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A NEW NONLOCAL MODEL FOR THE RESTORATION OF TEXTURED IMAGES

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Abstract In this paper, we focus on the mathematical and numerical study of a new nonlocal reaction-diffusion system for image denoising. This model is motivated by involving the decomposition approach of H^{-1} norm suggested by Meyer [25] which is more appropriate to represent the oscillatory patterns and small details in the textured image. Based on Schaeffer's fixed point theorem, we prove the existence and uniqueness of solution of the proposed model. To illustrate the efficiency and effectiveness of our model, we test the denoising experimental results as well we compare with some existing models in the literature.

Keywords Image denoising, nonlocal model, schaeffer's fixed point theorem, textured images, nonlocal p-Laplacian.

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1. Introduction

In these days, image denoising has been an attractive title for researchers in image processing and computer vision. The main goal is to seek the restored image $u(i)$ such that

$$f(i) = u(i) + \xi(i),$$

where $f(i)$ is a degraded image corrupted by the noise perturbation $\xi(i)$ at a pixel i which is often considered to be the stationary Gaussian with zero mean and variance σ^2. The challenge is to recover an image corrupted by the noise with preserving edges, fine details and textures. To handle this problem, many models are available such as variational models [2, 5, 21, 27–29], bilateral filtering [11] and wavelet thresholding [9]. Recently, some kind of nonlocal methods [12–14, 17] have been proved to be very powerful in the image denoising which are able to remove noise, preserve edges, take care of the fine structure, details and texture. This relatively new class of denoising methods originates from the nonlocal means, introduced by Buades and al. [8], and based on the work of Yaroslavsky filter [31]. Besides, the transform-based BM3D filter by Dabov and al. [10] relies both on nonlocal and local characteristics of natural images. Other developed versions have been proposed in [20, 22–24]. Furthermore, Kindermann, Osher and Jones [19] have presented the first variational understanding of the nonlocal p-Laplacian problems for deblurring and denoising images. Gilboa and Osher [13, 14] later formalized a systematical

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study for nonlocal image processing by introducing nonlocal operators. Generally, the restored image $u: \Omega \longrightarrow I\!R$ is computed from the following minimization problem:

$$\min_u F(u) = \mathcal{J}_p(u) + \lambda ||f-u||_X^2, \tag{1.1}$$

where $\Omega \subset I\!R^N$ (N equal to 2 or 3 in practical situations) is an open and bounded domain with smooth boundary Γ, $\lambda > 0$ is a weight parameter, X is a Banach space, $||f-u||_X^2$ is the fidelity term and $\mathcal{J}_p(u)$ is a regularizing term to remove the noise. In this work, we are interested in nonlinear nonlocal regularization of the form

$$\mathcal{J}_p(u) = \frac{1}{2p}\int_\Omega\int_\Omega J(x,y)|u(y)-u(x)|^p dydx$$

where J is a given function and $1 < p < \infty$. The problem (1.1) with $X = L^2(\Omega)$ has been studied by many authors ([3, 6, 7]) and it represents an efficient and effective tool for image denoising. However, smaller details, such as texture, are destroyed. To overcome this, Meyer [25] proposed to change the $L^2(\Omega)$ norm of $(u-f)$ by a weaker norm more appropriate to represent textured or oscillatory patterns. Subsequently, in ([15, 16, 26] and [1]) the authors decomposed the image into two components: the first is a smoothed original image and the second is the texture or noise information. In other word, they used a weaker norm $X = H^{-1}$ for oscillatory functions. Since this norm is defined as $||.||_{H^{-1}}^2 = \int_\Omega |\nabla\Delta^{-1}(.)|^2$, the minimization problem (1.1) is formally associated to the Euler-Lagrange equation, which can be formally written as:

$$\mathcal{L}_p(u) - 2\lambda\Delta^{-1}(f-u) = 0 \quad \text{in } \Omega, \tag{1.2}$$

where $\mathcal{L}_p(u) = \int_\Omega J(x,y)|u(y)-u(x)|^{p-2}(u(y)-u(x))dy$ is the nonlocal p-Laplacian operator with homogeneous Neumann boundary conditions. Setting $v = \Delta^{-1}(f-u)$ and using the previous arguments in the nonlocal framework, we propose the following nonlocal system

$$\begin{cases} \mathcal{L}_p(u) - 2\lambda v = 0 & \text{in } \Omega, \\ \mathcal{L}_2(v) - (f-u) = 0 & \text{in } \Omega. \end{cases} \tag{1.3}$$

The nonlocal evolutional reaction diffusion system associated to (1.3) can be written as:

$$(P)\begin{cases} u_t(t,x) = \int_\Omega J(x,y)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))dy - 2\lambda v(t,x) & \text{in } Q_T, \\ v_t(t,x) = \int_\Omega K(x,y)(v(t,y)-v(t,x))dy - (f(x)-u(t,x)) & \text{in } Q_T, \\ u(0,x) = u_0(x) = f(x), \qquad v(0,x) = v_0(x) = 0 & \text{in } \Omega. \end{cases}$$

Here, $Q_T := (0,T)\times\Omega$, the kernel $J: \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and $K: \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ are nonnegative smooth functions with compact support contained in $\Omega \times B(0,d) \subset \mathbb{R}^N \times \mathbb{R}^N$ with

$$0 < \sup_{y\in B(0,d)} J(x,y) = R(x) \in L^\infty(\Omega),$$

$$0 < \sup_{y \in B(0,d)} K(x,y) = A(x) \in L^\infty(\Omega). \tag{1.4}$$

Furthermore, J and K are symmetric functions satisfying:

$$\int_{\mathbb{R}^N} J(x,y)dx = \int_{\mathbb{R}^N} K(x,y)dx = 1, \quad J(x,y) = J(y,x), \quad K(x,y) = K(y,x).$$

We recall the following integration by parts formula (see [3] for instance). For every $u, \xi \in L^p(Q_T)$, $1 < p < \infty$, we have

$$\begin{aligned} &-\int_\Omega \int_\Omega J(x,y)|u(t,y) - u(t,x)|^{p-2}(u(t,y) - u(t,x))dy\xi(t,x)dx \\ &= \frac{1}{2}\int_\Omega \int_\Omega J(x,y)|u(t,y) - u(t,x)|^{p-2}(u(t,y) - u(t,x))(\xi(t,y) - \xi(t,x))dydx. \end{aligned} \tag{1.5}$$

The rest of this paper is structured as follows. In section 2, we prove the existence and uniqueness of the solution to the proposed model (P). At last, section 3 is devoted to numerical results and comparative experiments to improve our model.

2. Existence

In this section we prove the following existence and uniqueness theorem.

Theorem 2.1. *Let $\lambda > 0$, $1 < p < \infty$ and $f \in L^\infty(\Omega)$ be given. Then there exists a unique couple $(u, v) \in \left[C\Big([0,T]; L^1(\Omega)\Big) \cap W^{1,1}\Big((0,T); L^1(\Omega)\Big)\right]^2$ solution of (P) satisfying $u(0,x) = u_0(x)$, $v(0,x) = v_0(x)$ a.e. $x \in \Omega$ and*

$$u_t(t,x) = \int_\Omega J(x,y)|u(t,y) - u(t,x)|^{p-2}(u(t,y) - u(t,x))dy - 2\lambda v(t,x) \quad \text{in } Q_T,$$

$$v_t(t,x) = \int_\Omega K(x,y)(v(t,y) - v(t,x))dy - (f(x) - u(t,x)) \quad \text{in } Q_T.$$

To prove the existence result of the problem (P), firstly we approximate the system (P) by a suitable problem (P^ε), we prove the existence of the solution to the problem (P^ε) based on Schaeffer's Fixed Point Theorem, then we pass to the limit which proves that the solution of (P^ε) converges to the solution of problem (P). Finally, we close the demonstration by proving the uniqueness of the solution.

2.1. Approximate problem

In this subsection, we present the approximate system (P^ε) of the problem (P):

$$(P^\varepsilon)\begin{cases} u_t(t,x) = \int_\Omega J(x,y)|u(t,y) - u(t,x)|^{p-2}(u(t,y) - u(t,x))dy - 2\lambda v(t,x) & \text{in } Q_T, \\ v_t(t,x) = \varepsilon\Delta v(t,x) + \int_\Omega K(x,y)(v(t,y) - v(t,x))dy - (f(x) - u(t,x)) & \text{in } Q_T, \\ \nabla v.\vec{n} = 0 & \text{on } \Gamma_T, \\ u(0,x) = u_0(x) = f(x), \quad v(0,x) = v_0(x) = 0 & \text{in } \Omega, \end{cases}$$

where $\vec{n}$ is the unit outward normal, $\Gamma_T = (0,T)\times\Gamma$ and $\varepsilon > 0$ is a fixed parameter.

To prove the existence result of the problem (P^ε), we shall use Schaeffer's Fixed Point Theorem. In other words, we solve the decoupled problem and we seek for the estimates that allow us to pass to the limit.

