Journal of Applied Analysis and Computation
Volume 8, Number 4, August 2018, 1170–1185

Website:http://jaac-online.com/
DOI:10.11948/2018.1170

SOLVABILITY FOR NONLINEAR SINGULAR FRACTIONAL DIFFERENTIAL SYSTEMS WITH MULTI-ORDERS*

Yige Zhao

Abstract In this paper, we consider the existence of positive solutions for a class of nonlinear singular fractional differential systems with multi-orders. Our analysis relies on fixed point theorems on cones. Some sufficient conditions for the existence of at least one or two positive solutions for boundary value problem of nonlinear singular fractional differential systems with multi-orders are established. As an application, an example is presented to illustrate the main results.

Keywords Positive solution, singular system, fractional Green's function, fixed point theorem.

MSC(2010) 34A08, 34B18.

1. Introduction

Fractional calculus has been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications; see [11]. Recently, there have appeared a large number of papers dealing with the existence of solutions of nonlinear fractional differential equations by the use of techniques of nonlinear analysis; see [2, 3, 13, 15, 17, 18, 20–22].

Yu et al. [17] examined the existence of positive solutions for the following problem

$$D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$

$$u(0) = u(1) = u'(0) = 0,$$

where $2 < \alpha \leq 3$ is a real number, $f \in C([0, 1] \times [0, +\infty), (0, +\infty))$ and D_{0+}^{α} is the Riemann-Liouville fractional differentiation. By using the properties of the Green function, some existence criteria for one or two positive solutions for singular and nonsingular boundary value problems were obtained by means of the Krasnosel'skii fixed point theorem and a mixed monotone method.

Xu et al. [15] considered the existence of positive solutions for the following problem

$$D_{0+}^{\alpha}u(t) = f(t, u(t)), \quad 0 < t < 1,$$

Email address: zhaoeager@126.com (Y. Zhao)
School of Mathematical Sciences, University of Jinan, Jinan, Shandong 250022, China

*The authors were supported by the National Natural Science Foundation of China (G61703180, G61374065, G61374002, G61573215, G61601197), and the Scientific Research Foundation of University of Jinan (1008399).

$$u(0) = u(1) = u'(0) = u'(1) = 0,$$

where $3 < \alpha \leq 4$ is a real number, $f \in C([0,1] \times [0,+\infty),(0,+\infty))$ and D^{α}_{0+} is the Riemann-Liouville fractional differentiation. By using the properties of the Green function, some multiple positive solutions for singular and nonsingular boundary value problems were given by means of Leray-Schauder nonlinear alternative, a fixed point theorem on cones and a mixed monotone method.

On the other hand, the study of singular and nonsingular systems involving fractional differential equations is also important as such systems occur in various problems; see [1,4,5,9,10,12,14,16,19].

Bai et al. [1] considered the existence of positive solutions of singular coupled system

$$\begin{cases} D^s u = f(t,v), & 0 < t < 1, \\ D^p v = g(t,u), & 0 < t < 1, \end{cases}$$

where $0 < s,\ p < 1$, and $f,\ g : [0,1) \times [0,+\infty) \to [0,+\infty)$ are two given continuous functions, $\lim_{t\to 0^+} f(t,\cdot) = +\infty$, $\lim_{t\to 0^+} g(t,\cdot) = +\infty$ and D^s, D^p are two standard Riemann-Liouville fractional derivatives. The existence results of positive solutions were established by a nonlinear alternative of Leray-Schauder type and Krasnosel'skii fixed point theorem on a cone.

Su [12] discussed a boundary value problem for a coupled differential system of fractional order

$$\begin{cases} D^{\alpha} u(t) = f(t, v(t), D^{\mu} v(t)), & 0 < t < 1, \\ D^{\beta} v(t) = g(t, u(t), D^{\nu} u(t)), & 0 < t < 1, \\ u(0) = u(1) = v(0) = v(1) = 0, \end{cases}$$

where $1 < \alpha,\ \beta \leq 2,\ \mu,\ \nu > 0,\ \alpha - \nu \geq 1,\ \beta - \mu \geq 1,\ f,\ g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions and D is the standard Riemann-Liouville fractional derivative. By means of Schauder fixed point theorem, an existence result for the solution was obtained.

Zhao et al. [19] examined the existence of positive solutions for a coupled system of nonlinear differential equations of mixed fractional orders

$$\begin{cases} -D^{\alpha}_{0+} u(t) = f(t, v(t)), & 0 < t < 1, \\ D^{\beta}_{0+} v(t) = g(t, u(t)), & 0 < t < 1, \\ u(0) = u(1) = u'(0) = v(0) = v(1) = v'(0) = v'(1) = 0, \end{cases} \tag{1.1}$$

where $2 < \alpha \leq 3,\ 3 < \beta \leq 4,\ D^{\alpha}_{0+},\ D^{\beta}_{0+}$ are the standard Riemann-Liouville fractional derivative, and $f,\ g : [0,1] \times [0,+\infty) \to [0,+\infty)$ are given continuous functions, $f(t,0) \equiv 0$, $g(t,0) \equiv 0$. Their analysis relied on fixed point theorems on cones. Some sufficient conditions for the existence of at least one or two positive solutions for the boundary value problem were established.

From the above works, we can see a fact, although the coupled systems of fractional boundary value problems have been investigated by some authors, the singular coupled systems due to multi-order fractional orders are seldom considered. On the one hand, the orders α and β of the nonlinear singular fractional differential systems witch are considered in the existing papers belong to the same interval $(n, n+1]$ $(n \in \mathbb{N}^+)$. On the other hand, in **Remark 3.2** ([19]), conditions $f(t,0) \equiv 0$ and $g(t,0) \equiv 0$ are too strong for the boundary value problem (1.1). Therefore, we will give some new existence criteria for the boundary value problem (1.1) without conditions $f(t,0) \equiv 0$ and $g(t,0) \equiv 0$ in this paper. Our results in this paper improve some known results in [19].

Motivated by all the works above, in this paper we investigate the existence of positive solutions for the boundary value problem (1.1) of the nonlinear singular fractional systems with multi-orders, where $f,\ g : (0,1] \times [0,+\infty) \to [0,+\infty)$ are continuous, and $\lim_{t\to 0^+} f(t,\cdot) = +\infty$, $\lim_{t\to 0^+} g(t,\cdot) = +\infty$ (that is, f and g are singular at $t = 0$). Our analysis relies on fixed point theorems on cones. Some sufficient conditions for the existence of at least one or two positive solutions for boundary value problem of the nonlinear singular fractional systems with multi-orders are established. Finally, we present an example to demonstrate our results.

