Journal of Applied Analysis and Computation
Volume 8, Number 4, August 2018, 1159–1169 Website:http://jaac-online.com/
DOI:10.11948/2018.1159

A COMPACT DIFFERENCE SCHEME FOR FOURTH-ORDER FRACTIONAL SUB-DIFFUSION EQUATIONS WITH NEUMANN BOUNDARY CONDITIONS*

Zhongsheng Yao and Zhibo Wang†

Abstract In this paper, a compact finite difference scheme with global convergence order $O(\tau^2 + h^4)$ is derived for fourth-order fractional sub-diffusion equations subject to Neumann boundary conditions. The difficulty caused by the fourth-order derivative and Neumann boundary conditions is carefully handled. The stability and convergence of the proposed scheme are studied by the energy method. Theoretical results are supported by numerical experiments.

Keywords Fourth-order fractional sub-diffusion equation, compact difference scheme, energy method.

MSC(2010) 65M06, 65M12, 65M15, 35R11.

1. Introduction

In the last few decades, fractional calculus, the subject of differentiation and integration with arbitrary order, has been extensively studied. For more details, we refer readers to [10, 12, 16]. Fractional differential equations (FDEs) have been applied in many physical and engineering problems. To simulate and model these systems, solving the FDEs is necessary. The exact solutions of FDEs are not available under most circumstances. Therefore, effective numerical techniques for solving FDEs become necessary and important.

We review briefly here some works that are related to our current study. In [19], Sun and Wu constructed a difference scheme to solve the fractional diffusion-wave equation, which has been proved to be unconditionally stable. Based on the Grünwald-Letnikov approximation, Yuste and Acedo [25] constructed an explicit difference scheme for fractional diffusion equations, and investigated the stability using the von Neumann method. By the finite difference method in time and the Legendre spectral method in space, a high order efficient scheme was established in [13]. Zhuang et al. [27] investigated the stability and convergence of an implicit numerical method for the anomalous sub-diffusion equation by the energy

†The corresponding author. Email address: wzbmath@gdut.edu.cn(Z. Wang)
School of Applied Mathematics, Guangdong University of Technology, Guangzhou 510006, China

*The second author was supported by the National Natural Science Foundation of China (No. 11701103 & No. 11626067), Natural Science Foundation of Guangdong Province (No. 2017A030310538), the Project of Department of Education of Guangdong Province (No. 2017KTSCX062) and the grant from Guangdong University of Technology (No. 220413131 & No. 220413550).

method. A compact finite difference scheme for this equation was then presented by Cui in [2], where the local truncation error and the stability were studied by the Fourier method. By a transformation of this problem, Gao and Sun [4] proposed a high order scheme to improve the temporal convergence order. Compared with the Dirichlet boundary conditions, the spatial approximation is different between the boundary and interior points for the Neumann boundary conditions, thus the discretization must be dealt with more carefully to match the global accuracy. Based on [4], Ren et al. established a compact scheme for fractional diffusion equations with Neumann boundary conditions in [17]. By using the weighted and shifted Grünwald difference operator, Wang and Vong established schemes with $O(\tau^2+h^4)$ convergence order in [24], then the method was applied to solve fractional diffusion-wave equations with Neumann boundary conditions [22]. Lately, in [1], Alikhanov proposed a new numerical differentiation formula, called the $L2$-1_σ formula, to approximate the Caputo fractional derivative at a special point. The $L2$-1_σ formula has been extensively used in the literature, see [3, 5, 14, 20, 21].

All the results mentioned above concern with equations having second-order space derivatives. However, in some practical applications, the problem must be modeled by fourth-order space derivatives, such as the modeling formation of grooves on a flat surface ([15, 18]) and the propagation of intense laser beams in a bulk medium with Kerr nonlinearity ([9]). In this paper, we study the high order finite difference method for the following fourth-order fractional problem:

$$ {}_0^C D_t^\alpha u(x,t) + \frac{\partial^4 u(x,t)}{\partial x^4} + qu(x,t) = f(x,t), \quad 0<x<L, \quad 0<t\le T, \tag{1.1} $$

subject to the initial conditions

$$ u(x,0) = \phi(x), \quad 0<x<L, \tag{1.2} $$

and the Neumann boundary conditions

$$ \frac{\partial u(0,t)}{\partial x} = \beta_0(t),\ \frac{\partial u(L,t)}{\partial x} = \beta_1(t),\ \frac{\partial^3 u(0,t)}{\partial x^3} = \gamma_0(t),\ \frac{\partial^3 u(L,t)}{\partial x^3} = \gamma_1(t),\ 0\le t\le T, \tag{1.3} $$

where q is a positive constant,

$$ {}_0^C D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha} $$

is the Caputo fractional derivative of order $\alpha \in (0,1)$ with $\Gamma(\cdot)$ being the gamma function. FDEs with fourth-order space derivatives were also studied in [6–8, 23, 26], but the boundary conditions under consideration are different from the one studied in this paper.