2.2. Schaeffer's fixed–point method

We assume that $\underline{u} \in L^2\big(Q_T\big)$ is fixed, and thanks to [18] for a constant function p equals to 2, there exists a unique solution $v \in C\Big([0,T];L^1(\Omega)\Big)\cap L^2\Big((0,T);H^1(\Omega)\Big)$ of the following problem

$$\begin{cases} v_t(t,x) = \varepsilon\Delta v(t,x) + \displaystyle\int_\Omega K(x,y)(v(t,y)-v(t,x))dy - (f(x)-\underline{u}(t,x)) & \text{in } Q_T,\\ \nabla v.\vec{n} = 0 & \text{on } \Gamma_T,\\ v(x,0)=0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

in the sense that

$$-\Big\langle \frac{\partial\varphi}{\partial t},\, v\Big\rangle_{Q_\tau} - \int_0^\tau\int_\Omega\int_\Omega K(x,y)(v(t,y)-v(t,x))dy\varphi dxdt$$
$$+\,\varepsilon\int_0^\tau\int_\Omega \nabla v(t,x)\nabla\varphi dxdt + \int_0^\tau\int_\Omega (f(x)-\underline{u}(t,x))\varphi dxdt + \int_\Omega v(t,x)\varphi dx\Big|_0^\tau = 0,$$

for every $\tau\in(0,T]$ and every test-function

$$\varphi\in L^2\Big((0,T);H^1(\Omega)\Big),\ \frac{\partial\varphi}{\partial t}\in L^2\Big((0,T);(H^1(\Omega))'\Big),$$

where $< .\ ,\ . >$ denotes the duality bracket between $L^2\Big((0,T);(H^1(\Omega))\Big)$ and $L^2\Big((0,T);(H^1(\Omega))'\Big)$.

Now, let $v\in C\Big([0,T];L^1(\Omega)\Big)\cap L^2\Big((0,T);H^1(\Omega)\Big)$ be given, thanks to Remark 2.6 of [3], there exists a unique solution $u\in C\Big([0,T];L^1(\Omega)\Big)\cap W^{1,1}\Big((0,T);L^1(\Omega)\Big)$ to the following problem

$$\begin{cases} u_t(t,x) = \displaystyle\int_\Omega J(x,y)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))dy - 2\lambda v(t,x) & \text{in } Q_T,\\ u(0,x)=f(x) & \text{in } \Omega. \end{cases} \tag{2.2}$$

As it is done in theorem 6.40 (page 161) of the reference [4], one can show that $u\in L^2\big(Q_T\big)$, since $u_0\in L^2\big(\Omega\big)$ and $v\in L^2\big(Q_T\big)$. For further details, we refer to [4] and the references therein.

Now, we introduce a mapping $\mathcal{F}$ defined as

$$\mathcal{F}: L^2\big(Q_T\big)\longrightarrow L^2\big(Q_T\big)$$
$$\underline{u}\quad\longrightarrow \mathcal{F}(\underline{u}) = u,$$

where $\mathcal{F}(\underline{u})$ is the solution of the problem (2.2) for a given v the solution of the problem (2.1).

In the next, we show that $\mathcal{F}$ satisfies the hypotheses of Schaeffer's fixed point theorem which includes two steps.

First step: let us show that $\mathcal{F}$ is a continuous and compact mapping. Let $(\underline{u}^n)_n$ be a bounded sequence in $L^2(Q_T)$ and $\underline{u} \in L^2(Q_T)$ such that

$$\underline{u}^n \longrightarrow \underline{u} \quad \text{weakly} \quad \text{in} \quad L^2(Q_T) \quad \text{as} \quad n \longrightarrow \infty.$$

Define $u^n = \mathcal{F}(\underline{u}^n)$, i.e. u^n is the solution of (2.2) associated with $\underline{u}^n$ and v^n is the solution of (2.1). We shall prove that Γ is a continuous mapping. i.e.

$$\mathcal{F}(\underline{u}^n) \longrightarrow \mathcal{F}(\underline{u}) \text{ in } L^2(Q_T) \text{ as } n \longrightarrow \infty.$$

Let (u^n, v^n) be a solution of the following system

$$\begin{cases} v_t^n(t,x) = \varepsilon \triangle v^n(t,x) + \displaystyle\int_\Omega K(x,y)(v^n(t,y) - v^n(t,x))dy - (f(x) - \underline{u}^n(t,x)), & \text{in } Q_T, \\ \nabla v.\vec{n} = 0 & \text{on } \Gamma_T, \\ v(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{2.3}$$

$$\begin{cases} u_t^n(t,x) = \displaystyle\int_\Omega J(x,y)|u^n(t,y) - u^n(t,x)|^{p-2}(u^n(t,y) - u^n(t,x))dy - 2\lambda v^n(t,x) & \text{in } Q_T, \\ u(0,x) = f(x) & \text{in } \Omega. \end{cases} \tag{2.4}$$

Now, to estimate the approximate solution, we state the following lemma.

Lemma 2.1. *Let (u^n, v^n) be the solution of (2.3)-(2.4). We have*

$$\begin{aligned} &||v^n(t,x)||^2_{L^\infty(0,T;L^2(\Omega))} + \varepsilon \int_0^\tau \int_\Omega |\nabla v^n(t,x)|^2 dxdt \\ &\qquad + \frac{1}{2} \int_0^\tau \int_\Omega \int_\Omega K(x,y)|v^n(t,y) - v^n(t,x)|^2 dydxdt \le C_1, \\ &||u^n(t,x)||^2_{L^\infty(0,T;L^2(\Omega))} + \frac{1}{2} \int_0^\tau \int_\Omega \int_\Omega J(x,y)|u^n(t,y) - u^n(t,x)|^p dydxdt \le C_2, \end{aligned} \tag{2.5}$$

where the constants C_1 and C_2 are independent of n and ε.

Proof. Let $\tau < T$, taking v^n as a test function in (2.3), multiplying (2.4) by u^n and integrating over $\Omega \times [0, \tau]$, we get

$$\begin{aligned} &\frac{1}{2} \int_\Omega (v^n(\tau,x))^2 dx - \int_0^\tau \int_\Omega \int_\Omega K(x,y)(v^n(t,y) - v^n(t,x))dy v^n(t,x) dxdt \\ &+ \varepsilon \int_0^\tau \int_\Omega |\nabla v^n(t,x)|^2 dxdt + \int_0^\tau \int_\Omega (f(x) - \underline{u}^n(t,x)) v^n(t,x) dxdt \\ =&\frac{1}{2} \int_\Omega (v^n(0,x))^2 dx, \end{aligned} \tag{2.6}$$

$$- \int_0^\tau \int_\Omega \int_\Omega J(x,y)|u^n(t,y) - u^n(t,x)|^{p-2}(u^n(t,y) - u^n(t,x))dyu^n(t,x)dxdt$$
$$+ \frac{1}{2}\int_\Omega (u^n(\tau,x))^2 dx + 2\lambda \int_0^\tau \int_\Omega v^n(t,x)u^n(t,x)dxdt = \frac{1}{2}\int_\Omega (u^n(0,x))^2 dx. \quad (2.7)$$

Thanks to (1.5) and using the initial condition $u^n(0,x) = f(x)$ and $v^n(0,x) = 0$, then the equations (2.6) and (2.7) become

$$\frac{1}{2}\int_\Omega (v^n(\tau,x))^2 dx + \frac{1}{2}\int_0^\tau \int_\Omega \int_\Omega K(x,y)|v^n(t,y) - v^n(t,x)|^2 dydxdt$$
$$+ \varepsilon \int_0^\tau \int_\Omega |\nabla v^n(t,x)|^2 dxdt + \int_0^\tau \int_\Omega (f(x) - \underline{u}^n(x,t))v^n(t,x)dxdt = 0, \quad (2.8)$$

$$\frac{1}{2}\int_\Omega (u^n(\tau,x))^2 dx + \frac{1}{2}\int_0^\tau \int_\Omega \int_\Omega J(x,y)|u^n(t,y) - u^n(t,x)|^p dydxdt$$
$$+ 2\lambda \int_0^\tau \int_\Omega v^n(t,x)u^n(t,x)dxdt = \frac{1}{2}\int_\Omega f(x)^2 dx. \quad (2.9)$$

Then, we have

$$\frac{1}{2}\int_\Omega (v^n(\tau,x))^2 dx + \frac{1}{2}\int_0^\tau \int_\Omega \int_\Omega K(x,y)|v^n(t,y) - v^n(t,x)|^2 dydxdt$$
$$+ \varepsilon \int_0^\tau \int_\Omega |\nabla v^n(t,x)|^2 dxdt \leq \int_0^\tau \int_\Omega |(f(x) - \underline{u}^n(x,t))v^n(t,x)|dxdt, \quad (2.10)$$

$$\frac{1}{2}\int_\Omega (u^n(\tau,x))^2 dx + \frac{1}{2}\int_0^\tau \int_\Omega \int_\Omega J(x,y)|u^n(t,y) - u^n(t,x)|^p dydxdt$$
$$\leq 2\lambda \int_0^\tau \int_\Omega |v^n(t,x)u^n(t,x)|dxdt + \frac{1}{2}\int_\Omega f(x)^2 dx. \quad (2.11)$$

Now, applying Young's inequality, we obtain

$$\int_0^\tau \int_\Omega |(f(x) - \underline{u}^n(t,x))v^n(t,x)|dxdt \leq \frac{1}{2}\int_0^\tau \int_\Omega (f(x) - \underline{u}^n(t,x))^2 dxdt$$
$$+ \frac{1}{2}\int_0^\tau \int_\Omega (v^n(t,x))^2 dxdt, \quad (2.12)$$

$$2\lambda \int_0^\tau \int_\Omega v^n(t,x)u^n(t,x)dxdt \leq \lambda^2 \int_0^\tau \int_\Omega (v^n(t,x))^2 dxdt + \int_0^\tau \int_\Omega (u^n(t,x))^2 dxdt. \quad (2.13)$$