The plan of the paper is as follows. In Section 2, we shall give some definitions and lemmas to prove our main results. In Section 3, using Leray-Schauder nonlinear alternative theorem and Guo-Krasnosel'skii fixed point theorem, we obtain some new existence criteria for boundary value problem (1.1) of nonlinear singular fractional systems with multi-orders. Section 4 gives an illustrative example to support our new results, which is followed by a brief conclusion in Section 5.

2. Preliminaries

For the convenience of readers, we give some background materials from fractional calculus theory to facilitate analysis of the problem (1.1). These materials can be found in the recent literatures; see [6–8, 15, 17].

Definition 2.1 ([8]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0,+\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0,+\infty)$.

Definition 2.2 ([8]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,+\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided that the right side is pointwise defined on $(0,+\infty)$.

From the definition of the Riemann-Liouville derivative, we can obtain the following statements.

Lemma 2.1 ([8]). *Let $\alpha > 0$. If we assume $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation*

$$D_{0^+}^{\alpha} u(t) = 0$$

has $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$, as unique solutions, where n is the smallest integer greater than or equals to α.

Lemma 2.2 ([8]). *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \text{ for some } c_i \in \mathbb{R},\ i = 1, 2, \cdots, n,$$

where n is the smallest integer greater than or equals to α.

In the following, we present the Green's function for the boundary value problem of fractional differential equations.

Lemma 2.3 ([17]). *Let $h_1 \in C[0,1]$ and $2 < \alpha \leq 3$. The unique solution of the problem*

$$-D_{0+}^{\alpha} u(t) = h_1(t), \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = u(1) = u'(0) = 0, \tag{2.2}$$

is

$$u(t) = \int_0^1 G_1(t,s) h_1(s) ds,$$

where

$$G_1(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.3}$$

here $G_1(t,s)$ is called the Green's function of the boundary value problem (2.1) *and* (2.2).

Lemma 2.4 ([17]). *The function $G_1(t,s)$ defined by* (2.3) *satisfies the following conditions:*

($A1$) $G_1(t,s) = G_1(1-s, 1-t)$, *for* $t,\ s \in (0,1)$;

($A2$) $t^{\alpha-1}(1-t)s(1-s)^{\alpha-1} \leq \Gamma(\alpha) G_1(t,s) \leq (\alpha-1)s(1-s)^{\alpha-1}$, *for* $t,\ s \in (0,1)$;

($A3$) $G_1(t,s) > 0$, *for* $t,\ s \in (0,1)$;

($A4$) $t^{\alpha-1}(1-t)s(1-s)^{\alpha-1} \leq \Gamma(\alpha) G_1(t,s) \leq (\alpha-1)(1-t)t^{\alpha-1}$, *for* $t,\ s \in (0,1)$.

Remark 2.1. Let $q_1(t) = t^{\alpha-1}(1-t),\ k_1(s) = s(1-s)^{\alpha-1}$. Then

$$q_1(t) k_1(s) \leq \Gamma(\alpha) G_1(t,s) \leq (\alpha-1) k_1(s).$$

Lemma 2.5 ([15]). *Let $h_2 \in C[0,1]$ and $3 < \beta \leq 4$. The unique solution of the problem*

$$D_{0+}^{\beta} u(t) = h_2(t), \quad 0 < t < 1 \tag{2.4}$$

$$u(0) = u(1) = u'(0) = u'(1) = 0, \tag{2.5}$$

is

$$u(t) = \int_0^1 G_2(t,s) h_2(s) ds,$$

where

$$G_2(t,s) = \begin{cases} \frac{(t-s)^{\beta-1} + (1-s)^{\beta-2} t^{\beta-2} [(s-t) + (\beta-2)(1-t)s]}{\Gamma(\beta)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\beta-2}(1-s)^{\beta-2} [(s-t) + (\beta-2)(1-t)s]}{\Gamma(\beta)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.6}$$

here $G_2(t,s)$ is called the Green's function of the boundary value problem (2.4) *and* (2.5).

Lemma 2.6 ([15]). *The function $G_2(t,s)$ defined by (2.6) satisfies the following conditions:*

(B1) $G_2(t,s) = G_2(1-s, 1-t), \quad$ *for* $t,\ s \in (0,1)$*;*

(B2) $(\beta-2)t^{\beta-2}(1-t)^2 s^2(1-s)^{\beta-2} \le \Gamma(\beta)G_2(t,s) \le M_0 s^2(1-s)^{\beta-2}, \quad$ *for* $t,\ s \in (0,1)$*;*

(B3) $G_2(t,s) > 0, \quad$ *for* $t,\ s \in (0,1)$*;*

(B4) $(\beta-2)s^2(1-s)^{\beta-2}t^{\beta-2}(1-t)^2 \le \Gamma(\beta)G_2(t,s) \le M_0 t^{\beta-2}(1-t)^2, \quad$ *for* $t,\ s \in (0,1)$,

here $M_0 = \max\{\beta - 1,\ (\beta-2)^2\}$.

Remark 2.2. Let $q_2(t) = t^{\beta-2}(1-t)^2,\ k_2(s) = s^2(1-s)^{\beta-2}$. Then

$$(\beta-2)q_2(t)k_2(s) \le \Gamma(\beta)G_2(t,s) \le M_0 k_2(s).$$

The following two lemmas are fundamental in the proofs of our main results.

Lemma 2.7 ([7]). *Let E be a Banach space, and let $P \subset E$ be a cone in E. Assume $\Omega_1,\ \Omega_2$ are open subsets of E with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $S : P \to P$ be a completely continuous operator such that, either*

(D1) $\|Sw\| \le \|w\|$, $w \in P \cap \partial\Omega_1$, $\|Sw\| \ge \|w\|$, $w \in P \cap \partial\Omega_2$, *or*

(D2) $\|Sw\| \ge \|w\|$, $w \in P \cap \partial\Omega_1$, $\|Sw\| \le \|w\|$, $w \in P \cap \partial\Omega_2$.

Then S has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Lemma 2.8 ([6]). *Let E be a Banach space with $C \subset E$ close and convex. Assume U is a relatively open subset of C with $0 \in U$ and $A : \overline{U} \to C$ is a continuous compact map. Then either*

(E1) *A has a fixed point in U; or*

(E2) *there exists a $u \in \partial U$, and a $\lambda \in (0,1)$ with $u = \lambda Au$.*

3. Main Results

In this section, we establish some new existence results for the boundary value problem (1.1) of the nonlinear singular fractional systems with multi-orders.

