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PAHSS-PTS ALTERNATING SPLITTING ITERATIVE METHODS FOR NONSINGULAR SADDLE POINT PROBLEMS*

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Abstract In this paper, we propose the PAHSS-PTS alternating splitting iterative methods for nonsingular saddle point problems. Convergence properties of the proposed methods are studied and corresponding convergence results are given under some suitable conditions. Numerical experiments are presented to confirm the theoretical results, which impliy that PAHSS-PTS iterative methods are effective and feasible.

Keywords Saddle point problems, PAHSS-PTS alternating splitting, iterative methods, convergence analysis, numerical experiment.

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1. Introduction

In this paper, we consider the following augmented linear system:

$$\begin{pmatrix} A & B \\ B^\top & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \tag{1.1}$$

where $A \in R^{m\times m}$ is a symmetric positive definite matrix, $B \in R^{m\times n}$ is a matrix of full column rank, $f \in R^m$ and $g \in R^n$ $(m \geq n)$ are two given vectors, $B^\top$ denotes the transpose of B, both A and B are usually large and sparse. These assumptions guarantee the existence and uniqueness of the solution of linear system (1.1). Linear systems of the form (1.1) are called saddle point problems. The linear system (1.1) arises in a variety of scientific and engineering applications, such as computational fluid dynamics, image processing, mixed finite element approximation of elliptic partial differential equation, electronic networks, constrained least-squares problem, see [2, 12, 14, 15, 17, 18, 20–24] and the references therein.

Iterative methods are more attractive than direct methods for the saddle point problems (1.1) in terms of storage requirements and computing time. If matrix B is rank deficient, then (1.1) is a singular linear system. Many efficient iterative methods have been proposed for singular saddle point problems in the literatures, including the HSS-like methods [3, 4, 25], Krylov subspace methods [26], matrix splitting iterative method [11], the inexact Uzawa methods [28] and so on. If rank

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$(B) = n$, then linear system (1.1) is called the nonsingular saddle point problem. Many efficient iterative methods have been studied in the literature, such as Uzawa-type methods [13], matrix splitting iterative method [9] and successive overrelaxation methods [6] and so on. Huang et al. proposed the preconditioned accelerated Hermitian and skew-Hermitian splitting (PAHSS) iterative method in [16]. Bai and Golub established a class of AHSS iterative methods [7], which is a special case of PAHSS iterative method. Bai and Wang studied the parameterized inexact Uzawa (PIU) methods for solving the nonsingular saddle point problems in [8]. They discussed the properties of eigenvalues distribution of the iterative matrix, the optimal iterative parameters and corresponding convergence factor. Moreover, Chen and Jiang [10] generalized the PIU methods and presented the generalized PIU methods, Liang and Zhang [19] presented VAPIU methods to solve nonsingular saddle point problems based on the SOR and SSOR splitting of coefficient matrix of linear equation (1.1).

In this paper, we propose the PAHSS-PTS alternating splitting iterative methods for linear system (1.1) by combining the PAHSS iterative method and the preconditioned triangular splitting (PTS) of coefficient matrix of nonsingular saddle point problem. We described the convergence of the proposed methods. Numerical experiments are provided to confirm the theoretical results and illustrate the effectiveness of new methods.

The paper is organized as follows. In Section 2, we propose the PAHSS-PTS alternating splitting iterative methods for nonsingular saddle point problems (1.1). The convergence of the PAHSS-PTS iterative methods are analysed in Section 3. Moreover, numerical experiments are presented to illustrate the effectiveness of new methods in Section 4. Finally, we draw the conclusions in Section 5.

2. The PAHSS-PTS methods

Without loss of generality, the linear system (1.1) can be rewritten as

$$\begin{pmatrix} A & B \\ -B^\top & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix}, \tag{2.1}$$

where $A \in R^{m\times m}$ is a symmetric positive definite matrix, $B \in R^{m\times n}$ is a matrix of full column rank. Let

$$\widehat{A} = \begin{pmatrix} A & B \\ -B^\top & O \end{pmatrix}, \qquad z = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad b = \begin{pmatrix} f \\ -g \end{pmatrix},$$

then (2.1) can be expressed as

$$\widehat{A}z = b. \tag{2.2}$$

In this section, we first introduce the PAHSS iterative method and preconditioned triangular splitting (PTS) of coefficient matrix of (2.1), then we propose the PAHSS-PTS iteration methods to solve the nonsingular linear system (1.1).