Thanks to $\big($2.12) (respectively (2.13)$\big)$, the equation $\big($2.10) (respectively (2.11)$\big)$ becomes

$$\frac{1}{2}\int_\Omega (v^n(\tau,x))^2 dx + \frac{1}{2}\int_0^\tau \int_\Omega \int_\Omega K(x,y)|v^n(t,y) - v^n(t,x)|^2 dydxdt$$
$$+\varepsilon \int_0^\tau \int_\Omega |\nabla v^n(t,x)|^2 dxdt \leq \frac{1}{2}\int_0^\tau \int_\Omega (f(x) - \underline{u}^n(x))^2 dxdt + \frac{1}{2}\int_0^\tau \int_\Omega (v^n(t,x))^2 dxdt, \quad (2.14)$$

$$\frac{1}{2}\int_{\Omega}(u^n(\tau,x))^2dx + \frac{1}{2}\int_0^\tau \int_\Omega\int_\Omega J(x,y)|u^n(t,y)-u^n(t,x)|^p dydxdt$$
$$\leq \lambda^2\int_0^\tau \int_\Omega (v^n(t,x))^2dxdt + \int_0^\tau\int_\Omega (u^n(t,x))^2dxdt + \frac{1}{2}\int_\Omega f(x)^2dx. \tag{2.15}$$

Now, setting $\Theta_n(\tau) = \displaystyle\int_\Omega v^n(\tau,x)^2dx$ and $\tilde{\Theta}_n(\tau) = \displaystyle\int_\Omega u^n(\tau,x)^2dx$, we have

$$0 \leq \frac{1}{2}\Theta_n(\tau) \leq \frac{1}{2}\int_0^\tau\int_\Omega (f(x)-\underline{u}^n(x))^2dxdt + \frac{1}{2}\int_0^\tau \Theta_n(t)dt$$

and

$$0 \leq \frac{1}{2}\tilde{\Theta}_n(\tau) \leq \lambda^2\int_0^\tau\int_\Omega (v^n(t,x))^2dxdt + \frac{1}{2}\int_\Omega f(x)^2dx + \int_0^\tau \tilde{\Theta}_n(t)dt.$$

Using Gronwall's inequality, we get

$$0 \leq \Theta_n(\tau) \leq \Big(\int_\Omega \frac{1}{2}(f(x)-\underline{u}^n(x))^2dx\Big)\exp(\frac{\tau}{2}) \qquad \text{and}$$

$$0 \leq \tilde{\Theta}_n(\tau) \leq \Big(\lambda^2\int_0^\tau\int_\Omega (v^n(t,x))^2dxdt + \frac{1}{2}\int_\Omega f(x)^2dx\Big)\exp(\tau), \quad \forall \tau\in[0,T]. \tag{2.16}$$

Since $\underline{u}^n$ is bounded in $L^2\big(Q_T\big)$, we have

$$\sup_{0<\tau<T}\int_\Omega v^n(\tau,x)^2dx \leq C_1 \text{ and } \sup_{0<\tau<T}\int_\Omega u^n(\tau,x)^2dx \leq C_2, \tag{2.17}$$

where the constants C_1 and C_2 depend only on T, λ, $\int_\Omega f(x)^2dx$ and $\int_0^\tau\int_\Omega \underline{u}^n(x)^2dxdt$.

Combining (2.14) and (2.15) with (2.17), we conclude that

$$\varepsilon\int_0^T\int_\Omega |\nabla v^n(t,x)|^2dxdt \leq C_1, \quad \int_0^T\int_\Omega\int_\Omega K(x,y)|v^n(y)-v^n(x)|^2dydxdt \leq C_1$$

$$\text{and} \quad \int_0^T\int_\Omega\int_\Omega J(x,y)|u^n(y)-u^n(x)|^p dydxdt \leq C_2.$$

Consequently, the Lemma is hold. □

Passage to the limit. Taking φ as a test function in (2.3), multiplying (2.4) by $\phi\in\mathcal{D}(\overline{Q_T})$ and integrating over $\Omega\times\big[0,\tau\big]$, we have

$$-\int_0^\tau\int_\Omega v^n(t,x)\varphi_t(t,x)dxdt + \int_\Omega v^n(t,x)\varphi(t,x)dx\Big|_{t=0}^{t=\tau}$$
$$-\int_0^\tau\int_\Omega\int_\Omega K(x,y)(v^n(t,y)-v^n(t,x))dy\varphi(t,x)dxdt$$
$$+\,\varepsilon\int_0^\tau\int_\Omega \nabla v^n(t,x)\nabla\varphi(t,x)dxdt + \int_0^\tau\int_\Omega (f(x)-\underline{u}^n(t,x))\varphi(t,x)dxdt = 0, \tag{2.18}$$

$$-\int_0^\tau\int_\Omega u^n(t,x)\phi_t(t,x)dxdt+2\lambda\int_0^\tau\int_\Omega v^n(t,x)\phi(t,x)dxdt+\int_\Omega u^n(t,x)\phi(t,x)dx\Big|_{t=0}^{t=\tau}$$
$$-\int_0^\tau\int_\Omega\int_\Omega J(x,y)|u^n(t,y)-u^n(t,x)|^{p-2}(u^n(t,y)-u^n(t,x))dy\phi(t,x)dxdt=0. \tag{2.19}$$

Using Lemma 2.1, there exist a subsequence of (u^n, v^n) will be denoted also by (u^n, v^n) and limit functions u, v such that

$$\begin{aligned} u^n &\longrightarrow u \quad \text{weakly} \quad \text{in } L^2\big(Q_T\big), \\ v^n &\longrightarrow v \quad \text{weakly} \quad \text{in } L^2\big((0,T);H^1(\Omega)\big). \end{aligned} \tag{2.20}$$

It is not difficult to see that

$$\int_\Omega K(x,y)(v^n(t,y)-v^n(t,x))dy \text{ is bounded in } L^2(Q_T).$$

By the equation (2.3), we have

$$v_t^n(t,x)=\varepsilon\triangle v^n(t,x)+\int_\Omega K(x,y)(v^n(t,y)-v^n(t,x))dy-(f(x)-\underline{u}^n(t,x)).$$

Since v^n is bounded in $L^2(0,T;H^1(\Omega))$ and $\int_\Omega K(x,y)(v^n(t,y)-v^n(t,x))dy-(f(x)-\underline{u}^n(t,x))$ is bounded in $L^2\big(Q_T\big)$, we deduce that v_t^n is bounded in $L^2(0,T;(H^1(\Omega))')+L^2\big(Q_T\big)$. Thanks to the Aubin-Lions-Simon lemma [30], we have

$$v^n \longrightarrow v \quad \text{in} \quad L^2\big(Q_T\big). \tag{2.21}$$

Now, taking $\phi=(u^n-u^m)$ as a test function in (2.19), we obtain

$$\begin{aligned} &\int_0^\tau\int_\Omega(u^n(t,x)-u^m(t,x))\,(u^n(t,x)-u^m(t,x))_t dxdt \\ &+2\lambda\int_0^\tau\int_\Omega(v^n(t,x)-v^m(t,x))(u^n(t,x)-u^m(t,x))dxdt \\ &-\int_0^\tau\int_\Omega\int_\Omega J(x,y)|u^n(t,y)-u^n(tx)|^{p-2}(u^n(t,y)-u^n(t,x))dy \\ &(u^n(t,x)-u^m(t,x))dxdt+\int_0^\tau\int_\Omega\int_\Omega J(x,y)|u^m(t,y)-u^m(t,x)|^{p-2} \\ &(u^m(t,y)-u^m(t,x))dy(u^n(t,x)-u^m(t,x))dxdt\le 0. \end{aligned} \tag{2.22}$$

Thanks to the monotonicity lemma 2.3 of [3], we get

$$\frac{1}{2}\int_\Omega(u^n(t,x)-u^m(t,x))^2dx\Big|_{t=0}^{t=\tau}\le 2\lambda\int_0^\tau\int_\Omega|(v^n(t,x)-v^m(t,x))(u^n(t,x)-u^m(t,x))|dxdt. \tag{2.23}$$

Using Young's inequality, we have

$$\int_\Omega(u^n(t,x)-u^m(t,x))^2dx\le 2\lambda^2\int_0^\tau\int_\Omega(v^n(t,x)-v^m(t,x))^2dxdt \tag{2.24}$$
$$+2\int_0^\tau\int_\Omega(u^n(\tau,x)-u^m(\tau,x))^2dxdt.$$

Applying Gronwall's inequality, we obtain

$$\int_\Omega (u^n(\tau,x)-u^m(\tau,x))^2dx \leq \Big(2\lambda^2\int_0^\tau\int_\Omega (v^n(t,x)-v^m(t,x))^2dxdt\Big)\exp(2\tau), \quad \forall\tau\in[0,T]. \tag{2.25}$$

We have proved in (2.21) that $v^n \longrightarrow v$ in $L^2\big(Q_T\big)$. Consequently, from (2.25), we have

$$u^n \longrightarrow u \text{ in } L^2\big(Q_T\big).$$

Passing to the limit in (2.18)-(2.19) and using the previous convergence results, we conclude that

$$\begin{aligned}
&-\int_0^\tau\int_\Omega v(t,x)\varphi_t(t,x)+\int_\Omega v(t,x)\varphi(t,x)dx\Big|_{t=0}^{t=\tau}\\
&-\int_0^\tau\int_\Omega\int_\Omega K(x,y)(v(t,y)-v(t,x))dy\varphi(t,x)dxdt\\
&+\varepsilon\int_0^\tau\int_\Omega \nabla v(t,x)\nabla\varphi(t,x)dxdt+\int_0^\tau\int_\Omega (f(x)-\underline{u}(t,x))\varphi(t,x)dxdt=0,
\end{aligned} \tag{2.26}$$

$$\begin{aligned}
&-\int_0^\tau\int_\Omega u(t,x)\phi_t(t,x)dxdt+2\lambda\int_0^\tau\int_\Omega v(t,x)\phi(t,x)dxdt+\int_\Omega u(t,x)\phi(t,x)dx\Big|_{t=0}^{t=\tau}\\
&-\int_0^\tau\int_\Omega\int_\Omega J(x,y)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))dy\phi(t,x)dxdt=0.
\end{aligned} \tag{2.27}$$