Consider the following coupled system of integral equations:

$$\begin{cases} u(t) = \int_0^1 G_1(t,s)f(s,v(s))ds, \\ v(t) = \int_0^1 G_2(t,s)g(s,u(s))ds. \end{cases} \tag{3.1}$$

Lemma 3.1. *Let $0 < \sigma_1 < 1$, $2 < \alpha \le 3$, $F_1 : (0,1] \to \mathbb{R}$ be continuous and $\lim_{t\to 0^+} F_1(t) = \infty$. Suppose that $t^{\sigma_1}F_1(t)$ is continuous function on $[0,1]$. Then the function*

$$H_1(t) = \int_0^1 G_1(t,s)F_1(s)ds$$

is continuous on $[0,1]$.

Proof. By the continuity of $t^{\sigma_1}F_1(t)$ and $H_1(t)=\int_0^1 G_1(t,s)s^{-\sigma_1}s^{\sigma_1}F_1(s)ds$. It is easily to check that $H_1(0)=0$. The proof is divided into three cases:

Case 1: $t_0=0,\ \forall\ t\in(0,1]$.

Since $t^{\sigma_1}F_1(t)$ is continuous in $[0,1]$, there exists a constant $M>0$, such that $\left|t^{\sigma_1}F_1(t)\right|\le M$, for $t\in[0,1]$. Hence,

$$
\begin{aligned}
&\Big|H_1(t)-H_1(0)\Big|\\
=&\Big|\int_0^1 G_1(t,s)F_1(s)ds\Big| = \Big|\int_0^1 G_1(t,s)s^{-\sigma_1}s^{\sigma_1}F_1(s)ds\Big|\\
=&\Big|\int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma_1}s^{\sigma_1}F_1(s)ds\\
&+\int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma_1}s^{\sigma_1}F_1(s)ds\Big|\\
=&\Big|\int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma_1}s^{\sigma_1}F_1(s)ds\\
&-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma_1}s^{\sigma_1}F_1(s)ds\Big|\\
\le&\Big|\int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma_1}s^{\sigma_1}F_1(s)ds\Big|\\
&+\Big|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma_1}s^{\sigma_1}F_1(s)ds\Big|\\
\le& M\int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma_1}ds + M\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma_1}ds\\
=&\frac{Mt^{\alpha-1}}{\Gamma(\alpha)}B(1-\sigma_1,\alpha)+\frac{Mt^{\alpha-\sigma_1}}{\Gamma(\alpha)}B(1-\sigma_1,\alpha)\\
=&\frac{M\Gamma(1-\sigma_1)}{\Gamma(1+\alpha-\sigma_1)}(t^{\alpha-1}+t^{\alpha-\sigma_1})\to 0 \quad (\text{as } t\to 0).
\end{aligned}
$$

where $B(\cdot,\cdot)$ denotes the beta function.

Case 2: $t_0\in(0,1),\ \forall\ t\in(t_0,1]$.

$$
\begin{aligned}
&\Big|H_1(t)-H_1(t_0)\Big|\\
=&\Big|\int_0^1 G_1(t,s)F_1(s)ds-\int_0^1 G_1(t_0,s)F_1(s)ds\Big|\\
=&\Big|\int_0^1 G_1(t,s)s^{-\sigma_1}s^{\sigma_1}F_1(s)ds-\int_0^1 G_1(t_0,s)s^{-\sigma_1}s^{\sigma_1}F_1(s)ds\Big|\\
=&\Big|\int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma_1}s^{\sigma_1}F_1(s)ds\\
&+\int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma_1}s^{\sigma_1}F_1(s)ds\\
&-\int_{t_0}^1 \frac{t_0^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma_1}s^{\sigma_1}F_1(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^{t_0} \frac{t_0^{\alpha-1}(1-s)^{\alpha-1} - (t_0-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma_1} s^{\sigma_1} F_1(s) ds \Big| \\
= &\Big| \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma_1} s^{\sigma_1} F_1(s) ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma_1} s^{\sigma_1} F_1(s) ds \\
&- \int_0^1 \frac{t_0^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma_1} s^{\sigma_1} F_1(s) ds + \int_0^{t_0} \frac{(t_0-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma_1} s^{\sigma_1} F_1(s) ds \Big| \\
= &\Big| \int_0^1 \frac{(t^{\alpha-1} - t_0^{\alpha-1})(1-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma_1} s^{\sigma_1} F_1(s) ds - \int_{t_0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma_1} s^{\sigma_1} F_1(s) ds \\
&- \int_0^{t_0} \frac{(t-s)^{\alpha-1} - (t_0-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma_1} s^{\sigma_1} F_1(s) ds \Big| \\
\leq &\frac{M(t^{\alpha-1} - t_0^{\alpha-1})}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} s^{-\sigma_1} ds + \frac{M}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} s^{-\sigma_1} ds \\
&+ \frac{M}{\Gamma(\alpha)} \int_0^{t_0} [(t-s)^{\alpha-1} - (t_0-s)^{\alpha-1}] s^{-\sigma_1} ds \\
= &\frac{M(t^{\alpha-1} - t_0^{\alpha-1})}{\Gamma(\alpha)} B(1-\sigma_1, \alpha) + \frac{M(t^{\alpha-\sigma_1} - t_0^{\alpha-\sigma_1})}{\Gamma(\alpha)} B(1-\sigma_1, \alpha) \\
= &\frac{M\Gamma(1-\sigma_1)(t^{\alpha-1} - t_0^{\alpha-1})}{\Gamma(1+\alpha-\sigma_1)} + \frac{M\Gamma(1-\sigma_1)(t^{\alpha-\sigma_1} - t_0^{\alpha-\sigma_1})}{\Gamma(1+\alpha-\sigma_1)} \\
= &\frac{M\Gamma(1-\sigma_1)}{\Gamma(1+\alpha-\sigma_1)} (t^{\alpha-1} - t_0^{\alpha-1} + t^{\alpha-\sigma_1} - t_0^{\alpha-\sigma_1}) \to 0 \quad (\text{as } t \to t_0).
\end{aligned}
$$

Case 3: $t_0 \in (0, 1]$, $\forall\, t \in [0, t_0)$. The proof is similar to that of **Case 2**, so we omit it.

The proof is completed. □

Lemma 3.2. *Let* $0 < \sigma_2 < 1$, $3 < \beta \leq 4$, $F_2 : (0, 1] \to \mathbb{R}$ *be continuous and* $\lim_{t\to 0^+} F_2(t) = \infty$. *Suppose that* $t^{\sigma_2} F_2(t)$ *is continuous function on* $[0, 1]$. *Then the function*

$$H_2(t) = \int_0^1 G_2(t, s) F_2(s) ds$$

is continuous on $[0, 1]$.

