Journal of Applied Analysis and Computation
Volume 8, Number 4, August 2018, 1122–1142
Website:http://jaac-online.com/
DOI:10.11948/2018.1122

THE OPTIMAL EXECUTION STRATEGY OF EMPLOYEE STOCK OPTIONS*

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Abstract In this paper, we develop an optimal execution strategy for employee stock options by means of the fluid model, in which a voluntary turnover is considered. We show that the value function is the viscosity solution of the Hamilton-Jacobi-Bellman variational inequality and prove the uniqueness of the viscosity solution. Finally, we present numerical illustrative examples and numerical solutions of optimal strategies which are computed by the finite difference method.

Keywords Optimal execution strategy, ESO, HJBVI, viscosity solution.

MSC(2010) 39A14, 93E20.

1. Introduction

An employee stock option (ESO) is an individually awarded call option on the common stock of a company, granted by the company to an employee as a part of the employee's remuneration package. The objective of ESOs is to give employees an incentive to improve a company's market value by benefitting themselves from a higher market price of the company's shares [1].

Employee stock options are characterized by the following differences from standardized exchange-traded financial options. (i) ESOs are non-transferable. Employees are not allowed to sell their employee stock options. They can exercise the options if the circumstances are appropriate. (ii) Unlike exchange-traded options, ESOs are considered to be a private contract between the employer and the employee. (iii) ESOs are usually characterized by a long holding period, or, a maturity far from their issuance. (iv) ESOs are usually characterized by a vesting period. During the vesting period, the options cannot be exercised (The vesting period can e.g. be as long as four years.). (v) ESOs are American-style options. Once vested, ESOs can be exercised at any time before expiration. (vi) ESOs have a departure risk. Employees will lose a part or the total value of the unexecuted ESOs, if they are laid off or leave the company voluntarily.

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*The authors were supported by MOE (Ministry of Education in China) Youth Project of Humanities and Social Sciences (Project No.17YJCZH044),The National Natural Science Foundation for Young Scientists of China (Project No.11701377), The National Natural Science Foundation of China (Project No. 11671292, 11631004).

Due to the wide use of employee stock options, there exists a large amount of research on their features, valuation and execution strategies, respectively. Research has e.g. been done on the effects of the non-tradability, hedging restrictions and early exercise. Hull and White (2004) [13] discuss practical issues in valuing employee stock options by means of the binomial method. Referring to the likelihood of an early exercise, Cvitanic, Wiener and Zapatero (2008) [10] derive an analytic formula for the price of an ESO. León and Vaello-Sebastià (2009) [15] value typical long-dated American style ESOs with the volatility of the underlying asset assumed to be time-varying under a GARCH framework. Monyios and Ng(2011) [17] focus on American-style feature of ESOs to maximize the expected discounted payoff over all stopping times. Henderson et al. (2012) [12] use a utility-based valuation approach to study strategies of managing a portfolio which contains ESOs with different strikes and maturities, respectively. Chang et al. (2015) [5] provide empirical evidence on the positive effect of non-executive employee stock options on corporate innovation. Chong et al. (2015) [6] extend the approach of option pricing for executive stock options (ESOs) on the basis of stochastic volatility (SV) models. Kyng et al. (2016) [14] extend the exercise multiple approach of Hull and White (2004) [13] and use exit probabilities obtained from empirically determined multiple decrement or life tables to model involuntary early exercise or forfeiture.

Other research has focused on the repricing and vesting feature. Brenner et al. (2000) [3] deal with properties of repriceable ESOs and develop a corresponding valuation model. By using the utility maximization approach, Corrado et al. (2001) [7] extend this model such that multiple times of repricing are allowed. By means of these methods, Leung and Kwok (2008) [16] study various repricing mechanisms based on some forms of Brownian functionals of the stock price process. In their model, the firms exercise repricing only when the stock price falls below some target barrier level for a certain period of time. Wu et al. (2016) [20] conduct a more general fair value estimation based on attaching performance targets to option vesting. Callaghan et al.(2016) [4] show that firms do retain managers when they reprice their options compared to when they do not.

In contrast to the previously mentioned valuation models for stand-alone ESOs, Bian et al.(2015) [2] assume the number of options to be continuous. Bian et al. adopt a fluid model to characterize the exercise process and restrict the exercise rate not to exceed an upper boundary. They aim thus to maximize the overall discounted exercise return instead of pursuing on utility maximization.

Following [2], we develop in this paper an optimal execution strategy for employee stock options with a term of resignation. More precisely, we use control variable to indicate the execution rate during the employment and use stopping time to indicate the time of voluntary turnovers. Although the ESOs can be executed all at once at leaving time, the corresponding payoff from exercising the option will be restricted by the resetting strike price and a penalty factor. Using the Dynamic Programming Principle (DPP) of [18], the Hamilton-Jacobi-Bellman variational inequality(HJBVI) is established. The strategy and the payoff of ESOs are obtained from the solution of this equation.

In general, the HJBVI equation method is an appropriate choice for a continuous execution strategy model, because the target (i.e. the value function) of the model is always a combination of a stochastic control problem and an optimal stopping problem. However, the HJBVI equation is usually not well-posed, and the value function is often not smooth enough to satisfy the preconditions of the HJBVI equation in

the classical sense. Therefore, it makes sense to ask for a weak solution such that the value function is unique even though it is not smooth. Such a weak solution is called *viscosity solution* [8] [9]. More properties of the weak solutions are discussed by Soner(1997) [19]. In this paper, the existence and the comparison principle are proved by using the DPP and the contradiction method. The uniqueness of the solution will be obtained, too.

This paper is organized as follows. In Section 2, we rigorously establish the mathematical model for the optimal execution strategy of an ESO with a limited execution rate. We show that the value function is the viscosity solution of the HJBVI equation. In Section 3, the comparison principle of the viscosity solutions is proved, which allows to draw the conclusion of the uniqueness. In Section 4, some numerical illustrative examples are analyzed and discussed.

2. Problem Formulation

We choose firstly an infinite time horizon to approximate the life period of the ESO, because the life period can generally be as long as e.g. four years. Let X_t denote the stock price at time t and follow the stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t dB_t,\ t \in [0, +\infty)\ X(0) = x\,,\ x \in (0, +\infty) \tag{2.1}$$

with positive μ and σ, where B_t is a standard Brownian motion.

We denote the number of ESOs yet held at time t by $Y(t)$. Then $Y(t)$ satisfies the following first-order differential equation

$$dY_t = -l_t dt,\ t \in [0, +\infty),\ Y(0) = y,\ y \in [0, +\infty), \tag{2.2}$$

where $l \in [0, \tilde{l}]$ is the ratio of execution, and the constant $\tilde{l}$ is the upper boundary of l. The vesting period is not considered in our model.

We deal thus with (X_t, Y_t) at any time $t, t \geq 0$, with the corresponding space $\bar{Q} = R^+ \times \bar{R}^+$, where $\bar{R}^+ = [0, +\infty)$.

Definition 2.1. A control l is said to be admissible with respect to the initial values $(x, y) \in \bar{Q}$ if (i) l is an $\mathscr{F}_t = \sigma\{X_s : s < t\}$ adapted, (ii) $l \in [0, \tilde{l}]$ for all $t \geq 0$, (iii) For arbitrary l, $(X(t), Y(t)) \in \bar{Q}$, when $t \geq 0$.

We use $\mathscr{L} = \mathscr{L}(x, y)$ to denote the set of all admissible controls.

In this model, the vesting period is not contained. The employee can thus exercise the ESOs at any time t with a ratio of execution l_t, and the corresponding payoff is $l_t(X_t - K_1)^+$ with K_1 being the initial strike price. Moreover, we assume that once the employee quits the firm voluntarily, the ESOs will be exercised all at once with a constant stopping penalization c and a resetting strike price K_2, where $c \in (0, 1)$. Consequently, the payoff of the ESOs can be expressed as

$$J(x, y; \tau, l) = E[\int_0^{\tau\wedge\kappa} e^{-\rho t} l_t (X_t - K_1)^+ dt + c e^{-\rho(\tau\wedge\kappa)} Y_{\tau\wedge\kappa}(X_{\tau\wedge\kappa} - K_2)^+], \tag{2.3}$$

where

$$\kappa = \inf_{0 \leq s \leq \infty} \{s \,|Y_s \leq 0\},$$

ρ means the continuously market rate of return for discounting. τ is the leaving time (i.e., early exercise time or optimal stopping time, $\tau \in \mathscr{T}$), and $\mathscr{T}$ is the set of all stopping times in $[0, +\infty)$. Let $\mathscr{A} := \mathscr{T} \times \mathscr{L}$.