We have

$$\int_0^T\int_\Omega\int_\Omega J(x,y)|u^n(y)-u^n(x)|^p dydxdt \leq C_2.$$

Consequently $J(x,y)|u^n(t,y)-u^n(t,x)|^{p-2}(u^n(t,y)-u^n(t,x))$ is bounded in $L^q\big(\Omega\times Q_T\big)$ $(1/p+1/q=1)$ which implies for a subsequence that

$$J(x,y)|u^n(t,y)-u^n(t,x)|^{p-2}(u^n(t,y)-u^n(t,x))$$

converges weakly to some function M. Since u^n converges almost everywhere to u, we deduce that $M(x,y,t)=J(x,y)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))$. Which shows that

$$\int_0^\tau\int_\Omega\int_\Omega J(x,y)|u^n(t,y)-u^n(t,x)|^{p-2}(u^n(t,y)-u^n(t,x))dy\phi(t,x)dxdt$$

converges to

$$\int_0^\tau\int_\Omega\int_\Omega J(x,y)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))dy\phi(t,x)dxdt.$$

Therefore, we have shown $\mathcal{F}(\underline{u}^n)\longrightarrow\mathcal{F}(\underline{u})$ in $L^2\big(Q_T\big)$ as $n\to\infty$. Consequently, $\mathcal{F}$ is a compact application. □

Next, we will prove the second hypotheses of Scaeffer's fixed point theorem.

Second step: **let us prove that the set**

$$\mathcal{M}=\Big\{u\in L^2(Q_T): u=\alpha\mathcal{F}(u) \text{ for some } \alpha\in[0,1]\Big\} \text{ is bounded in } L^2(Q_T).$$

Assume that $u \in L^2(Q_T)$ and $u = \alpha\mathcal{F}(u)$ for some $\alpha \in [0,1]$, then $u/\alpha = \mathcal{F}(u)$ and

$$v_t(t,x) = \varepsilon \triangle v(t,x) + \int_\Omega K(x,y)(v(t,y) - v(t,x))dy - (f(x) - u(t,x)), \quad (2.28)$$

$$\frac{u_t(t,x)}{\alpha} = \frac{1}{\alpha^{p-1}} \int_\Omega J(x,y)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))dy - 2\lambda v(t,x). \quad (2.29)$$

Taking v as a test function in (2.28), multiplying (2.29) by u and integrating over $\Omega \times \big[0,\tau\big]$, we obtain

$$\frac{1}{2}\int_\Omega v(\tau,x)^2dx + \varepsilon\int_0^\tau\int_\Omega \nabla v(t,x)\nabla v(t,x)dxdt + \int_0^\tau\int_\Omega (f(x) - u(t,x))v(t,x)dxdt$$
$$-\int_0^\tau\int_\Omega\int_\Omega K(x,y)(v(t,y) - v(t,x))dyv(t,x)dxdt = \frac{1}{2}\int_\Omega v(0,x)^2dx, \quad (2.30)$$

$$\frac{1}{2\alpha}\int_\Omega u(\tau,x)^2dx + 2\lambda\int_0^\tau\int_\Omega v(x,t)u(t,x)dxdt - \frac{1}{2\alpha}\int_\Omega u(0,x)^2dx$$
$$-\frac{1}{\alpha^{p-1}}\int_0^\tau\int_\Omega\int_\Omega J(x,y)|u(t,y) - u(t,x)|^{p-2}(u(t,y) - u(t,x))dyu(t,x)dxdt = 0. \quad (2.31)$$

Multiplying (2.30) by 2λ and adding the previous equality, we have

$$\frac{1}{2\alpha}\int_\Omega u(\tau,x)^2dx + \lambda\int_\Omega v(\tau,x)^2dx + \frac{1}{\alpha^{p-1}}\int_0^\tau\int_\Omega\int_\Omega J(x,y)|u(t,y)-u(t,x)|^p dydxdt$$
$$+ 2\lambda\int_0^\tau\int_\Omega\int_\Omega K(x,y)(v(t,y) - v(t,x))^2dydxdt + 2\lambda\int_0^\tau\int_\Omega f(x)v(t,x)dxdt$$
$$+ 2\lambda\varepsilon\int_0^\tau\int_\Omega |\nabla v(t,x)|^2dxdt = \frac{1}{2\alpha}\int_\Omega f(x)^2dx.$$

Using Young's inequality, it follows that

$$\int_0^\tau\int_\Omega f(x)v(t,x)dxdt \leq \frac{1}{2}\int_0^\tau\int_\Omega f(x)^2dx + \frac{1}{2}\int_0^\tau\int_\Omega (v(t,x))^2dxdt$$
$$\leq \frac{T}{2}\int_\Omega f(x)^2dx + \frac{1}{2}\int_0^\tau\int_\Omega (v(t,x))^2dxdt. \quad (2.32)$$

Consequently,

$$\frac{1}{2\alpha}\int_\Omega u(\tau,x)^2dx + \lambda\int_\Omega v(\tau,x)^2dx \leq \lambda T\int_\Omega f(x)^2dx + \lambda\int_0^\tau\int_\Omega (v(t,x))^2dxdt$$
$$+ \frac{1}{2\alpha}\int_\Omega f(x)^2dx. \quad (2.33)$$

Then, we have

$$\frac{1}{2\alpha}\int_\Omega u(\tau,x)^2dx + \lambda\int_\Omega v(\tau,x)^2dx \leq \lambda\int_0^\tau\int_\Omega (v(t,x))^2dxdt + \Big((\frac{1}{2} + \lambda T)\int_\Omega f(x)^2dx\Big). \quad (2.34)$$

Using Gronwall's inequality, we obtain

$$\int_\Omega v(\tau,x)^2dx \le \Big[\Big(\frac{1}{2}+\lambda T\Big)\int_\Omega f(x)^2dx\Big]\exp(\lambda\tau), \quad \forall\tau\in[0,T],$$

and

$$\int_\Omega v(\tau,x)^2dx \le \tilde{C}_1, \tag{2.35}$$

where the constant $\tilde{C}_1$ depends only on T, λ and $\int_\Omega f(x)^2dx$.

On the other hand, from (2.34) and (2.35), we have

$$\int_\Omega u(\tau,x)^2dx \le \lambda\tilde{C}_1 + \Big((\frac{1}{2}+\lambda T)\int_\Omega f(x)^2dx\Big),$$

Therefore,

$$\int_\Omega u(\tau,x)^2dx \le \tilde{C}_2, \tag{2.36}$$

where the constant $\tilde{C}_2$ depends only on T, λ and $\int_\Omega f(x)^2dx$.

Consequently, $\mathcal{M}$ is bounded in $L^2(Q_T)$ where the constant $\tilde{C}_1$ and $\tilde{C}_1$ are independent of α. Thanks to Schaeffer's fixed point theorem, the existence of solution of (P^ε) has been proved.

2.3. Passage to the limit

Now, we will prove that the solution of the problem (P^ε) converges to the solution of the problem (P). For that, let $(u^\varepsilon, v^\varepsilon)$ be a solution of the problem (P^ε) in the following sense

$$\begin{aligned}&-\Big\langle\frac{\partial\varphi}{\partial t}, v^\varepsilon\Big\rangle_{Q_\tau} - \int_0^\tau\int_\Omega\int_\Omega K(x,y)(v^\varepsilon(t,y)-v^\varepsilon(t,x))dy\varphi dxdt\\&+\varepsilon\int_0^\tau\int_\Omega\nabla v^\varepsilon(t,x)\nabla\varphi dxdt+\int_0^\tau\int_\Omega(f(x)-u^\varepsilon(t,x))\varphi dxdt+\int_\Omega v^\varepsilon(t,x)\varphi dx\Big|_0^\tau=0,\end{aligned} \tag{2.37}$$

and

$$\begin{aligned}&-\int_0^\tau\int_\Omega u^\varepsilon(t,x)\phi_t(t,x)dxdt+2\lambda\int_0^\tau\int_\Omega v^\varepsilon(t,x)\phi(t,x)dxdt+\int_\Omega u^\varepsilon(t,x)\phi(t,x)dx\Big|_0^\tau\\&-\int_0^\tau\int_\Omega\int_\Omega J(x,y)|u^\varepsilon(y,t)-u^\varepsilon(t,x)|^{p-2}(u^\varepsilon(t,y)-u^\varepsilon(t,x))dy\phi(t,x)dxdt=0,\end{aligned} \tag{2.38}$$

for every $\tau\in(0,T]$ and every test-functions

$$\varphi\in L^2\Big((0,T);H^1(\Omega)\Big),\ \frac{\partial\varphi}{\partial t}\in L^2\Big((0,T);(H^1(\Omega))'\Big),\ \phi\in\mathcal{D}(\overline{Q_T}) \text{ and } \phi_t\in\mathcal{D}(\overline{Q_T}).$$

Thanks to the Lemma 2.1, we have that u^ε, v^ε and $\sqrt{\varepsilon}\nabla v^\varepsilon$ are bounded in $L^2\big(Q_T\big)$. Then,

$$u^\varepsilon \longrightarrow u \qquad \text{weakly} \quad \text{in } L^2\big(Q_T\big),$$

$$\begin{aligned} v^\varepsilon &\longrightarrow v \quad \text{weakly} \quad \text{in } L^2\big(Q_T\big),\\ \varepsilon\nabla u^\varepsilon &\longrightarrow 0 \quad \text{weakly} \quad \text{in } L^2\big(Q_T\big). \end{aligned} \tag{2.39}$$