Proof. This proof is similar to that of Lemma 3.1, so is omitted. □

Lemma 3.3. *Let* $2 < \alpha \leq 3$, $3 < \beta \leq 4$, *and* $f,\ g : (0, 1] \times [0, +\infty) \to [0, +\infty)$ *be continuous functions satisfying* $\lim_{t\to 0^+} f(t, \cdot) = +\infty$, $\lim_{t\to 0^+} g(t, \cdot) = +\infty$. *Assume that* $0 < \sigma_1, \sigma_2 < 1$, *and* $t^{\sigma_1} f(t, y)$ *and* $t^{\sigma_2} g(t, y)$ *are two continuous functions on* $[0, 1] \times [0, +\infty)$. *Then system* (1.1) *is equivalent to system* (3.1).

Proof. This proof is similar to that of Lemma 2.2 in [1], so is omitted. □

Let $E = X \times X$. Define the cone $P \subset E$ by

$$P = \{(u, v) \in E : u(t) \geq 0,\ v(t) \geq 0,\ t \in [0, 1]\}.$$

By Lemma 3.3, let $T : P \to E$ be the operator defined as in Section 3,

$$
\begin{aligned}
T(u, v)(t) &= \left(\int_0^1 G_1(t, s) f(s, v(s)) ds, \int_0^1 G_2(t, s) g(s, u(s)) ds \right) \\
&=: (T_1 v(t), T_2 u(t)), \quad t \in I.
\end{aligned}
$$

Lemma 3.4. *Let $2 < \alpha \le 3$, $3 < \beta \le 4$, and $f,\ g : (0,1] \times [0,+\infty) \to [0,+\infty)$ be continuous functions satisfying $\lim_{t\to 0^+} f(t,\cdot) = +\infty$, $\lim_{t\to 0^+} g(t,\cdot) = +\infty$. Assume that $0 < \sigma_1, \sigma_2 < 1$, and $t^{\sigma_1} f(t,y)$ and $t^{\sigma_2} g(t,y)$ are two continuous functions on $[0,1] \times [0,+\infty)$. Then the operator $T : P \to P$ is completely continuous.*

Proof. For each $(u,v) \in P$, we have that $u,\ v \in P_1 = \{y \in X : y(t) \ge 0,\ t \in [0,1]\}$. Since $T_1 v(t) = \int_0^1 G_1(t,s) f(s,v(s))ds$. By Lemma 3.1 and the fact that $f,\ G_1(t,s)$ are nonnegative, we have $T_1 : P_1 \to P_1$.

For any given $v_0 \in P$ with $\|v_0\| = C_0$, if $v \in P_1$ and $\|v - v_0\| < 1$, then $\|v\| < 1 + C_0 = C$. By the continuity of $t^{\sigma_1} f(t,v)$, we know that $t^{\sigma_1} f(t,v)$ is uniformly continuous on $[0,1] \times [0,C]$.

Thus, $\forall\ \epsilon > 0$, there exists $\delta > 0$ ($\delta < 1$), such that $|t^{\sigma_1} f(t,v_2) - t^{\sigma_1} f(t,v_1)| < \epsilon$, for all $t \in [0,1]$, and $v_1,\ v_2 \in [0,C]$ with $\|v_2 - v_1\| < \delta$. Obviously, if $\|v - v_0\| < \delta$, then $v(t),\ v_0(t) \in [0,C]$ and $|v(t) - v_0(t)| < \delta$, for all $t \in [0,1]$. Hence, for all $t \in [0,1]$, $v \in P_1$, with $\|v - v_0\| < \delta$.

$$|t^{\sigma_1} f(t,v(t)) - t^{\sigma_1} f(t,v_0(t))| < \epsilon. \tag{3.2}$$

It follows from (3.2) that

$$\begin{aligned}
\|T_1 v - T_1 v_0\| &= \max_{0\le t\le 1} \left|T_1 v(t) - T_1 v_0(t)\right| \\
&\le \max_{0\le t\le 1} \int_0^1 G_1(t,s) s^{-\sigma_1} \left|s^{\sigma_1} f(s,v(s)) - s^{\sigma_1} f(s,v_0(s))\right| ds \\
&< \epsilon \int_0^1 G_1(t,s) s^{-\sigma_1} ds \\
&\le \epsilon \int_0^1 \frac{(\alpha-1)k_1(s)}{\Gamma(\alpha)} s^{-\sigma_1} ds \\
&\le \frac{\epsilon(\alpha-1)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} s^{1-\sigma_1} ds \\
&= \frac{\epsilon(\alpha-1)}{\Gamma(\alpha)} B(2-\sigma_1,\alpha) = \frac{\epsilon(\alpha-1)\Gamma(2-\sigma)}{\Gamma(2+\alpha-\sigma_1)}.
\end{aligned}$$

By the arbitrariness of v_0, $T_1 : P_1 \to P_1$ is continuous. Similarly, by Lemma 3.2, we have

$$\begin{aligned}
\|T_2 u - T_2 u_0\| &\le \max_{0\le t\le 1} \int_0^1 G_2(t,s) s^{-\sigma_2} \left|s^{\sigma_2} g(s,u(s)) - s^{\sigma_2} g(s,u_0(s))\right| ds \\
&= \frac{\epsilon M_0}{\Gamma(\beta)} B(3-\sigma_2,\beta-1).
\end{aligned}$$

Then, $T_2 : P_2 \to P_2$ is continuous. That is, we get the the operator $T : P \to P$ is continuous.

Let $M \subset P$ be bounded, i.e., there exists a positive constant b such that $\|(u,v)\| \le b,\ \forall\ (u,v) \in M$. Since $t^{\sigma_1} f(t,y)$ and $t^{\sigma_2} g(t,y)$ are continuous in $[0,1] \times [0,+\infty)$, let

$$L = \max_{0\le t\le 1,(u,v)\in M} \{t^{\sigma_1} f(t,v(t)), t^{\sigma_2} g(t,u(t))\} + 1.$$

For each $(u,v)\in M$, then we have

$$\begin{aligned}\left|T_1v(t)\right| &\le \int_0^1 G_1(t,s)s^{-\sigma_1}\left|s^{\sigma_1}f(s,v(s))\right|ds\\ &\le L\int_0^1 \frac{(\alpha-1)k_1(s)}{\Gamma(\alpha)}s^{-\sigma_1}ds=\frac{L(\alpha-1)\Gamma(2-\sigma_1)}{\Gamma(2+\alpha-\sigma_1)}.\end{aligned}$$

Thus,

$$\|T_1v\|=\max_{0\le t\le 1}\left|T_1v(t)\right|\le \frac{L(\alpha-1)\Gamma(2-\sigma_1)}{\Gamma(2+\alpha-\sigma_1)}.$$

Similarly, we have

$$\|T_2u\|=\max_{0\le t\le 1}\left|T_2u(t)\right|\le \frac{LM_0}{\Gamma(\beta)}B(3-\sigma_2,\beta-1).$$

So,

$$\begin{aligned}\|T(u,v)\|&=\max\{\|T_1v\|,\|T_2u\|\}\\ &\le \max\left\{\frac{(\alpha-1)\Gamma(2-\sigma_1)}{\Gamma(2+\alpha-\sigma_1)},\frac{M_0}{\Gamma(\beta)}B(3-\sigma_2,\beta-1)\right\}L.\end{aligned}$$

Hence, $T(M)$ is bounded.