To maximize the present value (2.3), we define the value function as follows:

$$W(x, y) = \sup_{(\tau,l)\in\mathscr{A}(\mathrm{x,y})} J(x, y; \tau, l), \ (x, y) \in \bar{Q}. \tag{2.4}$$

We discuss now some properties of (2.4) for deducing the existence and uniqueness of the viscosity solution.

Lemma 2.1. *The value function $W(x, y)$ (2.4) satisfies a quadratic growth in $\overline{Q}$. This means that there exists a finite positive constant C_p such that for any (x, y) in $\overline{Q}$ holds*

$$|W(x, y)| \leq C_p(1 + x^2 + y^2). \tag{2.5}$$

Proof. We replace the optimal control l by the optimal control $\tilde{l}$ in (2.4) and get

$$\begin{aligned} |W(x, y)| &\leq \frac{\tilde{l}}{\rho - \mu} x + \frac{1}{2} c(y^2 + x^2) \\ &\leq C_p(1 + x^2 + y^2). \end{aligned}$$

□

Remark 2.1. In the following, finite positive constants are always denoted by C_p.

Lemma 2.2. *Assume $\rho > \mu$. Then the following assertions hold:*
(a) For each x, $W(x, y)$ is non-decreasing in y.
(b) $W(x, y)$ is continuous in $(x, y) \in \bar{Q}$.

Proof. (a) Note that for $0 \leq y_1 \leq y_2$, $\mathscr{A}(x, y_1) \subset \mathscr{A}(x, y_2)$. Then $l \in \mathscr{A}(x, y_1)$ implies $(\tau, l) \in \mathscr{A}(x, y_2)$. We have

$$\begin{aligned} W(x, y_2) &\geq J(x, y_2; \tau, l) \\ &\geq J(x, y_1; \tau, l) \end{aligned}$$

for any x. This implies $W(x, y_2) \geq W(x, y_1)$.

(b) Note that

$$X_{i,t} = x_i \exp(\int_0^t \mu - \frac{1}{2}\sigma^2 ds + \sigma B_t), \quad (i = 1, 2).$$

Then we have $\mathscr{A}(x_1, y, z) = \mathscr{A}(x_2, y)$ for any $x_1 \geq 0, x_2 \geq 0$ and $y \in \bar{R}$.

For any $(\tau, l) \in \mathscr{A}(x_1, y) = \mathscr{A}(x_2, y)$ holds

$$\begin{aligned} &|J(x_2, y; \tau, l) - J(x_1, y; \tau, l)| \\ =&E[\int_0^{\tau\wedge\kappa} e^{-\rho t} l_t (X_{1,t} - K_1)^+ dt - \int_0^{\tau\wedge\kappa} e^{-\rho t} l_t (X_{2,t} - K_1)^+ dt] \\ &+ cE[e^{-\rho(\tau\wedge\kappa)} Y_{\tau\wedge\kappa}(X_{1,\tau\wedge\kappa} - K_2)^+ - e^{-\rho(\tau\wedge\kappa)} Y_{\tau\wedge\kappa}(X_{2,\tau\wedge\kappa} - K_2)^+], \end{aligned} \tag{2.6}$$

where

$$\begin{aligned}
&E[\int_0^{\tau\wedge\kappa} e^{-\rho t} l_t (X_{1,t}-K_1)^+ dt - \int_0^{\tau\wedge\kappa} e^{-\rho t} l_t (X_{2,t}-K_1)^+ dt] \\
&\leq \tilde{l} E[\int_0^{\tau\wedge\kappa} e^{-\rho t} \left|\mathrm{x}_1-\mathrm{x}_2\right| e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma B_t} dt] \\
&\leq \frac{\tilde{l}\left|\mathrm{x}_1-\mathrm{x}_2\right|}{\rho-\mu},
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
&cE[e^{-\rho(\tau\wedge\kappa)} Y_{\tau\wedge\kappa}(X_{1,\tau\wedge\kappa}-K_2)^+ - e^{-\rho(\tau\wedge\kappa)} Y_{\tau\wedge\kappa}(X_{2,\tau\wedge\kappa}-K_2)^+] \\
&\leq y\cdot E[e^{-\rho(\tau\wedge\kappa)} \left|\mathrm{x}_1-\mathrm{x}_2\right| e^{(\mu-\frac{1}{2}\sigma^2)(\tau\wedge\kappa)+\sigma B_{\tau\wedge\kappa}}] \\
&\leq cy\cdot\left|x_1-x_2\right|.
\end{aligned} \tag{2.8}$$

According to

$$\begin{aligned}
&\left|W(x_2,y)-W(x_1,y)\right| \\
&\leq \sup_{(\tau,l)} \left|J(x_2,y;\tau,l)-J(x_1,y;\tau,l)\right|,
\end{aligned} \tag{2.9}$$

the inequality (2.6) – (2.8) implies the continuity of $W(x,y)$ with respect to x.

Next we show that $W(x,y)$ is continuous in y. Referring again to (a), it suffices now to show that for $0\leq y_1\leq y_2<+\infty$, $W(x,y_1)\leq W(x,y_2)$. Let $(\tau_2,l_2)\in\mathscr{A}(x,y_1)$, such that

$$y_2=\int_0^\infty l_{2,s}ds, \quad \text{and} \quad W(x,y_2)\leq J(x,y_2;\tau_2,l_2)+\left|\mathrm{y}_2-\mathrm{y}_1\right|.$$

Let $\kappa_i=inf\{t>0:\int_0^t l_{i,s}ds=y_i\},\ i\in\{1,2\}$. Define

$$l_1=\begin{cases} l_2, & t\in[0,\kappa_1] \\ 0, & t\in(\kappa_1,+\infty) \end{cases}, \quad \tau_1=\inf\{t\geq 0|\int_{\tau_1\wedge\kappa_1}^{\tau_2\wedge\kappa_2}\tilde{l}ds\leq\left|y_2-y_1\right|, t\leq\tau_2\}.$$

It is now obvious that $(\tau_1,l_1)\in\mathscr{A}(x,y_1)$. Thus, we have

$$\begin{aligned}
&J(x,y_2;\tau_2,l_2)-J(x,y_1;\tau_1,l_1) \\
=&E[\int_0^{\tau_2\wedge\kappa_2} e^{-\rho t} l_{2,t}(X_t-K_1)^+ dt - \int_0^{\tau_1\wedge\kappa_1} e^{-\rho t} l_{1,t}(X_t-K_1)^+ dt] \\
&+cE[e^{-\rho(\tau_2\wedge\kappa_2)} Y_{\tau_2\wedge\kappa_2}(X_{\tau_2\wedge\kappa_2}-K_2)^+ - e^{-\rho(\tau_2\wedge\kappa_2)} Y_{\tau\wedge\kappa_1}(X_{\tau_2\wedge\kappa_2}-K_2)^+] \\
&+cE[e^{-\rho(\tau_2\wedge\kappa_2)} Y_{\tau_1\wedge\kappa_1}(X_{\tau_2\wedge\kappa_2}-K_2)^+ - e^{-\rho(\tau_2\wedge\kappa_2)} Y_{\tau_1\wedge\kappa_1}(X_{\tau_1\wedge\kappa_1}-K_2)^+] \\
&+cE[e^{-\rho(\tau_2\wedge\kappa_2)} Y_{\tau_1\wedge\kappa_1}(X_{\tau_1\wedge\kappa_1}-K_2)^+ - e^{-\rho(\tau_1\wedge\kappa_1)} Y_{\tau_1\wedge\kappa_1}(X_{\tau_1\wedge\kappa_1}-K_2)^+] \\
=&(a)+(b)+(c)+(d).
\end{aligned} \tag{2.10}$$

Using integration by parts for the first component of (2.10) we get

$$\begin{aligned}
(a) &= E[\int_0^{\tau_2\wedge\kappa_2} e^{-\rho t} l_{2,t}(X_t-K_1)^+dt - \int_0^{\tau_1\wedge\kappa_1} e^{-\rho t} l_{1,t}(X_t-K_1)^+dt] \\
&= E[\int_{\tau_1\wedge\kappa_1}^{\tau_2\wedge\kappa_2} e^{-\rho t} l_{2,t}(X_t-K_1)^+dt \\
&\le E[\int_{\tau_1\wedge\kappa_1}^{\tau_2\wedge\kappa_2} e^{-\rho t} l_{2,t}X_t dt \\
&= E[\int_{\tau_1\wedge\kappa_1}^{\tau_2\wedge\kappa_2} e^{-\rho t} X_t d\int_{\tau_1\wedge\kappa_1}^{t} l_{2,s}ds \\
&= E[e^{-\rho t}X_t d\int_{\tau_1\wedge\kappa_1}^{t} l_{2,s}ds|_{\tau_1\wedge\kappa_1}^{\tau_2\wedge\kappa_2} + \int_{\tau_1\wedge\kappa_1}^{\tau_2\wedge\kappa_2}\int_{\tau_1\wedge\kappa}^{t} l_{2,s}ds\cdot e^{-\rho t}(\rho-\mu)X_t dt] \\
&\le (x+1)\left|y_2-y_1\right|,
\end{aligned} \tag{2.11}$$

