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HOPF BIFURCATION AND NEW SINGULAR ORBITS COINED IN A LORENZ-LIKE SYSTEM*

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Abstract We seize some new dynamics of a Lorenz-like system: $\dot{x} = a(y-x)$, $\dot{y} = dy - xz$, $\dot{z} = -bz + fx^2 + gxy$, such as for the Hopf bifurcation, the behavior of non-isolated equilibria, the existence of singularly degenerate heteroclinic cycles and homoclinic and heteroclinic orbits. In particular, our new discovery is that the system has also two heteroclinic orbits for $bg = 2a(f+g)$ and $a > d > 0$ other than known $bg > 2a(f+g)$ and $a > d > 0$, whose proof is completely different from known case. All the theoretical results obtained are also verified by numerical simulations.

Keywords Lorenz-like system, singularly degenerate heteroclinic cycle, heteroclinic orbit, Lyapunov-like function.

MSC(2010) 34C37, 34D20, 37C29.

1. Introduction

In this paper, we give some new insights into the following Lorenz-like system

$$\begin{cases} \dot{x} = a(y-x), \\ \dot{y} = dy - xz, \\ \dot{z} = -bz + fx^2 + gxy, \end{cases} \tag{1.1}$$

where $a > 0$, $f, g \geq 0$, $f + g > 0$ and $b, d \in \mathbb{R}$, expand and complement the previous results obtained in [13] by Li and Ou. They first proposed the system and considered its local and global dynamical behaviors as more as possible. More importantly, system (1.1) is proved to have two and only heteroclinic orbits when $bg > 2a(f+g)$ and $a > d > 0$. We here find that the statement also holds for $bg = 2a(f+g)$ and $a > d > 0$. Furthermore, the Hopf bifurcation of its equilibria $S_{\pm} = (\pm\sqrt{\frac{bd}{f+g}}, \pm\sqrt{\frac{bd}{f+g}}, d)$, the existence of its singularly degenerate heteroclinic cycle and homoclinic orbit, etc. are deeply considered by combining theoretical analysis and numerical technique. For the results of some special cases of system (1.1), refer also to [4].

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The rest of the paper is organized as follows. In Section 2, using the formulation in [8] to do some direct computations, the conditions of Hopf bifurcation at $S_\pm$ are obtained. Section 3 formulates the dynamics of non-isolated equilibria S_z. The proof for the existence of singularly degenerate heteroclinic cycle is given in Section 4. In Section 5, system (1.1) has been proved to have no homoclinic orbits when $bg \geq 2a(f+g)$ and $a > d > 0$. For $bg = 2a(f+g)$ and $a > d > 0$, system (1.1) is shown to have two and only two heteroclinic orbits, that is presented in Section 6. Furthermore, some other heteroclinic orbits have been observed to exist in other parameter region via numerical simulations. Finally, some conclusions are drawn in Section 7.

2. Hopf bifurcation analysis for $S_\pm$

At the beginning of this section, we first review the Projection Method described in [8] for the calculation of the first Lyapunov coefficient associated to Hopf bifurcations, denoted by l_1. This method can be extended to the calculation of the other Lyapunov coefficients. See [8] for the calculation of the second, the third and the fourth Lyapunov coefficients [22, 23], respectively. Then, we study the Hopf bifurcation of system (1.1) at $S_\pm$ by using this method. For related to work, see [1, 6, 7, 16, 17, 19, 21, 28, 32].

2.1. Projection method for computing the first Lyapunov coefficient

Consider the following nonlinear system:

$$\dot{x} = Ax + F(x, \xi), \quad x \in \mathbb{R}^n, \quad \xi \in \mathbb{R},$$

where $F(x, \xi) = O(\|x^2\|)$ is a smooth function and for $\xi = 0$ it can be expanded as

$$F(x, 0) = \frac{1}{2}B(x, x) + \frac{1}{6}C(x, x, x) + O(\|x^4\|,$$

where $B(x, x)$ and $C(x, x, x)$ are bilinear and trilinear functions, respectively.

Suppose that A has a simple pair of complex eigenvalues on the imaginary axis: $\lambda_{1,2} = \pm\omega_0 i$, $\omega_0 > 0$ and these eigenvalues are the only eigenvalues with $Re(\lambda) = 0$. Let $q \in C^n$ be a complex eigenvector associated with λ_1, i.e.,

$$Aq = i\omega_0 q, \quad A\bar{q} = -i\omega_0 \bar{q}.$$

Also, introduce the adjoint eigenvector $p \in C^n$ having the following property:

$$Ap = -i\omega_0 p, \quad A\bar{p} = i\omega_0 \bar{p},$$

where $\langle p, q\rangle = 1$.

The first Lyapunov coefficient at the origin can be written as [8]

$$l_1 = \frac{1}{2\omega_0} Re[\langle p, C(q, q, \bar{q})\rangle + 2\langle p, B(q, A^{-1}B(q, \bar{q}))\rangle + \langle p, B(\bar{q}, (2i\omega_0 E - A)B(q, q))\rangle].$$

This formula is very convenient for analyzing a Hopf bifurcation in an n-dimensional system with $n > 2$ [8]. Therefore, it is convenient to realize the computations of the first Lyapunov coefficient by using computer algebraic system like in Matlab, Mathematica, or Maple, etc.

2.2. Hopf bifurcation for $S_\pm$

Referring to [13], the equilibria $S_0 = (0,0,0)$ and $S_z = (0,0,z)$ for any $z \in \mathbb{R}$ are not Hopf point of system (1.1). Therefore, one only considers $S_\pm$. Due to the symmetry of the system, it suffices to consider S_+. The characteristic equation of Jaccobian matrix of system (1.1) at S_+ is given by

$$\lambda^3 + (a+b-d)\lambda^2 + (ab - \frac{bdf}{f+g})\lambda + 2abd = 0 \tag{2.1}$$

and all of its corresponding principal minors are as follows [13, p. 262]:

$$\Delta_1 = a+b-d, \quad \Delta_2 = (a+b-d)(ab - \frac{bdf}{f+g}) - 2abd, \quad \Delta_3 = 2abd\Delta_2.$$

Notice that the parameters a, b, d, f and g belong to the set

$$W = \{(a,b,d,f,g) \in \mathbb{R}^5 | a > 0, bd > 0, f \geq 0, g \geq 0, f+g > 0\}.$$

For convenience of discussion in the sequel, define the sets W_1 and W_2 as follows:

$$W_1 = \{(a,b,d,f,g) \in W : b < 0, d < 0\}, \quad W_2 = \{(a,b,d,f,g) \in W : b > 0, d > 0\}.$$

Then $W = W_1 \cup W_2$. The set W_2 can be further rewritten as $W_2 = W_{21} \cup W_{22} \cup W_{23}$, where

$$W_{21} = \{(a,b,d,f,g) \in W_2 : b \leq d-a\},$$

$$W_{22} = \{(a,b,d,f,g) \in W_2 : b > d-a, a > \tfrac{df}{f+g}\},$$

$$W_{23} = \{(a,b,d,f,g) \in W_2 : b > d-a, a \leq \tfrac{df}{f+g}\}.$$

Furthermore, divide the set W_{22} into the following subsets:

$$W_{22}^1 = \{(a,b,d,f,g) \in W_{22} : b < b_0\},$$

$$W_{22}^2 = \{(a,b,d,f,g) \in W_{22} : b = b_0\},$$

$$W_{22}^3 = \{(a,b,d,f,g) \in W_{22} : b > b_0\},$$

where $b_0 = d - a + \frac{2ad(f+g)}{a(f+g)-df}$. By the Routh-Hurwitz criterion, it is easy to get the following results.

