Journal of Applied Analysis and Computation
Volume 8, Number 2, April 2018, 558–572
Website:http://jaac-online.com/
DOI:10.11948/2018.558

A MORE ACCURATE MULTIDIMENSIONAL HARDY-HILBERT'S INEQUALITY*

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Abstract In this paper, by the use of the weight coefficients, the transfer formula, Hermite-Hadamard's inequality and the technique of real analysis, a more accurate multidimensional Hardy-Hilbert's inequality with multi-parameters and a best possible constant factor is given, which is an extension of some published results. Moreover, the equivalent forms and the operator expressions are considered.

Keywords Hardy-Hilbert's inequality, weight coefficient, Hermite-Hadamard's inequality, equivalent form, operator.

MSC(2010) 26D15, 47A05.

1. Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0, a = \{a_m\}_{m=1}^{\infty} \in l^p, b = \{b_n\}_{n=1}^{\infty} \in l^q$, $||a||_p = (\sum_{m=1}^{\infty} a_m^p)^{\frac{1}{p}} > 0$, $||b||_q > 0$, then we have the following well known Hardy-Hilbert's inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}||a||_p||b||_q, \tag{1.1}$$

and the following more accurate Hardy-Hilbert's inequality with the same best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [7], Theorem 315, Theorem 323):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n-1} < \frac{\pi}{\sin(\pi/p)}||a||_p||b||_q. \tag{1.2}$$

Inequalities (1.1) and (1.2) are important in analysis and its applications (cf. [7, 19, 25]).

Assuming that $\{\mu_m\}_{m=1}^{\infty}$ and $\{\upsilon_n\}_{n=1}^{\infty}$ are positive sequences,

$$U_m = \sum_{i=1}^{m}\mu_i, V_n = \sum_{j=1}^{n}\upsilon_j \ (m, n \in \mathbf{N} = \{1, 2, \ldots\}),$$

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*The author was supported by National Natural Science Foundation of China (No. 61370186, No. 61640222) and Appropriative Researching Fund for Professors and Doctors, Guangdong University of Education (No. 2015ARF25).

we have the following Hardy-Hilbert-type inequality (cf. [7], Theorem 321):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{U_m+V_n}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}\frac{a_m^p}{\mu_m^{p-1}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{b_n^q}{\upsilon_n^{q-1}}\right)^{\frac{1}{q}}. \tag{1.3}$$

For $\mu_i = \upsilon_j = 1$ $(i, j \in \mathbf{N})$, inequality (1.3) reduces to (1.1).

In 2015, by using the transfer formula, Yang [26] gave the following multidimensional Hilbert-type inequality: For $i_0, j_0 \in \mathbf{N}, \alpha, \beta > 0$,

$$||x||_\alpha := \left(\sum_{k=1}^{i_0}|x^{(k)}|^\alpha\right)^{\frac{1}{\alpha}} \quad (x=(x^{(1)},\ldots,x^{(i_0)})\in\mathbf{R}^{i_0}),$$
$$||y||_\beta := \left(\sum_{k=1}^{j_0}|y^{(k)}|^\beta\right)^{\frac{1}{\beta}} \quad (y=(y^{(1)},\ldots,y^{(j_0)})\in\mathbf{R}^{j_0}),$$

$0<\lambda_1\le i_0, 0<\lambda_2\le j_0, \lambda_1+\lambda_2=\lambda$, $a_m, b_n \ge 0$, we have

$$\sum_n\sum_m\frac{1}{||m||_\alpha^\lambda+||n||_\beta^\lambda}a_m b_n$$
$$<K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}\left[\sum_m ||m||_\alpha^{p(i_0-\lambda_1)-i_0}a_m^p\right]^{\frac{1}{p}}\left[\sum_n ||n||_\beta^{q(j_0-\lambda_2)-j_0}b_n^q\right]^{\frac{1}{q}}, \tag{1.4}$$

where, $\sum_m = \sum_{m_{i_0}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}$, $\sum_n = \sum_{n_{j_0}=1}^{\infty}\cdots\sum_{n_1=1}^{\infty}$, the series in the right hand side of (1.4) are positive values, and the best possible constant factor $K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}$ is indicated by

$$K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}=\left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}}\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}}\frac{\pi}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})}.$$

For $i_0 = j_0 = \lambda = \alpha = \beta = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, inequality (1.4) reduces to (1.1). Some other results on this type of inequalities and multiple inequalities were provided by [3, 6, 9–11, 14, 16, 20–22].

Recently, by using the weight coefficients, Yang [27] gave an extension of (1.3) as follows: For $\eta > 0$, $0<\lambda_1\le 1, 0<\lambda_2\le 1$, $\lambda_1+\lambda_2=\lambda$, $a_m, b_n\ge 0$,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(U_m^\eta+V_n^\eta)^{\lambda/\eta}}$$
$$<\frac{1}{\eta}B(\frac{\lambda_1}{\eta},\frac{\lambda_2}{\eta})\left[\sum_{m=1}^{\infty}\frac{U_m^{p(1-\lambda_1)-1}a_m^p}{\mu_m^{p-1}}\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}\frac{V_n^{q(1-\lambda_2)-1}b_n^q}{\upsilon_n^{q-1}}\right]^{\frac{1}{q}}, \tag{1.5}$$

where, the constant factor $\frac{1}{\eta}B(\frac{\lambda_1}{\eta},\frac{\lambda_2}{\eta})$ is the best possible (the series in the right hand side of (1.5) are positive values). Another results on Hardy-Hilbert-type inequalities and Hilbert-type inequalities were given by [2–5, 8, 12, 13, 17, 18, 23, 24, 28, 29].

In this paper, by the use of the weight coefficients, the transfer formula, Hermite-Hadamard's inequality and the technique of analysis, a more accurate multidimensional Hardy-Hilbert's inequality with multi-parameters and a best possible constant factor is given, which is an extension of (1.2), (1.4) and (1.5). Moreover, the equivalent forms and the operator expressions are considered.

