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GLOBAL REGULARITY FOR 3D GENERALIZED HALL MAGNETO-HYDRODYNAMICS EQUATIONS

Baoquan Yuan† and Chaoying Li

Abstract For the 3D incompressible Hall magneto-hydrodynamics equations, global regularity of the weak solutions is not established so far. The major difficulty is that the dissipation given by the Laplacian operator is insufficient to control the nonlinearities. Wan obtained the global regularities of the 3D generalized Hall-MHD equations with critical and subcritical hyperdissipation in (*Global regularity for generalized Hall-magnetohydrodynamics systems*, Electron. J. Differential Equations, 2015, 2015(179), 1–18). We improve this slightly by making logarithmic reductions in the dissipation and still obtain the global regularity.

Keywords Hall magneto-hydrodynamics equations, global regularity, hyperdissipation, Littlewood-Paley decomposition.

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1. Introduction

This paper is concerned with the global regularity problem to the generalized incompressible Hall-MHD system of the form:

$$\begin{cases} \partial_t u + u\cdot\nabla u + \zeta_1^2 u = -\nabla p + b\cdot\nabla b, \\ \partial_t b + u\cdot\nabla b + \zeta_2^2 b = b\cdot\nabla u - \nabla\times((\nabla\times b)\times b), \\ \nabla\cdot u = \nabla\cdot b = 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x). \end{cases} \tag{1.1}$$

Here $t \geq 0, x \in \mathbb{R}^3$, $u(x,t), p(x,t), b(x,t)$ stand for the velocity vector, the scalar pressure and the magnetic vector, respectively. Where the Laplacian $-\Delta$ in the dissipation terms have been replaced by general multiplier operators with symbols given by m_1 and m_2, namely $\widehat{\zeta_1 u}(\xi) = m_1(\xi)\widehat{u}(\xi)$ and $\widehat{\zeta_2 b}(\xi) = m_2(\xi)\widehat{b}(\xi)$. When $\zeta_1^2 u = -\Delta u$, $\zeta_2^2 b = -\Delta b$, (1.1) becomes the standard incompressible Hall-MHD equations.

Hall-MHD system was derived strictly from Euler-Maxwell equations or kinetic model in [1], which played an important role in many physical problems, such as magnetic reconnection in space plasmas [11], star formation [5] and also neutron stars [20]. Hall-MHD system is known as the key to solving magnetic reconnection

†the corresponding author.
Email address: bqyuan@hpu.edu.cn (B. Yuan), cyli916@163.com (C. Li)
School of Mathematics and Information Science, Henan Polytechnic University, Century Avenue, Jiaozuo, Henan, 454000, China

happening in the case of large magnetic shear. Since the topological structure changed a lot, the affect of the Hall term due to the Ohm's law must be taken into account. For the physical background of the Hall-MHD, readers refer to [18] and references therein. Incompressible Hall-MHD equations play a dominant role in mathematical theory as well as physical applications. As early as 1960, Lighthill studied the Hall-MHD equations systematically in [18]. Afterwards, Acheritogaray et al. not only deduced the Hall-MHD system from Euler-Maxwell equations or kinetic model but also obtained the global existence of weak solutions in the periodic domain in [1]. Later, Chae, Degond and Liu in [8] established the global existence of weak solutions and the local well-posedness of classical solutions in the whole space to the Hall-MHD, as well as the blow-up criterion of the classical solutions and the global existence of the solutions for small initial data. Chae and Lee [9] improved the relevant results in [8], and acquired Serrin type blow-up criteria and the blow-up criterion in BMO space. Other scholars have also given a lot of regularity criteria in Lebesgue space, BMO space or Besov space in the works [12, 13, 16, 26, 30, 32] and references therein. Wang and Zuo studied the Hall-MHD system with partial viscosity in [27], and Fei and Xiang researched the Hall-MHD equations with horizontal dissipation in [14]. As same as the 3D Navier-Stokes equations, the regularity and uniqueness of weak solutions for the 3D Hall-MHD equations remain completely open. For lack of global well-posedness theory, the development of regularity criteria is a significant topic no matter in theory or in practice.

Here we consider the generalized case with $\alpha, \beta \geq 1$. Indeed, the investigations of these fractional operators have a long history in fluid mechanics. As regards related works, readers refer to [10, 23, 28, 29, 31] and references therein. To date, in the work [23], Wan obtained the global regularity of (1.1) with $\alpha \geq \frac{5}{4}, \beta \geq \frac{7}{4}$ for sufficiently smooth initial data, which was optimal by combining the scaling invariance with energy estimates for generalized Navier-Stokes system ($b = 0$) and simple Hall problem($u = 0$). Pan and Zhu [19] obtained a new regularity criterion for the generalized Hall-MHD system with $b \in (1/2, 1]$. Wu, Yu and Tang described the asymptotic behavior of the generalized Hall-MHD equations in [25]. Can we make a reduction to the optimal dissipation obtained by Wan [23] and still guarantee the global regularity? Tao [22] examined the hyperdissipative Navier-Stokes equations involving general Fourier multiplier operators, and he made a logarithmic reduction in the dissipation and still obtained a unique global solution. Tao's result was later improved by Barbato et al. [4]. Bian and Yuan [6] reduced the logarithmical supercritical dissipation in [22] by $\frac{(-\Delta)^{\frac{5}{4}}}{\log(2-\Delta)}u$ with the condition $\int_0^t \|u\|^2_{B^{\alpha}_{2,\infty}} \mathrm{d}s < +\infty$, and still established the global regularity of a generalized Navier-Stokes equations. Wu [24] generalized Tao's result [22] to the GMHD equations in which the condition $\beta \geq \frac{1}{2} + \frac{d}{4}$ was not required and was replaced by $\alpha \geq \frac{1}{2} + \frac{d}{4}, \beta > 0, \alpha + \beta \geq 1 + \frac{d}{2}$.

Motivated by the references mentioned above, we consider that whether the similar result can be derived for a generalized Hall-MHD equations. The answer is positive, and the key idea is to bring in the logarithmic factor. However, the condition that $\beta \geq \frac{7}{4}$ is indispensable because of the Hall term. Now we state our result.

Theorem 1.1. *Consider (1.1) with $\alpha \geq \frac{5}{4}$, $\beta \geq \frac{7}{4}$. Assume the initial data $(u_0, b_0) \in H^s(\mathbb{R}^3)$ with $s > \frac{5}{2}$. Assume the symbols m_1 and m_2 satisfy*

$$m_1(\xi) \geq \frac{|\xi|^\alpha}{g_1(\xi)} \qquad \text{and} \qquad m_2(\xi) \geq \frac{|\xi|^\beta}{g_2(\xi)},$$

and $g_1 \geq 1$ and $g_2 \geq 1$ are radially symmetric, nondecreasing and satisfy

$$\int_1^\infty \frac{1}{s(g_1^2(s)+g_2^2(s))^2}\,ds = +\infty.$$

Then the generalized Hall-MHD equations (1.1) has a unique global classical solution (u,b).

Remark 1.1. In particular, the result applies to the logarithmical supercritical dissipation

$$m_1(\xi) \geq \frac{|\xi|^\alpha}{[\log(2+|\xi|^2)]^{\frac14}} \qquad \text{and} \qquad m_2(\xi) \geq \frac{|\xi|^\beta}{[\log(2+|\xi|^2)]^{\frac14}}.$$

The article is organized as follows, in the second section, we give some notations and preliminaries on functional settings and some important inequalities. In the third section, we prove Theorem 1.1.