For the coefficient matrix $\widehat{A}$ of the linear Eq.(2.1), we make the following matrix splitting:

$$\widehat{A} = (\Lambda + J) - (\Lambda - K), \tag{2.3}$$

$$= (\Lambda + K) - (\Lambda - J),$$

where

$$J = \frac{1}{2}\left(\widehat{A} + \widehat{A}^\top\right) = \begin{pmatrix} A & O \\ O & O \end{pmatrix}, \qquad K = \frac{1}{2}\left(\widehat{A} - \widehat{A}^\top\right) = \begin{pmatrix} O & B \\ -B^\top & O \end{pmatrix}.$$

Let Λ be a block diagonal matrix as

$$\Lambda = \begin{pmatrix} \omega P & O \\ O & \tau Q \end{pmatrix},$$

where the parameters ω and τ are positive real, $P \in R^{m\times m}$ and $Q \in R^{n\times n}$ are symmetric positive definite matrices and $PA = AP$. Obviously, matrix $\Lambda + J$ and $\Lambda + K$ are invertible.

The preconditioned accelerated Hermitian and skew-Hermitian splitting (PAHSS) iterative method [25] for solving the saddle point problems is as follows:

The PAHSS method. Suppose $z^{(0)} \in R^{m+n}$ is an initial vector, α and β are two positive real parameters , for $k = 0, 1, 2, \cdots$, when the iteration sequence $\{z^k\}$ converges to the exact solution of the Eq.(2.1), we compute

$$\begin{cases} (\Lambda + J)z^{(k+\frac{1}{2})} = (\Lambda - K)z^{(k)} + b, \\ (\Lambda + K)z^{(k+1)} = (\Lambda - J)z^{(k+\frac{1}{2})} + b. \end{cases}$$

When $P = I_m$ and $Q = I_n$, Bai and Golub [7] established the AHSS iterative method. When $\omega = \tau$, the PAHSS iterative method reduces to the PHSS iterative method [5].

Denote

$$M = \begin{pmatrix} A & B \\ O & Q \end{pmatrix}, \qquad N = \begin{pmatrix} O & O \\ -B^\top & -Q \end{pmatrix},$$

then the coefficient matrix $\widehat{A}$ have the preconditioned triangular splitting is as follows:

$$\widehat{A} = (\Lambda + M) - (\Lambda - N) = M(\omega, \tau) - N(\omega, \tau). \tag{2.4}$$

So we can induce the PTS iterative scheme by splitting (2.4)

$$z^{(k+1)} = H(\omega, \tau)z^{(k)} + M(\omega, \tau)^{-1}b,$$

where

$$\begin{aligned} M(\omega, \tau)^{-1} &= \begin{pmatrix} (1+\omega)A & B \\ O & (1+\tau)Q \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{1}{1+\omega}A^{-1} & -\frac{1}{(1+\omega)(1+\tau)}A^{-1}BQ^{-1} \\ O & \frac{1}{1+\tau}Q^{-1} \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned}
H(\omega,\tau) &= M(\omega,\tau)^{-1}N(\omega,\tau)\\
&= (\Lambda+M)^{-1}(\Lambda-N)\\
&= \begin{pmatrix} \frac{1}{1+\omega}\left(\omega I - \frac{1}{1+\tau}A^{-1}BQ^{-1}B^\top\right) & -\frac{1}{1+\omega}A^{-1}B \\ \frac{1}{1+\tau}Q^{-1}B^\top & I \end{pmatrix}.
\end{aligned}$$

In this paper, let $P = A$, i.e:

$$\Lambda = \begin{pmatrix} \omega A & O \\ O & \tau Q \end{pmatrix}.$$

Combining with the two splittings of coefficient matrix $\widehat{A}$, i.e, matrix splitting (2.3) and (2.4), we can induce the PAHSS-PTS alternating splitting iterative methods for nonsingular saddle point problem is as follows:

$$\begin{cases} (\Lambda+J)z^{(k+\frac{1}{2})} = (\Lambda-K)z^{(k)}+b, \\ (\Lambda+M)z^{(k+1)} = (\Lambda-N)z^{(k+\frac{1}{2})}+b. \end{cases}$$