Theorem 2.1. *S_+ is unstable when $(a,b,d,f,g) \in W_1 \cup W_{21} \cup W_{23} \cup W_{22}^1$ and is asymptotically stable when $(a,b,d,f,g) \in W_{22}^3$.*

One notices that there exists a bifurcation occurrence in system (1.1) when $(a,b,d,f,g) \in W_{22}^2$. Therefore, the kind of bifurcation and its stability are discussed in the rest of the section. First of all, the following lemma holds.

Lemma 2.1. *Consider b as a bifurcation parameter. Then, for $(a,b,d,g,f) \in W_{22}^2$, system (1.1) undergoes a Pioncaré–Andronov–Hopf bifurcation (or simply a Hopf bifurcation) at S_+.*

Proof. For $(a, b, d, g, f) \in W_{22}^2$, it follows that Eq. (2.1) has one negative real root $\lambda_1 = -\frac{2ad(f+g)}{a(f+g)-df}$ and a pair of conjugate purely imaginary roots $\lambda_{2,3} = \pm\omega_0 i$ with $\omega_0 = \sqrt{\frac{a(3d-a)(f+g)+df(a-d)}{f+g}}$. Taking into account that $Re(\lambda_{2,3}) = 0$ at $b = b_0$, one obtains

$$\left.\frac{dRe(\lambda_{2,3})}{db}\right|_{b=b_0} = -\frac{\omega_0^2}{2[\omega_0^2 + (\frac{2ad(f+g)}{a(f+g)-df})^2]} < 0.$$

The above inequality reads that, for $(a, b, d, g, f) \in W_{22}^2$, the transversal condition always holds. Also, $Re(\lambda_1) = \lambda_1 < 0$. Therefore, all conditions for Hopf bifurcation [8] to occur are met. Consequently, the Hopf bifurcation happens at S_+.

The proof for this lemma is over. □

Next, one studies the stability of the periodic orbit bifurcated from S_+ for the parameters $(a, b, d, g, f) \in W_{22}^2$, by using Projection Method. The following result is true.

Lemma 2.2. *Consider the parameters $(a, b, d, g, f) \in W_{22}^2$. Then the first Lyapunov coefficient of system (1.1) at S_+ is given by*

$$l_1(a, d, f, g) = -\frac{N}{D} \tag{2.2}$$

where

$$\begin{aligned} N = {} & a^2(a-d)(f+g)(a^2f + a^2g + d^2f - 4adf - 3adg)(af + ag - df) \\ & (6a^4f^4 + 24a^4f^3g + 36a^4f^2g^2 + 24a^4fg^3 + 6a^4g^4 - 34a^3df^4 \\ & -117a^3df^3g - 147a^3df^2g^2 - 79a^3dfg^3 - 15a^3dg^4 + 54a^2d^2f^4 \\ & +141a^2d^2f^3g + 120a^2d^2f^2g^2 + 33a^2d^2fg^3 - 34ad^3f^4 - 61ad^3f^3g \\ & -29ad^3f^2g^2 - 2ad^3fg^3 + 6d^4f^4 + 5d^4f^3g) \end{aligned}$$

and

$$\begin{aligned} D = {} & 2d(a^4f^3 + 3a^4f^2g + 3a^4fg^2 + a^4g^3 - 6a^3df^3 - 15a^3df^2g - 12a^3dfg^2 \\ & -3a^3dg^3 + 6a^2d^2f^3 + 5a^2d^2f^2g - 5a^2d^2fg^2 - 4a^2d^2g^3 - 6ad^3f^3 \\ & -5ad^3f^2g + d^4f^3)(a^4f^3 + 3a^4f^2g + 3a^4fg^2 + a^4g^3 - 6a^3df^3 \\ & -15a^3df^2g - 12a^3dfg^2 - 3a^3dg^3 + 9a^2d^2f^3 + 14a^2d^2f^2g + 4a^2d^2fg^2 \\ & -a^2d^2g^3 - 6ad^3f^3 - 5ad^3f^2g + d^4f^3). \end{aligned}$$

Proof. When $b = b_0$, one has $S_+ = (\sqrt{\frac{b_0d}{f+g}}, \sqrt{\frac{b_0d}{f+g}}, d)$. Take the change of the variables $x_1 = x - \sqrt{\frac{b_0d}{f+g}}$, $x_2 = y - \sqrt{\frac{b_0d}{f+g}}$ and $x_3 = z - d$, which transforms S_+

to S_0 and system (1.1) into

$$\begin{cases} \dot{x_1} = a(x_2 - x_1), \\ \dot{x_2} = -dx_1 + dx_2 - \sqrt{\frac{b_0 d}{f+g}} x_3 - x_1 x_3, \\ \dot{x_3} = (2f+g)\sqrt{\frac{b_0 d}{f+g}} x_1 + g\sqrt{\frac{b_0 d}{f+g}} x_2 - b_0 x_3 + f x_1^2 + g x_1 x_2. \end{cases}$$

Denoting $u = \sqrt{\frac{b_0 d}{f+g}}$ and noticing $\omega_0 = \sqrt{\frac{a(3d-a)(f+g)+df(a-d)}{f+g}}$, one has

$$A = \begin{pmatrix} -a & a & 0 \\ -d & d & u \\ (2f+g)u & gu & -b_0 \end{pmatrix}.$$

It is easy to derive that A has a real eigenvalue $\lambda_1 = -\frac{2ad(f+g)}{a(f+g)-df}$ and a pair of purely imaginary eigenvalues $\lambda_{2,3} = \pm\omega_0 i$.