2. Preliminaries

First, we introduce the Littlewood-Paley decomposition and the definition of Besov spaces. Let $\mathcal{B} = \{\xi \in \mathbb{R}^d, |\xi| \leq \frac43\}$ and $\mathcal{C} = \{\xi \in \mathbb{R}^d, \frac34 \leq |\xi| \leq \frac83\}$. Choose two nonnegative radial functions $\chi, \varphi \in C_0^\infty(\mathbb{R}^d)$ supported in $\mathcal{B}$ and $\mathcal{C}$, let $\varphi(\xi) = \chi(\frac{\xi}{2}) - \chi(\xi)$, $\chi_j(\xi) = \chi(2^{-j}\xi), \varphi_j(\xi) = \varphi(2^{-j}\xi)$ for $j \in \mathbb{Z}$, such that

$$\begin{aligned}\chi(\xi) + \sum_{j\geq 0}\varphi(2^{-j}\xi) =&1, \qquad \xi \in \mathbb{R}^d,\\ \sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi) =&1, \qquad \xi \in \mathbb{R}^d\backslash\{0\}.\end{aligned}$$

We denote $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$, where $\mathcal{F}^{-1}$ stands for the inverse Fourier transform. Write $h_j(x) = 2^{jd}h(2^jx), \tilde{h}_j(x) = 2^{jd}\tilde{h}(2^jx)$. As a consequence, for any $f \in \mathcal{S}'$, we have the Littlewood-Paley decomposition

$$\begin{aligned}f(x) =&\tilde{h} * f(x) + \sum_{j\geq 0} h_j * f(x),\\ f(x) =& \sum_{j=-\infty}^{\infty} h_j * f(x).\end{aligned}$$

Where $\mathcal{S}'$ denotes the class of Schwartz temperate distribution functions.

Define the Littlewood-Paley projection operators $\triangle_j$ and S_j as follows

$$\begin{aligned}\triangle_j f(x) =&\varphi(2^{-j}D)f = 2^{jd}\int_{\mathbb{R}^d} h(2^jy)f(x-y)\mathrm{d}y \qquad \text{for} \quad j \in \mathbb{Z},\\ S_jf(x) =& \sum_{k\leq j-1}\triangle_k f = \chi(2^{-j}D)f = 2^{jd}\int_{\mathbb{R}^d}\tilde{h}(2^jy)f(x-y)\mathrm{d}y \qquad \text{for} \quad j \in \mathbb{Z}.\end{aligned}$$

Naturally, $\triangle_j = S_j - S_{j-1}$ is a frequency projection to annulus $\{C_1 2^j \le |\xi| \le C_2 2^j\}$, and S_j is a frequency projection to the ball $\{|\xi| \le C2^j\}$. One can easily verify that with our choice of φ,

$$\triangle_j \triangle_k f = 0 \quad \text{if} \quad |j-k| \ge 2 \qquad \text{and} \qquad \triangle_j (S_{k-1} f \triangle_k f) = 0 \quad \text{if} \quad |j-k| \ge 5.$$

With the introduction of $\triangle_j$ and S_j, we recall the definition of the homogeneous Besov space.

Definition 2.1. Let $s \in \mathbb{R}$, $(p,q) \in [1,\infty]^2$, the homogeneous space $\dot{B}^s_{p,q}$ is defined by

$$\dot{B}^s_{p,q} = \{f \in \mathcal{S}'_0 : \|f\|_{\dot{B}^s_{p,q}} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{jsq} \|\triangle_j f\|^q_{L^p}\right)^{\frac{1}{q}}, & \text{for } 1 \le q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js} \|\triangle_j f\|_{L^p}, & \text{for } q = \infty. \end{cases}$$

Here $\mathcal{S}'_0$ denotes the dual space of

$$\mathcal{S}_0 = \{f \in \mathcal{S}(\mathbb{R}^d) : \partial^\alpha \hat{f}(0) = 0 : \forall \alpha \in \mathbb{N}^d multi-index\},$$

and can be identified by the quotient space of $\mathcal{S}'/\mathcal{P}$ with the polynomials space $\mathcal{P}$. In other words, two distributions in $\mathcal{S}'$ are identified as the same in $\mathcal{S}'_0$ if their difference is a polynomial. For details, readers refer to [3, 21].

Next, we state the definition of the inhomogeneous Besov space. Set

$$\triangle_j f = \begin{cases} 0, & \text{if } j \le -2, \\ \tilde{h} * f & \text{if } j = -1, \\ h_j * f, & \text{if } j = 0, 1, 2, \cdots \end{cases}$$

We caution that $\triangle_j$ with $j \le -1$ associated with the homogeneous Besov space $\dot{B}^s_{p,q}$ are different from those associated with the inhomogeneous Besov space $B^s_{p,q}$.

Definition 2.2. For $s > 0$, and $(p,q) \in [1,\infty]^2$, the inhomogeneous Besov space $B^s_{p,q}$ is defined as follows

$$B^s_{p,q} = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{B^s_{p,q}} < \infty\},$$

where

$$\|f\|_{B^s_{p,q}} = \begin{cases} \left(\sum_{j=-1}^{\infty} 2^{sjq} \|\triangle_j f\|^q_{L^p}\right)^{\frac{1}{q}}, & \text{for} \quad 1 \le q < \infty, \\ \sup_{-1 \le j < \infty} 2^{sj} \|\triangle_j f\|_{L^p}, & \text{for} \quad q = \infty. \end{cases}$$

If $s > 0$, then $B^s_{p,q} = L^p \bigcap \dot{B}^s_{p,q}$, whose norm is equivalent to

$$\|f\|_{B^s_{p,q}} \approx \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,q}}.$$

Additionally, when $p = q = 2$, the Besov space and Sobolev space are equivalent. That is

$$\dot{H}^s \approx \dot{B}^s_{2,2}, \quad H^s \approx B^s_{p,q}.$$

Bernstein's inequalities are useful tools in dealing with Fourier localized functions and these inequalities trade integrability for derivatives. The following proposition provides Bernstein type inequalities for fractional derivatives. The proof is an immediate consequence of Young's inequality (see [7] for details).

Proposition 2.1. *Let* $\alpha \geq 0,\ 1 \leq p \leq q \leq \infty$.

(i) If f satisfies

$$supp\hat{f} \subset \{\xi \in \mathbb{R}^d : |\xi| \leq K2^j\}$$

for some integer j and a constant $K > 0$, then

$$\|(-\triangle)^{\alpha} f\|_{L^q(\mathbb{R}^d)} \leq C_1 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)};$$

(ii) If f satisfies

$$supp\hat{f} \subset \{\xi \in \mathbb{R}^d : K_1 2^j \leq |\xi| \leq K_2 2^j\}$$

for some integer j and constants $0 < K_1 \leq K_2$, then

$$C_1 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \leq \|(-\triangle)^{\alpha} f\|_{L^q(\mathbb{R}^d)} \leq C_2 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 and C_2 are constants depending on α, p and q only.

For more details about Besov space such as some useful embedding relations, readers refer to [2,15]. Thanks to the Proposition 2.1, we can see that for any $s > 0$

$$\|f\|_{H^s} \approx \|f\|_{L^2} + (\sum_{j\geq 0} 2^{2js} \|\triangle_j f\|_{L^2}^2)^{\frac{1}{2}}, \tag{2.1}$$

which will be used in our proof.

To prove our theorem, we need a bound for a special type of commutators.