2. Some lemmas

If $\mu_i^{(k)} > 0\,, 0 \leq \widetilde{\mu}_i^{(k)} \leq \frac{1}{2}\mu_i^{(k)}$ $(k = 1, \ldots, i_0; i = 1, \ldots, m), \upsilon_j^{(l)} > 0$, $0 \leq \widetilde{\upsilon}_j^{(l)} \leq \frac{1}{2}\upsilon_j^{(l)}$ $(l = 1, \ldots, j_0; j{=}1, \ldots, n)$, then we set

$$\begin{aligned} U_m^{(k)} &: = \sum_{i=1}^{m} \mu_i^{(k)}, \widetilde{U}_m^{(k)} := U_m^{(k)} - \widetilde{\mu}_m^{(k)} \ (k = 1, \ldots, i_0), \\ V_n^{(l)} &: = \sum_{j=1}^{n} \upsilon_j^{(l)}\ , \widetilde{V}_n^{(l)} := V_n^{(l)} - \widetilde{\upsilon}_n^{(l)} \ (l = 1, \ldots, j_0), \\ U_m &: = (U_m^{(1)}, \ldots, U_m^{(i_0)}), \widetilde{\mu}_m := (\widetilde{\mu}_m^{(1)}, \ldots, \widetilde{\mu}_m^{(i_0)}), \\ \widetilde{U}_m &: = (\widetilde{U}_m^{(1)}, \ldots, \widetilde{U}_m^{(i_0)}) = U_m - \widetilde{\mu}_m, \\ V_n &: = (V_n^{(1)}, \ldots, V_n^{(j_0)}), \widetilde{\upsilon}_n := (\widetilde{\upsilon}_n^{(1)}, \ldots, \widetilde{\upsilon}_n^{(j_0)})\ , \\ \widetilde{V}_n &: = (\widetilde{V}_n^{(1)}, \ldots, \widetilde{V}_n^{(j_0)}) = V_n - \widetilde{\upsilon}_n \ (m, n \in \mathbf{N}). \end{aligned} \tag{2.1}$$

We also set functions $\mu_k(t) := \mu_m^{(k)}, t \in (m - \frac{1}{2}, m + \frac{1}{2}]$ $(m \in \mathbf{N})$; $\upsilon_l(t) := \upsilon_n^{(l)}$, $t \in (n - \frac{1}{2}, n + \frac{1}{2}]$ $(n \in \mathbf{N})$, and

$$\begin{aligned} U_k(x) &: = \int_{\frac{1}{2}}^{x} \mu_k(t)dt \ (k = 1, \ldots, i_0), \\ V_l(y) &: = \int_{\frac{1}{2}}^{y} \upsilon_l(t)dt \ (l = 1, \ldots, j_0), \qquad (2.2) \\ U(x) &: = (U_1(x), \ldots, U_{i_0}(x)), \\ V(y) &: = (V_1(y), \ldots, V_{j_0}(y)) \ (x, y \geq \frac{1}{2}). \qquad (2.3) \end{aligned}$$

It follows that

$$\begin{aligned} U_k(m) &= \int_{\frac{1}{2}}^{m} \mu_k(t)dt = \int_{\frac{1}{2}}^{m+\frac{1}{2}} \mu_k(t)dt - \frac{1}{2}\mu_m^{(k)} \\ &\leq \widetilde{U}_m^{(k)} \leq U_k(m + \frac{1}{2}) \ (k = 1, \ldots, i_0; m \in \mathbf{N}), \\ V_l(n) &\leq \widetilde{V}_n^{(l)} \leq V_l(n + \frac{1}{2}) \ (l = 1, \ldots, j_0; n \in \mathbf{N}), \end{aligned}$$

and for $x \in (m - \frac{1}{2}, m + \frac{1}{2})$, $U_k'(x) = \mu_k(x) = \mu_m^{(k)}$ $(k = 1, \ldots, i_0; m \in \mathbf{N})$; for $y \in (n - \frac{1}{2}, n + \frac{1}{2})$, $V_l'(y) = \upsilon_l(y) = \upsilon_n^{(l)}$ $(l = 1, \ldots, j_0; n \in \mathbf{N})$.

Lemma 2.1 (cf. [28])**.** *Suppose that $g(t)$ (> 0) is strictly decreasing and strictly convex in $(\frac{1}{2}, \infty)$, satisfying $\int_{\frac{1}{2}}^{\infty} g(t)dt \in \mathbf{R}_+$. We have the following Hermite-*

Hadamard's inequality

$$\int_{n}^{n+1} g(t)dt < g(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} g(t)dt \ (n \in \mathbf{N}), \tag{2.4}$$

and then

$$\int_{1}^{\infty} g(t)dt < \sum_{n=1}^{\infty} g(n) < \int_{\frac{1}{2}}^{\infty} g(t)dt. \tag{2.5}$$

Lemma 2.2. *If* $i_0 \in \mathbf{N}, \alpha, M > 0, \Psi(u)$ *is a nonnegative measurable function on* $(0, 1]$, *and*

$$D_M := \left\{ x = (x_1, \ldots, x_{i_0}) \in \mathbf{R}_+^{i_0}; u = \sum_{i=1}^{i_0} \left(\frac{x_i}{M} \right)^{\alpha} \leq 1 \right\}, \tag{2.6}$$

then we have the following transfer formula (cf. [11]):

$$\int \cdots \int_{D_M} \Psi\left(\sum_{i=1}^{i_0} \left(\frac{x_i}{M} \right)^{\alpha} \right) dx_1 \cdots dx_s = \frac{M^{i_0} \Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0} \Gamma(\frac{i_0}{\alpha})} \int_0^1 \Psi(u) u^{\frac{i_0}{\alpha}-1} du. \tag{2.7}$$

Lemma 2.3. *If* $i_0, j_0 \in \mathbf{N}, \alpha, \beta, \varepsilon > 0, \mu_m^{(k)} \geq \mu_{m+1}^{(k)}$ $(m \in \mathbf{N}; k = 1, \ldots, i_0)$, $\upsilon_n^{(l)} \geq \upsilon_{n+1}^{(l)}$ $(n \in \mathbf{N}; l = 1, \ldots, i_0)$, $b := \min_{1 \leq i \leq i_0, 1 \leq j \leq j_0} \{\mu_1^{(i)}, \upsilon_1^{(j)}\} (> 0)$, *then we have*

$$\sum_m |\widetilde{U}_m||_{\alpha}^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} \leq \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon b^{\varepsilon} i_0^{\varepsilon/\alpha} \alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} + O(1), \tag{2.8}$$

$$\sum_n ||\widetilde{V}_n||_{\beta}^{-j_0-\varepsilon} \prod_{k=1}^{j_0} \upsilon_n^{(k)} \leq \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\varepsilon b^{\varepsilon} j_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + \widetilde{O}(1) \ (\varepsilon \to 0^+). \tag{2.9}$$

Proof. For $M > b i_0^{1/\alpha}$, we set

$$\Psi(u) = \begin{cases} 0, 0 < u < \frac{b^{\alpha} i_0}{M^{\alpha}}, \\ \frac{1}{(M u^{1/\alpha})^{i_0+\varepsilon}}, \frac{b^{\alpha} i_0}{M^{\alpha}} \leq u \leq 1. \end{cases}$$