It follows from some tedious calculations that

$$p = \frac{1}{L}\begin{pmatrix} \frac{b_0 d + \omega_0^2 + \omega_0(b_0 - d)i}{a} - \frac{b_0 dg}{a(f+g)} \\ -b_0 + \omega_0 i \\ u \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} a \\ a + \omega_0 i \\ \frac{\omega_0^2 + \omega_0(a-d)i}{u} \end{pmatrix}$$

satisfy $Aq = i\omega_0 q, \quad A^T p = -i\omega_0 p, \quad \langle p, q\rangle = \sum_{i=1}^3 \bar{p}_i q_i = 1$, where

$$L = b_0(d-a) + 3\omega_0^3 - \frac{b_0 dg}{f+g} + 2\omega_0(a - b_0 - d)\omega_0 i.$$

By some further computations we have

$$h_{11} = \begin{pmatrix} -\frac{a^2}{u} - \frac{ab_0\omega_0^2}{u^3(f+g)} \\ -\frac{a^2}{u} - \frac{ab_0\omega_0^2}{u^3(f+g)} \\ -\frac{2a\omega_0^2}{u^2} \end{pmatrix}$$

and

$$h_{20} = \frac{1}{M}\begin{pmatrix} -auK_1 - a(b_0 + 2\omega_0 i)K_2 \\ -(a + 2\omega_0 i)(b + 2\omega_0 i)K_2 - u(a + 2\omega_0 i)K_1 \\ -2u(af + ag + g\omega_0 i)K_2 - 2\omega_0[(d-a)i + 2\omega_0]K_1 \end{pmatrix},$$

where

$$M = 4(d - b_0 - a)\omega_0^2 + 2au^2(f+g) + 2(gu^2 - b_0 d + ab_0 - 4\omega_0^2)\omega_0 i,$$

$$K_1 = 2a(af + ag + g\omega_0 i), \quad K_2 = \frac{2a\omega_0(\omega_0 - ai + di)}{u}.$$

Carrying out some calculations as in [8], one gets the first Lyapunov coefficient which is just given by (2.2).

It remains only to verify the transversal condition of the Hopf bifurcation, which holds in view of Lemma 2.1. $\square$

Now denote

$$(W_{22}^2)^+ = \{(a,d,f,g) \in W_{22}^2 | l_1(a,d,f,g) > 0\},$$

$$(W_{22}^2)^0 = \{(a,d,f,g) \in W_{22}^2 | l_1(a,d,f,g) = 0\},$$

$$(W_{22}^2)^- = \{(a,d,f,g) \in W_{22}^2 | l_1(a,d,f,g) < 0\}.$$

Noticing that $l_1(1,2,1,12) = 1.7033 > 0$, $l_1(1,1,1,1) = 0$, $l_1(3,2,1,12) = -9.9606 < 0$ and the continuation of $l_1(a,d,f,g)$ in W_{22}^2, it is easy to obtain that $(W_{22}^2)^+$, $(W_{22}^2)^0$ and $(W_{22}^2)^-$ are all nonempty, see Figs. 1–3.

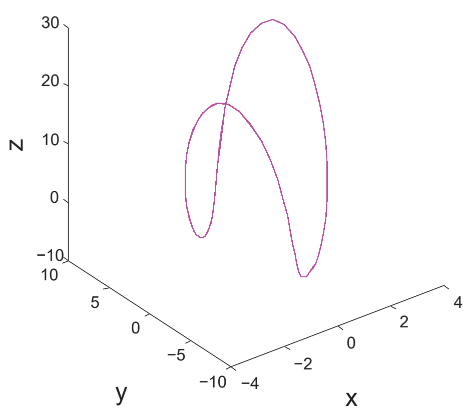


Figure 1. Phase portrait of system (1.1) when $(a,b,d,f,g) = (1,2,6,2,18)$ and $(x_0,y_0,z_0) = (-2.0618, -4.7017, 25.6000)$. This figure illustrates that system (1.1) has a unstable limit cycle bifurcating from $S_\pm$ when $(a,d,f,g) \in W_{22}^2$ and $l_1 > 0$.

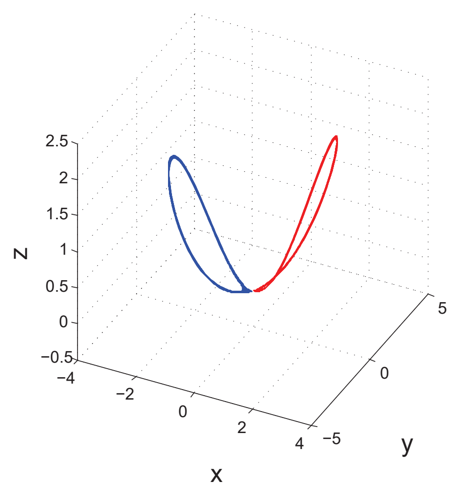


Figure 2. Phase portrait of system (1.1) when $(a,b,d,f,g) = (1,1,4,1,1)$ and $(x_0,y_0,z_0) = (\pm 3.14 \times 1e-4, \pm 1.618 \times 1e-4, \pm 2.718 \times 1e-4)$. This figure shows that system (1.1) has a degenerate limit cycle bifurcating from $S_\pm$ when $(a,d,f,g) \in W_{22}^2$ and $l_1 = 0$.

Summarizing the above discussions one obtains the main result of this section as follows.

Theorem 2.2. *System (1.1) undergoes a Pioncaré–Andronov–Hopf bifurcation at* S_+ *for* $(a,d,f,g) \in W_{22}^2$. *More precisely, for* $(a,d,f,g) \in (W_{22}^2)^-$, *the Hopf bifurcation is stable; for* $(a,d,f,g) \in (W_{22}^2)^+$, *the Hopf bifurcation is unstable. Namely, for each* $b > b_0$, *but close to* b_0, *there exists an unstable closed orbit near the asymptotically stable equilibrium point* S_+; *for* $(a,d,f,g) \in (W_{22}^2)^0$, *the Hopf bifurcation is degenerate, and the second or the third or even more higher order Lyapunov coefficients needs computing to determine the stability of the bifurcated periodic orbit. Symmetrically, there are the same results at* S_- *for* $(a,d,f,g) \in W_{22}^2$.

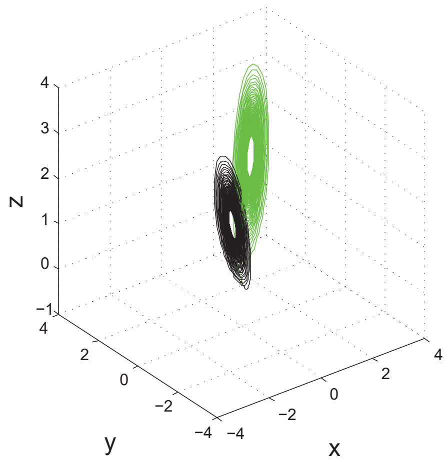


Figure 3. Phase portrait of system (1.1) when $(a, b, d, f, g) = (2.4, 2, 5.6, 2, 3)$ and $(x_0, y_0, z_0) = (\pm 3.14 \times 1e-4, \pm 1.618 \times 1e-4, \pm 2.718 \times 1e-4)$. This figure implies that system (1.1) has a stable limit cycle bifurcating from $S_\pm$ when $(a, d, f, g) \in W_{22}^2$ and $l_1 < 0$.