Lemma 2.1. *For any $j \geq -1$, $p \in [1, \infty]$,*

$$\|[\triangle_j, f\cdot\nabla]g\|_{L^p} \leq C 2^{j(-1+d(1-\frac{1}{\sigma}))} \|\nabla f\|_{L^q} \|\nabla g\|_{L^r} \|xh\|_{L^\sigma},$$

where $[\triangle_j, f\cdot\nabla]g$ denotes $\triangle_j(f\cdot\nabla g) - f\cdot\nabla\triangle_j g$, and q, r and σ satisfy

$$q, r, \sigma \in [1, \infty], \quad 1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r} + \frac{1}{\sigma}, \quad \frac{1}{r} + \frac{1}{\sigma} + \frac{1}{d} > 1.$$

In particular,

(i) for any $p \in [1, \infty]$,

$$\|[\triangle_j, f\cdot\nabla]g\|_{L^p} \leq C 2^{-j} \|\nabla f\|_{L^\infty} \|\nabla g\|_{L^p} \|xh\|_{L^1}; \tag{2.2}$$

(ii) for any $p \in [1, \infty]$, $r' \leq p$ and $\frac{1}{r} + \frac{1}{r'} = 1$,

$$\|[\triangle_j, f\cdot\nabla]g\|_{L^p} \leq C 2^{j(-1+\frac{d}{r})} \|\nabla f\|_{L^p} \|\nabla g\|_{L^r} \|xh\|_{L^{r'}}.$$

Remark 2.1. The proof of Lemma 2.1 can be found in Wu [24] and Hmidi et al. [17].

For convenience we recall the definition of Bony's para-product formula which gives the decomposition of the product $f \cdot g$ of two functions $f(x)$ and $g(x)$.

Definition 2.3. The para-product of f by g is defined by

$$T_g f = \sum_{j\in\mathbb{Z}} \sum_{i\le j-2} \triangle_i g \triangle_j f = \sum_{j\in\mathbb{Z}} S_{j-1} g \triangle_j f.$$

The remainder of f and g is defined by

$$R(f,g) = \sum_{j\in\mathbb{Z}} \sum_{|i-j|\le 1} \triangle_i g \triangle_j f.$$

Then Bony's para-product formula is

$$f\cdot g = T_g f + T_f g + R(f,g).$$

Throughout the paper, C stands for a real positive constant which may be different in each occurrence.

3. Proof of Theorem 1.1

Proof. First, we give the L^2 estimate. Taking the inner product of system (1.1) with u and b respectively in $L^2(\mathbb{R}^3)$, integrating and adding the resulting equations together, we get the following energy inequality

$$\|(u(t),b(t))\|_{L^2}^2 + 2\int_0^t \|\zeta_1 u\|_{L^2}^2 \mathrm{d}\tau + 2\int_0^t \|\zeta_2 b\|_{L^2}^2 \mathrm{d}\tau \le \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 \tag{3.1}$$

for almost every $t\ge 0$. Here we have used the cancelation property $\big((\nabla\times b)\times b\big)\cdot(\nabla\times b)=0$.

Next we establish the energy estimate in H^s. Applying the operator $\triangle_j$ to (1.1), one can write

$$\begin{aligned}
&\partial_t \triangle_j u + \zeta_1^2 \triangle_j u = -\triangle_j(\nabla p) - \triangle_j(u\cdot\nabla u) + \triangle_j(b\cdot\nabla b),\\
&\partial_t \triangle_j b + \zeta_2^2 \triangle_j b = -\triangle_j(u\cdot\nabla b) + \triangle_j(b\cdot\nabla u) - \triangle_j\big(\nabla\times\big((\nabla\times b)\times b\big)\big).
\end{aligned}\tag{3.2}$$

Taking the inner product of (3.2) with $\triangle_j u$ and $\triangle_j b$ respectively in $L^2(\mathbb{R}^3)$, by the divergence free condition and integrating by parts formula, we derive that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\triangle_j u\|_{L^2}^2 + \|\triangle_j b\|_{L^2}^2) + \|\zeta_1\triangle_j u\|_{L^2}^2 + \|\zeta_2\triangle_j b\|_{L^2}^2\\
&= -\int_{\mathbb{R}^3}[\triangle_j, u\cdot\nabla]u\cdot\triangle_j u\,\mathrm{d}x + \int_{\mathbb{R}^3}[\triangle_j, b\cdot\nabla]b\cdot\triangle_j u\,\mathrm{d}x - \int_{\mathbb{R}^3}[\triangle_j, u\cdot\nabla]b\cdot\triangle_j b\,\mathrm{d}x\\
&\quad + \int_{\mathbb{R}^3}[\triangle_j, b\cdot\nabla]u\cdot\triangle_j b\,\mathrm{d}x + \int_{\mathbb{R}^3}[\triangle_j, b\times](\nabla\times b)\cdot\triangle_j(\nabla\times b)\,\mathrm{d}x\\
&\le \|[\triangle_j, u\cdot\nabla]u\|_{L^2}\|\triangle_j u\|_{L^2} + \|[\triangle_j, b\cdot\nabla]b\|_{L^2}\|\triangle_j u\|_{L^2} + \|[\triangle_j, u\cdot\nabla]b\|_{L^2}\|\triangle_j b\|_{L^2}\\
&\quad + \|[\triangle_j, b\cdot\nabla]u\|_{L^2}\|\triangle_j b\|_{L^2} + \|[\triangle_j, b\times](\nabla\times b)\|_{L^2}\|\triangle_j(\nabla\times b)\|_{L^2}\\
&\triangleq \sum_{i=1}^5 L_i(t).
\end{aligned}\tag{3.3}$$

By the Bony's para-product decomposition, Hölder and Bernstein's inequalities and the commutator estimate (2.2) in Lemma 2.1, we deduce that

$$
\begin{aligned}
|L_1(t)| \leq & \sum_{|k-j|\leq 4} \|\triangle_j(S_{k-1}u\cdot\nabla\triangle_k u) - S_{k-1}u\cdot\nabla\triangle_j\triangle_k u\|_{L^2}\|\triangle_j u\|_{L^2} \\
& + \sum_{|k-j|\leq 4} \|\triangle_j(\triangle_k u\cdot\nabla S_{k-1}u) - \triangle_k u\cdot\nabla\triangle_j S_{k-1}u\|_{L^2}\|\triangle_j u\|_{L^2} \\
& + \sum_{k\geq j-3} \|\triangle_j\big(\nabla\cdot(\triangle_k u\otimes\tilde{\triangle}_k u)\big) - \nabla\cdot(\triangle_j\triangle_k u\otimes\tilde{\triangle}_k u)\|_{L^2}\|\triangle_j u\|_{L^2} \\
\leq & C2^{-j}\|\nabla S_{j-1}u\|_{L^\infty}\|\nabla\triangle_j u\|_{L^2}\|xh\|_{L^1}\|\triangle_j u\|_{L^2} \\
& + C2^j\sum_{k\geq j-3} 2^{-j}\|\nabla\triangle_k u\|_{L^\infty}\|\tilde{\triangle}_k u\|_{L^2}\|xh\|_{L^1}\|\triangle_j u\|_{L^2} \\
\leq & C\|\nabla S_{j-1}u\|_{L^\infty}\|\triangle_j u\|_{L^2}^2 + C\sum_{k\geq j-3}\|\nabla\triangle_k u\|_{L^\infty}\|\tilde{\triangle}_k u\|_{L^2}\|\triangle_j u\|_{L^2}, \quad (3.4)
\end{aligned}
$$

where $\tilde{\triangle}_k \triangleq \triangle_{k-1} + \triangle_k + \triangle_{k+1}$. Arguing similarly to the above inequality (3.4), we obtain

$$
\begin{aligned}
|L_2(t)| \leq & C\|\nabla S_{j-1}b\|_{L^\infty}\|\triangle_j b\|_{L^2}\|\triangle_j u\|_{L^2} \\
& + C\sum_{k\geq j-3}\|\nabla\triangle_k b\|_{L^\infty}\|\tilde{\triangle}_k b\|_{L^2}\|\triangle_j u\|_{L^2}, \quad (3.5)
\end{aligned}
$$