It is equivalent to:

$$\begin{aligned}
z^{(k+1)} &= (\Lambda+M)^{-1}(\Lambda-N)(\Lambda+J)^{-1}(\Lambda-K)z^{(k)}\\
&\quad +(\Lambda+M)^{-1}[(\Lambda-N)(\Lambda+J)^{-1}+I]b\\
&= T(\omega,\tau)z^{(k)} + \bar{M}(\omega,\tau)^{-1}b,
\end{aligned}$$

where

$$\begin{aligned}
\bar{M}(\omega,\tau) &= (\Lambda+J)(2\Lambda+J-N)^{-1}(\Lambda+M)\\
&= \begin{pmatrix} (1+\omega)A & O \\ O & \tau Q \end{pmatrix} \begin{pmatrix} (1+2\omega)A & O \\ B^\top & (1+2\tau)Q \end{pmatrix}^{-1} \begin{pmatrix} (1+\omega)A & B \\ O & (1+\tau)Q \end{pmatrix}\\
&= \begin{pmatrix} \frac{(1+\omega)^2}{1+2\omega}A & \frac{1+\omega}{1+2\omega}B \\ -\frac{\tau(1+\omega)}{(1+2\tau)(1+2\omega)}B^\top & \frac{\tau(1+\tau)}{1+2\tau}Q - \frac{\tau}{(1+2\tau)(1+2\omega)}B^\top A^{-1}B \end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
T(\omega,\tau) &= (\Lambda+M)^{-1}(\Lambda-N)(\Lambda+J)^{-1}(\Lambda-K)\\
&= \begin{pmatrix} (1+\omega)A & B \\ O & (1+\tau)Q \end{pmatrix}^{-1} \begin{pmatrix} \frac{1+\omega}{\omega}I_m & O \\ -\frac{\tau}{\omega(1+\tau)}B^\top A^{-1} & \frac{\tau}{1+\tau}I_n \end{pmatrix}^{-1} \begin{pmatrix} \omega A & -B \\ B^\top & \tau Q \end{pmatrix}. \quad (2.5)
\end{aligned}$$

Here $T(\omega,\tau)$ is the iterative matrix of the PAHSS-PTS iterative methods. Let

$$\begin{aligned}
\bar{N}(\omega,\tau) &= \bar{M}(\omega,\tau) - \widehat{A}\\
&= \begin{pmatrix} \frac{\omega^2}{1+2\omega}A & -\frac{\omega}{1+2\omega}B \\ \left(1-\frac{\tau(1+\omega)}{(1+2\tau)(1+2\omega)}\right)B^\top & \frac{\tau(1+\tau)}{1+2\tau}Q - \frac{\tau}{(1+2\tau)(1+2\omega)}B^\top A^{-1}B \end{pmatrix},
\end{aligned}$$

then

$$\widehat{A} = \bar{M}(\omega,\tau) - \bar{N}(\omega,\tau), \tag{2.6}$$

is a splitting of coefficient matrix $\widehat{A}$, we can induce the PAHSS-PTS iterative methods by splitting (2.6). In the following, we will provide the specific algorithmic procedures of the PAHSS-PTS iterative methods.

Specific algorithmic procedures of the PAHSS-PTS methods. Let $Q \in R^{n\times n}$ be a symmetric and positive definite matrix, $x^{(0)} \in R^m$ and $y^{(0)} \in R^n$. Moreover, give an initial guess $z^{(0)} = ({x^{(0)}}^\top, {y^{(0)}}^\top)^\top$ and two positive real relaxation parameters ω, τ. For $k = 0, 1, 2, \cdots$, until the iteration sequence $\{({x^{(k)}}^\top, {y^{(k)}}^\top)^\top\}$ converges to the exact solution of the nonsingular saddle point problems (2.1), we according to the following procedures to compute the iterate $z^{(k+1)} = ({x^{(k+1)}}^\top, {y^{(k+1)}}^\top)^\top$:

$$\begin{cases} x^{(k+\frac{1}{2})} = \frac{\omega}{1+\omega}x^{(k)} + \frac{1}{1+\omega}A^{-1}(f - By^{(k)}), \\ y^{(k+\frac{1}{2})} = y^{(k)} + \frac{1}{\tau}Q^{-1}(B^\top x^{(k)} - g), \\ y^{(k+1)} = y^{(k+\frac{1}{2})} + \frac{1}{1+\tau}Q^{-1}(B^\top x^{(k+\frac{1}{2})} - g), \\ x^{(k+1)} = \frac{\omega}{1+\omega}x^{(k+\frac{1}{2})} + \frac{1}{1+\omega}A^{-1}(f - By^{(k+1)}). \end{cases} \tag{2.7}$$

3. Convergence analysis of the PAHSS-PTS methods

In this section, we turn to study the convergence properties of the PAHSS-PTS iterative methods. Moreover, we will propose the sufficient conditions for the convergence of the PAHSS-PTS methods.

