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DYNAMICS OF A STOCHASTIC SIR MODEL WITH BOTH HORIZONTAL AND VERTICAL TRANSMISSION*

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Abstract A stochastic mathematical model with both horizontal and vertical transmission is proposed to investigate the dynamical behavior of SIR disease. By employing theories of stochastic differential equation and inequality techniques, the threshold associating on extinction and persistence of infectious diseases is deduced for the case of the small noise. Our results show that the threshold completely depends on the stochastic perturbation and the basic reproductive number of the corresponding deterministic model. Moreover, we find that large noise is conducive to control the spread of diseases and the persistent disease in deterministic model may eliminate ultimately due to the effect of large noise. Finally, numerical simulations are performed to illustrate the theoretical results.

Keywords Stochastic SIR epidemic model, vertical transmission, extinction, persistence, threshold.

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1. Introduction

Infectious diseases threaten human health and bring huge disaster to human beings. People have realized the importance of quantitative studies on the spread of infectious diseases to predict and control them. Mathematical models have been confirmed to be an effective and valuable approach to understand the dynamical behavior of infectious disease, then a large number of mathematical models have been constructed to investigate the dynamical behavior of propagation and evolution rule of infectious diseases [2]. A classical epidemic model known as SIR model was proposed and investigated firstly by Kermack and Mckendric in 1927 [12]. In

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SIR model, population is divided into three classes, the susceptible, the infectious and the recovered or removed. SIR models play a crucial role in studying the evolution of infectious diseases and have received widely attentions (see, for examples, [26,34,36,39–41,44]). SIR models mainly described the horizontal propagation and the evolution of diseases between members of the same population. However, researchers have shown that some diseases such as AIDS [28,31,35] and Hepatitis B or Hepatitis C [29,30], are transmitted horizontally, i.e., these kinds of diseases are directly transmitting from the mother to an embryo, fetus, or baby during pregnancy or childbirth. As it is important to study the dynamical properties emerge by the horizontal transmission, Lu et al. [22] proposed an SIR epidemic model incorporating both horizontal and vertical transmission as below:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{dt} = -bS(t) - \beta S(t)I(t) + pdI(t) + b(S(t) + R(t)), \\ \dfrac{\mathrm{d}I(t)}{dt} = \beta S(t)I(t) - dI(t) - \gamma I(t) + qdI(t), \\ \dfrac{\mathrm{d}R(t)}{dt} = \gamma I(t) - bR(t), \end{cases} \tag{1.1}$$

where $S(t), I(t)$ and $R(t)$ represent the members of the susceptible, the infectious and the removed members from infection, respectively. b is the birth and death rate. β is the contact rate. The constant $p(0 < p < 1)$ is the proportion of the offspring of infective parents who are susceptible for the disease. γ is the recovery rate of the infective individuals and $p+q = 1$. Note that the total population size is normalized to one and the basic reproductive number of model (1.1) is defined as $R_0 = \beta/(pd+\gamma)$. The authors pointed out that if $R_0 < 1$, the infection-free equilibrium $P_0(1, 0, 0)$ is globally asymptotically stable while if $R_0 > 1$, the infection-free equilibrium P_0 is unstable, and the endemic equilibrium $P^*(S^*, I^*, R^*)$ is globally asymptotically stable.

It is now well known that stochastic noise is widely present in biological systems [5,17,18,23,33,38,43] and stochastic noise factors play an important role in transmission of infectious diseases, because it can provide an additional degree of realism in comparison to their deterministic counterparts. Therefore, many scholars have studied the effect of stochasticity on epidemic models [1,3,25,27], different stochastic perturbation approaches have been introduced into epidemic models and excellent results have been obtained. The authors of the articles [6,10,15] have considered epidemic models including the environment noise and have analyzed the dynamical behavior by using method of time Markov chain. The transmission coefficient perturbation in epidemic models induced by environment white noise has been investigated [9,16,21,32,42], in which the stochastic perturbation of system can be offset by summation. In the research articles [7,14], the authors have extensively studied the environmental noise which was proportional to the variables. Stochastic epidemic models with a complex type of noises described by the combination of parameter perturbation and proportion of the variables were investigated in [8,20]. In the research articles [37,45], the authors have paid their attention on the epidemic models with Lévy jump noise, and in the articles [4,19], the authors focused on stochastic perturbation around the positive equilibria of deterministic models.