Moreover, using Holder inequality, we have

$$\frac{1}{2}\int_0^T \int_\Omega\int_\Omega J(x,y)|u^\varepsilon(y)-u^\varepsilon(x)|^p dydxdt \le C_2. \tag{2.40}$$

Hence, for any measurable subset $E \subset \Omega\times\Omega$, we see that

$$\begin{aligned} L &= \Big|\int\int_E J(x,y)|u^\varepsilon(y)-u^\varepsilon(x)|^{p-2}(u^\varepsilon(y)-u^\varepsilon(x))dydx\Big|\\ &\le \int\int_E J(x,y)|u^\varepsilon(y)-u^\varepsilon(x)|^{p-1}dydx \le C_2'|E|^{\frac{1}{p}}. \end{aligned}$$

Applying Dunfort-Pettis Theorem, there exists $\vartheta(x,y)\in L^1(Q_T\times Q_T)$ with $\vartheta(x,y) = -\vartheta(y,x)$ such that

$$J(x,y)|u^\varepsilon(y)-u^\varepsilon(x)|^{p-2}(u^\varepsilon(y)-u^\varepsilon(x)) \rightharpoonup J(x,y)\vartheta(x,y) \text{ weakly in } L^1(Q_T\times Q_T). \tag{2.41}$$

Letting $\varepsilon\to 0$ in (2.37–2.38) and using the previous convergence (2.39) and (2.41), we obtain

$$\begin{aligned} &-\int_0^\tau\int_\Omega v(t,x)\varphi_t(t,x)dxdt + \int_0^\tau\int_\Omega (f(x)-u(t,x))\varphi(t,x)dxdt\\ &-\int_0^\tau\int_\Omega\int_\Omega K(x,y)(v(t,y)-v(t,x))dy\varphi(t,x)dxdt + \int_\Omega v(t,x)\varphi(t,x)dx\Big|_0^\tau = 0, \end{aligned} \tag{2.42}$$

and

$$\begin{aligned} &-\int_0^\tau\int_\Omega u(t,x)\phi_t(t,x)dxdt + 2\lambda\int_0^\tau\int_\Omega v(t,x)\phi(t,x)dxdt + \int_\Omega u(t,x)\phi(t,x)dx\Big|_0^\tau\\ &-\int_0^\tau\int_\Omega\int_\Omega J(x,y)\vartheta(x,y)dy\phi(t,x)dxdt = 0. \end{aligned} \tag{2.43}$$

Now, let us prove that

$$\begin{aligned} &\int_0^\tau\int_\Omega\int_\Omega J(x,y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))dyu(x,t)dxdt\\ &=\int_0^\tau\int_\Omega\int_\Omega J(x,y)\vartheta(x,y)dyu(x,t)dxdt. \end{aligned} \tag{2.44}$$

Taking $\varphi = v^\varepsilon$ in (2.37) (respectively $\phi = u^\varepsilon$ in (2.38)), multiplying (2.37) by 2λ, and adding the two equations, we get

$$\begin{aligned} &\frac{1}{2}\int_\Omega (u^\varepsilon(\tau,x))^2dx + \lambda\int_\Omega (v^\varepsilon(\tau,x))^2dxdt + 2\lambda\varepsilon\int_0^\tau\int_\Omega |\nabla v^\varepsilon(t,x)|^2dxdt\\ &-\int_0^\tau\int_\Omega\int_\Omega J(x,y)|u^\varepsilon(t,y)-u^\varepsilon(t,x)|^{p-2}(u^\varepsilon(t,y)-u^\varepsilon(t,x))dyu^\varepsilon(t,x)dxdt\\ &-2\lambda\int_0^\tau\int_\Omega\int_\Omega K(x,y)(v^\varepsilon(t,y)-v^\varepsilon(t,x))dyv^\varepsilon(t,x)dxdt + 2\lambda\int_0^\tau\int_\Omega v^\varepsilon(t,x)f(x)dxdt \end{aligned}$$

$$= \frac{1}{2}\int_\Omega (u^\varepsilon(0,x))^2 dx + \lambda \int_\Omega (v^\varepsilon(0,x))^2 dx. \tag{2.45}$$

We consider a sequence of mollifiers $(\rho^k)_k$ satisfying $\int_{\mathbb{R}^N} \rho(x)dx = 1$ and $\rho^k(x) = k^N\rho(kx)$. Let $v^k = \rho^k * (\rho^k * v)$ and $u^k = \rho^k * (\rho^k * u)$ be a regularization sequences of v and u, respectively, such that

$$v^k \longrightarrow v \text{ strongly in } L^2(Q_T) \qquad \text{and} \qquad u^k \longrightarrow u \text{ strongly in } L^2(Q_T). \tag{2.46}$$

By choosing $\varphi = v^k$ in (2.42) (respectively $\phi = u^k$ for (2.43)), we obtain

$$\begin{aligned}&\frac{1}{2}\int_\Omega (\rho^k * v(\tau,x))^2 dx - \int_0^\tau \int_\Omega \int_\Omega K(x,y)(v(t,y) - v(t,x))dyv^k(t,x)dxdt \\ &+ \int_0^\tau \int_\Omega (f(x) - u(t,x))v^k(t,x)dxdt = \frac{1}{2}\int_\Omega (\rho^k * v(0,x))^2 dx,\end{aligned} \tag{2.47}$$

$$\begin{aligned}&\frac{1}{2}\int_\Omega (\rho^k * u(\tau,x))^2 dx + 2\lambda \int_0^\tau \int_\Omega v(t,x)u^k(t,x)dxdt \\ &- \int_0^\tau \int_\Omega \int_\Omega J(x,y)\vartheta(x,y)dyu^k(t,x)dxdt = \frac{1}{2}\int_\Omega (\rho^k * u(0,x))^2 dx.\end{aligned} \tag{2.48}$$

Letting $k \to \infty$, multiplying (2.47) by 2λ and adding the previous equalities, we have

$$\begin{aligned}&\frac{1}{2}\int_\Omega (u(\tau,x))^2 dx + \lambda \int_\Omega (v(\tau,x))^2 dx - \int_0^\tau \int_\Omega \int_\Omega J(x,y)\vartheta(x,y)dyu(t,x)dxdt \\ &+ 2\lambda \int_0^\tau \int_\Omega v(t,x)f(x)dxdt - 2\lambda \int_0^\tau \int_\Omega \int_\Omega K(x,y)(v(t,y) - v(t,x))dyv(t,x)dxdt \\ &= \frac{1}{2}\int_\Omega (u(0,x))^2 dx + \lambda \int_\Omega (v(0,x))^2 dx.\end{aligned} \tag{2.49}$$

Combining (2.45) and (2.49), we deduce that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\Big[\int_0^\tau \int_\Omega \int_\Omega J(x,y)|u^\varepsilon(y,t) - u^\varepsilon(x,t)|^{p-2}(u^\varepsilon(y,t) - u^\varepsilon(x,t))dyu^\varepsilon(x,t)dxdt \\ &-\int_0^\tau\int_\Omega\int_\Omega J(x,y)\vartheta(x,y)dyu(x,t)dx\Big] = \lim_{\varepsilon\to 0}\Big[\frac{1}{2}\int_\Omega (u^\varepsilon(\tau,x))^2dx - \frac{1}{2}\int_\Omega (u(\tau,x))^2dx \\ &+\lambda\int_\Omega (v^\varepsilon(\tau,x))^2dx - 2\lambda\varepsilon\int_0^\tau\int_\Omega |\nabla v^\varepsilon(t,x)|^2dxdt - \lambda\int_\Omega (v(\tau,x))^2dx \\ &-2\lambda\int_0^\tau\int_\Omega\int_\Omega K(x,y)(v^\varepsilon(t,y) - v^\varepsilon(t,x))dyv^\varepsilon(t,x)dxdt + 2\lambda\int_0^\tau\int_\Omega v^\varepsilon(t,x)f(x)dxdt \\ &-2\lambda\int_0^\tau\int_\Omega v(t,x)f(x)dxdt + 2\lambda\int_0^\tau\int_\Omega\int_\Omega K(x,y)(v(t,y) - v(t,x))dyv(t,x)dxdt\Big] \geq 0.\end{aligned} \tag{2.50}$$

By (1.5), we remark that

$$-2\int_0^\tau\int_\Omega\int_\Omega K(x,y)(v^\varepsilon(t,y) - v^\varepsilon(t,x))dyv^\varepsilon(t,x)dxdt$$

$$= \int_0^\tau \int_\Omega \int_\Omega K(x,y)(v^\varepsilon(t,y) - v^\varepsilon(t,x))^2 dydxdt.$$

By passing to the limit

$$\limsup_{\varepsilon\to 0} -2\int_0^\tau \int_\Omega \int_\Omega K(x,y)(v^\varepsilon(t,y) - v^\varepsilon(t,x))dyv^\varepsilon(t,x)dxdt$$
$$\geq \int_0^\tau \int_\Omega \int_\Omega K(x,y)(v(t,y) - v(t,x))^2 dydxdt,$$

we deduce that

$$\lim_{\varepsilon\to 0}\Big[\int_0^\tau \int_\Omega \int_\Omega J(x,y)|u^\varepsilon(y,t) - u^\varepsilon(x,t)|^{p-2}(u^\varepsilon(y,t)u^\varepsilon(x,t))dyu^\varepsilon(x,t)dxdt$$
$$- \int_0^\tau \int_\Omega \int_\Omega J(x,y)\vartheta(x,y)dyu(x,t)dx\Big] \geq 0.$$