By (2.7), it follows that

$$\int_{\{x \in \mathbf{R}_+^{i_0}; x_i \geq b\}} \frac{dx}{||x||_{\alpha}^{i_0+\varepsilon}} = \lim_{M \to \infty} \int \cdots \int_{D_M} \Psi\left(\sum_{i=1}^{i_0} \left(\frac{x_i}{M} \right)^{\alpha} \right) dx_1 \cdots dx_{i_0}$$
$$= \lim_{M \to \infty} \frac{M^{i_0} \Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0} \Gamma(\frac{i_0}{\alpha})} \int_{b^{\alpha} i_0 / M^{\alpha}}^{1} \frac{u^{\frac{i_0}{\alpha}-1}}{(M u^{1/\alpha})^{i_0+\varepsilon}} du = \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon b^{\varepsilon} i_0^{\varepsilon/\alpha} \alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})}.$$

Then by (2.5) and the above result, in view of $U_k(m) \le \widetilde{U}_m^{(k)}$, we find

$$\begin{aligned}
0 &< \sum_{\{m\in\mathbf{N}^{i_0};m_i\ge 2\}} ||\widetilde{U}_m||_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)} \\
&\le \sum_{\{m\in\mathbf{N}^{i_0};m_i\ge 2\}} \int_{\{x\in\mathbf{R}_+^{i_0};m_i-\frac{1}{2}\le x_i<m_i+\frac{1}{2}\}} ||U(m)||_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)}dx \\
&< \sum_{\{m\in\mathbf{N}^{i_0};m_i\ge 2\}} \int_{\{x\in\mathbf{R}_+^{i_0};m_i-\frac{1}{2}\le x_i<m_i+\frac{1}{2}\}} ||U(x)||_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_k(x)dx \\
&= \int_{\{x\in\mathbf{R}_+^{i_0};x_i\ge\frac{3}{2}\}} ||U(x)||_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_k(x)dx \\
&\overset{v=U(x)}{=} \int_{\{v\in\mathbf{R}_+^{i_0};v_i\ge\mu_1^{(i)}\}} ||v||_\alpha^{-i_0-\varepsilon}dv \le \int_{\{v\in\mathbf{R}_+^{i_0};v_i\ge b\}} ||v||_\alpha^{-i_0-\varepsilon}dv \\
&= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon b^\varepsilon i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}.
\end{aligned}$$

For $i_0 = 1$, $0 < \sum_{\{m\in\mathbf{N}^{i_0};m_i=1\}} ||\widetilde{U}_m||_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)} \le (\mu_1^{(1)})^{-\varepsilon} < \infty$; for $i_0 \ge 2$, we set

$$H_i := \sum_{\{m\in\mathbf{N}^{i_0};m_i=1\}} ||\widetilde{U}_m||_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)}(i = 1,\dots,i_0).$$

Without loss of generality, we estimate H_{i_0} as follows:

$$\begin{aligned}
H_{i_0} &\le \sum_{\{m\in\mathbf{N}^{i_0};m_{i_0}=1\}} ||U(m)||_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)} \\
&= \mu_1^{(i_0)}\sum_{m\in\mathbf{N}^{i_0-1}} \frac{\prod_{k=1}^{i_0-1}\mu_m^{(k)}}{[\sum_{i=1}^{i_0-1}U_i^\alpha(m)+(\frac{1}{2}\mu_1^{(i_0)})^\alpha]^{\frac{1}{\alpha}(i_0+\varepsilon)}} \\
&< \sum_{m\in\mathbf{N}^{i_0-1}} \int_{\{x\in\mathbf{R}_+^{i_0-1};m_i-\frac{1}{2}\le x_i<m_i+\frac{1}{2}\}} \frac{\mu_1^{(i_0)}\prod_{k=1}^{i_0-1}\mu_k(x)dx}{[\sum_{i=1}^{i_0-1}U_i^\alpha(x)+(\frac{1}{2}\mu_1^{(i_0)})^\alpha]^{\frac{1}{\alpha}(i_0+\varepsilon)}} \\
&= \mu_1^{(i_0)}\int_{\{x\in\mathbf{R}^{i_0-1};x_i\ge\frac{1}{2}\}} \frac{\prod_{k=1}^{i_0-1}\mu_k(x)}{[\sum_{i=1}^{i_0-1}U_i^\alpha(x)+(\frac{1}{2}\mu_1^{(i_0)})^\alpha]^{\frac{1}{\alpha}(i_0+\varepsilon)}}dx \\
&\overset{v=U(x)}{\le} \mu_1^{(i_0)}\int_{\mathbf{R}_+^{i_0-1}} \frac{1}{[M^\alpha\sum_{i=1}^{i_0-1}(\frac{v_i}{M})^\alpha+(\frac{1}{2}\mu_1^{(i_0)})^\alpha]^{\frac{1}{\alpha}(i_0+\varepsilon)}}dv.
\end{aligned}$$

By (2.7), we find

$$\begin{aligned}
0 < H_{i_0} &\le \mu_1^{(i_0)}\lim_{M\to\infty}\frac{M^{i_0-1}\Gamma^{i_0-1}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0-1}{\alpha})}\int_0^1\frac{u^{\frac{i_0-1}{\alpha}-1}du}{[M^\alpha u+(\frac{1}{2}\mu_1^{(i_0)})^\alpha]^{\frac{i_0+\varepsilon}{\alpha}}} \\
&\overset{t=\frac{M^\alpha u}{(\frac{1}{2}\mu_1^{(i_0)})^\alpha}}{=} \frac{2^{1+\varepsilon}}{(\mu_1^{(i_0)})^\varepsilon}\frac{\Gamma^{i_0-1}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0-1}{\alpha})}\int_0^\infty\frac{t^{\frac{i_0-1}{\alpha}-1}dt}{(t+1)^{\frac{i_0+\varepsilon}{\alpha}}}
\end{aligned}$$

$$= \frac{2^{1+\varepsilon}}{(\mu_1^{(i_0)})^{\varepsilon}} \frac{\Gamma^{i_0-1}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0-1}{\alpha})} B(\frac{i_0-1}{\alpha}, \frac{1+\varepsilon}{\alpha}) < \infty,$$

namely, $H_{i_0} = O_{i_0}(1)$. Hence, we have

$$\sum_m ||\widetilde{U}_m||_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} \leq \sum_{\{m\in\mathbf{N}^{i_0}; m_i\geq 2\}} ||\widetilde{U}_m||_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} + \sum_{i=1}^{i_0} H_i$$
$$\leq \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon b^\varepsilon i_0^{\varepsilon/\alpha} \alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} + \sum_{i=1}^{i_0} O_i(1) \ (\varepsilon \to 0^+),$$

and then (2.8) follows. In the same way, we have (2.9). □

Definition 2.1. For $0 < \alpha, \beta, \eta \leq 1, 0 < \lambda_1 \leq i_0, 0 < \lambda_2 \leq j_0, \lambda_1 + \lambda_2 = \lambda$, we define two weight coefficients $w(\lambda_1, n)$ and $W(\lambda_2, m)$ as follows:

$$w(\lambda_1, n) := \sum_m \frac{1}{(||\widetilde{U}_m||_\alpha^\eta + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}} \frac{||\widetilde{V}_n||_\beta^{\lambda_2}}{||\widetilde{U}_m||_\alpha^{i_0-\lambda_1}} \prod_{k=1}^{i_0} \mu_m^{(k)}, \tag{2.10}$$

$$W(\lambda_2, m) := \sum_n \frac{1}{(||\widetilde{U}_m||_\alpha^\eta + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}} \frac{||\widetilde{U}_m||_\alpha^{\lambda_1}}{||\widetilde{V}_n||_\beta^{j_0-\lambda_2}} \prod_{l=1}^{j_0} \upsilon_n^{(l)}. \tag{2.11}$$