In this paper, motivated by the above works, we introduce transmission coefficient perturbation with white noises into model (1.1) i.e., we replace β by $\beta+\sigma \mathrm{d}B(t)$,

where $B(t)$ is a standard Brownian motion with intensity $\sigma^2 > 0$. Then the resultant model turns into the following form:

$$\begin{cases} \mathrm{d}S(t) = (-bS(t) - \beta S(t)I(t) + pdI(t) + b(S(t) + R(t)))\mathrm{d}t - \sigma S(t)I(t)\mathrm{d}B(t), \\ \mathrm{d}I(t) = (\beta S(t)I(t) - dI(t) - \gamma I(t) + qdI(t))\mathrm{d}t + \sigma S(t)I(t)\mathrm{d}B(t), \\ \mathrm{d}R(t) = (\gamma I(t) - bR(t))\mathrm{d}t. \end{cases} \tag{1.2}$$

This paper is organized as follows. In Section 2, we will give some notations, definitions and lemmas which will be used to obtain our main results. In addition, the conditions leading to the extinction of the infectious disease will be given in Section 3. In Section 4, we will deduce the condition for the disease being persistent. In Section 5, we will give a brief conclusion to summary of the paper and also give some numerical simulations to illustrate the theoretical results.

2. Preliminaries

Throughout this paper, for convenience let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions such as increasing and right continuous and $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets. Further, $B(t)$ represents a scalar Brownian motion defined on the complete probability space Ω and $R^3_+ = \{x_i > 0, i = 1, 2, 3\}$. For a continuous function f, we define $\langle f(t)\rangle = \frac{1}{t}\int_0^t f(\tau)\mathrm{d}\tau$, then we have

Definition 2.1. For model (1.2),

(i) the diseases $I(t)$ is said to be extinctive if $\lim\limits_{t\to+\infty} I(t) = 0$;

(ii) the diseases $I(t)$ is said to be permanent in mean if there exist a positive constant λ such that $\liminf\limits_{t\to+\infty}\langle I(t)\rangle \geq \lambda$.

The following lemmas indicate the global existence, non-negativity and invariance of unique solution of model (1.2).

Lemma 2.1. *For any initial value $(S_0, I_0, R_0) \in R^3_+$, there exists a unique solution $(S(t), I(t), R(t))$ to model (1.2) on $t \geq 0$, and the solution remains in R^3_+ with probability one, namely, $(S(t), I(t), R(t)) \in R^3_+$ for all $t \geq 0$ a.s.*

Proof. Firstly, we know that, for any initial values $(S_0, I_0, R_0) \in R^3_+$, because the coefficients of model (1.2) are locally Lipschitz continuous, then there exists a unique local solution on $[0, \tau_\epsilon)$ where τ_ϵ is the explosion time. To prove that this solution is global, we need to show $\tau_\epsilon = \infty$ a.s. To do it, let $\epsilon_0 > 0$ such that $S_0 > \epsilon_0, I_0 > \epsilon_0, R_0 > \epsilon_0$. For any positive ϵ satisfying $\epsilon \leq \epsilon_0$, define the stopping time τ_ϵ by

$$\tau_\epsilon = \inf\{t \in [0, \tau_\epsilon) : S(t) \leq \epsilon \ \text{ or } \ I(t) \leq \epsilon\},$$

with the traditional setting $\inf \varnothing = \infty$, where $\varnothing$ denotes the empty set. Clearly, τ_ϵ is increasing as $\epsilon \to 0$. Set $\tau_0 = \lim_{\epsilon\to 0}$, then $\tau_0 \leq \tau_\epsilon$ a.s, hence we need only to prove $\tau_0 = \infty$ a.s. Otherwise, then there exists a pair of constants $T > 0$ and $\delta \in (0, 1)$ such that $P\{\tau_0 \leq T\} > \delta$. Hence, there exists a positive constant $\epsilon_1 \leq \epsilon_0$ such that $P\{\tau_0 \leq T\} > \delta$ for any positive $\epsilon \leq \epsilon_1$.