By monotonicity and by putting, we have $U^\varepsilon(t,y,x) = u^\varepsilon(t,y) - u^\varepsilon(t,x)$ and $U(t,y,x) = u(t,y) - u(t,x)$,

$$\int_0^\tau \int_\Omega \int_\Omega J(x,y)(|U^\varepsilon(t,y,x)|^{p-2}(U^\varepsilon(t,y,x) - |U(t,y,x))|^{p-2}U(t,y,x))$$
$$(U^\varepsilon(t,y,x) - U(t,y,x))dydxdt \geq 0.$$

Thanks to (2.50) and the fact that

$$\int_0^\tau \int_\Omega \int_\Omega J(x,y)\vartheta(x,y)dyu(x,t)dxdt = -\int_0^\tau \int_\Omega \int_\Omega J(x,y)dxu(y,t)dxdt,$$

we deduce that

$$\int_0^\tau \int_\Omega \int_\Omega J(x,y)(|U^\varepsilon(t,y,x)|^{p-2}(U^\varepsilon(t,y,x) - |U(t,y,x))|^{p-2}U(t,y,x))$$
$$(U^\varepsilon(t,y,x) - U(t,y,x))dydxdt$$

converges to 0. Consequently, up to a subsequence, we have

$$J(x,y)^{1/p}U^\varepsilon(t,y,x) \to J(x,y)^{1/p}U(t,y,x) \text{ a.e. } (t,y,x) \in (0;T)\times\Omega\times\Omega$$

which shows that $J(x,y)\vartheta(x,y) = J(x,y)|u(t,y) - u(t,x)|^{p-2}(u(t,y) - u(t,x))$ a.e. $(t,y,x) \in (0;T)$.

Finally, by letting ε to 0 in (P_ε), we have proven that for $\varepsilon \to 0$ the solution of the system (P^ε) converges to the solution of (P) and the existence of the solution of the problem (P) have been proven. □

2.4. Uniqueness

In this subsection, we prove the uniqueness of the solution of the problem (P). For that, let us assume that the problem (P) admits two solutions (u_1, v_1) and (u_2, v_2) and taking $(u_1 - u_2, v_1 - v_2)$ as a test function in the definition of solutions. Then, we have

$$\int_0^\tau \int_\Omega (u_1 - u_2)_\tau (u_1 - u_2)dxdt + 2\lambda \int_0^\tau \int_\Omega (v_1 - v_2)(u_1 - u_2)dxdt$$

$$+\int_0^\tau\int_\Omega\int_\Omega J(x,y)\Big[|u_1(y)-u_1(x)|^{p-2}(u_1(y)-u_1(x))$$
$$-|u_2(y)-u_2(x)|^{p-2}(u_2(y)-u_2(x))\Big]dy(u_1(x)-u_2(x))dxdt=0, \quad (2.51)$$

$$\int_0^\tau\int_\Omega(v_1-v_2)_\tau(v_1-v_2)dxdt+\int_0^\tau\int_\Omega(u_2-u_1)(v_1-v_2)dxdt$$
$$+\int_0^\tau\int_\Omega\int_\Omega K(x,y)\Big[(v_1(y)-v_1(x))-(v_2(y)-v_2(x))\Big]dy(v_1(x)-v_2(x))dxdt=0. \quad (2.52)$$

Using the monotonicity and adding $2\lambda\times$(2.52) to (2.51), we have

$$\int_0^\tau\int_\Omega(u_1-u_2)_\tau(u_1-u_2)dxdt+2\lambda\int_0^\tau\int_\Omega(v_1-v_2)_\tau(v_1-v_2)dxdt\le 0, \quad (2.53)$$

then we get

$$\frac{1}{2}\int_\Omega(u_1-u_2)^2dx+\lambda\int_\Omega(v_1-v_2)^2dx\le 0. \quad (2.54)$$

Consequently, we obtain $u_1(t,x)=u_2(t,x)$ and $v_1(t,x)=v_2(t,x)$ a.e. in Q_T. This completes the proof of the Theorem. □

3. Numerical aspects and results

In this section, we present the numerical results and comparative experiments obtained by implementing our proposed model. To compute numerically the problem (P), we choose the weight function of the form $\exp\Big(-\frac{d(x,y)}{\sigma^2}\Big)$ where $d(x,y)$ is the distance between patches located at x and y, σ is a positive constant which acts as a scale parameter. Let u_i be the value of a pixel i in the image, $J_{i,j}$ and $K_{i,j}$ are respectively the sparsely discrete version of the weight functions $J(x,y)$ and $K(x,y)$. Now, using the explicit Euler method with Neumann boundary condition, the discrete iterative schemes of the problem (P) can be written as:

$$\begin{cases}\dfrac{v_i^{k+1}-v_i^k}{\tau}=\displaystyle\sum_{j\in\mathcal{N}_i}K_{ij}(v_j^k-v_i^k)-(f-u_i^k),\\ \dfrac{u_i^{k+1}-u_i^k}{\tau}=\displaystyle\sum_{j\in\mathcal{N}_i}J_{ij}|u_j^k-u_i^k|^{p-2}(u_j^k-u_i^k)-2\lambda v_i^{k+1},\\ u_i^0=f_i, \qquad v_i^0=0,\end{cases} \quad (3.1)$$

where $\mathcal{N}_i$ is the neighbors set, τ is the time step size and k is the iteration number.

For the numerical experiments, we set $\lambda=0.001$, $p=1.0001$ and the time step size $\tau=0.1$. For computing the weight function, we take $d(x,y)=\int_\Omega G_a(z)|f(x+z)-f(y+z)|^2dz$ where G_a is a Gaussian function with standard deviation a and we choose a patches size of 11×11 (i.e. $P=5$), a search window of 23×23 (i.e. $N_w=11$) and $a=2$.

We start by the denoising tests (cf. Figs. 1-2). Secondly, we prove the ability of our algorithm to denoise images based on different noise (cf. Figs. 3-4). Then a comparison test will be done to show the advantages of using nonlocal instead of the local (cf. Figs. 5-6–7). At last another comparison test is conducted to justify that the proposed method performs better than other existing nonlocal methods in regarding both quality (cf. Figs. 8-9-10) and computation time (cf. Figs. 11). We applied statistical measure in order to evaluate the quality of the restoration results which are the peak signal to noise ratio (PSNR) and the signal–to–noise ratio (SNR) that can expressed by:

$$\mathrm{PSNR} = 10\log_{10}\Big[\frac{255^2 MN}{||u_0 - u||_2^2}\Big] dB,$$

where u_0, u and $M \times N$ are the original image, the restored image and the size of the image, respectively, and

$$\mathrm{SNR} = \log_{10}\big[\frac{\sigma_u}{\sigma_n}\big] dB,$$

where σ_u and σ_n are the signal and noise standard deviations, respectively.

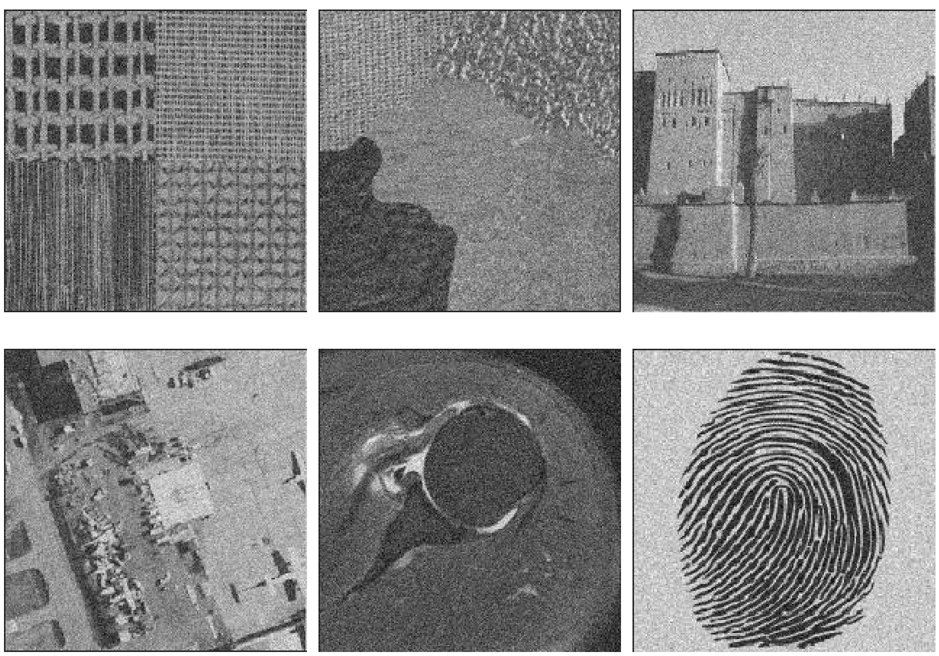


Figure 1. Images corrupted by Gaussian noise with zero mean and variance $\sigma^2 = 0.025$

We notice that our proposed model can clearly denoise the images corrupted by the additive noise (cf. Figs. 1-2). To show the effectiveness of our proposed model, we will present in Figure 3 and Figure 4 the results of denoising Lena and House images by applying additive gaussian noise with different noise levels $\sigma^2 \in \Big\{0.016; 0.025; 0.030; 0.035\Big\}$.