This paper is organized as follows. A high order compact scheme is proposed in the next section by using discretization formulas developed in [1]and [17]. The stability and convergence of the compact scheme are analyzed in Section 3. In Section 4, numerical experiments are carried out to justify the theoretical results. The article ends with a brief conclusion.

2. The proposed compact finite difference scheme

In order to derive a high order compact scheme for (1.1)–(1.3), we first introduce an equivalent form of the problem. Let

$$v(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}.$$

Equations (1.1)–(1.3) can then be transformed into

$$ {}_0^C D_t^\alpha u(x,t) + \frac{\partial^2 v(x,t)}{\partial x^2} + qu(x,t) = f(x,t), \quad 0 < x < L, \quad 0 < t \le T, \tag{2.1}$$

$$v(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < L, \quad 0 < t \le T. \tag{2.2}$$

$$u(x,0) = \phi(x), \quad 0 < x < L, \tag{2.3}$$

$$\frac{\partial u(0,t)}{\partial x} = \beta_0(t), \frac{\partial u(L,t)}{\partial x} = \beta_1(t), \frac{\partial v(0,t)}{\partial x} = \gamma_0(t), \frac{\partial v(L,t)}{\partial x} = \gamma_1(t), 0 \le t \le T. \tag{2.4}$$

To propose a compact scheme for (2.1)–(2.4), we let $h = \frac{L}{M}$ and $\tau = \frac{T}{N}$ be the spatial and temporal step sizes respectively, where M and N are two given positive integers. For $i = 0, 1, \ldots, M,\ n = 0, 1, \ldots, N$, denote $x_i = ih,\ t_n = n\tau$. For a grid function $u = \{u_i | 0 \le i \le M\}$, we introduce the following notations:

$$\delta_x u_{i-\frac{1}{2}} = \tfrac{1}{h}(u_i - u_{i-1}),$$

$$\delta_x^2 u_i = \begin{cases} \frac{2}{h}\delta_x u_{\frac{1}{2}}, & i = 0, \\ \frac{1}{h}(\delta_x u_{i+\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}), & 1 \le i \le M-1, \\ -\frac{2}{h}\delta_x u_{M-\frac{1}{2}}, & i = M, \end{cases}$$

$$\mathcal{H} u_i = \begin{cases} \frac{1}{6}(5u_0 + u_1), & i = 0, \\ \frac{1}{12}(u_{i-1} + 10u_i + u_{i+1}), & 1 \le i \le M-1, \\ \frac{1}{6}(u_{M-1} + 5u_M), & i = M. \end{cases}$$

For any grid function u, v, we further denote the following discrete inner products and norms:

$$(u,v) = h\Big(\tfrac{1}{2}u_0 v_0 + \sum_{i=1}^{M-1} u_i v_i + \tfrac{1}{2}u_M v_M\Big),$$

$$\langle u, v\rangle = h\sum_{i=0}^{M-1} \delta_x u_{i+\frac{1}{2}} \delta_x v_{i+\frac{1}{2}}, \quad \|u\|^2 = (u,u).$$

We then present a sequence $\{c_m^k\}$ defined in [1], which will be used later. Let $0 < \alpha < 1,\ \sigma = 1 - \frac{\alpha}{2}$, and

$$a_0 = \sigma^{1-\alpha},\ a_l = (l+\sigma)^{1-\alpha} - (l-1+\sigma)^{1-\alpha},\ l \ge 1,$$

$$b_l = \frac{1}{2-\alpha}[(l+\sigma)^{2-\alpha} - (l-1+\sigma)^{2-\alpha}] - \frac{1}{2}[(l+\sigma)^{1-\alpha} - (l-1+\sigma)^{1-\alpha}],\ l \ge 1.$$

For $k=0$, $c_0^k = a_0$. For $k \geq 1$,

$$c_m^k = \begin{cases} a_0 + b_1, & m = 0, \\ a_m + b_{m+1} - b_m, & 1 \leq m \leq k-1, \\ a_k - b_k, & m = k. \end{cases}$$

The discretization in temporal direction is based on the following lemma in [1].

Lemma 2.1. *Suppose $u(t) \in \mathcal{C}^3[0,T]$. Then*

$${}_0^C D_t^\alpha u(t_{k+\sigma}) - \Delta_t^\alpha u(t_{k+\sigma}) = O(\tau^{3-\alpha}),$$

where $\Delta_t^\alpha u(t_{k+\sigma}) = \dfrac{1}{\tau^\alpha \Gamma(2-\alpha)} \sum_{s=0}^{k} c_{k-s}[u(t_{s+1}) - u(t_s)].$

The following two lemmas give theoretical support for truncation error of the proposed scheme.