$$
\begin{aligned}
|L_3(t)| \leq & C\|\nabla S_{j-1}u\|_{L^\infty}\|\triangle_j b\|_{L^2}^2 + \|\nabla S_{j-1}b\|_{L^\infty}\|\triangle_j u\|_{L^2}\|\triangle_j b\|_{L^2} \\
& + C\sum_{k\geq j-3}\|\nabla\triangle_k b\|_{L^\infty}\|\tilde{\triangle}_k u\|_{L^2}\|\triangle_j b\|_{L^2}, \quad (3.6)
\end{aligned}
$$

$$
\begin{aligned}
|L_4(t)| \leq & C\|\nabla S_{j-1}b\|_{L^\infty}\|\triangle_j u\|_{L^2}\|\triangle_j b\|_{L^2} + C\|\nabla S_{j-1}u\|_{L^\infty}\|\triangle_j b\|_{L^2}^2 \\
& + C\sum_{k\geq j-3}\|\nabla\triangle_k u\|_{L^\infty}\|\tilde{\triangle}_k b\|_{L^2}\|\triangle_j b\|_{L^2}, \quad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
|L_5(t)| \leq & C2^j\|\nabla S_{j-1}b\|_{L^\infty}\|\triangle_j b\|_{L^2}^2 \\
& + C2^j\sum_{k\geq j-3}\|\nabla\triangle_k b\|_{L^\infty}\|\tilde{\triangle}_k b\|_{L^2}\|\triangle_j b\|_{L^2}. \quad (3.8)
\end{aligned}
$$

Multiply both sides of the equation in (3.3) by 2^{2sj} and take the summation over $j \geq 0$ to obtain that

$$
\begin{aligned}
& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j\geq 0} 2^{2sj}\big(\|\triangle_j u\|_{L^2}^2 + \|\triangle_j b\|_{L^2}^2\big) + \sum_{j\geq 0} 2^{2sj}\|\zeta_1\triangle_j u\|_{L^2}^2 + \sum_{j\geq 0} 2^{2sj}\|\zeta_2\triangle_j b\|_{L^2}^2 \\
\leq & \sum_{j\geq 0} 2^{2sj}\big(\sum_{i=1}^{5} L_i(t)\big). \quad (3.9)
\end{aligned}
$$

Since the estimates of these five terms are similar, we only provide the details of the third and the fifth terms. We now evaluate the third term, by (3.6), we get

$$\begin{aligned}\sum_{j\geq 0}2^{2sj}|L_3(t)|\leq& C\sum_{j\geq 0}2^{2sj}\|\nabla S_{j-1}u\|_{L^\infty}\|\triangle_j b\|_{L^2}^2+\sum_{j\geq 0}2^{2sj}\|\nabla S_{j-1}b\|_{L^\infty}\|\triangle_j u\|_{L^2}\|\triangle_j b\|_{L^2}\\ &+C\sum_{j\geq 0}2^{2sj}\|\triangle_j b\|_{L^2}\sum_{k\geq j-3}\|\nabla\triangle_k b\|_{L^\infty}\|\tilde{\triangle}_k u\|_{L^2}\\ &\triangleq L_{31}+L_{32}+L_{33}. \end{aligned}\tag{3.10}$$

For $L_{31}(t)$ employing Bernstein, Hölder and Young's inequalities, we obtain

$$\begin{aligned}L_{31}=&\sum_{j\geq 0}2^{2sj}\|\nabla S_{j-1}u\|_{L^\infty}\|\triangle_j b\|_{L^2}^2\\ \leq& C\sum_{j\geq 0}2^{2sj}\|\triangle_j b\|_{L^2}^2\sum_{m\leq j-2}2^{\frac{5}{2}m}\|\triangle_m u\|_{L^2}\\ \leq& C\sum_{j\geq 0}g_2(2^{j+1})2^{sj}\frac{2^{\beta j}}{g_2(2^{j+1})}\|\triangle_j b\|_{L^2}2^{sj}\|\triangle_j b\|_{L^2}2^{-\beta j}\sum_{m\leq j-2}2^{\frac{5}{2}m}\|\triangle_m u\|_{L^2}\\ \leq&\frac{1}{24}\sum_{j\geq 0}2^{2sj}\frac{2^{2\beta j}}{g_2^2(2^{j+1})}\|\triangle_j b\|_{L^2}^2\\ &+C\sum_{j\geq 0}g_2^2(2^{j+1})2^{2sj}\|\triangle_j b\|_{L^2}^2 2^{-2\beta j}\Big(\sum_{m\leq j-2}2^{\frac{5}{2}m}\|\triangle_m u\|_{L^2}\Big)^2\\ \leq&\frac{1}{24}\sum_{j\geq 0}2^{2sj}\|\zeta_2\triangle_j b\|_{L^2}^2+C(L_{311}+L_{312}),\end{aligned}\tag{3.11}$$

where

$$\begin{aligned}L_{311}&=\sum_{0\leq j\leq N}g_2^2(2^{j+1})2^{2sj}\|\triangle_j b\|_{L^2}^2 2^{-2\beta j}\Big(\sum_{m\leq j-2}2^{\frac{5}{2}m}\|\triangle_m u\|_{L^2}\Big)^2,\\ L_{312}&=\sum_{j>N}g_2^2(2^{j+1})2^{2sj}\|\triangle_j b\|_{L^2}^2 2^{-2\beta j}\Big(\sum_{m\leq j-2}2^{\frac{5}{2}m}\|\triangle_m u\|_{L^2}\Big)^2\end{aligned}$$

for a natural number N large enough to be determined later.

$$\begin{aligned}L_{311}\leq& C\sum_{0\leq j\leq N}g_2^2(2^{j+1})2^{2sj}\|\triangle_j b\|_{L^2}^2\Bigg(\sum_{m\leq -1}2^{\frac{5}{2}m-\beta j}\|\triangle_m u\|_{L^2}\\ &+\sum_{0\leq m\leq j-2}2^{\beta(m-j)}2^{\alpha m}\|\triangle_m u\|_{L^2}\Bigg)^2\\ \leq& Cg_2^2(2^{N+1})\|b\|_{H^s}^2\|u\|_{L^2}^2\\ &+Cg_2^2(2^{N+1})\sum_{0\leq j\leq N}2^{2sj}\|\triangle_j b\|_{L^2}^2\Bigg(\sup_{0\leq j\leq N}\sum_{0\leq m\leq j-2}2^{\beta(m-j)}2^{\alpha m}\|\triangle_m u\|_{L^2}\Bigg)^2.\end{aligned}$$

Using Young's inequality of series convolution, we have

$$L_{311}\leq Cg_2^2(2^{N+1})\|b\|_{H^s}^2\|u\|_{L^2}^2$$

$$
\begin{aligned}
&+ Cg_2^2(2^{N+1})\|b\|_{H^s}^2 \sum_{j\in\mathbb{Z}} 2^{2\beta j}\mathbf{1}_{j\le -2} \sum_{0\le j\le N} g_1^2(2^{j+1}) \frac{2^{2\alpha j}}{g_1^2(2^{j+1})}\|\triangle_j u\|_{L^2}^2 \\
\le& Cg_2^2(2^{N+1})\|b\|_{H^s}^2\|u\|_{L^2}^2 + Cg_1^2(2^{N+1})g_2^2(2^{N+1})\|b\|_{H^s}^2\|\zeta_1 u\|_{L^2}^2.
\end{aligned}
$$