3. Behavior of S_z

In this section, one studies the dynamics of non-isolated equilibria $S_z = (0, 0, z)$ for any $z \in \mathbb{R}$. First of all, the characteristic polynomial of system (1.1) at any one S_z is

$$p(\lambda) = \lambda(\lambda^2 + (a-d)\lambda - a(d-z)).$$

Therefore the eigenvalues are given by

$$\lambda_1 = 0, \quad \lambda_{2,3} = \frac{d - a \pm \sqrt{(a-d)^2 + 4a(d-z)}}{2}$$

with the corresponding eigenvectors

$$v_1 = (0, 0, 1), \quad v_{2,3} = (a, \frac{a + d \pm \sqrt{(a-d)^2 + 4a(d-z)}}{2}, 0).$$

Hereout, the following statement holds.

Theorem 3.1. *When $b = 0$, system (1.1) has non-isolated equilibria S_z for any $z \in \mathbb{R}$. Moreover, the local dynamical behaviors of any one are given in the following Table 1.*

4. Singularly degenerate heteroclinic cycle

Recall that a singularly degenerate heteroclinic cycle consists of an invariant set formed by a line of equilibria together with a heteroclinic orbit connecting two of the equilibria. Kokubu and Roussarie [9] first studied this kind of cycle when considering the classic Lorenz system. In that known literature, they gave the following result.

Table 1. The dynamical behaviors of non-isolated equilibria E_z of system (1.1).

$a-d$	z	Property of E_z
<0	$(-\infty, d)$	a 1D W^s_{loc}, a 1D W^c_{loc} and a 1D W^u_{loc}
	d	a 2D W^c_{loc} and a 1D W^u_{loc}
	$(d, +\infty)$	a 1D W^c_{loc} and a 2D W^u_{loc}
$=0$	$(-\infty, d)$	a 1D W^s_{loc}, a 1D W^c_{loc} and a 1D W^u_{loc}
	d	a 3D W^c_{loc}
	$(d, +\infty)$	fold-Hopf bifurcation may occur
>0	$(-\infty, d)$	a 1D W^s_{loc}, a 1D W^c_{loc} and a 1D W^u_{loc}
	d	a 1D W^s_{loc} and a 2D W^c_{loc}
	$(d, +\infty)$	a 2D W^s_{loc} and a 1D W^c_{loc}

Lemma 4.1. *Consider the system*

$$\dot{x} = y, \dot{y} = Ax - By - xz - x^3, \dot{z} = x^2 - \beta z \quad \text{with} \quad \beta = 0. \tag{4.1}$$

Then, for any $B_* > 0$ which is smaller than some $B_0 \in (5/4,\ 2)$ and for sufficiently large $A_* > 0$, there exists $G_* = O(A_*, B_*)$ with $G_* = O(A_*)$ uniformly in B_* as $A_* \to \infty$, such that the system (4.1) with these parameter values possesses a singularly degenerate heteroclinic circle connecting $(0, 0, 0)$ and $(0, 0, G_*)$.

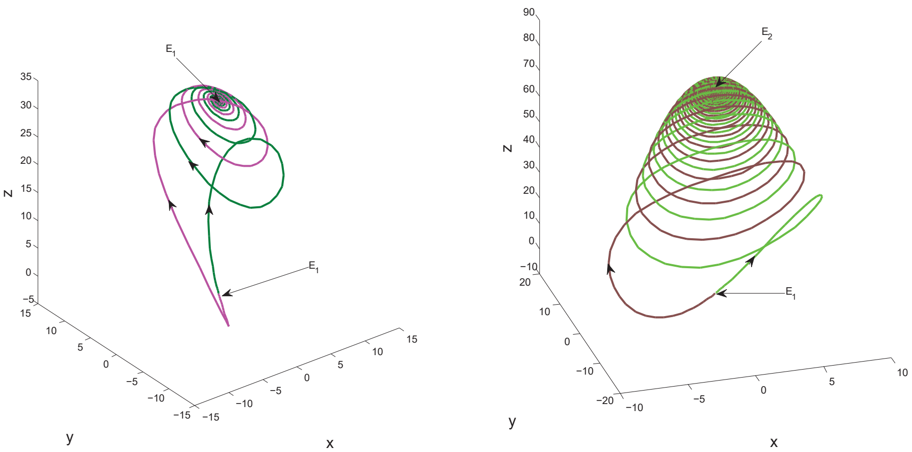
Later in 2009, Messias [20] further proved that the Lorenz system has a set of infinitely many singularly degenerate heteroclinic cycles combining analytical and numerical techniques. Employing these methods, many other chaotic systems [3,10,12,15,33,34] have been discovered to have such kind of property. Furthermore, numerical simulation illustrates that many chaotic attractors can be found near the singularly degenerate heteroclinic cycles, which maybe be another new route to chaos. Considering both the form and the existence of line equilibria S_z of system (1.1), one can not help asking the question "Does this kind of cycle exist in system (1.1)? " In this section, the positive answer will be given.

Firstly, numerical simulation shows that singularly degenerate heteroclinic cycle exits in system (1.1) with some appropriate choice of the parameters and initial values, see Fig. 4.

Secondly, using Lemma 4.1, the conclusion for system (1.1) is formulated on the existence of singularly degenerate heteroclinic cycle as follows.

Theorem 4.1. *Assume that $a > d$, $b = 0$ and $g > 0$. Then, for any $B_* = \frac{g(a-d)}{2a(f+g)}$ smaller than some $B_0 \in (5/4,\ 2)$ and $A_* = \frac{dg^2}{4a(f+g)^2}$ sufficiently large, there exists $G_* = O(A_*, B_*)$ with $G_* = O(A_*)$ uniformly in B_* as $A_* \to \infty$, such that system (1.1) with these parameter values possesses a singularly degenerate heteroclinic circle connecting $(0, 0, 0)$ and $(0, 0, G_*)$.*

Proof. For system (1.1) with these parameter values make the following change of variables $X = \mp\frac{g}{2a(f+g)}\sqrt{\frac{g}{2}}x$, $Y = \pm\frac{g^2}{4a(f+g)^2}\sqrt{\frac{g}{2}}(x-y)$, $Z = \frac{g^2}{4a(f+g)^2}z - \frac{g^3}{8a^2(f+g)^2}x^2$