Define a C^2 function $V : R^3_+ \to \overline{R}_+$ by

$$V(S(t), I(t), R(t)) = -\ln S - \ln I - \ln R.$$

Obviously, V is positive definite. Using Itô's formula, we obtain

$$\mathrm{d}V = LV\mathrm{d}t + \sigma(I - S)\mathrm{d}B,$$

where

$$LV = \beta I - pd\frac{I}{S} - b\frac{R}{S} - \beta S + d + \gamma - qd - \gamma\frac{I}{R} + b + \frac{1}{2}\sigma^2(S^2 + I^2).$$

Then, we have

$$LV \le \beta + b + pd + \gamma + \frac{1}{2}\sigma^2(S^2 + I^2) \le \beta + b + pd + \gamma + \sigma^2 := C.$$

Thus,

$$\mathrm{d}V \le C\mathrm{d}t + \sigma(I - S)\mathrm{d}B.$$

Integrating both sides from 0 to $\tau_e \wedge T$, and then taking expectations, yields

$$EV(S(\tau_\epsilon \wedge T), I(\tau_\epsilon \wedge T)) \le V(S_0, I_0, R_0) + CT.$$

Set $\Omega_\epsilon = \{\tau_e \le T\}$ for any positive $\epsilon \le \epsilon_1$ and then $P(\Omega_\epsilon) > \delta$. Note that for every $\omega \in \Omega_\epsilon$, there is at least one of $S(\tau_\epsilon, \omega), I(\tau_\epsilon, \omega), R(\tau_\epsilon, \omega)$ equals ϵ, then

$$V(S(\tau_\epsilon), I(\tau_\epsilon), R(\tau_\epsilon, \omega)) \ge -\ln\epsilon.$$

Consequently,

$$\begin{aligned} V(S_0, I_0, R_0) + CT &\ge E[I_{\Omega_\epsilon} V(S(\tau_\epsilon \wedge T), I(\tau_\epsilon \wedge T)R(\tau_\epsilon \wedge T))] \\ &= P(\Omega_\epsilon)V(S(\tau_\epsilon), I(\tau_\epsilon), R(\tau_\epsilon)) \\ &> -\delta\ln\epsilon, \end{aligned}$$

where I_{Ω_ϵ} is the indicator function of Ω_ϵ. Letting $\epsilon \to 0$ leads to the contradiction $\infty > V(S_0, I_0, R_0) + CT = \infty$. Therefore, we must have $\tau_0 = \infty$ a.s.

The proof of Lemma 2.1 is completed. □

Remark 2.1. By using the methods from Ji et al. [11], we can also prove, for any initial value $(S_0, I_0, R_0) \in \overline{R}^3_+$, there exists a unique solution $(S(t), I(t), R(t))$ to model (1.2) on $t \ge 0$, and the solution remains in $\overline{R}^3_+$ with probability one, namely, $(S(t), I(t), R(t)) \in \overline{R}^3_+$ for all $t \ge 0$ a.s.

Remark 2.2. By Lemma 2.1 and Remark 2.1, we can obtain the region

$$\Gamma = \{(S(t), I(t), R(t)) \in R^3_+ : S(t) + I(t) + R(t) \le 1\}$$

which is a positively invariant set, then through the paper, we always let the initial value $(S_0, I_0, R_0) \in \Gamma$.