The second part (b) becomes now

$$\begin{aligned}
(b) &= cE[e^{-\rho(\tau_2\wedge\kappa_2)}Y_{\tau_2\wedge\kappa_2}(X_{\tau_2\wedge\kappa_2}-K_2)^+ - e^{-\rho(\tau_2\wedge\kappa_2)}Y_{\tau_1\wedge\kappa_1}(X_{\tau_2\wedge\kappa_2}-K_2)^+] \\
&= cE[(Y_{\tau_2\wedge\kappa_2}-Y_{\tau_1\wedge\kappa_1})e^{-\rho(\tau_2\wedge\kappa_2)}(X_{\tau_2\wedge\kappa_2}-K_2)^+] \\
&\le cx\left|y_2-y_1\right|,
\end{aligned} \tag{2.12}$$

and for the third component (c) holds

$$\begin{aligned}
(c) &= cE[e^{-\rho(\tau_2\wedge\kappa_2)}Y_{\tau_1\wedge\kappa_1}(X_{\tau_2\wedge\kappa_2}-K_2)^+ - e^{-\rho(\tau_2\wedge\kappa_2)}Y_{\tau_1\wedge\kappa_1}(X_{\tau_1\wedge\kappa_1}-K_2)^+] \\
&\le cy_2\cdot E[e^{-\rho(\tau_2\wedge\kappa_2)}\left|X_{\tau_2\wedge\kappa_2}-X_{\tau_1\wedge\kappa_1}\right|].
\end{aligned} \tag{2.13}$$

Using the Itô-isometry, we obtain

$$\begin{aligned}
&E[e^{-\rho(\tau_2\wedge\kappa_2)}(X_{\tau_2\wedge\kappa_2}-X_{\tau_1\wedge\kappa_1})]^2 \\
\le& 2E(\int_{\tau_1\wedge\kappa_1}^{\tau_2\wedge\kappa_2}\mu e^{-\rho s}X_s ds)^2 + 2E[e^{-\rho(\tau_2\wedge\kappa_2)}\int_{\tau_1\wedge\kappa_1}^{\tau_2\wedge\kappa_2}\sigma X_s dB_s]^2 \\
\le& 2C_p(\tau_2\wedge\kappa_2-\tau_1\wedge\kappa_1)\cdot E(\int_{\tau_1\wedge\kappa_1}^{\tau_2\wedge\kappa_2}\left|e^{-\rho s}X_s\right|^2 ds) + 2C_pE[\int_{\tau_1\wedge\kappa_1}^{\tau_2\wedge\kappa_2}\left|e^{-\rho s}X_s\right|^2 ds] \\
\le& 2C_p\cdot x^2(\tau_2\wedge\kappa_2-\tau_1\wedge\kappa_1)^2 + 2C_p\cdot x^2(\tau_2\wedge\kappa_2-\tau_1\wedge\kappa_1).
\end{aligned}$$

Because $\int_{\tau_1\wedge\kappa_1}^{\tau_2\wedge\kappa_2}\tilde{l}ds \le |y_2-y_1|$, we have $|\tau_2\wedge\kappa_2-\tau_1\wedge\kappa_1| \le \frac{|y_2-y_1|}{\tilde{l}}$. Thus,

$$\begin{aligned}
&E[e^{-\rho(\tau_2\wedge\kappa_2)}\left|X_{\tau_2\wedge\kappa_2}-X_{\tau_1\wedge\kappa_1}\right|] \\
\le& E[(e^{-\rho(\tau_2\wedge\kappa_2)}(X_{\tau_2\wedge\kappa_2}-X_{\tau_1\wedge\kappa_1})^2]^{\frac{1}{2}} \\
\le& C_p\cdot x\sqrt{(\tau_2\wedge\kappa_2-\tau_1\wedge\kappa_1)^2+(\tau_2\wedge\kappa_2-\tau_1\wedge\kappa_1)} \\
\le& C_p\cdot x\sqrt{\left|y_2-y_1\right|^2+\left|y_2-y_1\right|}.
\end{aligned} \tag{2.14}$$

Together with (2.13) and (2.14), we obtain

$$\begin{aligned}(c) &= cE[e^{-\rho(\tau_2\wedge\kappa_2)}Y_{\tau_1\wedge\kappa_1}(X_{\tau_2\wedge\kappa_2}-K_2)^+ - e^{-\rho(\tau_2\wedge\kappa_2)}Y_{\tau_1\wedge\kappa_1}(X_{\tau_1\wedge\kappa_1}-K_2)^+] \\ &\le C_p\cdot xy_2\sqrt{|y_2-y_1|^2+|y_2-y_1|}.\end{aligned} \tag{2.15}$$

Consequently, we receive

$$\begin{aligned}(d) &= cE[e^{-\rho(\tau_2\wedge\kappa_2)}Y_{\tau_1\wedge\kappa_1}(X_{\tau_1\wedge\kappa_1}-K_2)^+ - e^{-\rho(\tau_1\wedge\kappa_1)}Y_{\tau_1\wedge\kappa_1}(X_{\tau_1\wedge\kappa_1}-K_2)^+] \\ &= cE[Y_{\tau_1\wedge\kappa_1}e^{-\rho(\tau_1\wedge\kappa_1)}(X_{\tau_1\wedge\kappa_1}-K_2)^+(e^{-\rho(\tau_2\wedge\kappa_2-\tau_1\wedge\kappa_1)}-e^0)] \\ &\le cy_2x(\tau_2\wedge\kappa_2-\tau_1\wedge\kappa_1) \\ &\le c\frac{xy_2\,|y_2-y_1|}{\tilde{l}}.\end{aligned} \tag{2.16}$$

With (2.10)–(2.16), we obtain now

$$\begin{aligned}&J(x,y_2;\tau_2,l_2)-J(x,y_1;\tau_1,l_1) \\ \le&(x+1)\,|y_2-y_1|+cx\,|y_2-y_1|+C_pxy_2\sqrt{|y_2-y_1|^2+|y_2-y_1|}+c\frac{xy_2\,|y_2-y_1|}{\tilde{l}} \\ \le&((\frac{cy_2}{\tilde{l}}+c+1)x+1)\,|y_2-y_1|+C_pxy_2\sqrt{|y_2-y_1|^2+|y_2-y_1|}.\end{aligned} \tag{2.17}$$

It follows now that

$$\begin{aligned}&W(x,y_1)\ge J(x,y_1;\tau_1,l_1) \\ \ge&J(x,y_2;\tau_2,l_2)-((\frac{cy_2}{\tilde{l}}+c+1)x+1)\,|y_2-y_1|+C_p\cdot xy_2\sqrt{|y_2-y_1|^2+|y_2-y_1|} \\ \ge&W(x,y_2)-((\frac{cy_2}{\tilde{l}}+c+2)x+1)\,|y_2-y_1|+C_p\cdot xy_2\sqrt{|y_2-y_1|^2+|y_2-y_1|}.\end{aligned} \tag{2.18}$$

This completes the proof of the continuity of $W(x,y)$ with respect to y. □

Referencing [18], we show now two DPPs of (2.4) for deducing the corresponding HJBVI equations.

Proposition 2.1. *For all $(x,y)\in\bar{Q}$ and any stopping time θ, we have [18].*

$$\begin{aligned}W(x,y)=\sup_{(\tau,l)\in\mathscr{A}(x,y)}E[&\int_0^{\tau\wedge\theta\wedge\kappa}e^{-\rho t}l_t(X_t-K_1)^+dt+1_{\{\tau<(\theta\wedge\kappa)\}}ce^{-\rho\tau}Y_\tau(X_\tau-K_2)^+ \\ &+1_{\{\tau\ge(\theta\wedge\kappa)\}}e^{-\rho(\theta\wedge\kappa)}W(X^x_{\theta\wedge\kappa},Y^y_{\theta\wedge\kappa})].\end{aligned}$$

Proposition 2.2. *Let $\varepsilon>0$. For all $(x,y)\in\bar{Q}$, and for each admissible control $l\in\mathscr{L}$ define the stopping time*

$$\tau^\varepsilon_{x,y,l}=\inf\{t\ge 0\,|W(X^x_t,Y^y_t)\le cY^y_t(X^x_t-K_2)^++\varepsilon\}.$$

Therefore, if $\tau\le\tau^\varepsilon_{x,y,l}$ for all $l\in\mathscr{L}$, we have [18].

$$W(x,y)=\sup_{l\in\mathscr{L}}E[\int_0^\tau e^{-\rho t}l_t(X_t-K_1)^+dt+e^{-\rho\tau}W(X^x_\tau,Y^y_\tau)].$$

According to the general hypothesis and the dynamic programming principle (DPP), we derive formally the equation for the value function (2.4):

$$\max\Big\{\sup_{l\in\mathscr{L}(x,y)}\{-l\frac{\partial W}{\partial y}+l(x-K_1)^+\}+\mu x\frac{\partial W}{\partial x}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 W}{\partial x^2}-\rho W, \\ cy(x-K_2)^+-W\Big\}=0,\quad (x,y)\in Q, \tag{2.19}$$

$$W(x,0)=0,\ \ x\in R^+, \tag{2.20}$$

where $Q=R^{+2}$. Since this is a combination of the Hamilton–Jacobi–Bellman (HJB) equation of stochastic control and the variational inequality (VI) of optimal stopping, we denote (2.19) from now by *HJBVI*.