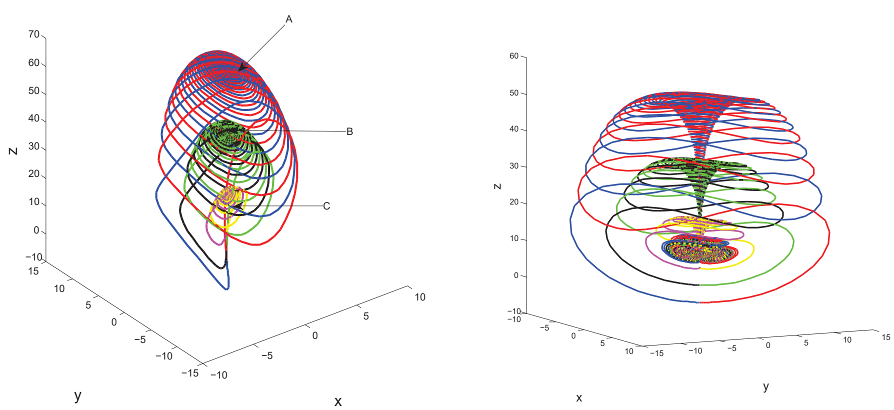
(a) $a = 190$ (b) $a = 204.9$

Figure 4. Phase portraits of system (1.1) when (a) $(a, d, b, f, g) = (9, 6, 0, 1, 0)$, (b) $(a, d, b, f, g) = (9, 8, 0, 1, 1)$, and initial values $(x_0, y_0, z_0) = (\pm 1.618 \times 1e-3, \pm 1.618 \times 1e-3, -1)$.

and $\tau = \frac{2a(f+g)}{g}t$. Then it becomes

$$\begin{cases} \dot{X} = \frac{dX}{d\tau} = Y, \\ \dot{Y} = \frac{dY}{d\tau} = \frac{dg^2}{4a(f+g)^2}X - \frac{g(a-d)}{2a(f+g)}Y - XZ - X^3, \\ \dot{Z} = \frac{dZ}{d\tau} = X^2. \end{cases} \tag{4.2}$$

It follows from Lemma 4.1 that system (4.2) with these parameter values, which is topologically equivalent to system (1.1), possesses a singularly degenerate heteroclinic circle connecting $(0, 0, 0)$ and $(0, 0, G_*)$. So, the proof is over. $\square$



(a) (b)

Figure 5. Phase portraits of system (1.1) when $(a, d, f, g) = (9, 8, -2, 4)$, (a) $b = 0$, (b) $b = 0.1$, and the initial values $(x_0, y_0, z_0) = (\pm 1.618 \times 1e-3, \pm 1.618 \times 1e-3, z)$, (A) $z = 5$, (B) $z = 0$, (C) $z = -5$. These figures illustrate that system (1.1) has infinite many singularly degenerate heteroclinic cycles and corresponding chaotic attractors.

Thirdly, combining Theorem 4.1 and the dynamics of S_z, one has the numerical result as follows.

Numerical Result 4.1. For $a > d$, $b = 0$ and $g > 0$, the 1D unstable manifold $W^u(E_1)$ of each normally hyperbolic saddle-like $E_1 = (0, 0, z_1)$ ($z_1 \in (-\infty, d)$) tends to one of the normal hyperbolic stable focus-like $E_2 = (0, 0, z_2)$ ($z_2 \in (d, +\infty)$) given in Theorem 3.1, forming singularly degenerate heteroclinic cycles (see Fig. 5 (a)).

Thereout, one has the following statement.

Theorem 4.2. *Assume that $a > d$, $b = 0$ and $g > 0$. Then system (1.1) has infinitely many singularly degenerate heteroclinic cycles.*

Finally, numerical simulation illustrates that chaotic attractors can bifurcate from the singularly degenerate heteroclinic cycles under a small perturbation of parameter b, see Fig. 5 (b).

5. Nonexistence of homoclinic orbit

In this section, we consider the nonexistence of homoclinic orbit of system (1.1). For a given solution (x, y, z) of system (1.1), set $Q = z - \frac{g}{2a}x^2$. Then $\dot{Q} = -bQ - \frac{bg-2a(f+g)}{2a}x^2$.

Theorem 5.1. *Assume that $bg > 2a(f + g)$ and $a > d > 0$. Then any one solution of system (1.1) is globally asymptotically stable. And hence there are no homoclinic orbits in system (1.1).*

Proof. $bg > 2a(f + g)$ implies $b > 0$ and $g > 0$. Let (x, y, z) be any one solution of system (1.1). Put

$$V = \dot{x}^2 + \frac{a^2}{b[bg - 2a(f + g)]}\dot{Q}^2 + \frac{a(f + g)}{2b}(x^2 - \frac{bd}{(f + g)})^2.$$

Then, noticing that $\ddot{x} = -(a-d)\dot{x}+ax(d-z)$, calculating the derivative of V w.r.t. time t along the solution of system (1.1) leads to

$$\frac{dV}{dt} = -2(a - d)\dot{x}^2 - \frac{2a^2}{bg - 2a(f + g)}\dot{Q}^2 \leq 0.$$

Furthermore,

$$\dot{V} = 0 \Leftrightarrow \dot{x} = \dot{Q} = 0 \Leftrightarrow \dot{x} = \dot{y} = \dot{z} = 0 \Leftrightarrow (x, y, z)$$

is an equilibrium solution. So, according to the LaSalle theorem [5], the solution is globally asymptotically stable. So, the homoclinic orbit does not exist in system (1.1). □

Theorem 5.2. *Assume that $bg = 2a(f + g)$ and $a > d > 0$. Then system (1.1) has no homoclinic orbits.*

Proof. Note $\dot{Q} = -\frac{2a(f+g)}{g}Q$ for $bg = 2a(f + g)$. Thus, $Q(t) = Q(0)e^{-\frac{2a(f+g)}{g}t}$. In particular, on γ^+ (which is defined in the sequel), $Q(t) \equiv 0$, so that

$$\ddot{x} + (a - d)\dot{x} = ax(d - \frac{g}{2a}x^2).$$

Set

$$V = \dot{x}^2 + \frac{a(f+g)}{2b}(x^2 - \frac{2ad}{g})^2.$$

Then $\dot{V} = -2(a-d)\dot{x}^2 \le 0$ and, therefore, $\gamma^+ \to S_+$ as $t \to +\infty$. (note that the set $\{(x,y,z)|V=0\}$ consists of two disjoint domains with S_+ and S_- in each of them). For any other trajectories, $Q(t) \to 0$ $(t \to +\infty)$. One can show that every trajectory has to tend to one of its stationary points and the details are omitted. □

Accordingly, combining with the results in [13], one can derive the main consequence in this section as follows.

Theorem 5.3. *Assume that $bg \ge 2a(f+g)$ and $a > d > 0$. Then system (1.1) has no homoclinic orbits associated with any stationary points. In addition, as $t \to +\infty$, every trajectory has a limit being one of the stationary points of system (1.1).*

6. Existence of herteroclinic orbit

In this section one employs the method in [2,11,13,14,16–18,24–27,29–31] to prove that system (1.1) has only two herteroclinic orbits when $bg = 2a(f+g)$ and $a > d > 0$.

6.1. Existence of herteroclinic orbit for the case $bg = 2a(f+g)$ and $a > d > 0$

For the convenience of statement, we introduce some more notations. Denote by $p(t;q_0) = (x(t;q_0), y(t;q_0), z(t;q_0))$ the solution of system (1.1) with the initial point $q_0 \in \mathbb{R}^3$. Let $p_+(t) = (x_+(t), y_+(t), z_+(t))$ be a solution of system (1.1) on W^u such that x_+ is positive for large negative t. Let p_- be the reflection of p_+ w.r.t. the z-axis, i.e., $p_-(t) = (-x_+(t), -y_+(t), z_+(t))$. Let $\gamma^+ = \{p_+(t)|t \in \mathbb{R}\}$ be the positive unstable manifold of system (1.1) at the origin and let $\gamma^- = \{p_-(t)|t \in \mathbb{R}\}$.