Lemma 2.2. *Let* $(S(t), I(t), R(t))$ *be a solution of model* (1.2) *with initial value* $(S_0, I_0, R_0) \in \Gamma$. *Then*

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \sigma S(\tau)dB(\tau) = 0.$$

Proof. Let $Z(t) = \int_0^t \sigma S(\tau)\mathrm{d}B(\tau)$ and $\theta > 2$. By the Burkholder-Davis-Gundy inequality in [24] and Lemma 2.1, we have

$$E\left[\sup_{0\le\tau\le t}|Z(\tau)|^\theta\right] \le C_\theta E\left[\int_0^t \sigma^2 S^2(\tau)\mathrm{d}\tau\right]^{\frac{\theta}{2}} \le C_\theta t^{\frac{\theta}{2}} E\left[\sup_{0\le\tau\le t}\sigma^\theta S^\theta(\tau)\right] \le C_\theta \sigma^\theta t^{\frac{\theta}{2}},$$

where C_θ. Then, for any $0 < \varepsilon < \frac{\theta}{2} - 1$, by Doob's martingale inequality [24],

$$\begin{aligned}\mathbb{P}\left\{\omega : \sup_{k\delta\le t\le(k+1)\delta}|Z(t)|^\theta > (k\delta)^{1+\varepsilon+\frac{\theta}{2}}\right\} &\le \frac{E\left(|Z((k+1)\delta)|^\theta\right)}{(k\delta)^{1+\varepsilon+\frac{\theta}{2}}}\\ &\le \frac{C_\theta\sigma^\theta[(k+1)\delta]^{\frac{\theta}{2}}}{(k\delta)^{1+\varepsilon+\frac{\theta}{2}}}\\ &\le \frac{2^{\frac{\theta}{2}}C_\theta\sigma^\theta}{(k\delta)^{1+\varepsilon}}\end{aligned}$$

holds. Thus by Borel-Cantelli lemma in [24], for almost all $\omega \in \Omega$, we get that

$$\sup_{k\delta\le t\le(k+1)\delta}|Z(t)|^\theta \le (k\delta)^{1+\varepsilon+\frac{\theta}{2}}$$

holds for all but finitely many k. Thus, there exists a positive $k_0(\omega)$, for almost all $\omega \in \Omega$ and $k \ge k_0(\omega)$, such that $\sup_{k\delta\le t\le(k+1)\delta}|Z(t)|^\theta \le (k\delta)^{1+\varepsilon+\frac{\theta}{2}}$. Hence, if $k \ge k_0(\omega)$ and $k\delta \le t \le (k+1)\delta$, then for almost all $\omega \in \Omega$,

$$\frac{\ln|Z(t)|^\theta}{\ln t} \le \frac{\left(1+\varepsilon+\frac{\theta}{2}\right)\ln(k\delta)}{\ln(k\delta)} = 1+\varepsilon+\frac{\theta}{2}.$$

Hence, we have

$$|Z(t)| \le t^{\frac{1}{2}+\frac{1+\varepsilon}{\theta}}.$$

Then, for the above ε, there exists a constant $T(\omega)$ and a set Ω_ϵ, such that $\mathbb{P}(\Omega_\epsilon) \ge 1-\epsilon$ and for $t \ge T(\omega)$, $\omega \in \Omega_\epsilon$,

$$0 \le \liminf_{t\to+\infty}\frac{|Z(t)|}{t} \le \limsup_{t\to+\infty}\frac{|Z(t)|}{t} \le \limsup_{t\to+\infty} t^{\frac{1+\varepsilon}{\theta}-\frac{1}{2}} = 0.$$

Then we have

$$\lim_{t\to+\infty}\frac{|Z(t)|}{t} = 0,$$

i.e.

$$\lim_{t\to+\infty}\frac{Z(t)}{t} = \lim_{t\to+\infty}\frac{1}{t}\int_0^t \sigma S(\tau)\mathrm{d}B(\tau) = 0.$$

This completes the proof of Lemma 2.2. □

3. Extinction

In this section, we deduce the condition which is crucially important to the disease to be died out. Let us consider

$$R_0^* = \frac{\beta}{pd+\gamma} - \frac{\sigma^2}{2(pd+\gamma)} = R_0 - \frac{\sigma^2}{2(pd+\gamma)},$$

and the we have the results mentioned in the following theorem

Theorem 3.1. *If* $\sigma^2 > \max\{\beta, \frac{\beta^2}{2(pd+\gamma)}\}$ *or* $\sigma^2 \leq \beta$ *and* $R_0^* < 1$, *then the infectious disease of model* (1.2) *goes to extinction almost surely. Moreover,*

$$\lim_{t\to+\infty} R(t) = 0, \lim_{t\to+\infty} S(t) = 1,$$

a.s.