Because the first term of (2.19) is linear in l, the optimal ESO–strategy can be described by the following table:

Table 1. Optimal Strategy of ESOs

Definition	**Strategy**	**Condition**
Holding Region(**HR**)	$l=0$	$-\frac{\partial W}{\partial y}+(x-K_1)^+>0,\ W<cy(x-K_2)^+$
Execution Line(**EL**)	—-	Between the HR and ER
Execution Region(**HR**)	$l=\tilde{l}$	$-\frac{\partial W}{\partial y}+(x-K_1)^+\leq 0,\ W<cy(x-K_2)^+$
Stopping Line(**SL**)	—-	Between the ER and SR
Stopping Region(**SR**)	Execute all	$W\geq cy(x-K_2)^+$

We will now prove that the value function defined in (2.4) is the viscosity solution of the HJBVI equation (2.19)–(2.20). We use now the following notations:

$$USC(\bar{Q}):=\{W:\bar{Q}\to\bar{R}^+|W \text{ is upper-semicontinuous in } \bar{Q}\}.$$

$$LSC(\bar{Q}):=\{W:\bar{Q}\to\bar{R}^+|W \text{ is lower-semicontinuous in } \bar{Q}\}.$$

Above all, we introduce the following notion of a viscosity solution [9].

Definition 2.2. Let $W:\overline{Q}\to R$ be locally bounded.

$$\begin{aligned}&F(x,y,W,\frac{\partial W}{\partial x},\frac{\partial W}{\partial y},\frac{\partial^2 W}{\partial x^2})\\ =&\max\Big\{\sup_{l\in\mathscr{L}(x,y)}\{-l\frac{\partial W}{\partial y}+l(x-K_1)^+\}\\ &\mu x\frac{\partial W}{\partial x}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 W}{\partial x^2}-\rho W, cy(x-K_2)^+-W\Big\}.\end{aligned} \tag{2.21}$$

(1) $W\in USC(\overline{Q})$ is a viscosity subsolution of (2.19) - (2.20) if

$$F(\overline{x},\overline{y},W,\frac{\partial\phi}{\partial x}(\overline{x},\overline{y}),\frac{\partial\phi}{\partial y}(\overline{x},\overline{y}),\frac{\partial^2\phi}{\partial x^2}(\overline{x},\overline{y}))\geq 0$$

for all $(\overline{x},\overline{y})\in Q$ and for all $\phi\in C^2(Q)$ such that $(\overline{x},\overline{y})$ is a maximum point of $W-\phi$.

(2) $W\in LSC(\overline{Q})$ is a viscosity supersolution of (2.19) - (2.20) if

$$F(\overline{x},\overline{y},W,\frac{\partial\phi}{\partial x}(\overline{x},\overline{y}),\frac{\partial\phi}{\partial y}(\overline{x},\overline{y}),\frac{\partial^2\phi}{\partial x^2}(\overline{x},\overline{y}))\leq 0$$

for all $(\overline{x},\overline{y}) \in Q$ and for all $\phi \in C^2(Q)$ such that $(\overline{x},\overline{y})$ is a minimum point of $W-\phi$.

Definition 2.3. W is said to be a viscosity solution of (2.19)–(2.20) if it is both a subsolution and supersolution of (2.19)–(2.20) .

Next, we will prove that the value function (2.4) is a viscosity solution which exhibits at most quadratic growth. This implies the existence for problem (2.19)–(2.20).

Theorem 2.1. *The value function* (2.4) *is a viscosity solution of the HJBVI–equation* (2.19)–(2.20).

Proof. The conditions (2.20) are obviously satisfied by (2.4). We prove now that the value function (2.4) is a viscosity subsolution of the HJBVI equation (2.19)–(2.20). Let $(\overline{x},\overline{y}) \in Q$ and let $\phi \in C^2(Q)$ be a test function, such that

$$0 = (W-\phi)(\overline{x},\overline{y}) = \max_{(x,y)\in Q}(W-\phi)(x,y). \tag{2.22}$$

This implies

$$W(x,y) \le \phi(x,y). \tag{2.23}$$

Because of the definition of (2.4), we have $W(\bar{x},\bar{y}) \ge c\bar{y}(\bar{x}-K_2)^+$. If $W(\bar{x},\bar{y}) = c\bar{y}(\bar{x}-K_2)^+$, the value function (2.4) is a viscosity subsolution of the HJBVI equation (2.19)–(2.20).

Assume $W(\bar{x},\bar{y}) > c\bar{y}(\bar{x}-K_2)^+$. For any admissible control l, define

$$\tau^\varepsilon \triangleq \tau^\varepsilon{}_{\bar{x},\bar{y},l} = \inf\{y \ge 0 \,\Big|\, W(X_t^{\bar{x}}, Y_t^{\bar{y}}) \le cY_t^{\bar{y}}(X_t^{\bar{x}}-K_2)^+ + \varepsilon\}. \tag{2.24}$$

Let $\theta > 0$ be a stopping time. We have

$$\begin{aligned} W(\bar{x},\bar{y}) = \sup_{l\in\mathscr{L}(\bar{x},\bar{y})} E[\int_0^{\theta\wedge\tau^\varepsilon\wedge\kappa} & e^{-\rho t} l_t (X_t^{\bar{x}}-K_1)^+ dt \\ & + e^{-\rho(\theta\wedge\tau^\varepsilon\wedge\kappa)} cY^{\bar{y}}_{\theta\wedge\tau^\varepsilon\wedge\kappa}(X^{\bar{x}}_{\theta\wedge\tau^\varepsilon\wedge\kappa}-K_2)^+]. \end{aligned} \tag{2.25}$$

Hence

$$\begin{aligned} W(\bar{x},\bar{y}) \le \sup_{l\in\mathscr{L}(\bar{x},\bar{y})} E[\int_0^{\theta\wedge\tau^\varepsilon\wedge\kappa} & e^{-\rho t} l_t (X_t^{\bar{x}}-K_1)^+ dt \\ & + e^{-\rho(\theta\wedge\tau^\varepsilon\wedge\kappa)} \phi(X^{\bar{x}}_{\theta\wedge\tau^\varepsilon\wedge\kappa}, Y^{\bar{y}}_{\theta\wedge\tau^\varepsilon\wedge\kappa})]. \end{aligned} \tag{2.26}$$

By using It$\hat{o}$'s formula, we get

$$0 \le \sup_{l\in\mathscr{L}(\bar{x},\bar{y})} E[\int_0^{\theta\wedge\tau^\varepsilon\wedge\kappa} e^{-\rho t} l_t (X_t^{\bar{x}}-K_1)^+ + (L_l-\rho)\phi(X_t^{\bar{x}}, Y_t^{\bar{y}}) dt, \tag{2.27}$$

where $L_l = -l\frac{\partial}{\partial y} + \mu x\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}$. Let $\theta \to 0$. We obtain

$$L_l\phi(\bar{x},\bar{y}) - \rho W(\bar{x},\bar{y}) + l(\bar{x}-K_1)^+ \ge 0 \tag{2.28}$$

by the mean value theorem. Thus,

$$F(\overline{x},\overline{y},W,\frac{\partial\phi}{\partial x}(\overline{x},\overline{y}),\frac{\partial\phi}{\partial y}(\overline{x},\overline{y}),\frac{\partial^2\phi}{\partial x^2}(\overline{x},\overline{y}))\geq 0. \tag{2.29}$$

It remains to show that the value function (2.4) is a viscosity supersolution of the HJBVI eqution (2.19)–(2.20). Let $(\overline{x},\overline{y})\in Q$, and let $\phi\in C^2(Q)$ be a test function such that

$$0=(W-\phi)(\overline{x},\overline{y})=\min_{(x,y)\in Q}(W-\phi)(x,y). \tag{2.30}$$