Example 2.1. With regards to the assumptions of Definition 2.1, we set $k_\lambda(x,y) = \frac{1}{(x^\eta+y^\eta)^{\lambda/\eta}}$ $(x, y > 0)$. Then for $\lambda > 0, 0 < \eta \leq 1$, we find

$$\frac{\partial}{\partial x} k_\lambda(x,y) = \frac{-\lambda x^{\eta-1}}{(x^\eta + y^\eta)^{(\lambda/\eta)+1}} < 0,$$
$$\frac{\partial^2}{\partial x^2} k_\lambda(x,y) = \frac{\lambda(1-\eta)x^{\eta-2}}{(x^\eta + y^\eta)^{(\lambda/\eta)+1}} + \frac{\lambda(\lambda+\eta)x^{2\eta-2}}{(x^\eta + y^\eta)^{(\lambda/\eta)+2}} > 0.$$

In the same way, for $0 < \lambda_1 \leq i_0, 0 < \lambda_2 \leq j_0$, we still can find that $k_\lambda(x,y)\frac{1}{x^{i_0-\lambda_1}}$ $(k_\lambda(x,y)\frac{1}{y^{j_0-\lambda_2}})$ is strictly decreasing and strictly convex in $x \in (0,\infty)$ $(y \in (0,\infty))$, satisfying

$$\frac{\partial}{\partial x}(k_\lambda(x,y)\frac{1}{x^{i_0-\lambda_1}}) < 0, \frac{\partial^2}{\partial x^2}(k_\lambda(x,y)\frac{1}{x^{i_0-\lambda_1}}) > 0;$$
$$\frac{\partial}{\partial y}(k_\lambda(x,y)\frac{1}{y^{j_0-\lambda_2}}) < 0, \frac{\partial^2}{\partial y^2}(k_\lambda(x,y)\frac{1}{y^{j_0-\lambda_2}}) > 0.$$

We obtain

$$k(\lambda_1) := \int_0^\infty k_\lambda(u,1)\frac{du}{u^{1-\lambda_1}} = \int_0^\infty \frac{u^{\lambda_1-1}}{(u^\eta+1)^{\lambda/\eta}} du$$
$$\overset{v=u^\eta}{=} \frac{1}{\eta}\int_0^\infty \frac{v^{(\lambda_1/\eta)-1}dv}{(v+1)^{\lambda/\eta}} = \frac{1}{\eta} B(\frac{\lambda_1}{\eta}, \frac{\lambda_2}{\eta}) \in \mathbf{R}_+. \tag{2.12}$$

(ii) If $(-1)^i h^{(i)}(t) > 0$ $(t > 0; i = 0, 1, 2)$, $A > 0$, $0 < \alpha \le 1$, then we have

$$\begin{aligned}
\frac{d}{dx} h((A + x^\alpha)^{\frac{1}{\alpha}}) &= h'((A + x^\alpha)^{\frac{1}{\alpha}})(A + x^\alpha)^{\frac{1}{\alpha}-1} x^{\alpha-1} < 0\,, \\
\frac{d^2}{dx^2} h((A + x^\alpha)^{\frac{1}{\alpha}}) &= h''((A + x^\alpha)^{\frac{1}{\alpha}})(A + x^\alpha)^{\frac{2}{\alpha}-2} x^{2\alpha-2} \\
&\quad + (1-\alpha) h'((A + x^\alpha)^{\frac{1}{\alpha}})(A + x^\alpha)^{\frac{1}{\alpha}-2} x^{2\alpha-2} \\
&\quad + (\alpha-1) h'((A + x^\alpha)^{\frac{1}{\alpha}})(A + x^\alpha)^{\frac{1}{\alpha}-1} x^{\alpha-2} \\
&= h''((A + x^\alpha)^{\frac{1}{\alpha}})(A + x^\alpha)^{\frac{2}{\alpha}-2} x^{2\alpha-2} \\
&\quad + A(\alpha-1) h'((A + x^\alpha)^{\frac{1}{\alpha}})(A + x^\alpha)^{\frac{1}{\alpha}-2} x^{\alpha-2} > 0 \ (x > 0).
\end{aligned}$$

Hence, by (2.4), for $m_i - \frac{1}{2} < x_i < m_i + \frac{1}{2}$ $(i = 1, \ldots, i_0; m \in \mathbf{N})$, we have $\prod_{k=1}^{i_0} \mu_m^{(k)} = \prod_{k=1}^{i_0} \mu_k(x)$ and

$$\begin{aligned}
&\frac{||U(m)||_\alpha^{\lambda_1 - i_0} \prod_{k=1}^{i_0} \mu_m^{(k)}}{(||U(m)||_\alpha^\eta + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}} \\
&< \int_{\{x \in (\frac{1}{2},\infty)^{i_0}; m_i - \frac{1}{2} < x_i < m_i - \frac{1}{2}\}} \frac{||U(x)||_\alpha^{\lambda_1 - i_0} \prod_{k=1}^{i_0} \mu_k(x)}{(||U(x)||_\alpha^\eta + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}} dx.
\end{aligned}$$

Lemma 2.4. *With regards to the assumptions of Definition 2.1, (i) we have*

$$w(\lambda_1, n) < K_\beta(\lambda_1) \ (n \in \mathbf{N}^{j_0}), \tag{2.13}$$

$$W(\lambda_2, m) < K_\alpha(\lambda_1) \ (m \in \mathbf{N}^{i_0}), \tag{2.14}$$

where,

$$K_\beta(\lambda_1) = \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} k(\lambda_1), K_\alpha(\lambda_1) = \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} k(\lambda_1); \tag{2.15}$$