Proof. Let $(S(t),I(t),R(t))$ be a solution of system (1.2) with initial value $(S_0,I_0,R_0)\in\Gamma$. Applying Itô's formula to the second equation of model (1.2) leads to

$$\mathrm{d}\ln I(t) = \left(\beta S(t) - (pd+\gamma) - \frac{\sigma^2}{2}S^2(t)\right)\mathrm{d}t + \sigma S(t)\mathrm{d}B(t). \tag{3.1}$$

Integrating both sides of (3.1) from 0 to t gives

$$\ln I(t) = \int_0^t \left(\beta S(\tau) - \frac{\sigma^2}{2}S^2(\tau)\right)\mathrm{d}\tau - (pd+\gamma)t + M(t) + \ln I(0), \tag{3.2}$$

where $M(t) = \int_0^t \sigma S(\tau)\mathrm{d}B(\tau)$ and $M(t)$ is the local continuous martingale with $M(0) = 0$. Next we have two cases to be discussed, depending on $\sigma^2 > \beta$.

If $\sigma^2 > \beta$, one can easily see from (3.2) that

$$\ln I(t) \leq \left(\frac{\beta^2}{2\sigma^2} - (pd+\gamma)\right)t + M(t) + \ln I(0). \tag{3.3}$$

Dividing both sides of (3.3) by $t(>0)$, we have

$$\frac{\ln I(t)}{t} \leq -\left(pd+\gamma-\frac{\beta^2}{2\sigma^2}\right) + \frac{M(t)}{t} + \frac{\ln I(0)}{t}. \tag{3.4}$$

By Lemma 2.2, we obtain $\lim_{t\to+\infty}\frac{M(t)}{t} = 0$, then taking the limit superior on both sides of (3.4) it leads to

$$\limsup_{t\to+\infty}\frac{\ln I(t)}{t} \leq -\left(pd+\gamma-\frac{\beta^2}{2\sigma^2}\right) < 0,$$

when $\sigma^2 > \frac{\beta^2}{2(pd+\gamma)}$, which implies that $\lim_{t\to+\infty} I(t) = 0$.

If $\sigma^2 \leq \beta$, we can similarly have

$$\ln I(t) \leq \left(\beta - (pd+\gamma) - \frac{\sigma^2}{2}\right)t + M(t) + \ln I(0). \tag{3.5}$$

Dividing both sides of (3.5) by $t(>0)$, we have

$$\frac{\ln I(t)}{t} \leq (pd+\gamma)\left[\frac{\beta}{pd+\gamma} - \frac{\sigma^2}{2(pd+\gamma)} - 1\right] + \frac{M(t)}{t} + \frac{\ln I(0)}{t}. \tag{3.6}$$

We take the superior limit on both sides of (3.6) and one has that

$$\limsup_{t\to+\infty}\frac{\ln I(t)}{t} \leq (pd+\gamma)(R_0^* - 1).$$

Then when $R_0^* < 1$, we obtain

$$\limsup_{t\to+\infty} \frac{\ln I(t)}{t} < 0,$$

which implies that $\lim_{t\to+\infty} I(t) = 0$.

Next, we prove that $\lim_{t\to+\infty} R(t) = 0$ and $\lim_{t\to+\infty} S(t) = 1$, a.s. Since $\lim_{t\to+\infty} I(t) = 0$, then, the third equation of model (1.2) lead to $\lim_{t\to+\infty} R(t) = 0$, notice that $S(t) + I(t) + R(t) = 1$, thus, we have $\lim_{t\to+\infty} S(t) = 1$. This completes the proof of Theorem 3.1. □

Remark 3.1. Theorem 3.1 shows that when $\sigma^2 > \max\{\beta, \beta^2/2(pd+\gamma)\}$, the infectious disease of model (1.2) goes to extinction almost surely, namely, large white noise stochastic disturbance is conducive to control infectious disease.