Lemma 2.2 ([17])**.** *Denote $\zeta(s) = (1-s)^3[5-3(1-s)^2]$.*

(I) *If $g(x) \in \mathcal{C}^6[x_0, x_1]$, then we have*

$$\begin{aligned} &\Big[\frac{5}{6} g''(x_0) + \frac{1}{6} g''(x_1)\Big] - \frac{2}{h}\Big[\frac{g(x_1) - g(x_0)}{h} - g'(x_0)\Big] \\ &= -\frac{h}{6} g'''(x_0) + \frac{h^3}{90} g^{(5)}(x_0) + \frac{h^4}{180} \int_0^1 g^{(6)}(x_0 + sh)\zeta(s)ds. \end{aligned}$$

(II) *If $g(x) \in \mathcal{C}^6[x_{M-1}, x_M]$, then we get*

$$\begin{aligned} &\Big[\frac{1}{6} g''(x_{M-1}) + \frac{5}{6} g''(x_M)\Big] - \frac{2}{h}\Big[g'(x_M) - \frac{g(x_M) - g(x_{M-1})}{h}\Big] \\ &= \frac{h}{6} g'''(x_M) - \frac{h^3}{90} g^{(5)}(x_M) + \frac{h^4}{180} \int_0^1 g^{(6)}(x_M - sh)\zeta(s)ds. \end{aligned}$$

(III) *If $g(x) \in \mathcal{C}^6[x_{i-1}, x_{i+1}]$, $1 \leq i \leq M-1$, then it holds that*

$$\begin{aligned} &\frac{1}{12}[g''(x_{i-1}) + 10g''(x_i) + g''(x_{i+1})] - \frac{1}{h^2}[g(x_{i-1}) - 2g(x_i) + g(x_{i+1})] \\ &= \frac{h^4}{360} \int_0^1 [g^{(6)}(x_i - sh) + g^{(6)}(x_i + sh)]\zeta(s)ds. \end{aligned}$$

We follow the idea in [17] to derive our scheme. Differentiating equation (2.1) with respect to x and letting $x \to 0^+$, one can obtain, by using the boundary condition (2.4),

$$\frac{\partial^3 v(0,t)}{\partial x^3} = -\big[{}_0^C D_t^\alpha \beta_0(t) + q\beta_0(t) - f_x(0,t)\big]. \tag{2.5}$$

Meanwhile, differentiating equation (2.1) three times with respect to x yields

$${}_0^C D_t^\alpha \frac{\partial^3 u(x,t)}{\partial x^3} + \frac{\partial^5 v(x,t)}{\partial x^5} + q\frac{\partial^3 u(x,t)}{\partial x^3} = f_{xxx}(x,t). \tag{2.6}$$

Then, differentiate equation (2.2) with respect to x, we get

$$\frac{\partial v(x,t)}{\partial x} = \frac{\partial^3 u(x,t)}{\partial x^3}. \tag{2.7}$$

Once again, letting $x \to 0^+$ in (2.6) and (2.7), noticing (2.4) again, we can achieve

$$\frac{\partial^5 v(0,t)}{\partial x^5} = -\big[{}_0^C D_t^\alpha \gamma_0(t) + q\gamma_0(t) - f_{xxx}(0,t)\big]. \tag{2.8}$$

We note that similar expressions hold at the other end of the boundary.

Furthermore, suppose $u(t) \in \mathcal{C}^3[0,T]$, by the Taylor expansion, it holds that

$$u(t_{n+\sigma}) = \sigma u(t_{n+1}) + (1-\sigma)u(t_n) + O(\tau^2).$$

Denote $u^{(\sigma_{k+1})} = \sigma u(t_{k+1}) + (1-\sigma)u(t_k)$, based on Lemma 2.1 and Lemma 2.2, we propose the following scheme for the problem (2.1)–(2.4):

$$\begin{aligned} &\mathcal{H}\Delta_t^\alpha u_0^{n+\sigma} + \frac{2}{h}[\delta_x v_{\frac{1}{2}}^{(\sigma_{n+1})} - \gamma_0(t_{n+\sigma})] \\ &\quad + \frac{h}{6}\big[{}_0^C D_t^\alpha \beta_0(t_{n+\sigma}) + q\beta_0(t_{n+\sigma}) - f_x(0,t_{n+\sigma})\big] \\ &\quad - \frac{h^3}{90}\big[{}_0^C D_t^\alpha \gamma_0(t_{n+\sigma}) + q\gamma_0(t_{n+\sigma}) - f_{xxx}(0,t_{n+\sigma})\big] + q\mathcal{H}u_0^{(\sigma_{n+1})} = \mathcal{H}f_0^{n+\sigma}, \end{aligned} \tag{2.9}$$