Lemma 3.1 ([27]). *Both roots of the real quadratic equation $x^2 - bx + c = 0$ are less 1 in modulus if and only if $|b| < 1 + c$ and $|c| < 1$.*

Lemma 3.2. *Let μ be an eigenvalue of $G = Q^{-1}B^\top A^{-1}B$ and λ be an eigenvalue of the iterative matrix $T(\omega,\tau)$, $w = (u^*, v^*)^* \in C^{m+n}$, with $u \in C^m$ and $v \in C^n$ be the corresponding eigenvector, then $u = 0$ if and only if $\lambda = -\frac{\omega}{1+\omega}$, $v = 0$ if and only if $\lambda = \frac{\omega^2}{(1+\omega)^2}$, moreover, if $v \neq 0$, $\lambda \neq -\frac{\omega}{1+\omega}$ and $\lambda \neq -\frac{\omega(1+\tau)}{3\omega\tau+2\tau+\omega+1}$, then λ satisfies*

$$\lambda^2 + \left[\frac{\mu}{1+\omega}\left(\frac{1}{\tau} + \frac{1}{1+\tau} + \frac{\omega}{(1+\omega)(1+\tau)}\right) - \frac{2\omega^2+2\omega+1}{(1+\omega)^2}\right]\lambda + \frac{\omega(\mu+\tau\omega)}{\tau(1+\omega)^2} = 0. \tag{3.1}$$

Proof. According to (2.5), we have:

$$\begin{pmatrix} \omega A & -B \\ B^\top & \tau Q \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} \frac{1+\omega}{\omega}I_m & O \\ -\frac{\tau}{\omega(1+\tau)}B^\top A^{-1} & \frac{\tau}{1+\tau}I_n \end{pmatrix}\begin{pmatrix} (1+\omega)A & B \\ O & (1+\tau)Q \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

it implies

$$\begin{cases} \omega Au - Bv = \frac{\lambda(1+\omega)^2}{\omega}Au + \frac{\lambda(1+\omega)}{\omega}Bv \\ B^\top u + \tau Qv = -\frac{\lambda\tau(1+\omega)}{\omega(1+\tau)}B^\top u + \tau\lambda Qv - \frac{\tau\lambda}{\omega(1+\tau)}B^\top A^{-1}Bv. \end{cases} \tag{3.2}$$

From the first equality of (3.2) we get

$$\left(\omega - \frac{\lambda(1+\omega)^2}{\omega}\right)Au = K_1 Au = \left(1 + \frac{\lambda(1+\omega)}{\omega}\right)Bv = K_2 Bv,$$

then we have $K_1Au = K_2Bv = 0$ when $v = 0$ or $u = 0$, it implies $\lambda = \frac{\omega^2}{(1+\omega)^2}$ or $\lambda = -\frac{\omega}{1+\omega}$ by definition of eigenvector, and vice versa. Moreover, if $v \neq 0$, $K_2 \neq 0$ that is $\lambda \neq -\frac{\omega}{1+\omega}$, then $K_1Au = K_2Bv \neq 0$, we can get $u = \frac{K_2}{K_1}A^{-1}Bv$, by substituting this equation into the second equality of (3.2) we get

$$(\lambda - 1)\tau Qv = \left[\frac{K_2}{K_1} + \frac{K_2\tau\lambda(1+\omega)}{K_1\omega(1+\tau)} + \frac{\tau\lambda}{\omega(1+\tau)}\right]B^\top A^{-1}Bv = K_3 B^\top A^{-1}Bv.$$