To estimate L_{312}, choose a positive real number δ_1 such that $0<\delta_1+\frac{1}{2}<\beta$, then $\beta-\delta_1>\frac{1}{2}$, we have

$$
\begin{aligned}
L_{312} =& \sum_{j>N} g_2^2(2^{j+1})2^{-2(\beta-\delta_1)j}2^{2sj}\|\triangle_j b\|_{L^2}^2 \Big(\sum_{m\le j-2} 2^{(m-j)\delta_1}2^{(\frac{5}{2}-\delta_1)m}\|\triangle_m u\|_{L^2}\Big)^2 \\
\le& Cg_2^2(2^{N+1})2^{-2N(\beta-\delta_1)}\sum_{j>N}2^{2sj}\|\triangle_j b\|_{L^2}^2\Big(\sup_{j>N}\sum_{m\le j-2}2^{(m-j)\delta_1}2^{(\frac{5}{2}-\delta_1)m}\|\triangle_m u\|_{L^2}\Big)^2 \\
\le& Cg_2^2(2^{N+1})2^{-2N(\beta-\delta_1)}\|b\|_{H^s}^2\sum_{m\in\mathbb{Z}}2^{2sm}\|\triangle_m u\|_{L^2}^2 \\
\le& Cg_2^2(2^{N+1})2^{-2N(\beta-\delta_1)}\|b\|_{H^s}^2\|u\|_{H^s}^2.
\end{aligned}
$$

Inserting the estimates of L_{311} and L_{312} into (3.11), we have

$$
\begin{aligned}
L_{31} \le& \frac{1}{24}\|\zeta_2 b\|_{H^s}^2 + Cg_2^2(2^{N+1})\|b\|_{H^s}^2\|u\|_{L^2}^2 + Cg_1^2(2^{N+1})g_2^2(2^{N+1})\|b\|_{H^s}^2\|\zeta_1 u\|_{L^2}^2 \\
&+ Cg_2^2(2^{N+1})2^{-2N(\beta-\delta_1)}\|b\|_{H^s}^2\|u\|_{H^s}^2.
\end{aligned}
$$

We shall use an argument similar to that used in deriving the estimate of L_{31} to obtain the bound of L_{32}. The only difference is that the term $\|\nabla S_{j-1}u\|_{L^\infty}$ in L_{31} is replaced by $\|\nabla S_{j-1}b\|_{L^\infty}$. Thus, for the lower frequency part, we enlarge $\frac{5}{2}$ to 2β; For the higher frequency part, it is handled similarly to the estimate of L_{312}. Then we see that

$$
\begin{aligned}
L_{32} \le& \frac{1}{24}\|\zeta_2 b\|_{H^s}^2 + Cg_2^2(2^{N+1})\|b\|_{L^2}^2\|u\|_{H^s}^2 + Cg_2^4(2^{N+1})\|u\|_{H^s}^2\|\zeta_2 b\|_{L^2}^2 \\
&+ Cg_2^2(2^{N+1})2^{-2N(\beta-\delta_1)}\|u\|_{H^s}^2\|b\|_{H^s}^2.
\end{aligned}
$$

For $L_{33}(t)$, it can be shown that

$$
\begin{aligned}
L_{33} \le& C\sum_{j\ge 0}2^{2sj}\|\triangle_j b\|_{L^2}\sum_{k\ge j-3}2^{\frac{5}{2}k}\|\triangle_k b\|_{L^2}\|\tilde{\triangle}_k u\|_{L^2} \\
\le& C\sum_{k\ge 0}2^{\frac{5}{2}k}\|\triangle_k b\|_{L^2}\|\tilde{\triangle}_k u\|_{L^2}\sum_{0\le j\le k+3}2^{2sj}\|\triangle_j b\|_{L^2} \\
=& C\sum_{j\ge 0}2^{\frac{5}{2}j}\|\triangle_j b\|_{L^2}\|\tilde{\triangle}_j u\|_{L^2}\sum_{0\le k\le j+3}2^{2sk}\|\triangle_k b\|_{L^2} \\
\triangleq& L_{331}+L_{332},
\end{aligned}
\tag{3.12}
$$

where

$$
\begin{aligned}
L_{331} =& \sum_{0\le j\le N}2^{\frac{5}{2}j}\|\triangle_j b\|_{L^2}\|\tilde{\triangle}_j u\|_{L^2}\sum_{0\le k\le j+3}2^{2sk}\|\triangle_k b\|_{L^2}, \\
L_{332} =& \sum_{j>N}2^{\frac{5}{2}j}\|\triangle_j b\|_{L^2}\|\tilde{\triangle}_j u\|_{L^2}\sum_{0\le k\le j+3}2^{2sk}\|\triangle_k b\|_{L^2}.
\end{aligned}
$$

$$L_{331} \leq \sum_{0\leq j\leq N} 2^{sj}\frac{2^{\beta j}}{g_2(2^{j+1})}\|\triangle_j b\|_{L^2} g_2(2^{j+1})2^{\alpha j}\|\tilde{\triangle}_j u\|_{L^2}\sum_{0\leq k\leq j+3} 2^{2sk-sj}\|\triangle_k b\|_{L^2}$$
$$\leq C\Big(\sum_{0\leq j\leq N} 2^{2sj}\frac{2^{2\beta j}}{g_2^2(2^{j+1})}\|\triangle_j b\|_{L^2}^2\Big)^{\frac{1}{2}}$$
$$\times\Big(\sum_{0\leq j\leq N} g_2^2(2^{j+1})2^{2\alpha j}\|\tilde{\triangle}_j u\|_{L^2}^2\big(\sum_{0\leq k\leq j+3} 2^{2sk-sj}\|\triangle_k b\|_{L^2}\big)^2\Big)^{\frac{1}{2}}.$$

By virtue of Hölder and Young's inequalities, one has

$$L_{331} \leq C\|\zeta_2 b\|_{H^s} g_2(2^{N+1})\Big(\sum_{0\leq j\leq N} g_1^2(2^{j+1})\frac{2^{2\alpha j}}{g_1^2(2^{j+1})}\|\tilde{\triangle}_j u\|_{L^2}^2\Big)^{\frac{1}{2}}$$
$$\sup_{0\leq j\leq N}\sum_{0\leq k\leq j+3} 2^{s(k-j)+sk}\|\triangle_k b\|_{L^2}$$
$$\leq C\|\zeta_2 b\|_{H^s} g_2(2^{N+1})g_1(2^{N+1})\|\zeta_1 u\|_{L^2}\|b\|_{H^s}$$
$$\leq \frac{1}{48}\|\zeta_2 b\|_{H^s}^2 + Cg_2^2(2^{N+1})g_1^2(2^{N+1})\|\zeta_1 u\|_{L^2}^2\|b\|_{H^s}^2,$$

$$L_{332} \leq \sum_{j>N} 2^{sj}\frac{2^{\beta j}}{g_2(2^{j+1})}\|\triangle_j b\|_{L^2} g_2(2^{j+1})2^{-\beta j}\|\tilde{\triangle}_j u\|_{L^2}\sum_{0\leq k\leq j+3} 2^{2sk}\|\triangle_k b\|_{L^2}$$
$$\leq\Big(\sum_{j>N} 2^{2sj}\frac{2^{2\beta j}}{g_2^2(2^{j+1})}\|\triangle_j b\|_{L^2}^2\Big)^{\frac{1}{2}}$$
$$\times\Big(\sum_{j>N} g_2^2(2^{j+1})2^{-2\beta j}2^{2sj}\|\tilde{\triangle}_j u\|_{L^2}^2\big(\sum_{0\leq k\leq j+3} 2^{2sk-sj}\|\triangle_k b\|_{L^2}\big)^2\Big)^{\frac{1}{2}}$$
$$\leq\|\zeta_2 b\|_{H^s} g_2(2^{N+1})2^{-\beta N}\Big(\sum_{j>N} 2^{2sj}\|\tilde{\triangle}_j u\|_{L^2}^2\Big)^{\frac{1}{2}}\sup_{j>N}\sum_{0\leq k\leq j+3} 2^{s(k-j)+sk}\|\triangle_k b\|_{L^2}$$
$$\leq\|\zeta_2 b\|_{H^s} g_2(2^{N+1})2^{-\beta N}\|u\|_{H^s}\|b\|_{H^s}$$
$$\leq\frac{1}{48}\|\zeta_2 b\|_{H^s}^2 + Cg_2^2(2^{N+1})2^{-2\beta N}\|u\|_{H^s}^2\|b\|_{H^s}^2.$$