Remark 3.2. Note that $R_0^* = R_0 - \sigma^2/2(pd+\gamma)$. Obviously, $R_0 < 1$ leads to $R_0^* < 1$, while the other side is not true. This implies that the condition for $I(t)$ going to extinction in the deterministic model is stronger than its stochastic counterpart due to the effect of the white noise disturbance.

4. Persistence in mean

Theorem 4.1. *If $R_0^* > 1$, then the infectious disease I is permanent in mean, moreover, I satisfies*

$$\liminf_{t\to+\infty} \langle I(t)\rangle \geq \frac{pd+\gamma}{\beta\left(1+\frac{\gamma}{b}\right)}(R_0^* - 1),$$

a.s.

Proof. Integrating from 0 to t and dividing by $t(>0)$ on both sides of the third equation of model (1.2) yields

$$\frac{R(t) - R(0)}{t} = \gamma\langle I(t)\rangle - b\langle R(t)\rangle \triangleq \Theta(t).$$

Notice that $\langle S(t)\rangle + \langle I(t)\rangle + \langle R(t)\rangle = 1$, then one can get

$$\langle S(t)\rangle = 1 + \frac{\Theta(t)}{b} - \left(1+\frac{\gamma}{b}\right)\langle I(t)\rangle.$$

Applying Itô's formula, it has that

$$\begin{aligned} \mathrm{d}\left(\ln I(t)\right) &= \left[\beta S(t) - (pd+\gamma) - \frac{\sigma^2}{2}S^2(t)\right]\mathrm{d}t + \sigma S(t)\mathrm{d}B(t) \\ &\geq \left[\beta S(t) - (pd+\gamma) - \frac{\sigma^2}{2}\right]\mathrm{d}t + \sigma S(t)\mathrm{d}B(t). \end{aligned} \tag{4.1}$$

Integrating from 0 to t and dividing by $t(>0)$ on both sides of (4.1) yields

$$\begin{aligned} \frac{\ln I(t) - \ln I(0)}{t} &\geq \beta\langle S(t)\rangle - \left[(pd+\gamma) + \frac{\sigma^2}{2}\right] + \frac{M(t)}{t}, \\ &= \beta\left(1 + \frac{\Theta(t)}{b} - \left(1+\frac{\gamma}{b}\right)\langle I(t)\rangle\right) \\ &\quad - \left[pd+\gamma+\frac{\sigma^2}{2}\right] + \frac{M(t)}{t}. \end{aligned} \tag{4.2}$$

From (4.2), we obtain

$$\langle I(t)\rangle \geq \frac{1}{\beta\left(1+\frac{\gamma}{b}\right)}\left[\beta\left(1+\frac{\Theta(t)}{b}\right)-pd-\gamma-\frac{\sigma^2}{2}-\frac{\ln I(t)-\ln I(0)}{t}+\frac{M(t)}{t}\right]. \tag{4.3}$$

Since $R(t), I(t) \leq 1$, then one has that $\lim_{t\to+\infty}\frac{R(t)}{t}=0$, $\lim_{t\to+\infty}\frac{\ln I(t)}{t}=0$ and $\lim_{t\to+\infty}\Theta(t)=0$. Notice that $\lim_{t\to+\infty}\frac{M(t)}{t}=0$, by taking the inferior limit of both sides of (4.3), we have

$$\liminf_{t\to+\infty}\langle I(t)\rangle \geq \frac{1}{\beta\left(1+\frac{\gamma}{b}\right)}\left[\beta-pd-\gamma-\frac{\sigma^2}{2}\right]=\frac{pd+\gamma}{\beta\left(1+\frac{\gamma}{b}\right)}(R_0^*-1).$$

This finishes the proof of Theorem 4.1. □

Remark 4.1. Theorems 3.1 and 4.1 show that when the white noise is not large, R_0^* is the threshold associated with the extinction of infectious disease. Moreover, the condition for the disease to go to extinction or persistence depend on the intensity of white noise disturbances strongly. In addition to that small white noise disturbances will be beneficial to long-term prevalence of the disease, conversely, large white noise disturbances may cause the epidemic disease to be died out.