$$\mathcal{H}\Delta_t^\alpha u_i^{n+\sigma} + \delta_x^2 v_i^{(\sigma_{n+1})} + q\mathcal{H}u_i^{(\sigma_{n+1})} = \mathcal{H}f_i^{n+\sigma}, \quad 1 \le i \le M-1, \tag{2.10}$$

$$\begin{aligned} &\mathcal{H}\Delta_t^\alpha u_M^{n+\sigma} + \frac{2}{h}[\gamma_1(t_{n+\sigma}) - \delta_x v_{M-\frac{1}{2}}^{(\sigma_{n+1})}] \\ &\quad - \frac{h}{6}\big[{}_0^C D_t^\alpha \beta_1(t_{n+\sigma}) + q\beta_1(t_{n+\sigma}) - f_x(L,t_{n+\sigma})\big] \\ &\quad + \frac{h^3}{90}\big[{}_0^C D_t^\alpha \gamma_1(t_{n+\sigma}) + q\gamma_1(t_{n+\sigma}) - f_{xxx}(L,t_{n+\sigma})\big] + q\mathcal{H}u_M^{(\sigma_{n+1})} = \mathcal{H}f_M^{n+\sigma}, \end{aligned} \tag{2.11}$$

$$\begin{aligned} \mathcal{H}v_0^{(\sigma_{n+1})} &= \frac{2}{h}[\delta_x u_{\frac{1}{2}}^{(\sigma_{n+1})} - \beta_0(t_{n+\sigma})] - \frac{h}{6}\gamma_0(t_{n+\sigma}) \\ &\quad - \frac{h^3}{90}\big[{}_0^C D_t^\alpha \beta_0(t_{n+\sigma}) + q\beta_0(t_{n+\sigma}) - f_x(0,t_{n+\sigma})\big], \end{aligned} \tag{2.12}$$

$$\mathcal{H}v_i^{(\sigma_{n+1})} = \delta_x^2 u_i^{(\sigma_{n+1})}, 1 \le i \le M-1, \tag{2.13}$$

$$\begin{aligned} \mathcal{H}v_M^{(\sigma_{n+1})} &= \frac{2}{h}[\beta_1(t_{n+\sigma}) - \delta_x u_{M-\frac{1}{2}}^{(\sigma_{n+1})}] + \frac{h}{6}\gamma_1(t_{n+\sigma}) \\ &\quad + \frac{h^3}{90}\big[{}_0^C D_t^\alpha \beta_1(t_{n+\sigma}) + q\beta_1(t_{n+\sigma}) - f_x(L,t_{n+\sigma})\big], \end{aligned} \tag{2.14}$$

$$u_i^0 = \phi(x_i), \quad 0 \le i \le M, \tag{2.15}$$

where $0 \le n \le N-1$. One can easily check that our proposed scheme has truncation error equal to $O(\tau^2 + h^4)$.

3. Stability and convergence of the compact scheme

In this section, the stability and convergence of the compact scheme are studied. To begin, we introduce some lemmas, which play a vital role in stability and convergence analysis.