Inserting the estimates of L_{331} and L_{332} into (3.12), then L_{33} is estimated by

$$L_{33} \leq\frac{1}{24}\|\zeta_2 b\|_{H^s}^2 + Cg_2^2(2^{N+1})g_1^2(2^{N+1})\|\zeta_1 u\|_{L^2}^2\|b\|_{H^s}^2 + Cg_2^2(2^{N+1})2^{-2\beta N}\|u\|_{H^s}^2\|b\|_{H^s}^2.$$

It is found that $\sum_{j\geq 0} 2^{2sj}|L_3(t)|$ is estimated by

$$\begin{aligned}\sum_{j\geq 0}2^{2sj}|L_3(t)| \leq&\frac{1}{8}\|\zeta_2 b\|_{H^s}^2 + Cg_2^2(2^{N+1})\|b\|_{H^s}^2\|u\|_{L^2}^2 + Cg_1^2(2^{N+1})g_2^2(2^{N+1})\|\zeta_1 u\|_{L^2}^2\|b\|_{H^s}^2\\ &+ Cg_2^2(2^{N+1})2^{-2N(\beta-\delta_1)}\|b\|_{H^s}^2\|u\|_{H^s}^2 + Cg_2^2(2^{N+1})\|u\|_{H^s}^2\|b\|_{L^2}^2\\ &+ Cg_2^4(2^{N+1})\|u\|_{H^s}^2\|\zeta_2 b\|_{L^2}^2 + Cg_1^2(2^{N+1})g_2^2(2^{N+1})\|\zeta_1 u\|_{L^2}^2\|b\|_{H^s}^2\\ &+ Cg_2^2(2^{N+1})2^{-2\beta N}\|u\|_{H^s}^2\|b\|_{H^s}^2.\end{aligned}\tag{3.13}$$

We now estimate $\sum_{j\geq 0} 2^{2sj}|L_5(t)|$. By (3.8), we have

$$\begin{aligned}\sum_{j\geq 0} 2^{2sj}|L_5(t)| \leq& C\sum_{j\geq 0} 2^{(2s+1)j}\|\triangle_j b\|_{L^2}^2 \sum_{m\leq j-2}\|\nabla\triangle_m b\|_{L^\infty}\\&+C\sum_{j\geq 0} 2^{(2s+1)j}\|\triangle_j b\|_{L^2}\sum_{k\geq j-3}\|\nabla\triangle_k b\|_{L^\infty}\|\tilde{\triangle}_k b\|_{L^2}\\ \triangleq& L_{51}(t)+L_{52}(t).\end{aligned}\tag{3.14}$$

Similarly to the estimate of $\sum_{j\geq 0} 2^{2sj}|L_3(t)|$, we obtain

$$\begin{aligned}|L_{51}| \leq& C\sum_{j\geq 0} 2^{(2s+1)j}\|\triangle_j b\|_{L^2}^2 \sum_{m\leq j-2} 2^{\frac{5}{2}m}\|\triangle_m b\|_{L^2}\\ \leq& C\sum_{j\geq 0} 2^{sj}\frac{2^{\beta j}}{g_2(2^{j+1})}\|\triangle_j b\|_{L^2} g_2(2^{j+1})2^{(s+1-\beta)j}\|\triangle_j b\|_{L^2}\sum_{m\leq j-2} 2^{\frac{5}{2}m}\|\triangle_m b\|_{L^2}\\ \leq& \frac{1}{16}\sum_{j\geq 0} 2^{2sj}\frac{2^{2\beta j}}{g_2^2(2^{j+1})}\|\triangle_j b\|_{L^2}^2\\&+C\sum_{j\geq 0} g_2^2(2^{j+1})2^{2sj}\|\triangle_j b\|_{L^2}^2 2^{2(1-\beta)j}\Big(\sum_{m\leq j-2} 2^{\frac{5}{2}m}\|\triangle_m b\|_{L^2}\Big)^2\\ \leq& \frac{1}{16}\|\zeta_2 b\|_{H^s}^2 + C(L_{511}(t)+L_{512}(t)),\end{aligned}\tag{3.15}$$

where

$$\begin{aligned}L_{511} =& \sum_{0\leq j\leq N} g_2^2(2^{j+1})2^{2sj}\|\triangle_j b\|_{L^2}^2 2^{2(1-\beta)j}\Big(\sum_{m\leq j-2} 2^{\frac{5}{2}m}\|\triangle_m b\|_{L^2}\Big)^2,\\ L_{512} =& \sum_{j>N} g_2^2(2^{j+1})2^{2sj}\|\triangle_j b\|_{L^2}^2 2^{2(1-\beta)j}\Big(\sum_{m\leq j-2} 2^{\frac{5}{2}m}\|\triangle_m b\|_{L^2}\Big)^2.\end{aligned}$$

$$\begin{aligned}L_{511} \leq& \sum_{0\leq j\leq N} g_2^2(2^{j+1})2^{2sj}\|\triangle_j b\|_{L^2}^2\left(\|b\|_{L^2}+\sum_{m\leq j-2} 2^{(2\beta-1)m+(1-\beta)j}\|\triangle_m b\|_{L^2}\right)^2\\ \leq& Cg_2^2(2^{N+1})\|b\|_{H^s}^2\|b\|_{L^2}^2\\&+Cg_2^2(2^{N+1})\sum_{0\leq j\leq N} 2^{2sj}\|\triangle_j b\|_{L^2}^2\left(\sup_{0\leq j\leq N}\sum_{m\leq j-2} 2^{(\beta-1)(m-j)}2^{\beta m}\|\triangle_m b\|_{L^2}\right)^2\\ \leq& Cg_2^2(2^{N+1})\|b\|_{H^s}^2\|b\|_{L^2}^2 + Cg_2^4(2^{N+1})\|b\|_{H^s}^2\|\zeta_2 b\|_{L^2}^2.\end{aligned}$$

To estimate L_{512}, let δ_2 be a positive number such that $0<\delta_2+\frac{3}{2}<\beta$, then $\beta-\delta_2>\frac{3}{2}$, one has

$$\begin{aligned}L_{512} =& \sum_{j>N} g_2^2(2^{j+1})2^{-2j(\beta-\delta_2-1)}2^{2sj}\|\triangle_j b\|_{L^2}^2\left(\sum_{m\leq j-2} 2^{(m-j)\delta_2}2^{(\frac{5}{2}-\delta_2)m}\|\triangle_m b\|_{L^2}\right)^2\\ \leq& Cg_2^2(2^{N+1})2^{-2N(\beta-\delta_2-1)}\sum_{j>N} 2^{2sj}\|\triangle_j b\|_{L^2}^2\left(\sup_{j>N}\sum_{m\leq j-2} 2^{(m-j)\delta_2}2^{(\frac{5}{2}-\delta_2)m}\|\triangle_m b\|_{L^2}\right)^2\\ \leq& Cg_2^2(2^{N+1})2^{-2N(\beta-\delta_2-1)}\|b\|_{H^s}^4.\end{aligned}$$