This implies

$$\phi(x,y)\leq W(x,y),(x,y)\in Q.$$

Let $\theta>0$ be a stopping time. We apply the dynamic programming principle (cf. proposition 2.1). For arbitrary $l(\cdot)$ and $\tau=+\infty$, we have

$$\begin{aligned}W(\bar{x},\bar{y})\geq\sup_{l\in\mathscr{L}(\bar{x},\bar{y})}E[&\int_0^{\theta\wedge\kappa}e^{-\rho t}l_t(X_t^{\bar{x}}-K_1)^+dt\\&+e^{-\rho(\theta\wedge\kappa)}W(X_{(\theta\wedge\kappa)}^{\bar{x}},Y_{(\theta\wedge\kappa)}^{\bar{y}})].\end{aligned} \tag{2.31}$$

Hence

$$\begin{aligned}W(\bar{x},\bar{y})\geq\sup_{l\in\mathscr{L}(\bar{x},\bar{y})}E[&\int_0^{\theta\wedge\kappa}e^{-\rho t}l_t(X_t^{\bar{x}}-K_1)^+dt\\&+e^{-\rho(\theta\wedge\kappa)}\phi(X_{(\theta\wedge\kappa)}^{\bar{x}},Y_{(\theta\wedge\kappa)}^{\bar{y}})].\end{aligned} \tag{2.32}$$

Applying It$\hat{o}$'s formula and dividing by θ, we obtain

$$\begin{aligned}0\geq\sup_{l\in\mathscr{L}(\bar{x},\bar{y})}E[&\int_0^{\theta\wedge\kappa}e^{-\rho t}l_t(X_t^{\bar{x}}-K_1)^+dt\\&+\int_0^{\theta\wedge\kappa}e^{-\rho t}(L_l\phi-\rho\phi)(X_t^{\bar{x}},Y_t^{\bar{y}})dt].\end{aligned} \tag{2.33}$$

We replace now the control l_t by any constant control l^*. With $\theta\to 0$, we have

$$L_{l^*}\phi(\bar{x},\bar{y})-\rho W(\bar{x},\bar{y})+l^*(\bar{x}-K_1)^+\leq 0 \tag{2.34}$$

by the mean value theorem. Because of the definition of (2.4), it is obvious that $W(\bar{x},\bar{y})\geq c\bar{y}(\bar{x}-K_2)^+$. Referring to the arbitrariness of l^* in (2.34), we have

$$F(\overline{x},\overline{y},W,\frac{\partial\phi}{\partial x}(\overline{x},\overline{y}),\frac{\partial\phi}{\partial y}(\overline{x},\overline{y}),\frac{\partial^2\phi}{\partial x^2}(\overline{x},\overline{y}))\leq 0. \tag{2.35}$$

Consequently, (2.4) is a viscosity subsolution of (2.19)–(2.20). Together with (2.31) and (2.35), we deduce that (2.4) is a viscosity solution of (2.19)–(2.20). This completes the proof. □

We choose the Euler transformation. Let $\xi = lnx$ and $u(\xi, y) = W(x, y)$. We have

$$\begin{aligned} &\max\Big\{ \sup_{l\in\mathscr{L}(\xi,y)} \{-l\frac{\partial u}{\partial y} + l(e^{\xi} - K_1)^+\} + (\mu - \frac{1}{2}\sigma^2)\frac{\partial u}{\partial \xi} + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial \xi^2} - \rho u, \\ &\qquad cy(e^{\xi} - K_2)^+ - u\Big\} = 0, \quad (\xi, y) \in Q^1, \end{aligned} \tag{2.36}$$

$$u(\xi, 0) = 0, \ \xi \in R, \tag{2.37}$$

where $Q^1 = R \times R^+$ and $\bar{Q}^1 = R \times \bar{R}^+$.

Lemma 2.3. *For all $(\xi, y) \in \bar{Q}^1$, $u(\xi, y)$ is the viscosity solution of* (2.36)–(2.37). *Furthermore, there exists a positive constant C_p such that*

$$u(\xi, y) \le C_p(e^{2\xi} + |y|^2 + 1).$$

3. Comparison Principle and Uniqueness

In this section, we proceed to develop the comparison principle and the uniqueness of the viscosity solutions to (2.36)–(2.37). Firstly, we introduce a formulation of viscosity sub- and supersolutions based on the so-called semijets.

Definition 3.1. Let

$$\begin{aligned} P^{2,+}U(\overline{u}) =&\{\big(\overline{q}, \overline{M}\big) \in R^2 \times S^2 \,|U(u) \le U(\overline{u}) + \overline{q}(u - \overline{u}) \\ &+ \frac{1}{2}\overline{M}(u - \overline{u})(u - \overline{u}) + o(|u - \overline{u}|^2)|Q^1 \ni u \to \bar{u}\} \end{aligned}$$

be the set of second–order superjet of a USC function U at point $\overline{u} \in Q^1$, where S^2 is 2×2 symmetric matrix, and $u = (x, y)$ and let

$$\begin{aligned} P^{2,-}V(\overline{u}) =&\{\big(\overline{q}, \overline{M}\big) \in R^2 \times S^2 \,|V(u) \ge V\ (\overline{u}) + \overline{q}(u - \overline{u}) \\ &+ \frac{1}{2}\overline{M}(u - \overline{u})(u - \overline{u}) + o(|u - \overline{u}|^2)|Q^1 \ni u \to \bar{u}\} \end{aligned}$$

be the set of second-order subjet of a LSC function V at point $\overline{u} \in Q^1$.

Superjet and subjet are both called semijet.

Then a point $\big(\overline{q}, \overline{M}\big) \in R^2 \times S^2$ is said to be in the closure of set $\overline{P}^{2,+}U(\overline{u})(\overline{P}^{2,-}V(\overline{u}))$ if there exists a sequence $\big(u^k, q^k, M^k\big) \in Q^1 \times R^2 \times S^2$ such that $\big(u^k, U(u^k), q^k, M^k\big) \to \big(\overline{u}, U(\overline{u}), \overline{q}, \overline{M}\big)$ as $k \to \infty$, where $\big(q^k, M^k\big) \in P^{2,+}U(\overline{u})\ \big(P^{2,-}V(\overline{u})\big)$ for all k.

Definition 3.2. (1) A locally bounded function $U \in USC\left(\overline{Q}^1\right)$ is said to be a viscosity subsolution of (2.36) if and only if $\forall(\xi, y) \in Q^1$ and $\forall(q, M) \in P^{2,+}U(\xi, y)$

$$\max\{ \sup_{l\in\mathscr{L}(\xi,y)} [pq + \frac{1}{2}tr\left(\Sigma\Sigma^T M\right) + l(e^{\xi} - K_1)^+] - \rho U, \ \ cy(e^{\xi} - K_2)^+ - U\} \ge 0, \tag{3.1}$$

where

$$p = \begin{pmatrix} \mu - \frac{1}{2}\sigma^2 \\ -l \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}.$$

(2) A locally bounded function $U \in LSC\left(\overline{Q}^1\right)$ is said to be a viscosity supersolution of (2.36) if and only if $\forall(\xi, y) \in Q^1$ and $\forall(q, M) \in P^{2,-}U(\xi, y)$

$$\max\{\sup_{l\in\mathscr{L}(\xi,y)}[pq+\frac{1}{2}tr\left(\Sigma\Sigma^T M\right)+l(e^{\xi}-K_1)^+]-\rho V,\ \ cy(e^{\xi}-K_2)^+-V\}\leq 0. \quad (3.2)$$

We refer now to the maximum principle for semicontinuous functions.

Lemma 3.1 (Ishii's lemma [9]). *Let $U_i \in USC\,(O_i)$ for $i = 1, \cdots, k$, where Q_i is a locally compact subset of R^N. Let ϕ be defined on an open neighbourhood of $O_1 \times \cdots \times O_k$ such that $\phi(x_1, \cdots, x_k)$ is twice continuously differentiable in $(x_1, \cdots, x_k) \in O_1 \times \cdots \times O_k$. Suppose*

$$(\widehat{x}_1, \cdots, \widehat{x}_k) \in O_1 \times \cdots \times O_k,$$

$$F(x_1, \cdots, x_k) = U_1(x_1) + \cdots + U_k(x_k) - \phi(x_1, \cdots, x_k) \leq F(\widehat{x}_1, \cdots, \widehat{x}_k),$$

for $(x_1, \cdots, x_k) \in O_1 \times \cdots \times O_k$. Then for each $\eta > 0$, there are $M_i \in S_{N_i}$ such that

$$\begin{cases} (D_{x_i}\phi(\widehat{x}_1, \cdots, \widehat{x}_k), M_i) \in \overline{P}^{2,+}U_i(\overline{x}_i)\ for\ i = 1, \cdots, k, \\ -\left(\frac{1}{\eta} + \|A\|\right) I \leq \begin{pmatrix} M_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_K \end{pmatrix} \leq A + \eta A^2, \end{cases} \quad (3.3)$$

where $A = (D_x^2\phi)(\widehat{x}_1, \cdots, \widehat{x}_k)$. The norm of a symmetric matrix A is defined as $\|A\| = \sup\{|\langle A\xi, \xi\rangle| : |\xi| \leq 1\}$.