We now study the case where the parameters a, b, d, f, g satisfy the conditions that $bg = 2a(f+g)$ and $a > d > 0$. We first have the following result.

Lemma 6.1. *Consider system (1.1). Assume that $bg = 2a(f+g)$ and $a > d > 0$. Set*

$$V(x,y,z) = \frac{2a(f+g)}{g}(y-x)^2 + \frac{1}{2a(f+g)}((f+g)x^2 - bd)^2.$$

Then:

(i) If $p(t;q_0)$ is bounded as $t \to -\infty$, then $x^2(t;q_0) \equiv \frac{2a}{g}z(t;q_0)$.

(ii) If $x^2(t;q_0) \equiv \frac{2a}{g}z(t;q_0)$, then the derivative of V along the solution $p(t;q_0)$ is

$$\frac{d}{dt}V(p(t;q_0)) = -\frac{4a(f+g)(a-d)}{g}[y(t;q_0) - x(t;q_0)]^2 \le 0.$$

(iii) If $x^2(t;q_0) \equiv \frac{2a}{g}z(t;q_0)$ and if there exist t_1 and t_2 with $t_1 < t_2$ such that $V(p(t_1;q_0)) = V(p(t_2;q_0))$, then q_0 is one of the equilibria of system (1.1).

(iv) If $\lim_{t\to-\infty} p(t;q_0) = S_0$ and $x(t_3;q_0) > 0$ for some t_3, then $V(S_0) > V(p(t;q_0))$ and $x(t;q_0) > 0$ for all $t \in \mathbb{R}$. Consequently, $q_0 \in \gamma^+$.

Proof. (i) Set $Q(x,y,z) = x^2 - \frac{2a}{g}z$. Then, from Eq. (1.1), one has the derivative of Q along the solution $p(t;q_0)$ as follows: $\frac{d}{dt}Q(p(t;q_0)) = -\frac{2a(f+g)}{g}Q(p(t;q_0))$. Consequently, one obtains

$$Q(p(t;q_0)) = Q(p(\tau;q_0))e^{-\frac{2a(f+g)}{g}(t-\tau)} \quad \text{for all} \quad \tau, t \in \mathbb{R}. \tag{6.1}$$

Since $p(\tau;q_0)$ is bounded as $\tau \to -\infty$, Eq. (6.1) yields to

$$Q(p(t;q_0)) \equiv 0, \quad i.e., \quad x^2(t;q_0) \equiv \frac{2a}{g}z(t;q_0).$$

(ii) The assertion follows from the equality $x^2(t;q_0) \equiv \frac{2a}{g}z(t;q_0)$ and Eq. (1.1).

(iii) Assertion (ii) implies that, for all $t \in (t_1,t_2)$, $\frac{d}{dt}V(p(t,q_0)) = 0$, i.e.

$$y(t;q_0) - x(t;q_0) = 0. \tag{6.2}$$

It follows from the first equation of (1.1), Eq. (6.2) and the equality $x^2(t;q_0) \equiv \frac{2a}{g}z(t;q_0)$ that

$$x'(t;q_0) \equiv y'(t;q_0) \equiv z'(t;q_0) \equiv 0$$

for all $t \in (t_1,t_2)$. Hence, q_0 is just one of the equilibria of the system.

(iv) We first prove $V(S_0) > V(p(t;q_0))$ for all $t \in \mathbb{R}$. Suppose that $V(S_0) \leq V(p(t_0;q_0))$ for some $t_0 \in \mathbb{R}$. Then the last three results imply that q_0 is one of the equilibria of this system. This contradicts the facts that $\lim_{t\to-\infty} p(t;q_0) = 0$ and $x(t_3;q_0) > 0$. Hence, it follows that $V(S_0) > V(p(t;q_0))$ for all $t \in \mathbb{R}$. Now, we prove $x(t;q_0) > 0$ for all t. Suppose $x(t_4;q_0) \leq 0$ for some $t_4 \in \mathbb{R}$. Since $x(t_3;q_0) > 0$, one has $x(t_5;q_0) = 0$ for some t_5. Using $V(S_0) > V(p(t;q_0))$ for all $t \in \mathbb{R}$, one gets

$$p(t_5;q_0) \in \{(x,y,z)|V(x,y,z) < V(S_0)\} \cap \{(x,y,z)|x = 0\}.$$

On the other hand,

$$\begin{aligned} &\{(x,y,z)|V(x,y,z) < V(S_0)\} \cap \{(x,y,z)|x = 0\} \\ &= \{(0,y,z)| \tfrac{2a(f+g)}{g}y^2 + \tfrac{b^2d^2}{2a(f+g)} < \tfrac{b^2d^2}{2a(f+g)}\} \\ &= \emptyset, \end{aligned}$$

which is a contradiction. Hence, $x(t;q_0) > 0$ for all $t \in \mathbb{R}$. □

Theorem 6.1. *Assume that $bg = 2a(f+g)$ and $a > d > 0$. If the negative orbit from a point q_0 is bounded, then the solution $p(t,q_0)$ approaches, as $t \to -\infty$, one of the equilibria of system (1.1). Consequently, system (1.1) has no closed orbits.*

Proof. Assertions (i) and (ii) of Lemma 6.1 imply that the limit of $V(p(t;q_0))$ as $t \to -\infty$, denoted by $\Psi(q_0)$, exists. Let $q \in \alpha(q_0)$, the α-limit set of the system from q_0, i.e. there exists a sequence $\{t_n\}$ such that

$$\lim_{n\to+\infty} t_n = -\infty \quad \text{and} \quad \lim_{n\to+\infty} p(t_n;q_0) = q.$$

Then, for all $t \in \mathbb{R}$, the relation

$$p(t;q) = \lim_{n\to+\infty} p(t;p(t_n;q_0)) = \lim_{n\to+\infty} p(t+t_n;q_0)$$

leads to

$$\begin{cases} p(t;q) \quad \text{is bounded on} \quad \mathbb{R}, \\ V(p(t;q)) = \lim_{n\to+\infty} V(p(t+t_n;q_0)) = \Psi(q_0). \end{cases} \tag{6.3}$$

It follows from Lemma 6.1 that $q \in \{S_-, S_0, S_+\}$. Hence,

$$\alpha(q_0) \subseteq \{S_-, S_0, S_+\}.$$

Due to $\alpha(q_0)$ being connected, one has $\alpha(q_0) = \{S_-\}$, or $\alpha(q_0) = \{S_0\}$, or $\alpha(q_0) = \{S_+\}$, which means that $p(t;q_0)$ approaches one of the equilibria of the system as $t \to -\infty$. □

Theorem 6.2. *Consider system (1.1). Assume that $bg = 2a(f+g)$ and $a > d > 0$. Then:*

(i) The system has no homoclinic orbits.

