 ds \leq C_9(|\nabla u(t)|^2 + |u(t)|^2 + \int_t^{t+1} (|u(s)|_{2p-2}^{2p-2} + |u(s)|^2) ds + |g(x)|^2).$$

By Lemmas 3.1 and 3.2, we obtain that there exists $T = T_B > 0$, $\forall t \geq T$,

$$\int_t^{t+1} |u_t(s)|^2 ds \leq C_{10}. \tag{3.17}$$

Differentiating Eq.(1.1) in time and denoting $v = u_t$, we get

$$v_t - \triangle v + f'(u)v + \lambda v = 0. \tag{3.18}$$

Multiplying the above equality by v, and using (1.5),(3.13), we obtain

$$\frac{1}{2}\frac{d}{dt}|v|^2 + |\nabla v|^2 + (\lambda - C_0)|v|^2 \leq 0,$$

using (3.17) and uniform Gronwall lemma, we get that for $s \geq T$,

$$|u_t(s)|^2 \leq C_{11}.$$

□

Theorem 3.5. *Assume $f(s)$ is a $\mathcal{C}^1$ function and (1.3)-(1.5) hold, $S(t)$ be the semigroup associated with Eq.(1.1). Then $\theta(x)$ is an exponential attractor of $S(t)$ in $H^1(\mathbb{R}^n)$.*

Proof. By (1.1) and (3.13), we deduce that

$$\begin{aligned}|\Delta u(t)|^2 &\le C_{11}(|u_t(s)|^2 + |u(t)|^2 + \int_{\mathbb{R}^n} |f(u)|^2 dx + |g(x)|^2)\\ &\le C_{12}(|u_t(s)|^2 + |u(t)|^2 + |u(t)|_{2p-2}^{2p-2} + |g(x)|^2)\\ &\le C_{13},\end{aligned}$$

for any $t \ge T_B$, which means that the semigroup $S(t)$ exists bounded absorbing set in $H^2(\mathbb{R}^n)$.

Let $u(t) = S(t)u_0(x)$, $u_0(x) \in B$, $v(t) = S(t)\theta(x)$, then

$$\begin{aligned}|\nabla(u(t) - v(t))|^2 &= (-\Delta(u(t) - v(t)), u(t) - v(t))\\ &\le |\Delta(u(t) - v(t)||u(t) - v(t)|\\ &\le C_{14}|u(t) - v(t)|,\end{aligned}$$

for any $t \ge T_B$.

By Theorem 3.3, we get

$$|\nabla(u(t) - \theta(x))|^2 \le C_{15}e^{-(\lambda - C_0)t}|u_0(x) - \theta(x)|,$$

which means that the semigroup $S(t)$ exists an exponential attractor in $H^1(\mathbb{R}^n)$, i.e., $\theta(x)$ exponentially attracts bounded set in $L^2(\mathbb{R}^n)$. □

Acknowledgements

The authors would like to express their sincere thanks to the referee for his/her valuable comments and suggestions.

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