We prove now a comparison principle for the viscosity sub- and super-solutions that satisfies quadratic growth.

Theorem 3.1. *Suppose $\rho \geq \frac{1}{2}\hat{\rho}$, and $U \in USC\left(\bar{Q}^1\right)$ is a viscosity subsolution of (2.36), $V \in LSC\left(\bar{Q}^1\right)$ is a viscosity supersolution of (2.36), satisfying $U(\xi, y) \leq V(\xi, y)$, for $(\xi, y) \in \partial Q^1$. Then we have $U(\xi, y) \leq V(\xi, y)$, for $(\xi, y) \in \overline{Q}^1$. It holds $\partial Q^1 = \bar{Q}^1 \setminus Q^1$, $\hat{\rho} = \max\{(2\mu + \lambda\sigma^2)(1 + \lambda), \varepsilon(2\mu + \sigma^2)\}$ and $\lambda > 1$.*

Proof. To begin with, we introduce the following notations and operators:
Let $P := (\xi, y)$, $P^* := (\xi^*, y^*)$, $P_1 := (\xi_1, y_1)$,$P_2 := (\xi_2, y_2)$. $P_{1,\alpha}$, $P_{2,\alpha}$, $P_{1,\varepsilon}$, $P_{2,\varepsilon}$, P_ε are defined in an analogous way.

Let $|P_1P_2| = \sqrt{(\xi_1 - \xi_2)^2 + (y_1 - y_2)^2}$, $\|P\|^n = e^{(1+n)\xi} + |\xi|^2 + |y|^{2n}, n \in R^+$.

We argue by contradiction, which yields to

$$U(P^*) \geq V(P^*) + 2\delta,\ \text{for some } P^* \in \overline{Q}^1 \quad (3.4)$$

with $\delta > 0$.

For any $0 < \varepsilon < 1$ and α, we introduce the following functions:

$$\Phi\left(P_1, P_2\right) = U(P_1) - V(P_2) - \phi(P_1, P_2),$$

$$\phi(P_1, P_2) = \frac{\alpha}{2}|P_1P_2|^2 + \varepsilon(\|P_1\|^\lambda + \|P_2\|^\lambda),$$

$1 < \lambda$.

Let

$$F_\alpha = \sup_{R\times\bar{R}^+\times R\times\bar{R}^+} \Phi\left(P_1, P_2\right). \tag{3.5}$$

From the growth condition and upper semicontinuity of $\Phi\left(P_1, P_2\right)$ we get that $F_\alpha < +\infty$ and there exists $(P_{1,\alpha}, P_{2,\alpha})$ such that

$$F_\alpha = \Phi\left(P_{1,\alpha}, P_{2,\alpha}\right). \tag{3.6}$$

Hence

$$F_\alpha \geq U(P^*) - V(P^*) - 2\varepsilon\|P^*\|^\lambda \geq \delta \tag{3.7}$$

for each ε small enough. This implies that $U(P_{1,\alpha}) \geq V(P_{2,\alpha}) + \delta$, for each $\alpha > 0$ and ε small enough.

Noting that $\Phi\left(0,0\right) \leq \Phi\left(P_{1,\alpha}, P_{2,\alpha}\right)$ and by the linear growth condition, we have

$$\begin{aligned} U(0) - V(0) - 2\varepsilon \leq & U(P_{1,\alpha}) - V(P_{2,\alpha}) \\ & - \frac{\alpha}{2}|P_{1,\alpha}P_{2,\alpha}|^2 - \varepsilon\left(\|P_{1,\alpha}\|^\lambda + \|P_{2,\alpha}\|^\lambda\right). \end{aligned}$$

Then we obtain

$$\begin{aligned} \frac{\alpha}{2}|P_{1,\alpha}P_{2,\alpha}|^2 + \varepsilon\left(\|P_{1,\alpha}\|^\lambda + \|P_{2,\alpha}\|^\lambda\right) - 2\varepsilon &\leq U(P_{1,\alpha}) - V(P_{2,\alpha}) - U(0) + V(0) \\ &\leq C_p\left(e^{2\xi_{1,\alpha}} + y_{1,\alpha}^2 + e^{2\xi_{2,\alpha}} + y_{2,\alpha}^2 + 1\right). \end{aligned}$$

This means that there exists a positive constant $C_{1,\varepsilon}$ such that $\|P_{1,\alpha}\|^\lambda + \|P_{2,\alpha}\|^\lambda \leq C_{1,\varepsilon}$, where $C_{1,\varepsilon}$ is determined by ε.

From the above arguments we deduce that there exists a subsequence, still denoted by $(P_{1,\alpha}, P_{2,\alpha})$, which converges toward some $(P_{1,\varepsilon}, P_{2,\varepsilon}) \in \bar{Q}^1 \times \bar{Q}^1$, as $\alpha \to \infty$ (for each fixed ε).

Furthermore, we can get $\frac{\alpha}{2}|P_{1,\alpha}P_{2,\alpha}|^2 \leq C_{2,\varepsilon}$ where $C_{2,\varepsilon}$ is a positive constant for any fixed ε .Thus, we have $\xi_{1,\alpha} - \xi_{2,\alpha} \to 0$, $y_{1,\alpha} - y_{2,\alpha} \to 0$, as $\alpha \to \infty$, and $P_{1,\varepsilon} = P_{2,\varepsilon}$. Let $P_\varepsilon = P_{1,\varepsilon} = P_{2,\varepsilon}$.

Considering that $\Phi\left(P_{1,\varepsilon}, P_{2,\varepsilon}\right) \leq \Phi\left(P_{1,\alpha}, P_{2,\alpha}\right)$, we get

$$\begin{aligned} \frac{\alpha}{2}|P_{1,\alpha}P_{2,\alpha}|^2 \leq & U(P_{1,\alpha}) - U(P_{1,\varepsilon}) - V(P_{2,\alpha}) + V(P_{2,\varepsilon}) \\ & + \varepsilon\left(\|P_{1,\varepsilon}\|^\lambda + \|P_{2,\varepsilon}\|^\lambda\right) - \varepsilon\left(\|P_{1,\alpha}\|^\lambda + \|P_{2,\alpha}\|^\lambda\right). \end{aligned}$$

The semicontinuity of U and V help us to yield $\frac{\alpha}{2}|P_{1,\alpha}P_{2,\alpha}|^2 \to 0$ as $\alpha \to \infty$ (for each fixed ε).

Since $\Phi\left(P^*, P^*\right) \leq \Phi\left(P_{1,\alpha}, P_{2,\alpha}\right)$ we have

$$\begin{aligned} \varepsilon\left(\|P_{1,\alpha}\|^\lambda + \|P_{2,\alpha}\|^\lambda\right) &\leq U(P_{1,\alpha}) - V(P_{2,\alpha}) - U(P^*) + V(P^*) + 2\varepsilon\|P^*\|^\lambda \\ &\leq U(P_{1,\alpha}) - V(P_{2,\alpha}) + 2\varepsilon\|P^*\|^\lambda. \end{aligned}$$

If $P_\varepsilon \in \partial Q^1$, then it is obvious that $U(P_{1,\alpha}) - V(P_{2,\alpha}) \geq F_\alpha \geq U(P^*) - V(P^*) - 2\varepsilon\|P^*\|^\lambda$. From the upper semicontinuity of $U - V$ and $U(P_\varepsilon) \leq V(P_\varepsilon)$, as $\alpha \to \infty$ and $\varepsilon \to 0$, we get $U(P^*) \leq V(P^*)$. This contradicts to (3.4). Therefore, $P_\varepsilon \in Q^1$.

We assume now $P_\varepsilon \in Q^1$ such that $P_{1,\alpha} \in Q^1,\ P_{2,\alpha} \in Q^1$, for any α being large enough. An application of Ishii's lemma yields that there exist M_α, N_α such that

$$(q_{1,\alpha}, M_\alpha) \in \bar{P}^{2,+}U(P_{1,\alpha}),$$
$$(q_{2,\alpha}, N_\alpha) \in \bar{P}^{2,-}V(P_{2,\alpha}),$$

where $q_{1,\alpha} = D_{P_{1,\alpha}}\phi, q_{2,\alpha} = D_{P_{2,\alpha}}\phi$.