Plugging the estimates of L_{511} and L_{512} into (3.15), we have the following estimate

$$\begin{aligned} L_{51} \le& \frac{1}{16}\|\zeta_2 b\|_{H^s}^2 + Cg_2^2(2^{N+1})\|b\|_{H^s}^2\|b\|_{L^2}^2 + Cg_2^4(2^{N+1})\|b\|_{H^s}^2\|\zeta_2 b\|_{L^2}^2 \\ &+ Cg_2^2(2^{N+1})2^{-2N(\beta-\delta_2-1)}\|b\|_{H^s}^4. \end{aligned}$$

Evaluating $L_{52}(t)$ similarly to the estimate of $L_{33}(t)$, we get

$$\begin{aligned} L_{52} \le& C\sum_{j\ge 0} 2^{(2s+1)j}\|\triangle_j b\|_{L^2}\sum_{k\ge j-3} 2^{\frac{5}{2}k}\|\triangle_k b\|_{L^2}\|\tilde{\triangle}_k b\|_{L^2} \\ =& C\sum_{j\ge 0} 2^{\frac{5}{2}j}\|\triangle_j b\|_{L^2}\|\tilde{\triangle}_k b\|_{L^2}\sum_{0\le k\le j+3} 2^{(2s+1)k}\|\triangle_k b\|_{L^2} \\ \triangleq& L_{521} + L_{522}, \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} L_{521} &= \sum_{0\le j\le N} 2^{\frac{5}{2}j}\|\triangle_j b\|_{L^2}\|\tilde{\triangle}_k b\|_{L^2}\sum_{0\le k\le j+3} 2^{(2s+1)k}\|\triangle_k b\|_{L^2}, \\ L_{522} &= \sum_{j>N} 2^{\frac{5}{2}j}\|\triangle_j b\|_{L^2}\|\tilde{\triangle}_k b\|_{L^2}\sum_{0\le k\le j+3} 2^{(2s+1)k}\|\triangle_k b\|_{L^2}. \end{aligned}$$

$$\begin{aligned} L_{521} \le& C\sum_{0\le j\le N} 2^{(2\beta-1)j}\|\triangle_j b\|_{L^2}\|\tilde{\triangle}_k b\|_{L^2}\sum_{0\le k\le j+3} 2^{(2s+1)k}\|\triangle_k b\|_{L^2} \\ \le& C\sum_{0\le j\le N} 2^{sj}\frac{2^{\beta j}}{g_2(2^{j+1})}\|\triangle_j b\|_{L^2} g_2(2^{j+1})2^{-(s+1)j}2^{\beta j}\|\tilde{\triangle}_j b\|_{L^2}\sum_{0\le k\le j+3} 2^{(2s+1)k}\|\triangle_k b\|_{L^2}. \end{aligned}$$

By Hölder and Young's inequalities, it follows that

$$\begin{aligned} L_{521} \le& C\Big(\sum_{0\le j\le N} 2^{2sj}\frac{2^{2\beta j}}{g_2^2(2^{j+1})}\|\triangle_j b\|_{L^2}^2\Big)^{\frac{1}{2}} \\ &\times\Big(\sum_{0\le j\le N} g_2^2(2^{j+1})2^{2\beta j}\|\tilde{\triangle}_j b\|_{L^2}^2\big(\sum_{0\le k\le j+3} 2^{(s+1)(k-j)+sk}\|\triangle_k b\|_{L^2}\big)^2\Big)^{\frac{1}{2}} \\ \le& C\|\zeta_2 b\|_{H^s} g_2^2(2^{N+1})\Big(\sum_{0\le j\le N}\frac{2^{2\beta j}}{g_2^2(2^{j+1})}\|\tilde{\triangle}_j b\|_{L^2}^2\Big)^{\frac{1}{2}} \\ &\times\sup_{0\le j\le N}\sum_{0\le k\le j+3} 2^{(s+1)(k-j)+sk}\|\triangle_k b\|_{L^2} \\ \le& C\|\zeta_2 b\|_{H^s} g_2^2(2^{N+1})\|\zeta_2 b\|_{L^2}\|b\|_{H^s} \\ \le& \frac{1}{32}\|\zeta_2 b\|_{H^s}^2 + Cg_2^4(2^{N+1})\|\zeta_2 b\|_{L^2}^2\|b\|_{H^s}^2, \end{aligned}$$

$$\begin{aligned} L_{522} \le& \sum_{j>N} 2^{sj}\|\triangle_j b\|_{L^2}^2\sum_{0\le k\le j+3} 2^{(2s+1)k}\|\triangle_k b\|_{L^2} \\ \le& \sum_{j>N} 2^{sj}\frac{2^{\beta j}}{g_2(2^{j+1})}\|\triangle_j b\|_{L^2} g_2(2^{j+1})2^{-\beta j}2^{(s+1)j}\|\triangle_j b\|_{L^2}\sum_{0\le k\le j+3} 2^{(2s+1)k-(s+1)j}\|\triangle_k b\|_{L^2} \end{aligned}$$

$$
\begin{aligned}
&\le\Big(\sum_{j>N}2^{2sj}\frac{2^{2\beta j}}{g_2^2(2^{j+1})}\|\triangle_j b\|_{L^2}^2\Big)^{\frac12}\\
&\quad\times\Big(\sum_{j>N}g_2^2(2^{j+1})2^{-2(\beta-1)j}2^{2sj}\|\triangle_j b\|_{L^2}^2\big(\sum_{0\le k\le j+3}2^{(s+1)(k-j)+sk}\|\triangle_k b\|_{L^2}\big)^2\Big)^{\frac12}\\
&\le C\|\zeta_2 b\|_{H^s}g_2(2^{N+1})2^{-(\beta-1)N}\Big(\sum_{j>N}2^{2sj}\|\triangle_j b\|_{L^2}^2\Big)^{\frac12}\\
&\quad\times\sup_{j>N}\sum_{0\le k\le j+3}2^{(s+1)(k-j)}2^{sk}\|\triangle_k b\|_{L^2}\\
&\le C\|\zeta_2 b\|_{H^s}g_2(2^{N+1})2^{-(\beta-1)N}\|b\|_{H^s}^2\\
&\le\frac{1}{32}\|\zeta_2 b\|_{H^s}^2+Cg_2^2(2^{N+1})2^{-2(\beta-1)N}\|b\|_{H^s}^4.
\end{aligned}
$$

Putting the estimates of L_{521} and L_{522} into (3.16) to yield that

$$
L_{52}\le\frac{1}{16}\|\zeta_2 b\|_{H^s}^2+Cg_2^4(2^{N+1})\|b\|_{H^s}^2\|\zeta_2 b\|_{L^2}^2+C2^{-2N(\beta-1)}g_2^2(2^{N+1})\|b\|_{H^s}^4.
$$

Thus by (3.14) it follows that

$$
\begin{aligned}
\sum_{j\ge0}2^{2sj}|L_5(t)|\le&\frac18\|\zeta_2 b\|_{H^s}^2+Cg_2^2(2^{N+1})\|b\|_{H^s}^2\|b\|_{L^2}^2+Cg_2^4(2^{N+1})\|\zeta_2 b\|_{L^2}^2\|b\|_{H^s}^2\\
&+Cg_2^2(2^{N+1})2^{-2N(\beta-\delta_2-1)}\|b\|_{H^s}^4\\
&+Cg_2^2(2^{N+1})2^{-2N(\beta-1)}\|b\|_{H^s}^4. \qquad (3.17)
\end{aligned}
$$