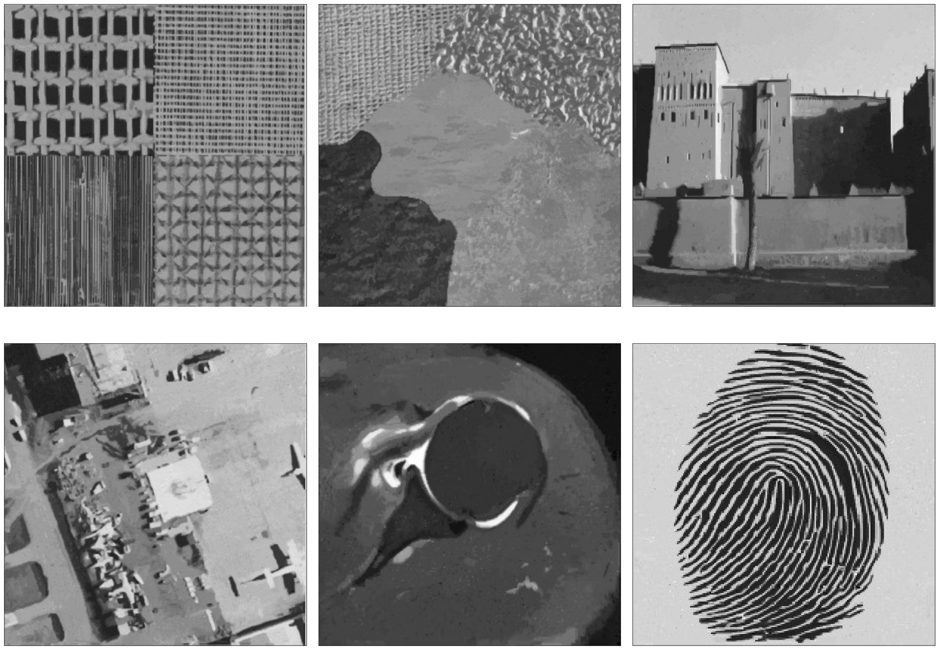


Figure 2. Restored images with the proposed model.

(a) σ^2=0.016 (b) σ^2=0.025 (c) σ^2=0.030 (d) σ^2=0.035

Figure 3. First row : Noisy Lena images with noise at zero mean and different variances. Second row : Restored images using the proposed algorithm.

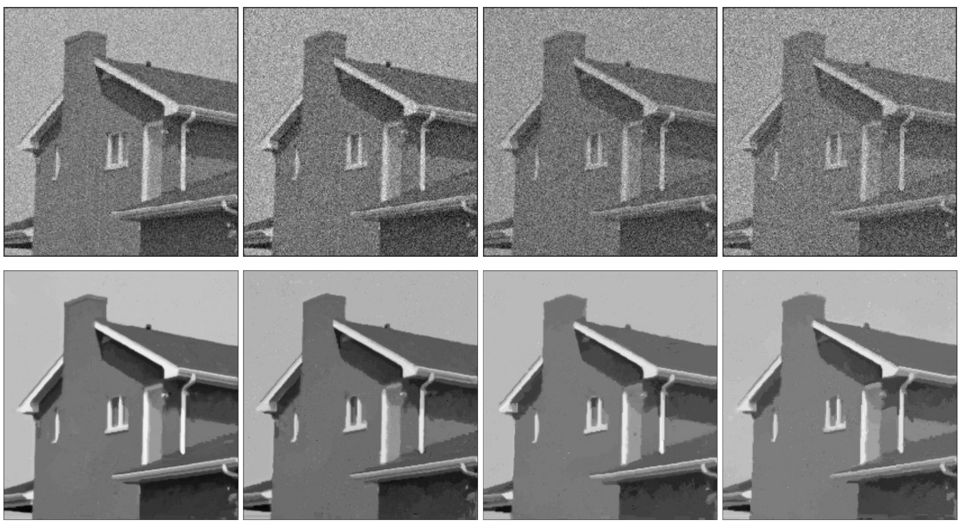
In the second experiment, we give a comparison between the proposed method and the local methods which is the local system (TVH) proposed in [16] where the authors have used total variation minimization and the H^{-1} norm (TVH). The Fig 5 indicate that our method could raises the PSNR data remarkably compared with the local methods.

Table 1. Noise performance parameters PSNR and SNR for Lena image with noise at zero mean and different variances.

Lena image	SNR		PSNR	
	Noisy Image	Restored Image	Noisy Image	Restored Image
$\sigma^2 = 0.016$	9.5232	17.7676	24.0557	32.3000
$\sigma^2 = 0.025$	5.6507	15.7410	20.1832	30.2734
$\sigma^2 = 0.030$	4.0241	14.8730	18.5566	29.4057
$\sigma^2 = 0.035$	2.7176	13.7395	17.2501	28.2714

Table 2. Noise performance parameters PSNR and SNR for House image with noise at zero mean and different variances.

House image	SNR		PSNR	
	Noisy Image	Restored Image	Noisy Image	Restored Image
$\sigma^2 = 0.016$	9.1522	17.9931	24.0237	32.8635
$\sigma^2 = 0.025$	5.3194	16.3607	20.1909	31.2313
$\sigma^2 = 0.030$	3.7248	15.6402	18.5964	30.5113
$\sigma^2 = 0.035$	2.4069	14.9337	17.2748	29.8019



(a) σ^2=0.016 (b) σ^2=0.025 (c) σ^2=0.030 (d) σ^2=0.035

Figure 4. First row : Noisy House images with noise at zero mean and different variances. Second row : Restored images using the proposed algorithm.

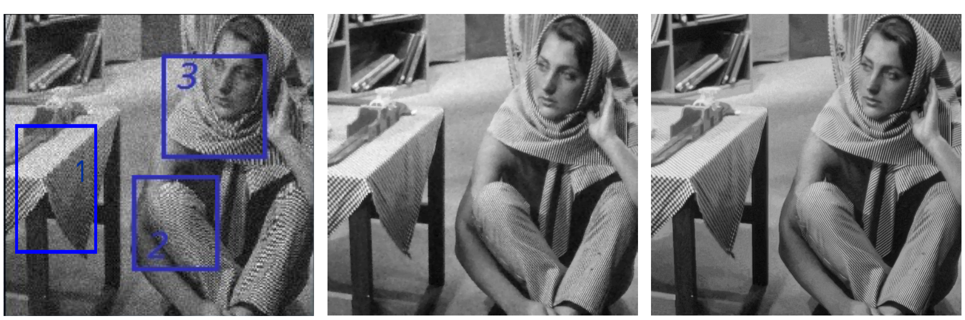


Figure 5. Left : Noisy Barbara image with $PSNR = 26.5544$, middle : Restored image by TVH with $PSNR = 29.0488$ and Right : Restored image by our model (NLH) with $PSNR = 32.5609$.

Furthermore, the texture is most favorable case of using the nonlocal concept. It is well known that the textured images have a large redundancy where the nonlocal method identifies automatically pixels with the same reflectance in the image. To demonstrate the benefits of using our new nonlocal model, we use the Barbara pictures which contain smooth and textured regions. In Fig. 6, we see an example in the zoomed region 1 (Barbara's tablecloth), region 2 (Barbara's pant) and region 3 (Barbara's scarf) where the ability of our model to preserve the main features even in the case of high frequencies compared with the local models. In Figure 7, we present the denoising error image.

Figure 6. Left column : Zoom of the results of TVH and right column : Zoom of the images restored using NLH.

(a) The smooth image using TVH model

(b) The texture of $u - f$ using TVH model

(c) The smooth image u using our model

(d) The texture of $u - f$ using our model

Figure 7. Here, we compare the texture of restored images between our model and TVH model.

In the next experiment, an another comparison emphasizes the efficiency of the proposed model as preserving texture and fine details compared with the non local p-Laplacian equation (NLPL) presented in [6] where the authors use the L^2 norm in fidelity term. We use the same parameters for both algorithms and we compare the results. Fig. 8 shows that using our model gives better results where the PSNR and SNR of the noisy images, restored ones by the NLPL and by our model are displayed in Table 3. To validate the ability of the proposed model to preserve the texture, we shall zoom the region containing the texture (cf. Fig 9) and we compare the NLPL and our denoising error image (Fig. 10). Our denoising error image contains many less details which proves that the proposed model is more robust in preserving texture and fine details.