Since A and Q are symmetric positive definite matrices, if $K_3 \neq 0$ that is $\lambda \neq -\frac{\omega(1+\tau)}{3\omega\tau+2\tau+\omega+1}$, we obtain

$$(\lambda - 1)\tau v = \left[\frac{K_2}{K_1} + \frac{K_2\tau\lambda(1+\omega)}{K_1\omega(1+\tau)} + \frac{\tau\lambda}{\omega(1+\tau)}\right]Gv,$$

it is equivalent to

$$(\lambda - 1)\tau = \left[\frac{K_2}{K_1} + \frac{K_2\tau\lambda(1+\omega)}{K_1\omega(1+\tau)} + \frac{\tau\lambda}{\omega(1+\tau)}\right]\mu,$$

or equivalently,

$$\lambda^2 + \left[\frac{\mu}{1+\omega}\left(\frac{1}{\tau} + \frac{1}{1+\tau} + \frac{\omega}{(1+\omega)(1+\tau)}\right) - \frac{2\omega^2+2\omega+1}{(1+\omega)^2}\right]\lambda + \frac{\omega(\mu+\tau\omega)}{\tau(1+\omega)^2} = 0.$$

This completes the proof. □

Theorem 3.1. *Suppose matrix $A \in R^{m\times m}$ and $Q \in R^{n\times n}$ are symmetric and positive definite, $B \in R^{m\times n}$ is a column full rank matrix. We denote the largest and smallest eigenvalue of the matrix $G = Q^{-1}B^\top A^{-1}B$ by $\mu_{\max}$ and $\mu_{\min}$. If $v = 0$ or $u = 0$, then the PAHSS-PTS iterative method is convergent. Moreover, if $v \neq 0$, $\lambda \neq -\frac{\omega}{1+\omega}$, $\lambda = -\frac{\omega(1+\tau)}{3\omega\tau+2\tau+\omega+1}$ or $\lambda \neq -\frac{\omega(1+\tau)}{3\omega\tau+2\tau+\omega+1}$, and parameters ω, τ satisfy the following conditions:*

(a) $2\tau\omega + \tau - \omega\mu_{\max} > 0$;

(b) $\mu_{\max} < \frac{4\tau\omega(1+\omega)+\tau^2\omega(4+3\omega)+2\tau(1+\tau)}{3\tau\omega+2\tau+1}$;

(c) $\mu_{\min} > \frac{\tau^2\omega^2}{3\omega\tau+2\tau+2\omega+1}$.

then the PAHSS-PTS iterative methods are convergent too.

Proof. From the proof of the above lemma we know $u = 0$ if and only if $\lambda = -\frac{\omega}{1+\omega}$, $v = 0$ if and only if $\lambda = \frac{\omega^2}{(1+\omega)^2}$, so we can see that $|\lambda| = \frac{\omega}{1+\omega} < 1$ or $|\lambda| = \frac{\omega^2}{(1+\omega)^2} < 1$, it means the PAHSS-PTS iterative methods are convergent.

If $v \neq 0$, $\lambda \neq -\frac{\omega}{1+\omega}$ and $\lambda = -\frac{\omega(1+\tau)}{3\omega\tau+2\tau+\omega+1}$, then $|\lambda| = \frac{1+\tau}{1+3\tau+\frac{1+2\tau}{\omega}} < 1$. Conversely, if $v \neq 0$, $\lambda \neq -\frac{\omega}{1+\omega}$ and $\lambda \neq -\frac{\omega(1+\tau)}{3\omega\tau+2\tau+\omega+1}$, by making use of Lemma 3.1 and real quadratic equation (3.1), we have $|\lambda| < 1$ if and only if

$$\left|\frac{\omega(\mu+\tau\omega)}{\tau(1+\omega)^2}\right| < 1,$$

and

$$\left|\frac{\mu}{1+\omega}\left(\frac{1}{\tau}+\frac{1}{1+\tau}+\frac{\omega}{(1+\omega)(1+\tau)}\right)-\frac{2\omega^2+2\omega+1}{(1+\omega)^2}\right| < 1+\frac{\omega(\mu+\tau\omega)}{\tau(1+\omega)^2}.$$

By calculating the above two inequalities, we have

$$\begin{cases} 2\tau\omega+\tau-\omega\mu > 0, \\ 2\tau\omega^2+2\tau\omega+\tau+\omega\mu > 0, \\ 4\tau\omega(1+\omega)+\tau^2\omega(4+3\omega)+2\tau(1+\tau)-\mu(3\tau\omega+2\tau+1) > 0, \\ \mu(3\omega\tau+2\tau+2\omega+1)-\tau^2\omega^2 > 0. \end{cases} \tag{3.3}$$