In the following, we proof that T is equicontinuous. In fact, $\forall\ \epsilon>0$, let

$$\delta<\min\left\{\frac{1}{2},\ \frac{\epsilon\Gamma(1+\alpha-\sigma_1)}{16L\Gamma(1-\sigma_1)},\ \delta_1\right\},$$

where

$$\begin{aligned}\delta_1<\min\Bigg\{&\frac{\epsilon\Gamma(\beta-1)}{12LB(2-\sigma_2,\beta-1)},\ \frac{\epsilon\Gamma(\beta)}{48LB(1-\sigma_2,\beta)},\\ &\frac{\epsilon\Gamma(\beta)}{24L[B(1-\sigma_2,\beta-1)+(\beta-2)B(2-\sigma_2,\beta-1)]}\Bigg\}.\end{aligned}$$

Then, for $\forall\ (u,v)\in M$, $t_1,\ t_2\in[0,1]$, with $t_1<t_2$, for $0<t_2-t_1<\delta$, we have

$$\begin{aligned}&\left|T_1v(t_2)-T_1v(t_1)\right|\\ =&\Big|\int_0^1 G_1(t_2,s)f(s,v(s))ds-\int_0^1 G_1(t_1,s)f(s,v(s))ds\Big|\\ =&\Big|\int_0^1 [G_1(t_2,s)-G_1(t_1,s)]s^{-\sigma_1}s^{\sigma_1}f(s,v(s))ds\Big|\\ =&\Big|\int_0^{t_2}\frac{t_2^{\alpha-1}(1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma_1}s^{\sigma_1}f(s,v(s))ds\\ &+\int_{t_2}^1\frac{t_2^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma_1}s^{\sigma_1}f(s,v(s))ds\\ &-\int_{t_1}^1\frac{t_1^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma_1}s^{\sigma_1}f(s,v(s))ds\\ &-\int_0^{t_1}\frac{t_1^{\alpha-1}(1-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma_1}s^{\sigma_1}f(s,v(s))ds\Big|\end{aligned}$$

$$
\begin{aligned}
&= \Big| \int_0^1 \frac{(t_2^{\alpha-1}-t_1^{\alpha-1})(1-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma_1} s^{\sigma_1} f(s,v(s))ds \\
&\quad - \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma_1} s^{\sigma_1} f(s,v(s))ds \\
&\quad - \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma_1} s^{\sigma_1} f(s,v(s))ds \Big| \\
&\le \frac{L(t_2^{\alpha-1}-t_1^{\alpha-1})}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} s^{-\sigma_1} ds + \frac{L}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} s^{-\sigma_1} ds \\
&\quad + \frac{L}{\Gamma(\alpha)} \int_0^{t_1} [(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}] s^{-\sigma_1} ds \\
&= \frac{L(t_2^{\alpha-1}-t_1^{\alpha-1})}{\Gamma(\alpha)} B(1-\sigma_1,\alpha) + \frac{L(t_2^{\alpha-\sigma_1}-t_1^{\alpha-\sigma_1})}{\Gamma(\alpha)} B(1-\sigma_1,\alpha) \\
&= \frac{L\Gamma(1-\sigma_1)(t_2^{\alpha-1}-t_1^{\alpha-1})}{\Gamma(1+\alpha-\sigma_1)} + \frac{L\Gamma(1-\sigma_1)(t_2^{\alpha-\sigma_1}-t_1^{\alpha-\sigma_1})}{\Gamma(1+\alpha-\sigma_1)} \\
&= \frac{L\Gamma(1-\sigma_1)}{\Gamma(1+\alpha-\sigma_1)} (t_2^{\alpha-1}-t_1^{\alpha-1}+t_2^{\alpha-\sigma_1}-t_1^{\alpha-\sigma_1}).
\end{aligned}
$$

In order to estimate $t_2^{\alpha-\sigma_1}-t_1^{\alpha-\sigma_1}$, and $t_2^{\alpha-1}-t_1^{\alpha-1}$, we can apply a method used in [2]. In the following, we divide the proof into three cases.

Case 1: $0 \le t_1 < \delta$, $t_2 < 2\delta$.

$$
\begin{aligned}
t_2^{\alpha-\sigma_1}-t_1^{\alpha-\sigma_1} &\le t_2^{\alpha-\sigma_1} < (2\delta)^{\alpha-\sigma_1} \le 2^{\alpha-\sigma_1}\delta < 8\delta, \\
t_2^{\alpha-1}-t_1^{\alpha-1} &\le t_2^{\alpha-1} < (2\delta)^{\alpha-1} \le 2^{\alpha-1}\delta < 4\delta.
\end{aligned}
$$

Case 2: $0 < t_1 < t_2 \le \delta$.

$$
\begin{aligned}
t_2^{\alpha-\sigma_1}-t_1^{\alpha-\sigma_1} &\le t_2^{\alpha-\sigma_1} < \delta^{\alpha-\sigma_1} \le (\alpha-\sigma_1)\delta < 8\delta, \\
t_2^{\alpha-1}-t_1^{\alpha-1} &\le t_2^{\alpha-1} < \delta^{\alpha-1} \le (\alpha-1)\delta < 4\delta.
\end{aligned}
$$

Case 3: $\delta \le t_1 < t_2 \le 1$.

$$
\begin{aligned}
t_2^{\alpha-\sigma_1}-t_1^{\alpha-\sigma_1} &\le (\alpha-\sigma_1)\delta < 8\delta, \\
t_2^{\alpha-1}-t_1^{\alpha-1} &\le (\alpha-1)\delta < 4\delta.
\end{aligned}
$$

Thus, we obtain

$$
\big|T_1v(t_2)-T_1v(t_1)\big| < \frac{\epsilon}{2}+\frac{\epsilon}{4} = \frac{3\epsilon}{4} < \epsilon.
$$

Similarly, we get

$$
\begin{aligned}
&\big|T_2u(t_2)-T_2u(t_1)\big| \\
&= \frac{L(t_2^{\beta-2}-t_1^{\beta-2})}{\Gamma(\beta-1)} B(2-\sigma_2,\beta-1) \\
&\quad + \frac{L(t_2^{\beta-1}-t_1^{\beta-1})}{\Gamma(\beta)} [B(1-\sigma_2,\beta-1)+(\beta-2)B(2-\sigma_2,\beta-1)] \\
&\quad + \frac{Lt_2^{\beta-\sigma_2}}{\Gamma(\beta)} B(1-\sigma_2,\beta) - \frac{Lt_1^{\beta-\sigma_2}}{\Gamma(\beta)} B(1-\sigma_2,\beta).
\end{aligned}
$$