Lemma 3.1 ([11]). *Let u be a grid function, then it holds that*

$$\frac{5}{12}\|u\|^2 \leq \|\mathcal{H}u\|^2 \leq \|u\|^2.$$

Lemma 3.2. *For any grid function $u,\ v$, we have*

$$(\delta_x^2 v, \mathcal{H}u) = (\delta_x^2 u, \mathcal{H}v).$$

Proof. When $1 \leq i \leq M-1$, noticing that

$$\mathcal{H}u_i = (1 + \frac{h^2}{12}\delta_x^2)u_i,$$

we then have

$$\begin{aligned}
&(\delta_x^2 v, \mathcal{H}u) - (\delta_x^2 u, \mathcal{H}v)\\
=&\ h\Big(\frac{1}{2}\delta_x^2 v_0 \mathcal{H}u_0 + \sum_{i=1}^{M-1}\delta_x^2 v_i \mathcal{H}u_i + \frac{1}{2}\delta_x^2 v_M \mathcal{H}u_M\Big)\\
&-h\Big(\frac{1}{2}\delta_x^2 u_0 \mathcal{H}v_0 + \sum_{i=1}^{M-1}\delta_x^2 u_i \mathcal{H}v_i + \frac{1}{2}\delta_x^2 u_M \mathcal{H}v_M\Big)\\
=&\ h\Big(\frac{1}{2}\delta_x^2 v_0 \mathcal{H}u_0 + \sum_{i=1}^{M-1}\delta_x^2 v_i u_i + \frac{1}{2}\delta_x^2 v_M \mathcal{H}u_M\Big)\\
&-h\Big(\frac{1}{2}\delta_x^2 u_0 \mathcal{H}v_0 + \sum_{i=1}^{M-1}\delta_x^2 u_i v_i + \frac{1}{2}\delta_x^2 u_M \mathcal{H}v_M\Big)\\
=&\ \delta_x v_{\frac{1}{2}}\Big(\frac{5}{6}u_0 + \frac{1}{6}u_1\Big) - h\sum_{i=1}^{M}\delta_x v_{i-\frac{1}{2}}\delta_x u_{i-\frac{1}{2}} - \delta_x v_{\frac{1}{2}}u_0 + \delta_x v_{M-\frac{1}{2}}u_M\\
&-\delta_x v_{M-\frac{1}{2}}\Big(\frac{5}{6}u_M + \frac{1}{6}u_{M-1}\Big) - \delta_x u_{\frac{1}{2}}\Big(\frac{5}{6}v_0 + \frac{1}{6}v_1\Big) + h\sum_{i=1}^{M}\delta_x v_{i-\frac{1}{2}}\delta_x u_{i-\frac{1}{2}}\\
&+\delta_x u_{\frac{1}{2}}v_0 - \delta_x u_{M-\frac{1}{2}}v_M + \delta_x u_{M-\frac{1}{2}}\Big(\tfrac{5}{6}v_M + \tfrac{1}{6}v_{M-1}\Big)\\
=&\ \frac{1}{6h}\delta_x u_{\frac{1}{2}}\delta_x v_{\frac{1}{2}} + \frac{1}{6h}\delta_x u_{M-\frac{1}{2}}\delta_x v_{M-\frac{1}{2}} - \frac{1}{6h}\delta_x u_{\frac{1}{2}}\delta_x v_{\frac{1}{2}} - \frac{1}{6h}\delta_x u_{M-\frac{1}{2}}\delta_x v_{M-\frac{1}{2}}\\
=&\ 0.
\end{aligned}$$

□

Lemma 3.3 ([1]). *Suppose that u is a grid function, $\sigma = 1 - \frac{\alpha}{2},\ \alpha \in (0,1)$. We have the following inequality*

$$[\sigma u^{k+1} + (1-\sigma)u^k]\Delta_t^\alpha u^{k+\sigma} \geq \frac{1}{2}\Delta_t^\gamma[(u^{k+\sigma})^2].$$

Lemma 3.4 ([1]). *For any $\alpha \in (0,1)$ and $c_m^k (0 \leq m \leq k, k \geq 1)$ defined in (2), it holds that*

(i) $c_m^k \geq \frac{1-\alpha}{2}(m+\sigma)^{-\alpha}$,

(ii) $c_0^k > c_1^k > c_2^k > \cdots > c_k^k$,

where $\sigma = 1 - \frac{\alpha}{2}$.

We now turn to the stability of the scheme.

Lemma 3.5. *Suppose that* $\{u_i^n\}$ *and* $\{v_i^n\}$ *be the solution of the following difference scheme*

$$\mathcal{H}\Delta_t^\alpha u_i^{n+\sigma} + \delta_x^2 v_i^{(\sigma_{n+1})} + q\mathcal{H}u_i^{(\sigma_{n+1})} = P_i^{n+\sigma}, \quad 0 \leq i \leq M, \quad 0 \leq n \leq N-1, \tag{3.1}$$

$$\mathcal{H}v_i^{(\sigma_{n+1})} = \delta_x^2 u_i^{(\sigma_{n+1})} + Q_i^{n+\sigma}, \quad 0 \leq i \leq M, \quad 0 \leq n \leq N-1, \tag{3.2}$$

$$u_i^0 = \phi(x_i), \quad 0 \leq i \leq M.$$

Then we have

$$\|\mathcal{H}u^n\|^2 \leq \|\mathcal{H}u^0\|^2 + T^\alpha\Gamma(1-\alpha)\Big(\frac{1}{q}\|P^{n+\sigma}\|^2 + \|Q^{n+\sigma}\|^2\Big), \ 1 \leq n \leq N.$$

Proof. Taking the inner product of (3.1) and (3.2) with $\mathcal{H}u^{(\sigma_{n+1})}$ and $\mathcal{H}v^{(\sigma_{n+1})}$, respectively, we then have

$$\begin{aligned}&(\mathcal{H}u^{(\sigma_{n+1})}, \mathcal{H}\Delta_t^\alpha u^{n+\sigma}) + (\mathcal{H}u^{(\sigma_{n+1})}, \delta_x^2 v^{(\sigma_{n+1})}) + q(\mathcal{H}u^{(\sigma_{n+1})}, \mathcal{H}u^{(\sigma_{n+1})})\\ =&(\mathcal{H}u^{(\sigma_{n+1})}, P^{n+\sigma}),\end{aligned} \tag{3.3}$$

and

$$(\mathcal{H}v^{(\sigma_{n+1})}, \mathcal{H}v^{(\sigma_{n+1})}) = (\mathcal{H}v^{(\sigma_{n+1})}, \delta_x^2 u^{(\sigma_{n+1})}) + (\mathcal{H}v^{(\sigma_{n+1})}, Q^{n+\sigma}). \tag{3.4}$$