(ii) for $\mu_m^{(k)} \ge \mu_{m+1}^{(k)}$ $(m \in \mathbf{N}), \upsilon_n^{(l)} \ge \upsilon_{n+1}^{(l)}$ $(n \in \mathbf{N})$, $U_\infty^{(k)} = V_\infty^{(l)} = \infty$ $(k = 1, \ldots, i_0, l = 1, \ldots, j_0)$, $0 < \lambda_1 \le i_0, \lambda_2 > 0$, we have

$$0 < K_\alpha(\lambda_1)(1 - \theta_\lambda(n)) < w(\lambda_1, n) \ (n \in \mathbf{N}^{j_0}), \tag{2.16}$$

where, for $c := \max_{1 \le k \le i_0}\{\mu_1^{(k)}\}$ (> 0),

$$\theta_\lambda(n) := \frac{1}{\eta k(\lambda_1)} \int_0^{c^\eta i_0^{\eta/\alpha}/||\widetilde{V}_n||_\beta^\eta} \frac{t^{(\lambda_1/\eta)-1} dt}{(t+1)^{\lambda/\eta}} = O\left(\frac{1}{||\widetilde{V}_n||_\beta^{\lambda_1}}\right). \tag{2.17}$$

Proof. (i) Since $||\widetilde{U}_m||_\alpha \ge ||U(m)||_\alpha$, by (2.5), (2.7) and Example 2.1(ii), for

$0 < \lambda_1 \leq i_0, \lambda > 0$, it follows that

$$\begin{aligned}
w(\lambda_1, n) &= \sum_m \frac{1}{(||\widetilde{U}_m||_\alpha^\eta + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}} \frac{||\widetilde{V}_n||_\beta^{\lambda_2}}{||\widetilde{U}_m||_\alpha^{i_0-\lambda_1}} \prod_{k=1}^{i_0} \mu_m^{(k)} \\
&\leq \sum_m \frac{1}{(||U(m)||_\alpha^\eta + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}} \frac{||\widetilde{V}_n||_\beta^{\lambda_2}}{||U(m)||_\alpha^{i_0-\lambda_1}} \prod_{k=1}^{i_0} \mu_m^{(k)} \\
&< \sum_m \int_{\{x\in \mathbf{R}_+^{i_0}; m_i - \frac{1}{2} < x_i \leq m_i + \frac{1}{2}\}} \frac{1}{(||U(x)||_\alpha^\eta + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}} \\
&\quad \times \frac{|\widetilde{V}_n||_\beta^{\lambda_2}}{||U(x)||_\alpha^{i_0-\lambda_1}} \prod_{k=1}^{i_0} \mu_k(x) dx \\
&= \int_{\{x\in \mathbf{R}_+^{i_0}; x_i > \frac{1}{2}\}} \frac{||U(x)||_\alpha^{\lambda_1 - i_0} ||\widetilde{V}_n||_\beta^{\lambda_2}}{(||U(x)||_\alpha^\eta + ||\widetilde{V}_n||_\beta^\eta)^\lambda} \prod_{k=1}^{i_0} \mu_k(x) dx \\
&\overset{v=U(x)}{=} \int_{\mathbf{R}_+^{i_0}} \frac{||v||_\alpha^{\lambda_1 - i_0} ||\widetilde{V}_n||_\beta^{\lambda_2}}{(||v||_\alpha^\eta + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}} dv \\
&= \lim_{M\to\infty} \int_{\mathbf{D}_M} \frac{M^{\lambda_1 - i_0} [\sum_{i=1}^{j_0} (\frac{v_i}{M})^\alpha]^{(\lambda_1 - i_0)/\alpha} ||\widetilde{V}_n||_\beta^{\lambda_2}}{(M^\eta [\sum_{i=1}^{i_0} (\frac{v_i}{M})^\alpha]^{\eta/\alpha} + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}} dv \\
&= \lim_{M\to\infty} \frac{M^{i_0} \Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0} \Gamma(\frac{i_0}{\alpha})} \int_0^1 \frac{M^{\lambda_1 - i_0} u^{(\lambda_1 - i_0)/\alpha} ||\widetilde{V}_n||_\beta^{\lambda_2}}{(M^\eta u^{\eta/\alpha} + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}} u^{\frac{i_0}{\alpha} - 1} du \\
&= \lim_{M\to\infty} \frac{M^{i_0} \Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0} \Gamma(\frac{i_0}{\alpha})} \int_0^1 \frac{(M u^{1/\alpha})^{\lambda_1 - i_0} ||\widetilde{V}_n||_\beta^{\lambda_2}}{(M^\eta u^{\eta/\alpha} + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}} u^{\frac{i_0}{\alpha} - 1} du \\
&\overset{t=\frac{M^\eta u^{\eta/\alpha}}{||\widetilde{V}_n||_\beta^\eta}}{=} \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0 - 1} \Gamma(\frac{i_0}{\alpha}) \eta} \int_0^\infty \frac{t^{(\lambda_1/\eta) - 1} dt}{(t+1)^{\lambda/\eta}} = \frac{\Gamma^{i_0}(\frac{1}{\alpha}) k(\lambda_1)}{\alpha^{i_0 - 1} \Gamma(\frac{i_0}{\alpha})} = K_\alpha(\lambda_1).
\end{aligned}$$

Hence, we have (2.13). In the same way, we have (2.14).

(ii) Since for $m_i \leq x_i < m_i + \frac{1}{2}, \mu_{m_i}^{(k)} \geq \mu_{m_i+1}^{(k)} = \mu_k(x + \frac{1}{2})$; for $m_i + \frac{1}{2} \leq x_i < m_i + 1$, $\mu_m^{(k)} = \mu_k(x + \frac{1}{2})$, by (2.5) and in the same way, for $c = \max_{1\leq k\leq i_0}\{\mu_1^{(k)}\}$ (> 0), we have

$$\begin{aligned}
w(\lambda_1, n) &\geq \sum_m \frac{||U(m + \frac{1}{2})||_\alpha^{\lambda_1 - i_0} ||\widetilde{V}_n||_\beta^{\lambda_2}}{(||U(m + \frac{1}{2})||_\alpha^\eta + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}} \prod_{k=1}^{i_0} \mu_m^{(k)} \\
&> \sum_m \int_{\{x\in \mathbf{R}_+^{i_0}; m_i \leq x_i < m_i + 1\}} \frac{||U(x + \frac{1}{2})||_\alpha^{\lambda_1 - i_0} ||\widetilde{V}_n||_\beta^{\lambda_2}}{(||U(x + \frac{1}{2})||_\alpha^\eta + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}} \prod_{k=1}^{i_0} \mu_k(x + \frac{1}{2}) dx \\
&= \int_{[1,\infty)^{i_0}} \frac{||U(x + \frac{1}{2})||_\alpha^{\lambda_1 - i_0} ||\widetilde{V}_n||_\beta^{\lambda_2}}{(||U(x + \frac{1}{2})||_\alpha^\eta + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}} \prod_{k=1}^{i_0} \mu_k(x + \frac{1}{2}) dx \\
&\overset{v=U(x+\frac{1}{2})}{\geq} \int_{[c,\infty)^{i_0}} \frac{||v||_\alpha^{\lambda_1 - i_0} ||\widetilde{V}_n||_\beta^{\lambda_2}}{(||v||_\alpha^\eta + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}} dv.
\end{aligned}$$