We can prove that if $t_1,\ t_2 \in [0,1]$ are such that $0 < t_2 - t_1 < \delta$, then we have

$$\left|T_2u(t_2) - T_2u(t_1)\right| < \epsilon.$$

Hence, for the Euclidean distance d on $\mathbb{R}^2$, we have that if $t_1,\ t_2 \in [0,1]$ are such that $0 < t_2 - t_1 < \delta$, then

$$d(T(u,v)(t_2), T(u,v)(t_1)) = \sqrt{(T_1v(t_2) - T_1v(t_1))^2 + (T_2u(t_2) - T_2u(t_1))^2} < \sqrt{2}\epsilon.$$

Therefore, $T(M)$ is equicontinuous. By means of the Arzela–Ascoli theorem, $\overline{T(M)}$ is compact. Thus, the operator $T : P \to P$ is completely continuous. This completes the proof. □

Theorem 3.1. *Let $2 < \alpha \leq 3$, $3 < \beta \leq 4$, and $f,\ g : (0,1] \times [0,+\infty) \to [0,+\infty)$ be continuous functions satisfying $\lim_{t\to 0^+} f(t,\cdot) = +\infty$, $\lim_{t\to 0^+} g(t,\cdot) = +\infty$. Let $0 < \sigma_1, \sigma_2 < 1$, and $t^{\sigma_1} f(t,y)$ and $t^{\sigma_2} g(t,y)$ are two continuous functions on $[0,1] \times [0,+\infty)$. Assume that there exist two positive constants $\rho,\ \mu$ with*

$$\rho > \max\left\{\frac{\alpha - 1}{n_1 l_1},\ \frac{M_0 B(3-\sigma_2, \beta - 1)}{(\beta - 2) n_2 l_2}\right\}\mu,$$

where

$$n_1 = \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^{\alpha-1} s^{1-\sigma_1} ds,\ l_1 = \min_{t\in[\frac{1}{4},\frac{3}{4}]} q_1(t),$$
$$n_2 = \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^{\beta-2} s^{2-\sigma_2} ds,\ l_2 = \min_{t\in[\frac{1}{4},\frac{3}{4}]} q_2(t),$$

such that

(H_7) *$t^{\sigma_1} f(t,\omega) \leq \rho\frac{\Gamma(2+\alpha-\sigma_1)}{(\alpha-1)\Gamma(2-\sigma_1)}$ and $t^{\sigma_2} g(t,\omega) \leq \rho\frac{\Gamma(\beta)}{M_0 B(3-\sigma_2,\beta-1)}$, for $(t,\omega) \in [0,1] \times [0,\rho]$;*

(H_8) *$t^{\sigma_1} f(t,\omega) \geq \mu\frac{\Gamma(\alpha)}{n_1 l_1}$ and $t^{\sigma_2} g(t,\omega) \geq \mu\frac{\Gamma(\beta)}{(\beta-2)n_2 l_2}$, for $(t,\omega) \in [0,1] \times [0,\mu]$.*

Then the boundary value problem (1.1) *has at least one positive solution.*

Proof. From Lemma 3.4, we have $T : P \to P$ is completely continuous. By assumptions of the theorem, we have

$$\rho > \max\left\{\frac{\alpha - 1}{n_1 l_1},\ \frac{M_0 B(3-\sigma_2, \beta - 1)}{(\beta - 2) n_2 l_2}\right\}\mu > \mu.$$

We divide the proof into the following two steps.

Step1: Let $\Omega_1 = \{(u,v) \in P : \|(u,v)\| < \mu\}$. For $(u,v) \in P \cap \partial\Omega_1$, we have

$0 \le u(t) \le \mu$, $0 \le v(t) \le \mu$, $\forall t \in [0,1]$. It follows from (H_8) that

$$\begin{aligned}
T_1 v(t) &= \int_0^1 G_1(t,s) f(s, v(s)) ds \\
&= \int_0^1 G_1(t,s) s^{-\sigma_1} s^{\sigma_1} f(s, v(s)) ds \\
&\ge \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(t,s) s^{-\sigma_1} s^{\sigma_1} f(s, v(s)) ds \\
&\ge \mu \frac{\Gamma(\alpha)}{n_1 l_1} \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{q_1(t) k_1(s)}{\Gamma(\alpha)} s^{-\sigma_1} ds \\
&\ge \mu \frac{\Gamma(\alpha)}{n_1 l_1} \frac{1}{\Gamma(\alpha)} \min_{t \in [\frac{1}{4}, \frac{3}{4}]} q_1(t) \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^{\alpha-1} s^{1-\sigma_1} ds \\
&= \mu,
\end{aligned}$$

and

$$\begin{aligned}
T_2 u(t) &= \int_0^1 G_2(t,s) g(s, u(s)) ds \\
&= \int_0^1 G_2(t,s) s^{-\sigma_2} s^{\sigma_2} g(s, u(s)) ds \\
&\ge \int_{\frac{1}{4}}^{\frac{3}{4}} G_2(t,s) s^{-\sigma_2} s^{\sigma_2} g(s, u(s)) ds \\
&\ge \mu \frac{\Gamma(\beta)}{(\beta-2) n_2 l_2} \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{(\beta-2) q_2(t) k_2(s)}{\Gamma(\beta)} s^{-\sigma_2} ds \\
&\ge \mu \frac{\Gamma(\beta)}{(\beta-2) n_2 l_2} \frac{\beta-2}{\Gamma(\beta)} \min_{t \in [\frac{1}{4}, \frac{3}{4}]} q_2(t) \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^{\beta-2} s^{2-\sigma_2} ds \\
&= \mu.
\end{aligned}$$

Hence,

$$\|T(u,v)\| \ge \mu = \|(u,v)\|, \quad \text{for } (u,v) \in P \cap \partial\Omega_1.$$