(ii) The system has only two heteroclinic orbits: γ^+ joining S_0 and S_+ and γ^- joining S_0 and S_-.

Proof. (i) We prove that system (1.1) has neither homoclinic orbits nor heteroclinic orbits joining S_- and S_+. Assume that $p(t) = (x(t), y(t), z(t))$ is a homoclinic orbit of the system or a heteroclinic orbits joining S_- and S_+, i.e. $p(t) = (x(t), y(t), z(t))$ is a solution of the system such that

$$\lim_{t\to-\infty} p(t) = s^- \quad \text{and} \quad \lim_{t\to+\infty} p(t) = s^+,$$

where s^- and s^+ satisfy either

$$s^- = s^+ \in \{S_-, S_0, S_+\}$$

or

$$\{s^-, s^+\} = \{S_-, S_+\}.$$

According to Lemma 6.1 and the relation $V(s^-) = V(s^+)$, $p(t)$ is just one of the equilibria of the system.

Therefore, the system has neither homoclinic orbits nor heteroclinic orbits joining S_- and S_+.

(ii) We first prove that if system (1.1) has a heteroclinic orbit joining S_0 and S_+, then this orbit is just γ^+.

Let $p_1(t) = (x_1(t), y_1(t), z_1(t))$ be a solution of system (1.1) such that

$$\lim_{t\to-\infty} p_1(t) = s_1^- \quad \text{and} \quad \lim_{t\to+\infty} p_1(t) = s_1^+,$$

where s_1^- and s_1^+ satisfy $\{s_1^-, s_1^+\} = \{S_0, S_+\}$. Then, for all $t \in \mathbb{R}$, the assertions (i) and (ii) of Lemma 6.1 imply

$$V(s_1^-) \geq V(p_1(t)) \geq V(s_1^+).$$

Notice that $V(S_0) > V(S_+)$. Then one gets $s_1^- = S_0$ and $s_1^+ = S_+$, i.e.

$$\lim_{t\to-\infty} p_1(t) = S_0 \quad \text{and} \quad \lim_{t\to+\infty} p_1(t) = S_+, \tag{6.4}$$

which implies that $p_1(t) \in \gamma^+$ from the assertion (iv) of Lemma 6.1.

Finally, it remains to prove that γ^+ is a heteroclinic orbit joining S_0 and S_+, i.e. $\lim_{t\to+\infty} p_1(t) = S_+$. Using Lemma 6.1, one can get

$$\begin{cases} x_+^2(t) = \frac{2a}{g} z_+(t), \\ \frac{d}{dt} V(p_+(t)) = -\frac{4a(f+g)(a-d)}{g}[y_+(t) - x_+(t)]^2, \\ V(p_+(t)) < V(S_0) \quad \text{for all} \quad t \in \mathbb{R}, \\ x_+(t) > 0 \quad \text{for all} \quad t \in \mathbb{R}. \end{cases} \tag{6.5}$$

The second equation of (6.5) implies that the limit of $V(p_+(t))$ exists as $t \to +\infty$. Denote this limit by v. The third relation in (6.5) implies that $x_+(t)$, $y_+(t)$ and $z_+(t)$ are all bounded on $[0, +\infty)$; therefore, the set $\{p_+(t)|t \geq 0\}$ is bounded. Denote by Ω the ω-limit set of solution $p_+(t)$. Let $q \in \Omega$, i.e. there exists a sequence $\{t_n\}$ such that $\lim_{n\to+\infty} t_n = +\infty$ and $\lim_{n\to+\infty} p_+(t_n) = q$. Then, for all $t \in \mathbb{R}$, the relations

$$p(t;q) = \lim_{n\to+\infty} p(t; p_+(t_n)) = \lim_{n\to+\infty} p_+(t + t_n)$$

and

$$\begin{cases} p(t;q) \quad \text{is bounded on} \quad \mathbb{R}, \\ V(p(t;q)) = \lim_{n\to+\infty} V(p_+(t+t_n)) = v, \end{cases} \tag{6.6}$$

together with the assertions (i) and (iii) of Lemma 6.1 lead to the conclusion that $q \in \{S_-, S_0, S_+\}$. Hence, $\Omega \subseteq \{S_-, S_0, S_+\}$. Due to Ω being connected, one has $\Omega = S_-$, or $\Omega = S_0$, or $\Omega = S_+$. It follows that $\Omega \neq S_0$ from the assertion (ii) of Lemma 6.1, and $\Omega \neq S_-$ from the fourth relation of system (6.5). Therefore, $\Omega = S_+$, i.e. $\lim_{n\to+\infty} p_+(t) = S_+$. □

The numerical simulation also verifies that Theorem 6.2 holds, see Fig. 6.

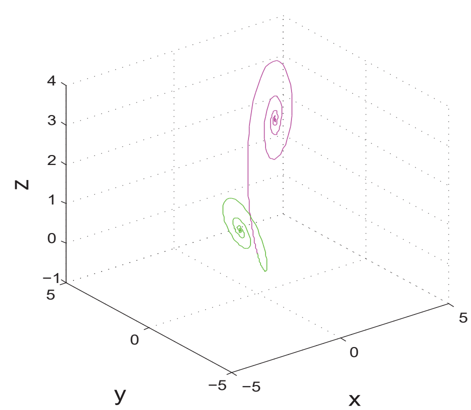


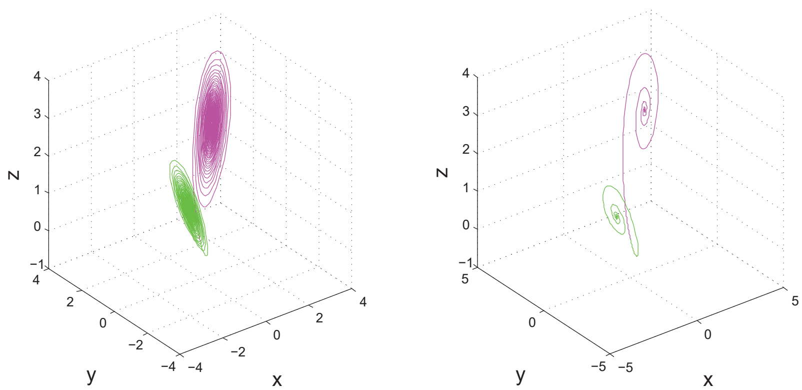
Fig. 6. Phase portrait of system (1.1) when $(a, b, d, f, g) = (3, 12, 2, 1, 1)$ and $(x_0, y_0, z_0) = (\pm 3.14 \times 1e-4, \pm 1.618 \times 1e-4, \pm 2.718 \times 1e-4)$. This figure illustrates that system (1.1) has two heteroclinic orbits for $bg = 2a(f+g)$ and $a > d > 0$.