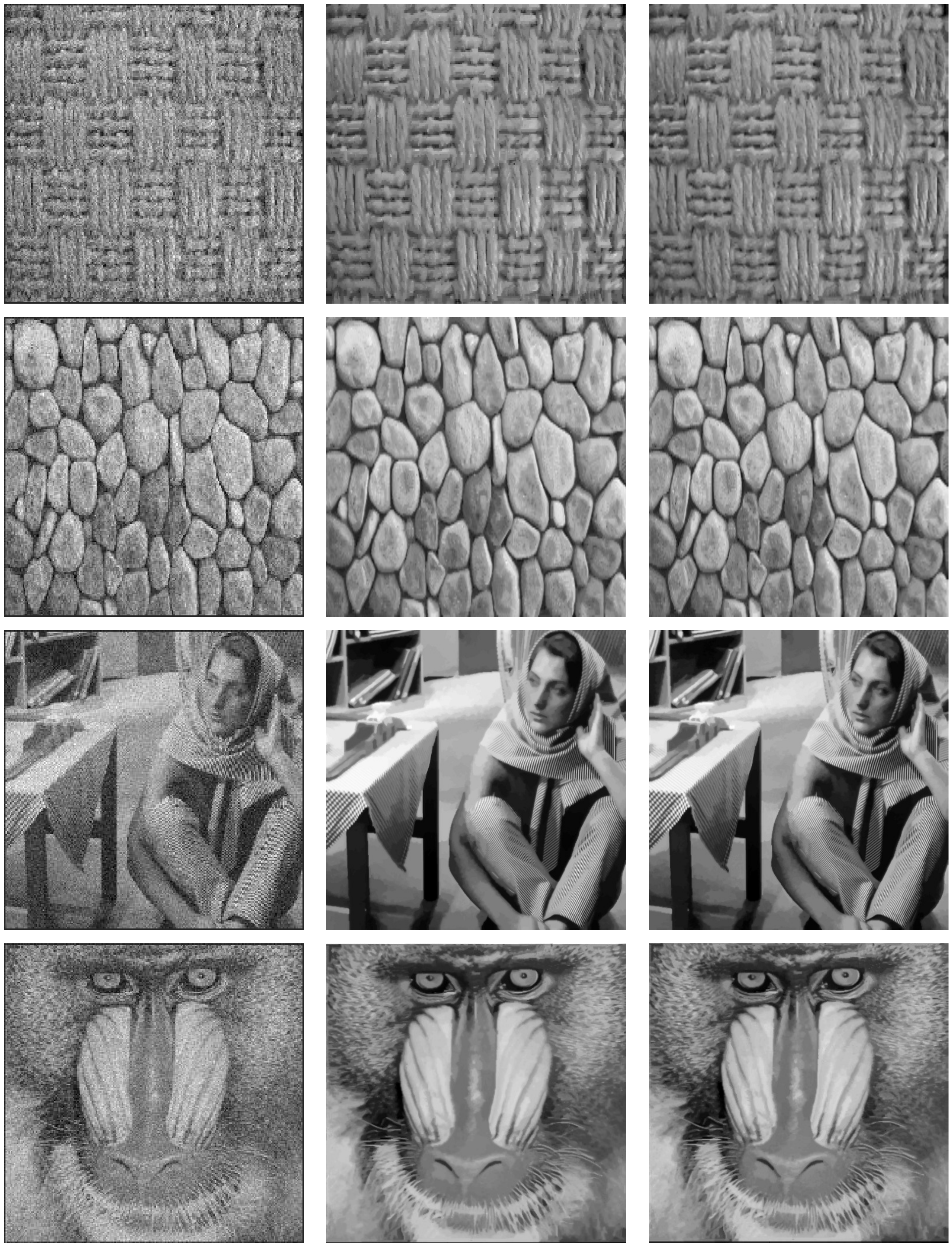


Figure 8. Left column: Top to Bottom: Noisy image A, Noisy image B, Noisy Barbara image and Noisy image C. Middle column: Restored images by NLPL and Right column: Restored images by our proposed model.

Now, we evaluate the performance and the efficiency of our proposed model compared to Nonlocal means presented in [8] by using CPU time. For the comparison we used a different textured test images and the same parameters to compute the weight function (patches size of 11×11 and a search window of 23×23) for both algorithms. Table shows that the proposed model delivers acceptable results as compared to the algorithm in [8] in a shorter amount of time. Figure 11 depicts,

Table 3. PSNR and SNR values of noisy images and the restored ones by NLPL and our model.

	PSNR Noisy	SNR Noisy	PSNR NLPL	SNR NLPL	PSNR NLH	SNR NLH
Image A	24.6100	4.5701	26.7278	6.6883	**27.0838**	**7.0442**
Image B	24.6135	9.3076	27.5997	12.2951	**27.7156**	**12.4105**
Barbara	24.6005	11.2148	29.2001	15.8145	**30.5295**	**17.1439**
Image C	24.6120	8.0629	28.5136	11.9647	**28.8496**	**12.3007**

Figure 9. Left column : Zoom of the result of NLPL and Right column : Zoom of the image restored by our model.

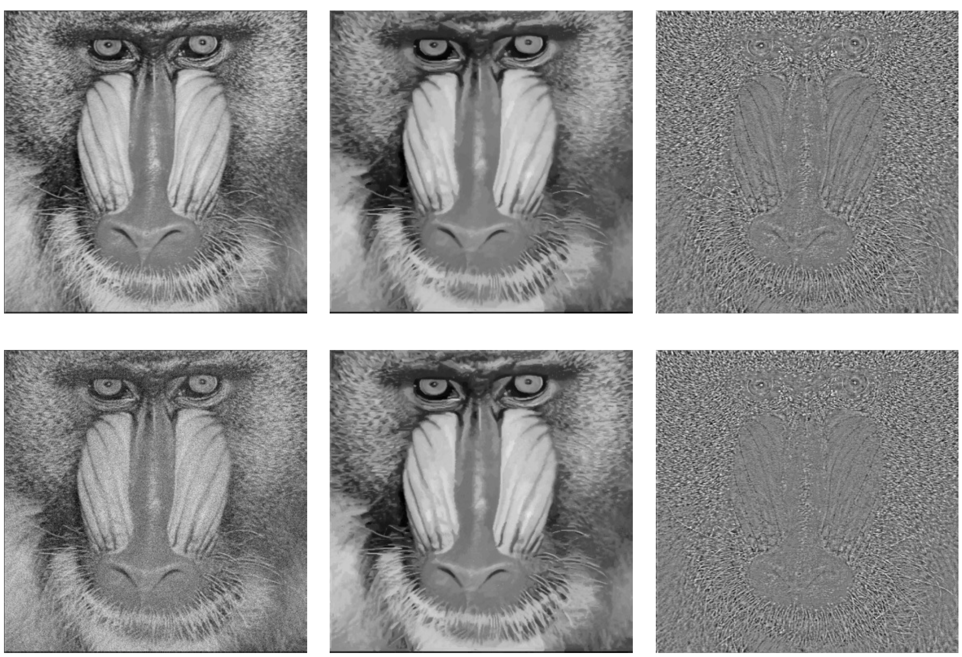


Figure 10. Left column: original image (top) and noisy image (bottom). Middle column: NLPL denoising result and error (PSNR=27.6702). Right column: our model denoising result and error (PSNR= 28.3045).

from left to right, the output image of NLmeans method, and the output of the proposed model. While NLmeans was able to recover a better image, the proposed method computed its output (on the right) in a shorter time as presented in Table 4. In fact, NLmeans does not even recover an image of comparable quality to that of the proposed method in more time in the experiment presented in Figure 11. This proves that can indeed be very efficient in regards to both quality and computation time.

Table 4. CPU time and PSNR of our method and NLmeans method, respectively, for textured images displayed in Figure (11)

	Image C		Lena image		Image D	
Image	PSNR	Time (s)	PSNR	Time (s)	PSNR	Time (s)
Proposed Model	28.9939	**59.7811**	32.4431	**149.2656**	12.6062	**17.8987**
NLmeans	28.8566	139.6565	32.2499	218.9331	12.6016	24.9257

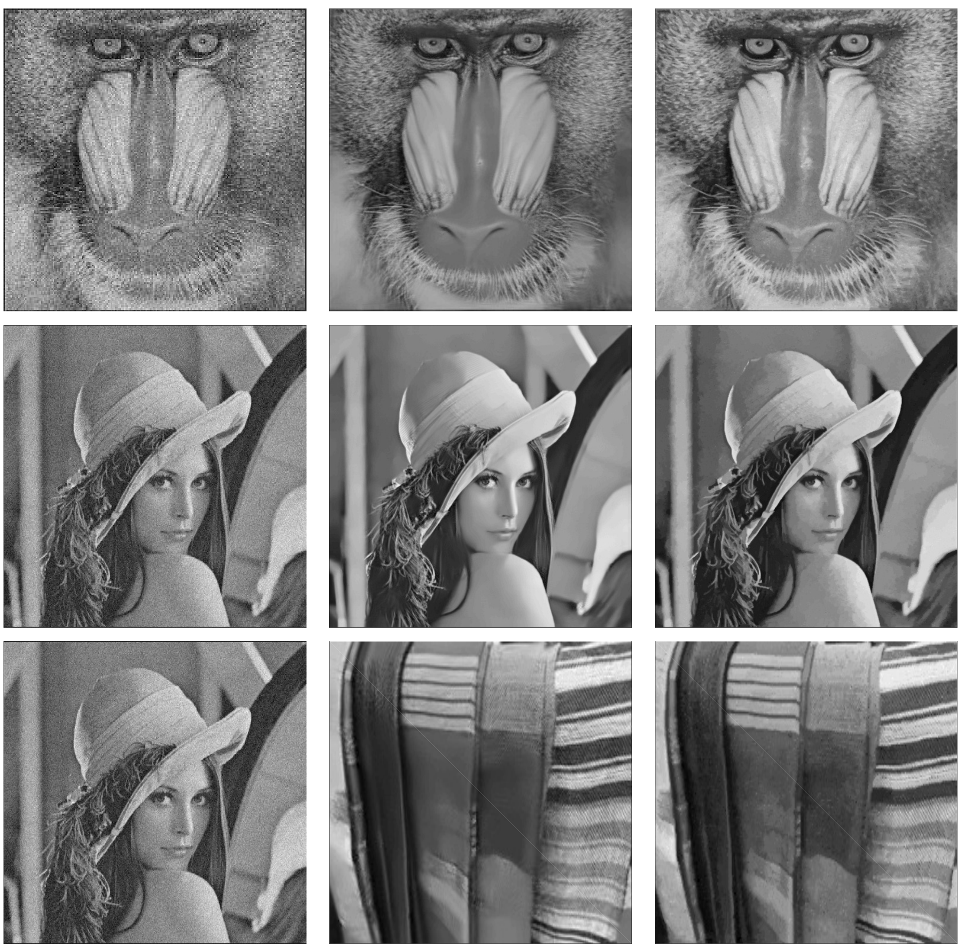


Figure 11. Left column: Top to Bottom: Noisy image C, Noisy Lena image, and Noisy image D. Middle column: Restored images by NLmeans and Right column: Restored images by our proposed model.

4. Conclusion

In this paper, we have proposed a new nonlocal reaction-diffusion system based on the decomposition approach of H^{-1} norm for filtering textured images. The proposed model is more robust than the local models in removing noise and specially in preserving small details and texture.

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