Step 2: Let $\Omega_2 = \{(u,v) \in P : \|(u,v)\| < \rho\}$. For $(u,v) \in P \cap \partial\Omega_2$, we have $0 \le u(t) \le \rho$, $0 \le v(t) \le \rho$, $\forall\ t \in [0,1]$. By assumption (H_7),

$$\begin{aligned}
T_1 v(t) &= \int_0^1 G_1(t,s) f(s, v(s)) ds \\
&= \int_0^1 G_1(t,s) s^{-\sigma_1} s^{\sigma_1} f(s, v(s)) ds \\
&\le \rho \frac{\Gamma(2+\alpha-\sigma_1)}{(\alpha-1)\Gamma(2-\sigma_1)} \int_0^1 \frac{(\alpha-1) k_1(s)}{\Gamma(\alpha)} s^{-\sigma_1} ds \\
&= \rho \frac{\Gamma(2+\alpha-\sigma_1)}{(\alpha-1)\Gamma(2-\sigma_1)} \frac{(\alpha-1)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} s^{1-\sigma_1} ds \\
&= \rho,
\end{aligned}$$

and

$$\begin{aligned}
T_2u(t) &= \int_0^1 G_2(t,s)g(s,u(s))ds \\
&= \int_0^1 G_2(t,s)s^{-\sigma_2}s^{\sigma_2}g(s,u(s))ds \\
&\le \rho\frac{\Gamma(\beta)}{M_0B(3-\sigma_2,\beta-1)}\int_0^1 \frac{M_0}{\Gamma(\beta)}k_2(s)s^{-\sigma_2}ds \\
&= \rho\frac{\Gamma(\beta)}{M_0B(3-\sigma_2,\beta-1)}\frac{M_0}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-2}s^{2-\sigma_2}ds \\
&= \rho.
\end{aligned}$$

Thus,

$$\|T(u,v)\| \le \rho = \|(u,v)\|, \quad \text{for } (u,v) \in P\cap\partial\Omega_2.$$

Therefore, by Lemma 2.7 and 3.3, we complete the proof. □

Theorem 3.2. *Let $2<\alpha\le 3$, $3<\beta\le 4$, and $f,\ g:(0,1]\times[0,+\infty)\to[0,+\infty)$ be continuous functions satisfying $\lim_{t\to 0^+} f(t,\cdot)=+\infty$, $\lim_{t\to 0^+} g(t,\cdot)=+\infty$. Let $0<\sigma_1,\sigma_2<1$, and $t^{\sigma_1}f(t,y)$ and $t^{\sigma_2}g(t,y)$ are two continuous functions on $[0,1]\times[0,+\infty)$. Assume that the following conditions are satisfied:*

(H_9) *there exist two continuous, nondecreasing function $\varphi,\ \psi:[0,+\infty)\to(0,\infty)$ with $t^{\sigma_1}f(t,\omega)\le\varphi(\omega)$ and $t^{\sigma_2}g(t,\omega)\le\psi(\omega)$, for $(t,\omega)\in[0,1]\times[0,+\infty)$;*

(H_{10}) *there exists $r>0$, with*

$$\frac{r}{\max\{\varphi(r),\ \psi(r)\}} > \max\left\{\frac{(\alpha-1)\Gamma(2-\sigma_1)}{\Gamma(2+\alpha-\sigma_1)},\ \frac{M_0B(3-\sigma_2,\beta-1)}{\Gamma(\beta)}\right\}.$$

Then the boundary value problem (1.1) has one positive solution.

Proof. Let $U=\{(u,v)\in P:\|(u,v)\|<r\}$, we have $U\subset P$. From Lemma 3.4, we know $A:\overline{U}\to P$ is completely continuous. If there exists $(u,v)\in\partial U$, $\lambda\in(0,1)$ such that

$$(u,v)=\lambda T(u,v). \tag{3.3}$$

By (H_9) and (3.3), for $t\in[0,1]$, then we have

$$\begin{aligned}
u(t) = \lambda T_1v(t) &= \lambda\int_0^1 G_1(t,s)f(s,v(s))ds \\
&\le \int_0^1 G_1(t,s)s^{-\sigma_1}s^{\sigma_1}f(s,v(s))ds \\
&\le \int_0^1 G_1(t,s)s^{-\sigma_1}\varphi(v(s))ds \\
&\le \varphi(\|v\|)\int_0^1 G_1(t,s)s^{-\sigma_1}ds \\
&\le \varphi(\|v\|)\int_0^1 \frac{(\alpha-1)k_1(s)}{\Gamma(\alpha)}s^{-\sigma_1}ds \\
&= \varphi(\|v\|)\frac{\alpha-1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}s^{1-\sigma_1}ds
\end{aligned}$$

$$
\begin{aligned}
&= \varphi(\|v\|)\frac{\alpha-1}{\Gamma(\alpha)}B(2-\sigma_1,\alpha)\\
&= \varphi(\|v\|)\frac{(\alpha-1)\Gamma(2-\sigma_1)}{\Gamma(2+\alpha-\sigma_1)}\\
&\leq \varphi(\|(u,v)\|)\frac{(\alpha-1)\Gamma(2-\sigma_1)}{\Gamma(2+\alpha-\sigma_1)}.
\end{aligned}
$$

Consequently,

$$
\|u\| \leq \varphi(\|(u,v)\|)\frac{(\alpha-1)\Gamma(2-\sigma_1)}{\Gamma(2+\alpha-\sigma_1)}. \tag{3.4}
$$

Similarly, we have

$$
\begin{aligned}
v(t) = \lambda T_2 u(t) &= \lambda\int_0^1 G_2(t,s)g(s,u(s))ds\\
&\leq \int_0^1 G_2(t,s)s^{-\sigma_2}s^{\sigma_2}g(s,u(s))ds \leq \int_0^1 G_2(t,s)s^{-\sigma_2}\psi(u(s))ds\\
&\leq \psi(\|u\|)\int_0^1 G_2(t,s)s^{-\sigma_2}ds \leq \psi(\|u\|)\int_0^1 \frac{M_0}{\Gamma(\beta)}k(s)s^{-\sigma_2}ds\\
&= \psi(\|u\|)\frac{M_0}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-2}s^{2-\sigma_2}ds = \psi(\|u\|)\frac{M_0}{\Gamma(\beta)}B(3-\sigma_2,\beta-1)\\
&\leq \psi(\|(u,v)\|)\frac{M_0}{\Gamma(\beta)}B(3-\sigma_2,\beta-1).
\end{aligned}
$$

Hence,

$$
\|v\| \leq \psi(\|(u,v)\|)\frac{M_0 B(3-\sigma_2,\beta-1)}{\Gamma(\beta)}. \tag{3.5}
$$

Combine (3.4) and (3.5), we obtain

$$
\frac{\|(u,v)\|}{\max\{\varphi(\|(u,v)\|),\ \psi(\|(u,v)\|)\}} \leq \max\left\{\frac{(\alpha-1)\Gamma(2-\sigma_1)}{\Gamma(2+\alpha-\sigma_1)},\ \frac{M_0 B(3-\sigma_2,\beta-1)}{\Gamma(\beta)}\right\}. \tag{3.6}
$$

Combining (H_{10}) and (3.6), then we have $\|(u,v)\| \neq r$, which is a contradiction with $(u,v) \in \partial U$. According to Lemma 2.8, T has a fixed point $(u,v) \in \overline{U}$. Therefore, the boundary value problem (1.1) has a positive solution. □

4. Example

In this section, as an application, an example is given to illustrate the main results.