Adding (3.3) and (3.4), by Lemma 3.2 and Lemma 3.3, we get

$$\begin{aligned}&\tfrac{1}{2}\Delta_t^\alpha\|\mathcal{H}u^{n+\sigma}\|^2 + \|\mathcal{H}v^{(\sigma_{n+1})}\|^2 + q\|\mathcal{H}u^{(\sigma_{n+1})}\|^2\\ \leq\ &(\mathcal{H}u^{(\sigma_{n+1})}, P^{n+\sigma}) + (\mathcal{H}v^{(\sigma_{n+1})}, Q^{n+\sigma}).\end{aligned}$$

Using Cauchy-Schwarz inequality, we obtain

$$\Delta_t^\alpha\|\mathcal{H}u^{n+\sigma}\|^2 \leq \frac{1}{2q}\|P^{n+\sigma}\|^2 + \frac{1}{2}\|Q^{n+\sigma}\|^2.$$

Multiplying the result by $\tau^\alpha\Gamma(2-\alpha)$, it follows that

$$\begin{aligned}c_0^{n+1}\|\mathcal{H}u^{n+1}\|^2 \leq &\sum_{k=1}^{n}(c_{n-k}^{n+1} - c_{n-k+1}^{n+1})\|\mathcal{H}u^k\|^2 + c_n^{n+1}\|\mathcal{H}u^0\|^2\\ &+\tau^\alpha\Gamma(2-\alpha)\Big(\tfrac{1}{2q}\|P^{n+\sigma}\|^2 + \tfrac{1}{2}\|Q^{n+\sigma}\|^2\Big).\end{aligned}$$

From Lemma 3.4,

$$c_n^{n+1} \geq \frac{1-\alpha}{2}(n+\sigma)^{-\alpha} = \frac{1-\alpha}{2}(n+1-\frac{\alpha}{2})^{-\alpha},$$

which gives

$$\begin{aligned}2c_n^{n+1}T^\alpha\Gamma(1-\alpha) &\geq T^\alpha(1-\alpha)\Gamma(1-\alpha)(n+1-\tfrac{\alpha}{2})^{-\alpha}\\ &\geq T^\alpha\Gamma(2-\alpha)N^{-\alpha} = \tau^\alpha\Gamma(2-\alpha),\end{aligned}$$

where $n \leq N-1$.

So

$$c_0^{n+1}\|\mathcal{H}u^{n+1}\|^2 \leq \sum_{k=1}^{n}(c_{n-k}^{n+1} - c_{n-k+1}^{n+1})\|\mathcal{H}u^k\|^2 + c_n^{n+1}\left[\|\mathcal{H}u^0\|^2 + T^\alpha\Gamma(1-\alpha)\Big(\frac{1}{q}\|P^{n+\sigma}\|^2 + \|Q^{n+\sigma}\|^2\Big)\right],$$

the desired result then follows by the induction. □

From Lemma 3.5, we can conclude the following stability statement.

Theorem 3.1. *The compact finite difference scheme* (2.9)–(2.15) *is unconditionally stable.*

With all the preparation, we can now give the convergence result of the scheme.

Theorem 3.2. *Assume that* $u(x,t) \in \mathcal{C}_{x,t}^{8,3}([0,L]\times[0,T])$ *is the solution of* (2.1)–(2.4) *and* $u^n = [u_0^n, u_1^n, \ldots, u_M^n]^T$, $v^n = [v_0^n, v_1^n, \ldots, v_M^n]^T$, $0 \leq n \leq N$, *is the solution of the finite difference scheme* (2.9)–(2.15), *respectively. Denote*

$$e_i^n = u(x_i,t_n) - u_i^n, \quad \epsilon_i^n = v(x_i,t_n) - v_i^n, \quad 0 \leq i \leq M, \quad 0 \leq n \leq N.$$

Then there exists a positive constant c such that

$$\|e^n\| \leq c(\tau^2 + h^4), \quad 0 \leq n \leq N.$$

Proof. We can easily get the following error equation

$$\begin{aligned}
&\mathcal{H}\Delta_t^\alpha e_i^{n+\sigma} + \delta_x^2\epsilon_i^{(\sigma_{n+1})} + q\mathcal{H}e_i^{(\sigma_{n+1})} = R_i^{n+\sigma}, \quad 0 \leq i \leq M,\ 0 \leq n \leq N-1,\\
&\mathcal{H}\epsilon_i^{(\sigma_{n+1})} = \delta_x^2 e_i^{(\sigma_{n+1})} + S_i^{n+\sigma}, \quad 0 \leq i \leq M,\ 0 \leq n \leq N-1,\\
&e_i^0 = 0, \quad 0 \leq i \leq M,
\end{aligned}$$

where $R_i^{n+\sigma} = O(\tau^2 + h^4)$, $S_i^{n+\sigma} = O(h^4)$.