5. Conclusion and numerical simulation

In this paper, a stochastic SIR model with both horizontal and vertical transmission is proposed and investigated. The threshold dynamics are explored when the stochastic noise is small. Our results show that the threshold completely depends on the stochastic perturbation and the basic reproductive number of the corresponding deterministic model. Moreover, we find there exists significant difference between the threshold of deterministic and that of the stochastic model due to the effect of stochastic noise, and large noise is conducive to control the spread of diseases.

In the following, by employing the Euler Maruyama (EM) method [13, 24], we make some numerical simulations to illustrate the extinction and persistence of the diseases in stochastic system and corresponding deterministic system for comparison.

For numerical simulations, we set parameters as $b = 0.008, \beta = 0.186, p = 0.04, d = 0.032, \gamma = 0.113$ in model (1.1). A simple computation shows that $R_0 = 1.6276 > 1$. Then model (1.1) has a unique stable positive equilibrium $P^*(0.6144, 0.0255, 0.3601)$, which implies that the disease of model (1.1) is permanent. (see Fig. 1)

Next, we consider the effect of stochastic white noise. Let $\sigma = 0.4$, obviously, $\sigma^2 > \max\{\beta, \beta^2/2(pd+\gamma)\}$, by Theorem 3.1, the disease dies out under a large white noise disturbance (see Fig. 2). If we change σ to 0.38, obviously, $\sigma^2 < \beta^2/2(pd+\gamma)$ and $R_0^* = 0.9791 < 1$, then by Theorem 3.1, the disease dies out (see Fig. 3). If we change σ to 0.1, obviously, $R_0^* = 1.3 > 1$, by Theorem 4.1, the disease is persistent (see Fig. 4). Fig 5 show the solution of the stochastic system oscillate around the positive equilibrium of the deterministic system.

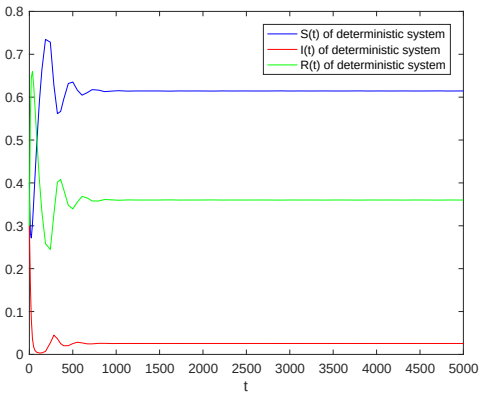
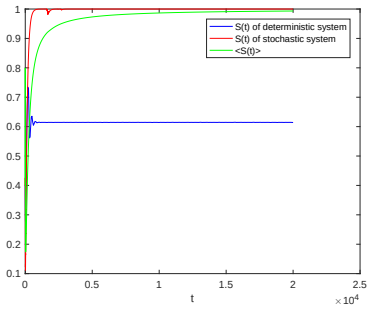
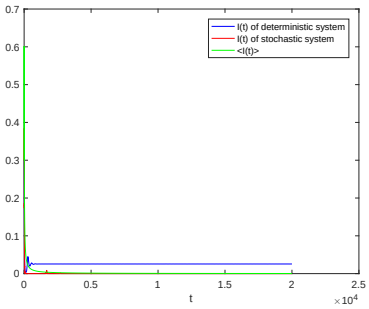


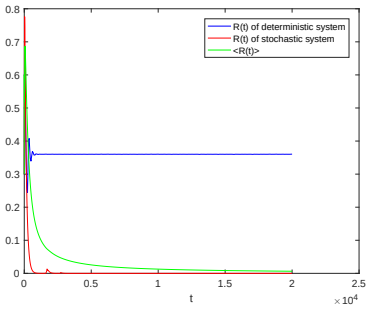
Figure 1. Time series for $S(t), I(t)$ and $R(t)$ of the deterministic system with $b = 0.008, \beta = 0.186, p = 0.04, d = 0.032, \gamma = 0.113$, where $R_0 = 1.6276 > 1$.