Example 4.1. Consider the following singular nonlinear fractional differential equations boundary value problem

$$
\begin{cases}
-D_{0^+}^{\frac{5}{2}}u(t) = \frac{(t-\frac{1}{2})^2\ln(2+v(t))}{\sqrt{t}}, \quad 0<t<1,\\
D_{0^+}^{\frac{7}{2}}v(t) = \frac{(t-\frac{1}{2})^2\ln(2+u(t))}{\sqrt{t}}, \quad 0<t<1,\\
u(0)=u(1)=u'(0)=v(0)=v(1)=v'(0)=v'(1)=0.
\end{cases} \tag{4.1}
$$

In this case, $f(t,v) = ((t-\frac{1}{2})^2\ln(2+v(t)))/\sqrt{t}$, $g(t,u) = ((t-\frac{1}{2})^2\ln(2+u(t)))/\sqrt{t}$,for $(t,v),\ (t,u)\in(0,1]\times[0,+\infty)$. Note that $f,\ g$ is continuous in $(0,1]\times[0,+\infty)$ and $\lim_{t\to0^+} f(t,\cdot)=+\infty$, $\lim_{t\to0^+} g(t,\cdot)=+\infty$. Choosing $\sigma_1=\sigma_2=1/2$ and $\varphi(\omega)=\psi(\omega)=\ln(2+\omega)$, then we have

$$\sqrt{t}\frac{(t-\frac{1}{2})^2\ln(2+\omega)}{\sqrt{t}} = (t-\frac{1}{2})^2\ln(2+\omega)\leq\ln(2+\omega),\ \text{for } (t,\omega)\in[0,1]\times[0,+\infty).$$

Also $\varphi,\ \psi:[0,+\infty)\to(0,\infty)$ are two continuous, nondecreasing functions, so the condition (H_9) in Theorem 3.2 holds. Next, set $r=1$, then the condition (H_{10}) in Theorem 3.3 holds. Therefore, the boundary value problem (4.1) has one positive solution.

5. Conclusion

In this paper, we have considered existence of positive solutions for a class of the boundary value problem of the nonlinear singular fractional differential systems with multi-orders. Some sufficient conditions for the existence of positive solutions for the boundary value problem of the nonlinear singular differential systems with multi-orders have been established by Leray-Schauder nonlinear alternative theorem and Guo-Krasnosel'skii fixed point theorem. The main results have been well illustrated with the help of examples.

Acknowledgements

The author would like to thank the referee(s) for their valuable suggestions to improve presentation of the paper.

References

[1] C. Bai and J. Fang, *The existence of a positive solution for a singular coupled system of nonlinear fractional differential equations*, Appl. Math. Comput., 2004, 150(3), 611–621.

[2] Z. Bai and H. Lü, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl., 2005, 311, 495–505.

[3] S. M. Ege and F. S. Topal, *Existence of positive solutions for fractional boundary value problems*, J. Appl. Anal. Comput., 2017, 7(2), 702–712.

[4] W. Feng, S. Sun, Z. Han and Y. Zhao, *Existence of solutions for a singular system of nonlinear fractional differential equations*, Comput. Math. Appl. 2011, 62(3), 1370–1378.

[5] L. Guo and C. Liu, *Existence of positive solutions for singular higher-order fractional differential equation via spectral analysis*, J. Appl. Math. Comput., 2017, 54(1-2), 357–379.

[6] G. Isac, *Leray-Schauder Type Alternatives Complemantarity Problem and Variational Inequalities*, Springer, America, 2006.

[7] M. A. Krasnoselskii, *Positive Solution of Operator Equation*, Noordhoff, Groningen, 1964.

[8] A. A. Kilbas, H. H. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B. V., Amsterdam, 2006.

[9] L. Liu, H. Li, C. Liu and Y. Wu, *Existence and uniqueness of positive solutions for singular fractional differential systems with coupled integral boundary conditions*, J. Nonlinear Sci. Appl., 2017, 10(1), 1–25.

[10] Q. Leng, J. Yin and P. Yan, *Existence and uniqueness of positive solutions for singular higher order fractional differential equations with infinite-point boundary value conditions*, J. Comput. Anal. Appl., 2018, 25(2), 302–310.

[11] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, Mathematics in Science and Engineering*, Vol. 198, Academic Press, Inc., San Diego, CA, 1999.

[12] X. Su, *Boundary value problem for a coupled system of nonlinear fractional differential equations*, Appl. Math. Lett., 2009, 22(1), 64–69.

[13] Y. Sun and K. Yan, *Existence of solutions for fractional differential equation three-point boundary value problems*, J. Appl. Anal. Comput., 2016, 6(4), 939–949.

[14] Y. Wang and J. Zhang, *Positive solutions for higher-order singular fractional differential system with coupled integral boundary conditions*, Adv. Difference Equ., 2016, 117, DOI: 10.1186/s13662-016-0844-0.

[15] X. Xu, D. Jiang and C. Yuan, *Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation*, Nonlinear Anal., 2009, 71(10), 4676–4688.

[16] S. Xie and Y. Xie, *Positive solutions of higher-order nonlinear fractional differential systems with nonlocal boundary conditions*, J. Appl. Anal. Comput., 2016, 6(4), 1211–1227.

[17] Y. Yu and D. Jiang, *Multiple Positive Solutions for the Boundary Value Problem of a Nonlinear Fractional Differential Equation*, Northeast Normal University, 2009.

[18] L. Zhang, J. J. Nieto and G. Wang, *Extremal solutions for a nonlinear impulsive differential equations with multi-orders fractional derivatives*, J. Appl. Anal. Comput., 2017, 7(3), 814–823.

[19] Y. Zhao, S. Sun, Z. Han and W. Feng, *Positive solutions for a coupled system of nonlinear differential equations of mixed fractional orders*, Adv. Difference Equ., 2011, 10, DOI: 10.1186/1687-1847-2011-10.

[20] Y. Zhao, S. Sun, Z. Han and Q. Li, *The existence of multiple positive solutions for boundary value problems of nonlinear fractional differential equations*, Commun. Nonlinear Sci. Numer. Simul., 2011, 16(4), 2086–2097.

[21] Y. Zhao, S. Sun, Z. Han and Q. Li, *Positive solutions to boundary value problems of nonlinear fractional differential equations*, Abstr. Appl. Anal., 2011, DOI: 10.1155/2011/390543.

[22] Y. Zhao, S. Sun, Z. Han and M. Zhang, *Positive solutions for boundary value problems of nonlinear fractional differential equations*, Appl. Math. Comput., 2011, 217(16), 6950–6958.