For $M > ci_0^{1/\alpha}$, we set

$$\Psi(u) = \begin{cases} 0, 0 < u < \frac{c^\alpha i_0}{M^\alpha}, \\ \frac{(Mu^{1/\alpha})^{\lambda_1 - i_0} ||\widetilde{V}_n||_\beta^{\lambda_2}}{(M^\eta u^{\eta/\alpha} + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}}, \frac{c^\alpha i_0}{M^\alpha} \leq u \leq 1. \end{cases}$$

By (2.7), it follows that

$$\begin{aligned} &\int_{\{x \in \mathbf{R}_+^{i_0}; x_i \geq c\}} \frac{||x||_\alpha^{\lambda_1 - i_0} ||\widetilde{V}_n||_\beta^{\lambda_2}}{(||x||_\alpha^\eta + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}} dx \\ &= \lim_{M \to \infty} \int \cdots \int_{D_M} \Psi\left(\sum_{i=1}^{i_0} \left(\frac{x_i}{M}\right)^\alpha\right) dx_1 \cdots dx_{i_0} \\ &= \lim_{M \to \infty} \frac{M^{i_0} \Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0} \Gamma(\frac{i_0}{\alpha})} \int_{c^\alpha i_0 / M^\alpha}^1 \frac{(Mu^{\frac{1}{\alpha}})^{\lambda_1 - i_0} ||\widetilde{V}_n||_\beta^{\lambda_2}}{(M^\eta u^{\frac{\eta}{\alpha}} + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}} u^{\frac{i_0}{\alpha} - 1} du \\ &\overset{t = \frac{M^\eta u^{\eta/\alpha}}{||\widetilde{V}_n||_\beta^\eta}}{=} \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0 - 1} \Gamma(\frac{i_0}{\alpha}) \eta} \int_{c^\eta i_0^{\eta/\alpha} / ||\widetilde{V}_n||_\beta^\eta}^\infty \frac{t^{(\lambda_1/\eta) - 1}}{(t+1)^{\lambda/\eta}} dt. \end{aligned}$$

Hence, we have

$$\begin{aligned} w(\lambda_1, n) &> \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0 - 1} \Gamma(\frac{i_0}{\alpha}) \eta} \int_{c^\eta i_0^{\eta/\alpha} / ||\widetilde{V}_n||_\beta^\eta}^\infty \frac{t^{(\lambda_1/\eta) - 1}}{(t+1)^{\lambda/\eta}} dt \\ &= K_\alpha(\lambda_1)(1 - \theta_\lambda(n)) > 0. \end{aligned}$$

We obtain

$$\begin{aligned} 0 < \theta_\lambda(n) &= \frac{1}{\eta k(\lambda_1)} \int_0^{c^\eta i_0^{\eta/\alpha} / ||\widetilde{V}_n||_\beta^\eta} \frac{t^{(\lambda_1/\eta) - 1}}{(t+1)^{\lambda/\eta}} dt \\ &\leq \frac{1}{\eta k(\lambda_1)} \int_0^{c^\eta i_0^{\eta/\alpha} / ||\widetilde{V}_n||_\beta^\eta} t^{\frac{\lambda_1}{\eta} - 1} dt = \frac{1}{\lambda_1 k(\lambda_1)} \left(\frac{c i_0^{1/\alpha}}{||\widetilde{V}_n||_\beta}\right)^{\lambda_1} \end{aligned}$$

and then (2.16) and (2.17) follow. □

3. Main results

Setting functions

$$\widetilde{\Phi}(m) := \frac{||\widetilde{U}_m||_\alpha^{p(i_0 - \lambda_1) - i_0}}{(\prod_{k=1}^{i_0} \mu_m^{(k)})^{p-1}}, \widetilde{\Psi}(n) := \frac{||\widetilde{V}_n||_\beta^{q(j_0 - \lambda_2) - j_0}}{(\prod_{l=1}^{j_0} \upsilon_n^{(l)})^{q-1}} \ (m \in \mathbf{N}^{i_0}, n \in \mathbf{N}^{j_0}),$$

and the following normed spaces

$$\begin{aligned} l_{p,\widetilde{\Phi}} &:= \left\{ a = \{a_m\}; ||a||_{p,\widetilde{\Phi}} := (\sum_m \widetilde{\Phi}(m) |a_m|^p)^{\frac{1}{p}} < \infty \right\}, \\ l_{q,\widetilde{\Psi}} &:= \left\{ b = \{b_n\}; ||b||_{q,\widetilde{\Psi}} := (\sum_n \widetilde{\Psi}(n) |b_n|^q)^{\frac{1}{q}} < \infty \right\}, \\ l_{p,\widetilde{\Psi}^{1-p}} &:= \left\{ c = \{c_n\}; ||c||_{p,\widetilde{\Psi}^{1-p}} := (\sum_n \widetilde{\Psi}^{1-p}(n) |c_n|^p)^{\frac{1}{p}} < \infty \right\}, \end{aligned}$$

we have

Theorem 3.1. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1,\ 0 < \alpha, \beta, \eta \leq 1, 0 < \lambda_1 \leq i_0, 0 < \lambda_2 \leq j_0$, $\lambda_1 + \lambda_2 = \lambda$, *then for* $a_m, b_n \geq 0, a = \{a_m\} \in l_{p,\widetilde{\Phi}}, b = \{b_n\} \in l_{q,\widetilde{\Psi}}, ||a||_{p,\widetilde{\Phi}}, ||b||_{q,\widetilde{\Psi}} > 0$, *we have the following equivalent inequalities:*

$$I := \sum_n \sum_m \frac{a_m b_n}{(||\widetilde{U}_m||_\alpha^\eta + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}} < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) ||a||_{p,\widetilde{\Phi}} ||b||_{q,\widetilde{\Psi}}, \tag{3.1}$$

$$J := \left\{ \sum_n \frac{\prod_{k=1}^{j_0} \upsilon_n^{(k)}}{||\widetilde{V}_n||_\beta^{j_0 - p\lambda_2}} \left[\sum_m \frac{a_m}{(||\widetilde{U}_m||_\alpha^\eta + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}} \right]^p \right\}^{\frac{1}{p}}$$
$$< K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) ||a||_{p,\widetilde{\Phi}}, \tag{3.2}$$

where,

$$K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} \frac{1}{\eta} B(\frac{\lambda_1}{\eta}, \frac{\lambda_2}{\eta}). \tag{3.3}$$