According to Lemma 3.5, we have

$$(\mathcal{H}e^n, \mathcal{H}e^n) \leq T^\alpha\Gamma(1-\alpha)\Big(\frac{1}{q}\|R^{n+\sigma}\|^2 + \|S^{n+\sigma}\|^2\Big).$$

It is easy to get the conclusion with Lemma 3.1. □

4. Numerical experiments

In this section, we carry out numerical experiments for the proposed compact finite difference scheme to illustrate our theoretical statements. All our tests were done in MATLAB. The L^2 norm errors between the exact and the numerical solutions

$$E_2(h,\tau) = \max_{0\leq n\leq N}\|e^n\|,$$

are shown in the following tables. Furthermore, the temporal convergence order, denoted by

$$\text{Rate1} = \log_2\left(\frac{E_2(h,2\tau)}{E_2(h,\tau)}\right),$$

for sufficiently small h, and the spatial convergence order, denoted by

$$\text{Rate2} = \log_2\left(\frac{E_2(2h,\tau)}{E_2(h,\tau)}\right),$$

when τ is sufficiently small, are reported.

Example 4.1. We consider the following problem:

$$\begin{aligned} &{}_0^C D_t^\alpha u(x,t) + \frac{\partial^4 u(x,t)}{\partial x^4} + u(x,t) = \frac{\Gamma(\alpha+3)}{2}\cos(\pi x)t^2 + (1+\pi^4)\cos(\pi x)t^{2+\alpha},\\ &\quad 0 < x < 1,\ 0 < t \le 1,\\ &u(x,0) = 0, \quad 0 < x < 1,\\ &\tfrac{\partial u(0,t)}{\partial x} = \tfrac{\partial u(L,t)}{\partial x} = \tfrac{\partial^3 u(0,t)}{\partial x^3} = \tfrac{\partial^3 u(L,t)}{\partial x^3} = 0, \quad 0 \le t \le 1. \end{aligned}$$

The exact solution for this problem is $u(x,t) = \cos(\pi x)t^{2+\alpha}$.

The convergence order in temporal direction with $h = \frac{1}{100}$ is reported in Table 1, while in Table 2, the convergence order in spatial direction with $\tau = \frac{1}{10000}$, $\alpha = 0.5$ is listed. The convergence orders of the numerical results match that of the theoretical ones.

Table 1. Numerical convergence orders in temporal direction with $h = \frac{1}{100}$.

τ	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 0.8$	
	$E_2(h,\tau)$	Rate1	$E_2(h,\tau)$	Rate1	$E_2(h,\tau)$	Rate1
1/10	8.3624e-4	*	2.4694e-3	*	4.2637e-3	*
1/20	2.0960e-4	1.9963	6.1976e-4	1.9944	1.0684e-3	1.9966
1/40	5.2463e-5	1.9983	1.5524e-4	1.9973	2.6740e-4	1.9984
1/80	1.3120e-5	1.9996	3.8841e-5	1.9988	6.6879e-5	1.9994
1/160	3.2767e-6	2.0014	9.7105e-6	2.0000	1.6719e-5	2.0001

Table 2. Numerical convergence orders in spatial direction with $\tau = \frac{1}{10000}$ when $\alpha = 0.5$.

h	$E_2(h,\tau)$	Rate2
1/4	2.7418e-3	*
1/8	1.5345e-4	4.1593
1/16	9.0536e-6	4.0831
1/32	5.4689e-7	4.0492
1/64	3.1319e-8	4.1261

5. Conclusions

In this article, we consider the numerical method for fourth-order fractional sub-diffusion equations under Neumann boundary conditions. After a transformation of the main equation, a compact finite difference scheme with global convergence order $O(\tau^2 + h^4)$ is successfully derived. The difficulty caused by the Neumann boundary conditions is also carefully handled. The stability and convergence of the proposed scheme are studied by the energy method. Theoretical results are supported by numerical experiments. Here we want remark that similar results can be obtained in the case of multi-term and distributed order time-fractional equation and accordingly the reference [3].

Acknowledgment

The authors would like to thank the editor and the referees for their helpful comments.

References

[1] Anatoly and A. Alikhanov, *A new difference scheme for the time fractional diffusion equation*, J. Comput. Phys., 2015, 280, 424–438.