Using an argument similar to that used in evaluating $\sum_{j\ge0}2^{2sj}|L_3(t)|$ and $\sum_{j\ge0}2^{2sj}|L_5(t)|$, we have

$$
\begin{aligned}
\sum_{j\ge0}2^{2sj}|L_1(t)|\le&\frac14\|\zeta_1 u\|_{H^s}^2+Cg_1^2(2^{N+1})\|u\|_{H^s}^2\|u\|_{L^2}^2+Cg_1^4(2^{N+1})\|u\|_{H^s}^2\|\zeta_1 u\|_{L^2}^2\\
&+Cg_1^2(2^{N+1})2^{-2N(\alpha-\delta_3)}\|u\|_{H^s}^4+Cg_1^2(2^{N+1})2^{-2\alpha N}\|u\|_{H^s}^4, \qquad (3.18)
\end{aligned}
$$

where δ_3 is a small positive number such that $0<\delta_3+\frac12<\alpha$, then $\alpha-\delta_3>\frac12$.

$$
\begin{aligned}
\sum_{j\ge0}2^{2sj}|L_2(t)|\le&\frac18\|\zeta_2 b\|_{H^s}^2+Cg_2^2(2^{N+1})\|u\|_{H^s}^2\|b\|_{L^2}^2+Cg_2^4(2^{N+1})\|u\|_{H^s}^2\|\zeta_2 b\|_{L^2}^2\\
&+Cg_2^2(2^{N+1})2^{-2N(\beta-\delta_1)}\|u\|_{H^s}^2\|b\|_{H^s}^2\\
&+Cg_2^2(2^{N+1})2^{-2\beta N}\|b\|_{H^s}^2\|u\|_{H^s}^2, \qquad (3.19)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{j\ge0}2^{2sj}|L_4(t)|\le&\frac18\|\zeta_2 b\|_{H^s}^2+\frac14\|\zeta_1 u\|_{H^s}^2+Cg_2^2(2^{N+1})\|u\|_{H^s}^2\|b\|_{L^2}^2\\
&+Cg_2^4(2^{N+1})\|u\|_{H^s}^2\|\zeta_2 b\|_{L^2}^2+Cg_2^2(2^{N+1})2^{-2N(\beta-\delta_1)}\|u\|_{H^s}^2\|b\|_{H^s}^2\\
&+Cg_2^2(2^{N+1})\|b\|_{H^s}^2\|u\|_{L^2}^2+Cg_1^2(2^{N+1})g_2^2(2^{N+1})\|b\|_{H^s}^2\|\zeta_1 u\|_{L^2}^2\\
&+Cg_1^2(2^{N+1})g_2^2(2^{N+1})\|b\|_{H^s}^2\|\zeta_2 b\|_{L^2}^2\\
&+g_1^2(2^{N+1})2^{-2\alpha N}\|b\|_{H^s}^4. \qquad (3.20)
\end{aligned}
$$

Plugging the estimates (3.13) and (3.17)-(3.20) into (3.9), by (3.1) and (2.1) we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u\|_{H^s}^2+\|b\|_{H^s}^2\big)+\frac{1}{2}\|\zeta_1 u\|_{H^s}^2+\frac{1}{2}\|\zeta_2 b\|_{H^s}^2\\
\le & Cg_1^2(2^{N+1})\|u\|_{H^s}^2\|u\|_{L^2}^2+Cg_1^4(2^{N+1})\|u\|_{H^s}^2\|\zeta_1 u\|_{L^2}^2\\
&+Cg_1^2(2^{N+1})2^{-2N(\alpha-\delta_3)}\|u\|_{H^s}^4+Cg_1^2(2^{N+1})2^{-2\alpha N}\|u\|_{H^s}^4,\\
&+Cg_2^2(2^{N+1})\|u\|_{H^s}^2\|b\|_{L^2}^2+Cg_2^4(2^{N+1})\|u\|_{H^s}^2\|\zeta_2 b\|_{L^2}^2\\
&+Cg_2^2(2^{N+1})2^{-2N(\beta-\delta_1)}\|u\|_{H^s}^2\|b\|_{H^s}^2+Cg_2^2(2^{N+1})2^{-2\beta N}\|b\|_{H^s}^2\|u\|_{H^s}^2,\\
&+Cg_2^2(2^{N+1})\|b\|_{H^s}^2\|u\|_{L^2}^2+Cg_1^2(2^{N+1})g_2^2(2^{N+1})\|\zeta_1 u\|_{L^2}^2\|b\|_{H^s}^2\\
&+Cg_1^2(2^{N+1})g_2^2(2^{N+1})\|b\|_{H^s}^2\|\zeta_2 b\|_{L^2}^2+g_1^2(2^{N+1})2^{-2\alpha N}\|b\|_{H^s}^4.\\
&+Cg_2^2(2^{N+1})\|b\|_{H^s}^2\|b\|_{L^2}^2+Cg_2^4(2^{N+1})\|\zeta_2 b\|_{L^2}^2\|b\|_{H^s}^2\\
&+Cg_2^2(2^{N+1})2^{-2N(\beta-\delta_2-1)}\|b\|_{H^s}^4+Cg_2^2(2^{N+1})2^{-2N(\beta-1)}\|b\|_{H^s}^4. \qquad (3.21)
\end{aligned}$$

Write $E_s(t)=\|u\|_{H^s}^2+\|b\|_{H^s}^2, A_1(t)=\|\zeta_1 u\|_{L^2}^2,\ A_2(t)=\|\zeta_2 b\|_{L^2}^2$. In view of the facts that

$$2^{-2\alpha N}\le 2^{-2N(\alpha-\delta_3)},\ 2^{-2\beta N}\le 2^{-2N(\beta-\delta_1)},\ 2^{-2(\beta-1)N}\le 2^{-2N(\beta-\delta_2-1)},$$

one can choose an integer N such that

$$2^{-2N(\alpha-\delta_3)}E_s<8,\ 2^{-2N(\beta-\delta_1)}E_s<8,\ 2^{-2N(\beta-\delta_2-1)}E_s<8.$$

Thus we set

$$N=[\log_2 E_s]-2\ge\max\left\{\left[\frac{\log_2 E_s-3}{2(\alpha-\delta_3)}\right]+1,\left[\frac{\log_2 E_s-3}{2(\beta-\delta_1)}\right]+1,\left[\frac{\log_2 E_s-3}{2(\beta-\delta_2-1)}\right]+1\right\},$$

then $2^{N+1}\le E_s$. By the definitions of $E_s(t),A_1(t),A_2(t)$ and the choice of N, (3.21) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}E_s(t)+\|\zeta_1 u\|_{H^s}^2+\|\zeta_2 b\|_{H^s}^2\le C\big(g_1^2(E_s)+g_2^2(E_s)\big)^2E_s\big(A_1(t)+A_2(t)+1\big).$$

Integrating on $[0,T]$ for any $T>0$, it leads to the following estimate,

$$\int_{E_s(0)}^{E_s(T)}\frac{\mathrm{d}E_s(t)}{E_s(t)\big(g_1^2(E_s)+g_2^2(E_s)\big)^2}\le\int_0^T C\big(A_1(t)+A_2(t)+1\big)\ \mathrm{d}t.$$

By the condition of $\int_1^\infty\frac{1}{s(g_1^2(s)+g_2^2(s))^2}\mathrm{d}s=+\infty$ and the boundedness of $\int_0^T C\big(A_1(t)+A_2(t)\big)\ \mathrm{d}t$ we can obtain that

$$\|u(t)\|_{H^s}^2+\|b(t)\|_{H^s}^2<\infty \text{ for any } 0<t<T,$$

and we thus complete the proof of Theorem 1.1. □

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