Proof. By Hölder's inequality with weight (cf. [15]), we have

$$\begin{aligned} I &= \sum_n \sum_m \frac{1}{(||\widetilde{U}_m||_\alpha^\eta + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}} \left[\frac{||\widetilde{U}_m||_\alpha^{\frac{i_0-\lambda_1}{q}}}{||\widetilde{V}_n||_\beta^{\frac{j_0-\lambda_2}{p}}} \frac{(\prod_{l=1}^{j_0} \upsilon_n^{(l)})^{\frac{1}{p}} a_m}{(\prod_{k=1}^{i_0} \mu_m^{(k)})^{\frac{1}{q}}} \right] \\ &\quad \times \left[\frac{||\widetilde{V}_n||_\beta^{\frac{j_0-\lambda_2}{p}}}{||\widetilde{U}_m||_\alpha^{\frac{i_0-\lambda_1}{q}}} \frac{(\prod_{k=1}^{i_0} \mu_m^{(k)})^{\frac{1}{q}}}{(\prod_{l=1}^{j_0} \upsilon_n^{(l)})^{\frac{1}{p}}} b_n \right] \\ &\leq \left[\sum_m W(\lambda_2, m) \frac{||\widetilde{U}_m||_\alpha^{p(i_0-\lambda_1)-i_0}}{(\prod_{k=1}^{i_0} \mu_m^{(k)})^{p-1}} a_m^p \right]^{\frac{1}{p}} \\ &\quad \times \left[\sum_n w(\lambda_1, n) \frac{||\widetilde{V}_n||_\beta^{q(j_0-\lambda_2)-j_0}}{(\prod_{l=1}^{j_0} \upsilon_n^{(l)})^{q-1}} b_n^q \right]^{\frac{1}{q}}. \end{aligned}$$

Then by (2.13) and (2.14), we have (3.1). We set

$$b_n := \frac{\prod_{l=1}^{j_0} \upsilon_n^{(l)}}{||\widetilde{V}_n||_\beta^{j_0-p\lambda_2}} \left[\sum_m \frac{1}{(||\widetilde{U}_m||_\alpha^\eta + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}} a_m \right]^{p-1}, \quad n \in \mathbf{N}^{j_0}.$$

Then we have $J = ||b||_{q,\widetilde{\Psi}}^{q-1}$. Since the right hand side of (3.2) is finite, it follows that $J < \infty$. If $J = 0$, then (3.2) is trivially valid; if $J > 0$, then by (3.1), we have

$$||b||_{q,\widetilde{\Psi}}^q = J^p = I < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) ||a||_{p,\widetilde{\Phi}} ||b||_{q,\widetilde{\Psi}},$$
$$||b||_{q,\widetilde{\Psi}}^{q-1} = J < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) ||a||_{p,\widetilde{\Phi}},$$

namely, (3.2) follows. On the other hand, assuming that (3.2) is valid, by Hölder's inequality (cf. [15]), we have

$$I=\sum_n \frac{(\prod_{l=1}^{j_0}\upsilon_n^{(l)})^{1/p}}{||\widetilde{V}_n||_\beta^{(j_0/p)-\lambda_2}}\sum_m \frac{1}{(||\widetilde{U}_m||_\alpha^\eta+||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}}a_m \times\frac{||\widetilde{V}_n||_\beta^{(j_0/p)-\lambda_2}}{(\prod_{l=1}^{j_0}\upsilon_n^{(l)})^{1/p}}b_n \le J||b||_{q,\widetilde{\Psi}}. \tag{3.4}$$

Then by (3.2), we have (3.1), which is equivalent to (3.2). □

Theorem 3.2. *With regards to the assumptions of Theorem 3.1, if* $\mu_m^{(k)}\ge\mu_{m+1}^{(k)}$ $(m\in\mathbf{N}),\upsilon_n^{(l)}\ge\upsilon_{n+1}^{(l)}$ $(n\in\mathbf{N})$, $U_\infty^{(k)}=V_\infty^{(l)}=\infty$ $(k=1,\dots,i_0,l=1,\dots,j_0)$, *then the constant factor* $K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)$ *in (3.1) and (3.2) is the best possible.*

Proof. For $0<\varepsilon<p\lambda_1$, $\widetilde{\lambda}_1=\lambda_1-\frac{\varepsilon}{p}$ $(\in(0,i_0)),\widetilde{\lambda}_2=\lambda_2+\frac{\varepsilon}{p}$ (>0), we set

$$\widetilde{a}=\{\widetilde{a}_m\},\widetilde{a}_m:=||\widetilde{U}_m||_\alpha^{-i_0+\widetilde{\lambda}_1}\prod_{k=1}^{i_0}\mu_m^{(k)}\ (m\in\mathbf{N}^{i_0}),$$

$$\widetilde{b}=\{\widetilde{b}_n\},\widetilde{b}_n:=||\widetilde{V}_n||_\beta^{-j_0+\widetilde{\lambda}_2-\varepsilon}\prod_{l=1}^{j_0}\upsilon_n^{(l)}\ (n\in\mathbf{N}^{j_0}).$$

Then by (2.8) and (2.9), we obtain

$$\begin{aligned}
&||\widetilde{a}||_{p,\widetilde{\Phi}}||\widetilde{b}||_{q,\widetilde{\Phi}}\\
&=\left[\sum_m\frac{||\widetilde{U}_m||_\alpha^{p(i_0-\lambda_1)-i_0}\widetilde{a}_m^p}{(\prod_{k=1}^{i_0}\mu_m^{(k)})^{p-1}}\right]^{\frac{1}{p}}\left[\sum_n\frac{||\widetilde{V}_n||_\beta^{q(j_0-\lambda_2)-j_0}\widetilde{b}_n^q}{(\prod_{l=1}^{j_0}\upsilon_n^{(l)})^{q-1}}\right]^{\frac{1}{q}}\\
&=\left(\sum_m||\widetilde{U}_m||_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)}\right)^{\frac{1}{p}}\left(\sum_n||\widetilde{V}_n||_\beta^{-j_0-\varepsilon}\prod_{l=1}^{j_0}\upsilon_n^{(j)}\right)^{\frac{1}{q}}\\
&\le\frac{1}{\varepsilon}\left(\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{b^\varepsilon i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}+\varepsilon O(1)\right)^{\frac{1}{p}}\left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{b^\varepsilon j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}+\varepsilon\widetilde{O}(1)\right)^{\frac{1}{q}}.
\end{aligned}$$

By (2.16) and (2.17), we find

$$\begin{aligned}
\widetilde{I}&:=\sum_n\left[\sum_m\frac{1}{(||\widetilde{U}_m||_\alpha^\eta+||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}}\widetilde{a}_m\right]\widetilde{b}_n\\
&=\sum_n w(\widetilde{\lambda}_1,n)||\widetilde{V}_n||_\beta^{-j_0-\varepsilon}\prod_{l=1}^{j_0}\upsilon_n^{(l)}\\
&>K_\alpha(\widetilde{\lambda}_1)\sum_n\left(1-O\left(\frac{1}{||\widetilde{V}_n||_\beta^{\widetilde{\lambda}_1}}\right)\right)||\widetilde{V}_n||_\beta^{-j_0-\varepsilon}\prod_{l=1}^{j_0}\upsilon_n^{(l)}\\
&=K_\alpha(\widetilde{\lambda}_1)\left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\varepsilon b^\varepsilon j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}+\widetilde{O}(1)-O_1(1)\right).
\end{aligned}$$