Since U and V are viscosity subsolutions and viscosity supersolutions of (2.36), there exists a constant l^*, such that

$$\begin{aligned}\max\Bigg\{&\begin{pmatrix}\mu-\frac{1}{2}\sigma^2\\ -l^*\end{pmatrix}^T\begin{pmatrix}\alpha(\xi_{1,\alpha}-\xi_{2,\alpha})+\varepsilon[(1+\lambda)e^{(1+\lambda)\xi_{1,\alpha}}+2\xi_{1,\alpha}]\\ \alpha(y_{1,\alpha}-y_{2,\alpha})+2\lambda\varepsilon y_{1,\alpha}^{2\lambda-1}\end{pmatrix}\\ &+\frac{1}{2}tr\left(\begin{pmatrix}\sigma&0\\0&0\end{pmatrix}\begin{pmatrix}\sigma&0\\0&0\end{pmatrix}^T M_\alpha\right)\Bigg\}+l^*(e^{\xi_{1,\alpha}}-K_1)^+-\rho U,\\ &cy_{1,\alpha}(e^{\xi_{1,\alpha}}-K_2)^+-U\Bigg\}\geq 0,\end{aligned}\tag{3.8}$$

$$\begin{aligned}\max\Bigg\{&\begin{pmatrix}\mu-\frac{1}{2}\sigma^2\\ -l^*\end{pmatrix}^T\begin{pmatrix}\alpha(\xi_{1,\alpha}-\xi_{2,\alpha})-\varepsilon[(1+\lambda)e^{(1+\lambda)\xi_{2,\alpha}}+2\xi_{2,\alpha}]\\ \alpha(y_{1,\alpha}-y_{2,\alpha})-2\lambda\varepsilon y_{2,\alpha}^{2\lambda-1}\end{pmatrix}\\ &+\frac{1}{2}tr\left(\begin{pmatrix}\sigma&0\\0&0\end{pmatrix}\begin{pmatrix}\sigma&0\\0&0\end{pmatrix}^T M_\alpha\right)\Bigg\}+l^*(e^{\xi_{2,\alpha}}-K_1)^+-\rho V,\\ &cy_{2,\alpha}(e^{\xi_{2,\alpha}}-K_2)^+-V\Bigg\}\leq 0.\end{aligned}\tag{3.9}$$

We compute now the difference of (3.8) and (3.9) and get

$$\max\{f_{a,1}, f_{a,2}\} - \max\{f_{b,1}, f_{b,2}\} \geq 0,\tag{3.10}$$

where $f_{a,1}$, $f_{a,2}$, $f_{b,1}$ and $f_{b,2}$ are corresponding expressions in (3.8) and (3.9). Thus, we have

$$f_{a,1} - f_{b,1} \geq 0,\tag{3.11}$$

or

$$f_{a,2} - f_{b,2} \geq 0.\tag{3.12}$$

Firstly, we prove the contradiction to (3.11).

From (3.11), we obtain

$$f_{a,1} - f_{b,1} = I + II - \rho\,(U-V) + l^*(e^{\xi_{1,\alpha}}-K_1)^+ - l^*(e^{\xi_{2,\alpha}}-K_1)^+ \geq 0,$$

where

$$\begin{aligned}I=&\begin{pmatrix}\mu-\frac{1}{2}\sigma^2\\ -l^*\end{pmatrix}^T\begin{pmatrix}\alpha(\xi_{1,\alpha}-\xi_{2,\alpha})+\varepsilon[(1+\lambda)e^{(1+\lambda)\xi_{1,\alpha}}+2\xi_{1,\alpha}]\\ \alpha(y_{1,\alpha}-y_{2,\alpha})+2\lambda\varepsilon y_{1,\alpha}^{2\lambda-1}\end{pmatrix}\\ &-\begin{pmatrix}\mu-\frac{1}{2}\sigma^2\\ -l^*\end{pmatrix}^T\begin{pmatrix}\alpha(\xi_{1,\alpha}-\xi_{2,\alpha})-\varepsilon[(1+\lambda)e^{(1+\lambda)\xi_{2,\alpha}}+2\xi_{2,\alpha}]\\ \alpha(y_{1,\alpha}-y_{2,\alpha})-2\lambda\varepsilon y_{2,\alpha}^{2\lambda-1}\end{pmatrix},\end{aligned}\tag{3.13}$$

$$II = \frac{1}{2} tr \left(\begin{pmatrix} CC^T & CD^T \\ DC^T & DD^T \end{pmatrix} \begin{pmatrix} M_\alpha & 0 \\ 0 & -N_\alpha \end{pmatrix} \right), \tag{3.14}$$

and $C = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} = D$.

From Ishii's lemma we conclude

$$\begin{pmatrix} M_\alpha & 0 \\ 0 & -N_\alpha \end{pmatrix} \leq A + \frac{1}{\alpha} A^2,$$

where $A = \alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \varepsilon \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$,

$$R_i = \begin{pmatrix} (1+\lambda)^2 e^{(1+\lambda)\xi_{i,\alpha}} + 2 & 0 \\ 0 & 2\lambda(2\lambda-1) y_{i,\alpha}^{2\lambda-2} \end{pmatrix}, i \in \{1, 2\},$$

and

$$A^2 = 2\alpha^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \varepsilon\alpha \begin{pmatrix} 2R_1 & -R_1 - R_2 \\ -R_1 - R_2 & 2R_2 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} R_1^2 & 0 \\ 0 & R_2^2 \end{pmatrix}.$$

After simplifying the above matrices, we get

$$\begin{aligned} A + \frac{1}{\alpha} A^2 =& 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \varepsilon \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} + \varepsilon \begin{pmatrix} 2R_1 & -R_1 - R_2 \\ -R_1 - R_2 & 2R_2 \end{pmatrix} \\ &+ \frac{\varepsilon^2}{\alpha} \begin{pmatrix} R_1^2 & 0 \\ 0 & R_2^2 \end{pmatrix}. \end{aligned} \tag{3.15}$$

Therefore

$$\begin{aligned} II &= \frac{1}{2} tr \left(\begin{pmatrix} CC^T & CD^T \\ DC^T & DD^T \end{pmatrix} \begin{pmatrix} M_\alpha & 0 \\ 0 & -N_\alpha \end{pmatrix} \right) \\ &\leq \frac{1}{2} tr \left(\begin{pmatrix} CC^T & CD^T \\ DC^T & DD^T \end{pmatrix} (A + \frac{1}{\alpha} A^2) \right) \end{aligned} \tag{3.16}$$

From (3.13)–(3.16), we obtain

$$\begin{aligned} &\lim_{\alpha \to \infty} I = 2\varepsilon(\mu - \frac{1}{2}\sigma^2)[(1+\lambda)e^{(1+\lambda)\xi_\varepsilon} + 2\xi_\varepsilon] - 4\varepsilon l^* \lambda y_\varepsilon^{2\lambda-1}, \\ &\lim_{\alpha \to \infty} II \leq \varepsilon\sigma^2[(1+\lambda)^2 e^{(1+\lambda)\xi_\varepsilon} + 2]. \end{aligned} \tag{3.17}$$

Consequently, from (3.11) – (3.17) we get

$$\begin{aligned}
0 &\le \varepsilon(2\mu-\sigma^2)(1+\lambda)e^{(1+\lambda)\xi_\varepsilon} + 4\varepsilon(\mu-\frac{1}{2}\sigma^2)\xi_\varepsilon + \varepsilon\sigma^2(1+\lambda)^2 e^{(1+\lambda)\xi_\varepsilon} + 2\varepsilon\sigma^2 \\
&\quad - \rho(U(P_\varepsilon)-V(P_\varepsilon)) \\
&\le \varepsilon(2\mu+\lambda\sigma^2)(1+\lambda)e^{(1+\lambda)\xi_\varepsilon} + \varepsilon(2\mu+\sigma^2)\xi_\varepsilon^2 + \varepsilon(2\mu+3\sigma^2) - \rho(U(P_\varepsilon)-V(P_\varepsilon)) \\
&\le \varepsilon(2\mu+3\sigma^2) + \varepsilon\max\{(2\mu+\lambda\sigma^2)(1+\lambda), \varepsilon(2\mu+\sigma^2)\}\|P_\varepsilon\|^\lambda - \rho(U(P_\varepsilon)-V(P_\varepsilon)) \\
&\le \varepsilon(2\mu+3\sigma^2) + \varepsilon\hat{\rho}\|P^*\|^\lambda - (\rho-\frac{1}{2}\hat{\rho})(U(P_\varepsilon)-V(P_\varepsilon)) \\
&\le \varepsilon(2\mu+3\sigma^2) + \varepsilon\hat{\rho}\|P^*\|^\lambda - (\rho-\frac{1}{2}\hat{\rho})\delta.
\end{aligned}$$

If ε is chosen sufficiently small, this leads to a contradiction of (3.11).

Next, from (3.8)–(3.12) we get

$$\lim_{\alpha\to\infty} f_{a,2} - f_{b,2} = -(U(P_\varepsilon)-V(P_\varepsilon)) < -\delta, \tag{3.18}$$

which leads to a contradiction of (3.12). This completes the proof. □

Corollary 3.1. *The value function $u(\xi, y)$ is the unique viscosity solution of* (2.36)–(2.37).