Since matrix $Q^{-1}B^\top A^{-1}B$ is similar to matrix $Q^{-\frac{1}{2}}B^\top A^{-1}BQ^{-\frac{1}{2}}$, so μ is positive real, then $2\tau\omega^2+2\tau\omega+\tau+\omega\mu > 0$ sets up directly. If parameters ω, τ satisfy the condition (a), then we have

$$2\tau\omega+\tau-\omega\mu > 2\tau\omega+\tau-\omega\mu_{\max} > 0.$$

The third and forth inequalities of (3.3) are equivalent to

$$\mu < \frac{4\tau\omega(1+\omega)+\tau^2\omega(4+3\omega)+2\tau(1+\tau)}{3\tau\omega+2\tau+1}, \quad \mu > \frac{\tau^2\omega^2}{3\omega\tau+2\tau+2\omega+1},$$

so when parameters ω, τ satisfy the conditions (b) and (c), then we have

$$\mu < \mu_{\max} < \frac{4\tau\omega(1+\omega)+\tau^2\omega(4+3\omega)+2\tau(1+\tau)}{3\tau\omega+2\tau+1},$$

and

$$\mu > \mu_{\min} > \frac{\tau^2\omega^2}{3\omega\tau+2\tau+2\omega+1},$$

then we can see that inequalities (3.3) is established.

The proof of theorem is completed. □

Based on the above analysis, we can obtain that the PAHSS-PTS iterative methods are convergent under certain conditions.

4. Numerical results

In this section, we will perform two numerical examples to show the effectiveness of the PAHSS-PTS iteration methods for solving the nonsingular saddle point problems (1.1). All numerical experiments are carried out on a PC using MATLAB

2014a under the AMD A8-4500M 1.9GHz CPU and 4G RAM Win7 operating system. In the following tables, IT and CPU stand for the number of iteration steps and the elapsed CPU time. In actual computations, we choose the right-hand-side vector $b \in R^{m+n}$ such that the exact solution of the nonsingular saddle point problems (1.1) is $z = (1, 1, \ldots, 1)^\top \in R^{m+n}$. Moreover, all runs are started from the zero vector and terminated if the iterations satisfy $ERR \leq 10^{-6}$ or the number of the prescribed iteration steps $k_{max} = 1000$ is exceeded, where the ERR is defined by

$$ERR = \frac{\sqrt{||p - Ax^{(k)} - By^{(k)}||_2^2 + ||q - B^\top x^{(k)}||_2^2}}{\sqrt{||p - Ax^{(0)} - By^{(0)}||_2^2 + ||q - B^\top x^{(0)}||_2^2}}.$$

Example 4.1 ([5]). Consider the following Stokes problems

$$\begin{cases} -\nu\Delta u + \nabla p = \widetilde{f}, & \text{in} \quad \Omega, \\ \nabla \cdot u = \widetilde{g}, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \\ \int_\Omega p(x)dx = 0. \end{cases} \tag{4.1}$$

Here $\Omega = [0, 1] \times [0, 1] \subset R^2$, $\partial\Omega$ is the boundary of Ω, ν and Δ denotes the viscous coefficient and Laplace operator of fluid, u and p is the velocity and pressure of fluid.

By discrete equation (4.1) with difference scheme, then we can obtain the following linear system

$$\begin{pmatrix} A & B \\ -B^\top & O \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix}, \tag{4.2}$$

where

$$A = \begin{pmatrix} I \otimes T + T \otimes I & O \\ O & I \otimes T + T \otimes I \end{pmatrix} \in R^{2l^2 \times 2l^2}, \quad B = \begin{pmatrix} I \otimes F \\ F \otimes I \end{pmatrix} \in R^{2l^2 \times l^2},$$

which

$$T = \frac{1}{h^2} Tridiag(-1, 2, -1) \in R^{l \times l}, \quad F = \frac{1}{h} Tridiag(-1, 1, 0) \in R^{l \times l},$$

the $\otimes$ signify the Kronecker product and $h = \frac{1}{l+1}$ the discretization mesh size. For this example, we set $m = 2l^2$ and $n = l^2$, then the total number of variables is $3l^2$.