If there exists a constant $K \leq K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)$, such that (3.1) is valid when replacing $K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)$ by K, then we have $\varepsilon\widetilde{I} < \varepsilon K||\widetilde{a}||_{p,\widetilde{\Phi}}||\widetilde{b}||_{q,\widetilde{\Phi}}$, namely,

$$
\begin{aligned}
&K_\alpha(\lambda_1 - \frac{\varepsilon}{p})\left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{b^\varepsilon j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \varepsilon\widetilde{O}(1) - \varepsilon O_1(1)\right)\\
<&K\left(\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{b^\varepsilon i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} + \varepsilon O(1)\right)^{\frac{1}{p}}\left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{b^\varepsilon j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \varepsilon\widetilde{O}(1)\right)^{\frac{1}{q}}.
\end{aligned}
$$

For $\varepsilon \to 0^+$, we find

$$
\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\frac{\Gamma^{i_0}(\frac{1}{\alpha})k(\lambda_1)}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \leq K\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{p}}\left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{q}},
$$

and then $K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1) \leq K$. Hence, $K = K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)$ is the best possible constant factor of (3.1). The constant factor in (3.2) is still the best possible. Otherwise, we would reach a contradiction by (3.4) that the constant factor in (3.1) is not the best possible. □

4. Operator expressions

With regards to the assumptions of Theorem 3.2, in view of

$$
\begin{aligned}
c_n &:= \frac{\prod_{k=1}^{j_0}\upsilon_n^{(k)}}{||\widetilde{V}_n||_\beta^{j_0-p\lambda_2}}\left[\sum_m \frac{1}{(||\widetilde{U}_m||_\alpha^\eta + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}}a_m\right]^{p-1}, \quad n \in \mathbf{N}^{j_0}\\
c &= \{c_n\}, ||c||_{p,\widetilde{\Psi}^{1-p}} = J < K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)||a||_{p,\widetilde{\Phi}} < \infty,
\end{aligned}
$$

we can set the following definition:

Definition 4.1. Define a multidimensional Hilbert's operator $T : l_{p,\widetilde{\Phi}} \to l_{p,\widetilde{\Psi}^{1-p}}$ as follows: For any $a \in l_{p,\widetilde{\Phi}}$, there exists a unique representation $Ta = c \in l_{p,\widetilde{\Psi}^{1-p}}$, satisfying

$$
Ta(n) := \sum_m \frac{1}{(||\widetilde{U}_m||_\alpha^\eta + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}}a_m \ (n \in \mathbf{N}^{j_0}). \tag{4.1}
$$

For $b \in l_{q,\widetilde{\Psi}}$, we define the following formal inner product of Ta and b as follows:

$$
(Ta, b) := \sum_n \left[\sum_m \frac{1}{(||\widetilde{U}_m||_\alpha^\eta + ||\widetilde{V}_n||_\beta^\eta)^{\lambda/\eta}}a_m\right]b_n. \tag{4.2}
$$

Then by Theorem 3.1, we have the following equivalent inequalities:

$$
(Ta, b) < K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)||a||_{p,\widetilde{\Phi}}||b||_{q,\widetilde{\Psi}}, \tag{4.3}
$$

$$
||Ta||_{p,\widetilde{\Psi}^{1-p}} < K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)||a||_{p,\widetilde{\Phi}}. \tag{4.4}
$$

It follows that T is bounded with

$$||T|| := \sup_{a(\neq\theta)\in l_{p,\widetilde{\Phi}}} \frac{||Ta||_{p,\widetilde{\Psi}^{1-p}}}{||a||_{p,\widetilde{\Phi}}} \leq K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1). \tag{4.5}$$

Since by Theorem 3.2, the constant factor $K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)$ in (4.4) is the best possible, we have

$$||T|| = K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}} k(\lambda_1). \tag{4.6}$$

Remark 4.1. (i) For $\widetilde{\mu}_i^{(k)} = 0$ $(k = 1, \ldots, i_0; i = 1, \ldots, m), \widetilde{\upsilon}_j^{(l)} = 0$ $(l = 1, \ldots, j_0;$ $j=1, \ldots, n)$, setting

$$\Phi(m) := \frac{||U_m||_\alpha^{p(i_0-\lambda_1)-i_0}}{(\prod_{k=1}^{i_0} \mu_m^{(k)})^{p-1}} \ (m \in \mathbf{N}^{i_0}),$$
$$\Psi(n) := \frac{||V_n||_\beta^{q(j_0-\lambda_2)-j_0}}{(\prod_{l=1}^{j_0} \upsilon_n^{(l)})^{q-1}} \ (n \in \mathbf{N}^{j_0}),$$

(3.1) and (3.2) reduce the following equivalent inequalities with the same best possible constant factor $K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)$:

$$\sum_n \sum_m \frac{a_m b_n}{(||U_m||_\alpha^\eta + ||V_n||_\beta^\eta)^{\lambda/\eta}} < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) ||a||_{p,\Phi} ||b||_{q,\Psi}, \tag{4.7}$$

$$\left\{\sum_n \frac{\prod_{k=1}^{j_0} \upsilon_n^{(k)}}{||V_n||_\beta^{j_0-p\lambda_2}} \left[\sum_m \frac{a_m}{(||U_m||_\alpha^\eta + ||V_n||_\beta^\eta)^{\lambda/\eta}}\right]^p\right\}^{\frac{1}{p}} < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) ||a||_{p,\Phi}. \tag{4.8}$$

Hence, (3.1) and (3.2) are more accurate extensions of (4.7) and (4.8).

(ii) For $\mu_i^{(k)} = 1$ $(k = 1, \ldots, i_0; i = 1, \ldots, m), \upsilon_j^{(l)} = 1$ $(l = 1, \ldots, j_0; j=1, \ldots, n)$, $\eta = \lambda$, (4.7) reduces to (1.4); for $i_0 = j_0 = 1$, (4.7) reduces to (1.5). Hence, (4.7) is an extension of (1.4) and (1.5); so is (3.1).

(iii) For $i_0 = j_0 = \lambda = \eta = 1, \mu_m^{(1)} = \upsilon_n^{(1)} = 1$, $\widetilde{\mu}_1^{(1)} = \widetilde{\upsilon}_1^{(1)} = \frac{1}{2}, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, (3.1) reduces to (1.2). Hence, (3.1) is an extension of (1.2).

Acknowledgements

The author would like to thank the referee for his useful propose in reforming the paper.

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