The uniqueness in corollary 3.1 is obvious after the conclusions of lemma 2.1 and theorem 3.1.

4. Numerical Simulation

In this section, we will give some numerical examples to illustrate our results.

4.1. Finite Difference Scheme

The finite difference method, which is discussed e.g. by [11], is used in the following simulation to obtain a positive coefficient discretization of equation (2.19). The main steps can be summarized as following:

(Step 1:) Consider the step size Δx and Δy for x, y. Define the infinite lattice

$$\Sigma_{Inf}^h = \{(x,y) = (i\Delta x, j\Delta y) | i, j = 0, 1, 2, \cdots\}.$$

For actual numerical calculations, Σ_{Inf}^h must be replaced by some finite lattice Σ_F^h as the subset of Σ_{Inf}^h. Denote

$$\Sigma_F^h = \left\{(x,y) \in \Sigma_{Inf}^h | 0 \le x \le M, 0 \le y \le N\right\},$$

where $M > 0, N > 0$ is large enough and $M = m\Delta x, N = n\Delta y$.

Let $i = 0, \cdots, m;\ k = 0, \cdots, n$. Then we have $x_i = i\Delta x,\ y_k = k\Delta y$. The equation (2.19) can be discretized using forward, backward or central differencing to obtain

$$\begin{aligned}
&-\frac{l}{\Delta y}(W_i^{k+1} - W_i^k) + l(i\Delta x - K_1)^+ + \mu i(W_i^{k+1} - W_{i-1}^{k+1}) \\
&+\frac{1}{2}\sigma^2 i^2(W_{i+1}^{k+1} - 2W_i^{k+1} + W_{i-1}^{k+1}) - \rho W_i^{k+1} = 0,
\end{aligned} \tag{4.1}$$

where l can be assigned any initial value.

(Step 2:) Calculate the vector $(W^{k+1})^1$ from W^k. If $\left|(W^{k+1})^1 - (W^{k+1})^0\right| < er$, the iteration stops, where er is the allowed error, and $(W^{k+1})^0 = W^k$. Otherwise, compute l^1 by $(W^{k+1})^1$ and replace l in (2.20).

(Step 3:) Repeat the step 3, until $\left|(W^{k+1})^n - (W^{k+1})^{n-1}\right| < er$.

(Step 4:) $W^{k+1} = max\{(W^{k+1})^n,\ ck\Delta y(i\Delta x - K_2)^+\}$.

4.2. Computational Examples

The parameters for this section are shown in Table 2.

Table 2. Parameters for this numerical simulation

Parameter	Value
μ	0.03
σ	0.2
ρ	0.08
$\tilde{l}$	1
K_1	3
K_2	4
c	0.6
M	20
N	32

Firstly, we show the the numerical values of the employee stock options obtained from numerical calculation by using the finite difference method discussed in section 4.1 (see Figure 1). It is obvious that W is monotonically increasing in X and Y. Secondly, we concentrate on the impacts of different parameters in our model.

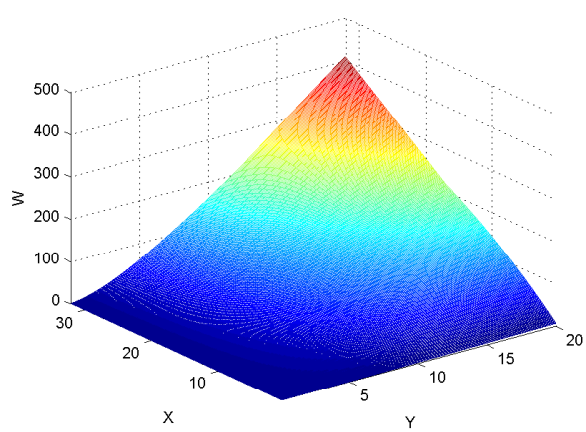

Figure 1. The present value

We observe from Figure 2 that the SL and EL with higher μ are respectively standing above those with lower μ. In fact, μ is the drift or growth factor of the price

process. The higher μ, the higher the price of the stock. To achieve a maximum payoff of the all ESOs, the employees will partly or totally exercise the ESOs with higher μ until the stock price rises.

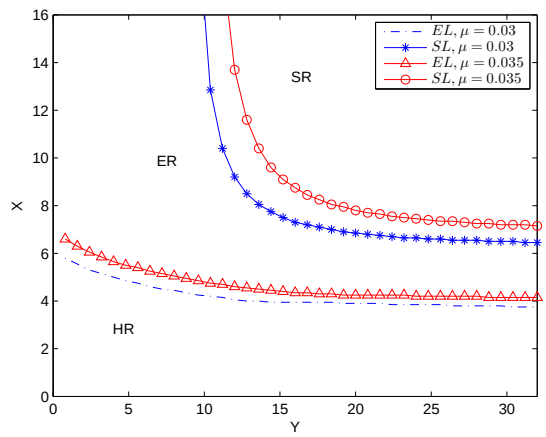

Figure 2. The strategy with different μ

In Figure 3, we show the plots of ER and SR with different ρ, ρ being the market rate of return. A higher ρ corresponds to better investment chances in the market. The employees will here be inspired to exercise more ESOs to invest other financial products in the market. Thus, the SL and EL with a higher ρ lie below those with a lower ρ, respectively.

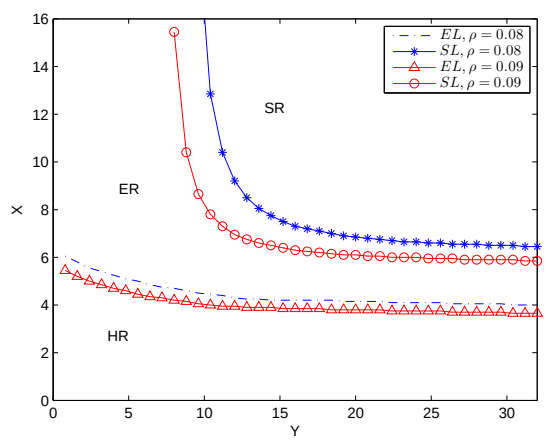

Figure 3. The strategy with different ρ

In Figure 4, we plot the strategy against different initial strike prices K_1. On the one hand, it is obvious that a higher K_1 corresponds to a higher stock price, which keeps the ESO in the money. On the other hand, a higher K_1 corresponds to a higher risk of the option, so the corresponding to the SL being situated lower.

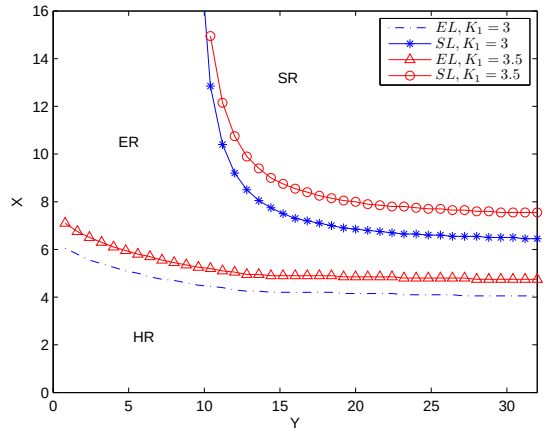

Figure 4. The strategy with different K_1

We also plot the strategy under varying c. The SL is sensitive to c because the value c effects directly the payoff at the stopping time τ. The employees will thus completely exercise the ESOs with a lower c at a higher price. The SL with the lower c is thus above their corresponding line. This means the employee will wait for a higher stock price to exercise the options with the lower c, because the waiting time cost will be compensated by the one-time execution returns.

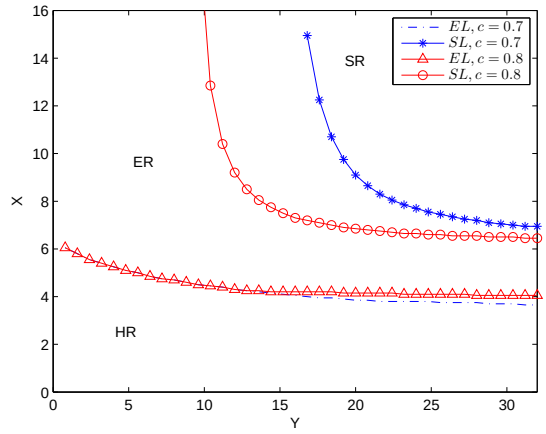

Figure 5. The strategy with different c

5. Conclusions

In this paper, we examine optimal strategies for exercising small numbers of ESOs over an infinite time horizon. In particular, a single–time execution is considered. The objective is to maximize the expected overall payoff. The value function for developing the optimal execution strategy is established, and a fully nonlinear

corresponding equation is derived by using the dynamic programming principle. The uniqueness and comparison principle of the viscosity solutions of the related HJBVI equation are proved. Finally, numerical examples are given and interpreted.

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