[2] M. Cui, *Compact finite difference method for the fractional diffusion equation*, J. Comput. Phys., 2009, 228(20), 7792–7804.

[3] G. Gao, A. A. Alikhanov and Z. Sun, *The temporal second order difference schemes based on the interpolation approximation for solving the time multi-term and distributed-order fractional sub-diffusion equations*, J. Sci. Comput., 2017, 73(1), 93–121.

[4] G. Gao and Z. Sun, *A compact finite difference scheme for the fractional sub-diffusion equations*, J. Comput. Phys., 2011, 230(3), 586–595.

[5] X. Gu, T. Huang, C. Ji, B. Carpentieri and A. A. Alikhanov, *Fast iterative method with a second-order implicit difference scheme for time-space fractional convection-diffusion equation*, J. Sci. Comput., 2017, 72(3), 957–985.

[6] L. Guo, Z. Wang and S. Vong, *Fully discrete local discontinuous Galerkin methods for some time-fractional fourth-order problems*, Int. J. Comput. Math., 2016, 93(10), 1665–1682.

[7] X. Hu and L. Zhang, *A new implicit compact difference scheme for the fourth-order fractional diffusion-wave system*, Int. J. Comput. Math., 2014, 91(10), 2215–2231.

[8] C. Ji, Z. Sun and Z. Hao, *Numerical algorithms with high spatial accuracy for the fourth-order fractional sub-diffusion equations with the first Dirichlet boundary conditions*, J. Sci. Comput., 2016, 66(3), 1148–1174.

[9] V. I. Karpman, *Stabilization of soliton instabilities by higher-order dispersion: fourth order nonlinear Schrödinger-type equations*, Phys. Rev. E., 1996, 53(2), 1336–1339.

[10] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, 2006.

[11] J. Li, Z. Sun and X. Zhao, *A three level linearized compact difference scheme for the Cahn-Hilliard equation*, Sci. China Math., 2012, 55(4), 805–826.

[12] C. Li and F. Zeng, *Numerical Methods for Fractional Calculus*, Chapman and Hall/CRC Press, 2015.

[13] Y. Lin and C. Xu, *Finite difference/spectral approximations for the time-fractional diffusion equation*, J. Comput. Phys., 2007, 225(2), 1533–1552.

[14] P. Lyu and S. Vong, *A linearized second-order scheme for nonlinear time fractional KleinGordon type equations*, Numer. Algor., 2017, doi:10.1007/s11075-017-0385-y.

[15] K. B. Oldhan and J. Spainer, *The Fractional Calculus*, Academic Press, New York, 1974.

[16] I. Podlubny, *Fractional Differential Equations*, Academic Press, 1999.

[17] J. Ren and Z. Sun, *Numerical algorithm with high spatial accuracy for the fractional diffusion-wave equation with Neumann boundary conditions*, J. Sci. Comput., 2013, 56(2), 381–408.

[18] I. N. Sneddon, *Fourier Transforms*, McGraw Hill, New York, 1951.

[19] Z. Sun and X. Wu, *A fully discrete scheme for a diffusion-wave system*, Appl. Numer. Math., 2006, 56(2), 193–209.

[20] S. Vong, P. Lyu, X. Chen and S. Lei, *High order finite difference method for time-space fractional differential equations with Caputo and Riemann-Liouville derivatives*, Numer. Algor., 2016, 72(1), 195–210.

[21] S. Vong, C. Shi and P. Lyu, *High-order compact schemes for fractional differential equations with mixed derivatives*, Numer. Methods Partial Differ. Equ., 2017, 33(6), 2141–2158.

[22] S. Vong and Z. Wang, *High order difference schemes for a time fractional differential equation with Neumann boundary conditions*, East Asian J. Appl. Math., 2014, 4(3), 222–241.

[23] S. Vong and Z. Wang, *Compact finite difference scheme for the fourth-order fractional subdiffusion system*, Adv. Appl. Math. Mech., 2014, 6(4), 419–435.

[24] Z. Wang and S. Vong, *Compact difference schemes for the modified anomalous fractional sub-diffusion equation and the fractional diffusion-wave equation*, J. Comput. Phys., 2014, 277, 1–15.

[25] S. B. Yuste and L. Acedo, *An explicit finite difference method and a new Von Neumann-type stability analysis for fractional diffusion equations*, SIAM J. Numer. Anal., 2005, 42(5), 1862–1874.

[26] P. Zhang and H. Pu, *A second-order compact difference scheme for the fourth-order fractional sub-diffusion equation*, Numer. Algor., 2017, 76(2), 573-598.

[27] P. Zhuang, F. Liu, V. Anh and I. Turner, *New solution and analytical techniques of the implicit numerical method for the anomalous subdiffusion equation*, SIAM J. Numer. Anal., 2008, 46(2), 1079–1095.