(a) Time series for $S(t)$

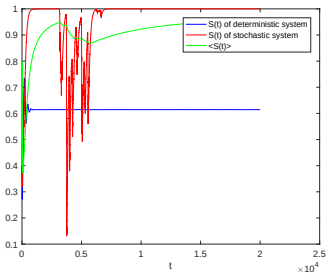


(b) Time series for $I(t)$

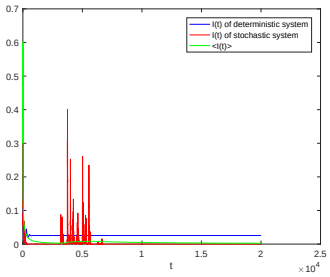


(c) Time series for $R(t)$

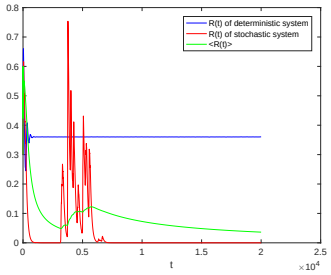
Figure 2. Comparison of the deterministic system and stochastic system with $b = 0.008, \beta = 0.186, p = 0.04, d = 0.032, \gamma = 0.113, \sigma = 0.4$, where $R_0 = 1.6276 > 1$.



(a) Time series for $S(t)$

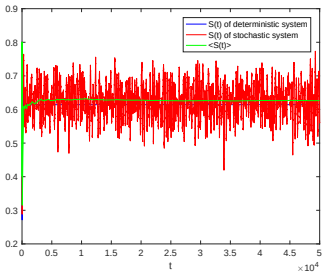


(b) Time series for $I(t)$

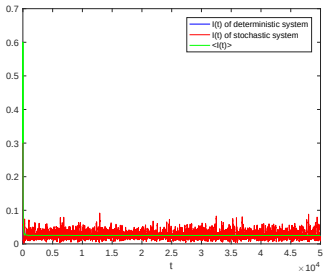


(c) Time series for $R(t)$

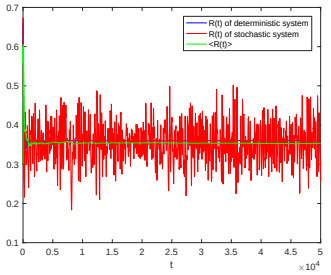
Figure 3. Comparison of the deterministic system and stochastic system, where $b = 0.008, \beta = 0.186, p = 0.04, d = 0.032, \gamma = 0.113, \sigma = 0.38$, where $R_0 = 1.6276 > 1, R_0^* = 0.9791 < 1$.



(a) Time series for $S(t)$

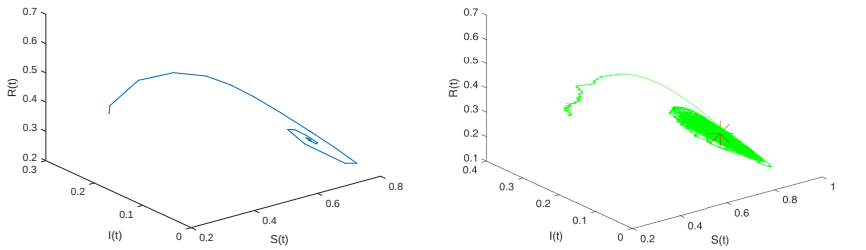


(b) Time series for $I(t)$



(c) Time series for $R(t)$

Figure 4. Comparison of the deterministic system and stochastic system, where $b = 0.008, \beta = 0.186, p = 0.04, d = 0.032, \gamma = 0.113, \sigma = 0.1$, where $R_0 = 1.6276 > 1, R_0^* = 1.5838 > 1$.



(a) 3D Phase diagram for $S(t), I(t)$ and $R(t)$ of the deterministic system.

(b) 3D Phase diagram for $S(t), I(t)$ and $R(t)$ of the stochastic system.

Figure 5. Comparison of the deterministic system and stochastic system, where $b = 0.008, \beta = 0.186, p = 0.04, d = 0.032, \gamma = 0.113, \sigma = 0.1$, where $R_0 = 1.6276 > 1, R_0^* = 1.5838 > 1$.

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