We compared the PAHSS-PTS iteration methods with the GSOR [6] methods and the PAHSS method [25]. the preconditioned matrix Q of the PAHSS-PTS methods is taken as θI ($\theta > 0$), the preconditioned matrix Q of the GSOR methods is taken as $\nu B^\top B$, where $\nu = \sqrt{\gamma_{min}\gamma_{max}}$, γ_{min} and γ_{max} represent the smallest and largest eigenvalues of the matrix A, and the preconditioned matrix for PAHSS methods is $B^\top B$. In Table 1-3, we list the numerical results of the GSOR, PAHSS-PTS and PAHSS methods for different sizes of the coefficient matrix.

In Table 1, we can see that PAHSS-PTS methods ($Q = 0.8I$) perform better than GSOR methods ($Q = \nu B^\top B$), since the former requires much less CPU time and

Table 1. Numerical results for GSOR and PAHSS-PTS ($\theta = 0.8$) methods

	GSOR					PAHSS-PTS				
l	ω_{opt}	τ_{opt}	ERR	CPU	IT	τ_{opt}	ω_{opt}	ERR	CPU	IT
8	0.5436	1.3467e+04	7.0297e-07	0.0519	46	0.82	0.29	8.5131e-07	0.0353	23
16	0.3419	5.0738e+04	8.9680e-07	1.0258	88	0.60	0.33	9.7907e-07	0.4504	31
24	0.2489	1.1145e+05	9.9151e-07	6.0532	130	0.57	0.35	9.4813e-07	2.5570	40
32	0.1956	1.9560e+05	9.6700e-07	28.5862	173	0.55	0.37	8.6375e-07	11.5300	49

Table 2. Numerical results for PAHSS and PAHSS-PTS ($\theta = 0.5$) methods

	PAHSS					PAHSS-PTS				
l	τ_{opt}	ω_{opt}	ERR	CPU	IT	τ_{opt}	ω_{opt}	ERR	CPU	IT
8	0.82	1.00	9.4109e-07	0.1186	12	1.41	0.33	7.1378e-07	0.0345	23
16	0.65	1.00	7.9088e-07	1.7347	15	1.13	0.34	8.9514e-07	0.4451	32
24	0.56	1.01	7.9122e-07	12.0434	17	1.01	0.36	9.2711e-07	2.4743	40
32	0.51	1.01	6.6752e-07	96.3265	19	1.00	0.37	9.9985e-07	10.9681	49

Table 3. Numerical results for PAHSS and PAHSS-PTS ($\theta = 1$) methods

	PAHSS					PAHSS-PTS				
l	τ_{opt}	ω_{opt}	ERR	CPU	IT	τ_{opt}	ω_{opt}	ERR	CPU	IT
8	0.41	1.17	9.1249e-07	0.1230	11	0.62	0.27	9.1601e-07	0.0343	23
16	0.30	1.47	8.9735e-07	1.7568	15	0.49	0.30	9.0963e-07	0.4669	32
24	0.25	1.58	6.1449e-07	12.8158	18	0.44	0.36	6.1956e-07	2.6680	40
32	0.22	1.65	4.9533e-07	101.1506	20	0.41	0.44	8.6141e-07	11.0944	50

IT to achieve the stopping criterion. Moreover, in Tables 2 and 3, we take $Q = 0.5I$ and $Q = I$ in this numerical experiment to show the PAHSS-PTS iteration methods perform very well while compared with the PAHSS method ($Q = B^{\top}B$). In Tables 2 and 3, we can see that PAHSS method requires less iteration steps than PAHSS-PTS methods, but it needs much more run-time. From Tables 1-3, one may also find that the PAHSS-PTS iteration methods are insensitive to the changes of θ, which indicates that slight variation of θ does not influent iteration counts too much.

5. Conclusions

In this paper, we provide the PAHSS-PTS iterative methods for nonsingular saddle point problems (1.1). The new algorithm is based on the PAHSS iterative method and preconditioned triangular splitting technique. The properties of convergence have been studied. Convergence behaviors of the PAHSS-PTS iterative methods are very efficient when the optimal parameters ω_{opt}, τ_{opt} are selected by computer. Moreover, future work will focus on estimating the optimal value of parameters τ , ω and choosing preconditioned matrix Q.

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