6.2. Numerical simulations for other heteroclinic orbits

In this subsection, we study the existence of heteroclinic orbits in other parameter domain than the one $bg \geq 2a(f+g)$ and $a > d > 0$. First of all, it follows from Theorem 2.1 that $S_{\pm}$ are asymptotically stable equilibrium points when $(a,b,d,f,g) \in W_{22}^3$. Therefore, we consider the following four subcases.

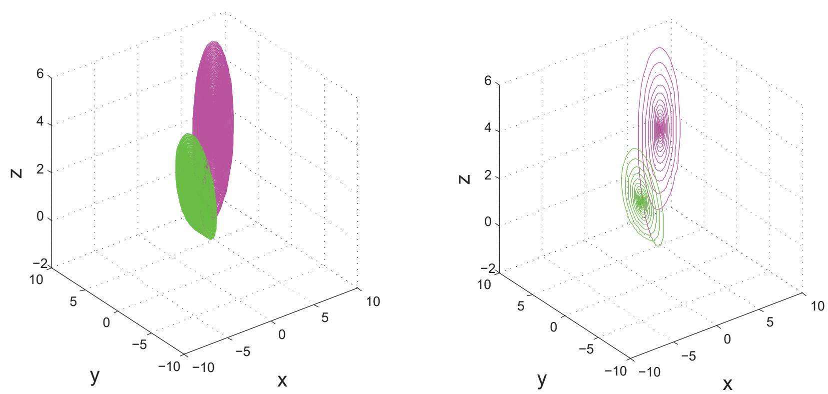
1. $a > d$, $bd > 0$, $\frac{2a(f+g)}{g} > b > \frac{2ad(f+g)}{(a-d)f+ag} + d - a > 0$,
2. $a = d$, $g > 0$, $b > \frac{2a(f+g)}{g}$,
3. $a < d$, $(a-d)f + ag > 0$, $b > 0$ and $b > \frac{2ad(f+g)}{(a-d)f+ag} + d - a > 0$,
4. $(a-d)f + ag > 0$, $\frac{2ad(f+g)}{(a-d)f+ag} + d - a \leq 0$.

The corresponding numerical simulations are shown in Figs. 7–10, implying that system (1.1) has two heteroclinic orbits to $S_{\pm}$ and S_0.



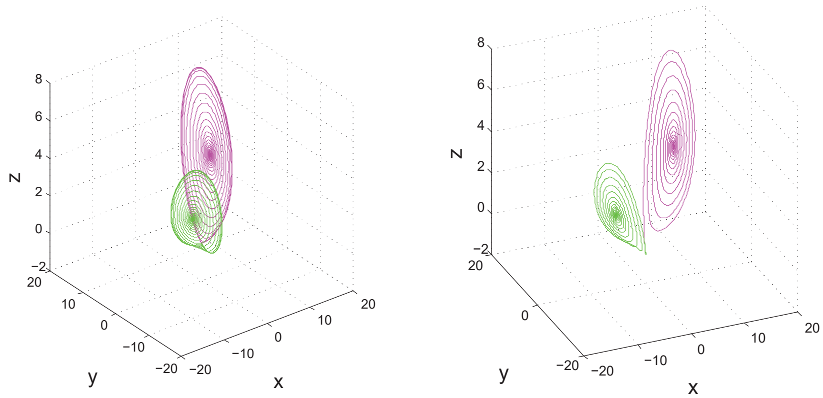
(a) $b = 5.1$ (b) $b = 11.9$

Figure 7. Phase portraits of system (1.1) when $(a,d,f,g) = (3,2,1,1)$ and b varies in $(5,12)$ with the initial conditions $(x_0, y_0, z_0) = (\pm 3.14 \times 1e-4, \pm 1.618 \times 1e-4, \pm 2.718 \times 1e-4)$.



(a) $b = 12.1$ (b) $b = 15$

Figure 8. Phase portraits of system (1.1) when $(a,d,f,g) = (3,3,1,1)$ and b varies in $(12,\infty)$ and $(x_0, y_0, z_0) = (\pm 3.14 \times 1e-4, \pm 1.618 \times 1e-4, \pm 2.718 \times 1e-4)$.



(a) $b = 47.73$

(b) $b = 60$

Figure 9. Phase portraits of system (1.1) when $(a, d, f, g) = (2, 3, 1, 1)$ and $b \in (47.73, \infty)$ and $(x_0, y_0, z_0) = (\pm 3.14 \times 1e - 4, \pm 1.618 \times 1e - 4, \pm 2.718 \times 1e - 4)$.

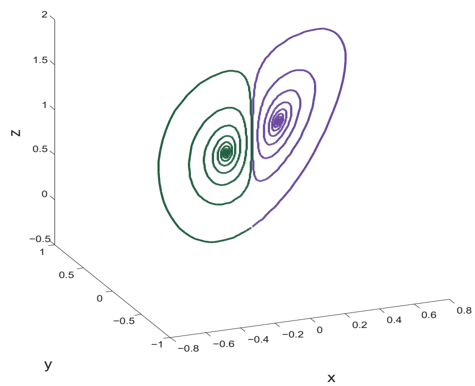


Figure 10. Phase portrait of system (1.1) with $(a, d, b, f, g) = (7, 1, 0.1, 1, 1)$ and $(x_0, y_0, z_0) = (\pm 3.14 \times 1e - 4, \pm 1.618 \times 1e - 4, \pm 2.718 \times 1e - 4)$.

6.3. Nonexistence of herteroclinic orbit

The above Theorem 6.2 answers the existence of herteroclinic orbit of system (1.1), then, a question naturally rises: When does system (1.1) has no herteroclinic orbits? One has the following result.

Theorem 6.3. *Assume that $bd(f + g) < 0$, then there are no heteroclinic orbits in system (1.1).*

Proof. When $bd(f + g) < 0$, system (1.1) has a single equilibrium point S_0. Therefore, there do not exist heteroclinic orbits of system (1.1). □

7. Conclusion

In this paper, a known Lorenz-like system is revisited and some of new and interesting dynamics are revealed. By using Project Method, the Hopf bifurcation at $S_\pm$

are deeper studied. The existence of infinitely many singularly heteroclinic cycles is rigorously proved combining theoretical analysis and numerical technique. For $bg \geq 2a(f+g)$ and $a > d > 0$, the system has been proved to have no homoclinic orbits. In particular, this system also has two and only two heteroclinic orbits in the case $bg = 2a(f+g)$ and $a > d > 0$, which complements the known result in [13]. What's more interesting, by numerical simulations, other heteroclinic orbits of this system are illustrated to exist in other parameter region than $bg \geq 2a(f+g)$ and $a > d > 0$. Hence, system (1.1) deserves further theoretical considering.

We hope, the present work will shed light on revealing of the true geometrical structure of the amazing original Lorenz attractor even on the forming mechanism of chaos.

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