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ON THE EQUIVALENCE OF TWO DIFFERENTIAL EQUATIONS BY MEANS OF REFLECTING FUNCTIONS COINCIDING*

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Abstract In this article, we discuss the equivalence of two differential systems by using the method of reflecting functions. We obtain some necessary and sufficient conditions under which certain differential equations are equivalent. Given these results, new types of differential systems equivalent to the given systems can be found. We also discussed the qualitative behavior of the periodic solutions of such differential systems. These results are new, in the sense that they generalize previous discussions on the equivalence of differential systems.

Keywords Reflecting function, reflecting integral, μ-integral, equivalence, qualitative behavior.

MSC(2010) 34A12, 34A34, 34C14.

1. Introduction

$$x' = X(t, x), \quad t \in \mathbb{R}, x \in D \subset \mathbb{R}^n, \tag{1.1}$$

which has a continuously differentiable right-hand side and general solution $\varphi(t; t_0, x_0)$. For each such system, the **reflecting function** [4] is defined as $F(t, x) := \varphi(-t, t, x)$. If system (1.1) is 2ω-periodic with respect to t, then $T(x) := F(-\omega, x)$ is the Poincaré mapping of (1.1) over the period $[-\omega, \omega]$. Thus, the solution $x = \varphi(t; -\omega, x_0)$ of (1.1) defined on $[-\omega, \omega]$ is 2ω-periodic if and only if x_0 is a fixed point of $T(x)$, and the character of stability of this periodic solution is the same as this of the fixed point.

In recent years, many scholars have been interested in studying the qualitative properties of the differential system (1.1) by applying the theory of reflecting function and obtained many interesting results [1–14]. Mironenko [4–9] has combined the theory of reflecting function with the integral manifolds theory to discuss the symmetry and other geometric properties of the solutions of some differential systems. Musafirov [10] has studied the case when a linear system has reflecting function which can be expressed as a product of three exponential matrices. Veresovich [11] and Maiorovskaya [3] have established the sufficient conditions under which the quadratic systems have linear reflecting function. Belsky [1, 2] has discussed when

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the first-order polynomial differential equation is equivalent to the Ricatti equation and Abel equation. Zhou [12–14] has discussed the structure of the reflecting function of some polynomial differential systems, and applied the obtained conclusions to study the qualitative behavior of solutions of such systems.

If the reflecting functions of two differential systems coincide in their common domain, then these systems are said to be **equivalent**. By this one can study the qualitative behavior of the solutions of a complex system by using a simple differential system with the same reflecting function. Unfortunately, in general, it is very difficult to find out the reflecting function of (1.1). How to judge two systems are equivalent when we do not know their reflecting functions? This is a very important and interesting problem! Mironenko in [7–9] has studied it and obtained some excellent results. In this paper, we will improve their conclusions and get some new necessary and sufficient condition under which the differential equations are equivalent.

Definition 1.1. If vector function $\Delta(t,x)$ is a non-zero solution of the differential system

$$\Delta_t(t,x) + \Delta_x(t,x)X(t,x) - X_x(t,x)\Delta(t,x) = \mu(t,x)\Delta(t,x), \tag{1.2}$$

then $\Delta(t,x)$ is said to be the $\mu-$ **integral** of (1.1). Where $\mu = \mu(t,x)$ is a continuously differentiable scalar function such that $\mu(t,x) + \mu(-t, F(t,x)) = 0$, $F(t,x)$ is the reflecting function of (1.1). In particular, if $\mu \equiv 0$, then $\Delta(t,x)$ is called the **Reflecting integral** [13] of (1.1).

By the Theorem 1 of [6], we know that if $\Delta(t,x)$ is the reflecting integral of (1.1), then the system (1.1) is equivalent to the system

$$x' = X(t,x) + \alpha(t)\Delta(t,x), \tag{1.3}$$

where $\alpha(t)$ is an arbitrary continuously differentiable scalar odd function.

In this paper, we will show that if $\Delta(t,x)$ is the μ-integral of (1.1), then the systems (1.1) and (1.3) are equivalent, too. At the same time, we will prove that when $n = 1$, the equations (1.1) and (1.3) are equivalent if and only if $\Delta(t,x)$ is the μ-integral of (1.1).

In the following, we will denote $\Delta = \Delta(t,x)$, $\bar{\Delta} = \Delta(-t, F(t,x))$, $\bar{X} = X(-t, F(t,x))$, $\bar{\mu} = \mu(t, F(t,x))$, $F = F(t,x)$, $\alpha = \alpha(t), \alpha_i = \alpha_i(t)$.

2. Main Results

Theorem 2.1. *If Δ is the μ-integral of* (1.1)*, then the systems* (1.1) *and* (1.3) *are equivalent. In addition, if systems* (1.1) *and* (1.3) *are 2ω-periodic with respect to t, then the initial conditions at $t = -\omega$ of their $2\omega-$ periodic solutions and their stability characters are the same.*

Proof. Let F be the reflecting function of (1.1), by its definition [4] we get

$$F_t + F_x X + \bar{X} = 0,\ F(0,x) = 0.$$

Then, F is also reflecting function of (1.3), if and only if,

$$F_t + F_x(X + \alpha(t)\Delta) + \bar{X} - \alpha(t)\bar{\Delta} = 0,\ F(0,x) = 0.$$

i.e.,

$$F_x\Delta - \bar{\Delta} = 0.$$

Denote $U := F_x\Delta - \bar{\Delta}$, so $U(0,x) = 0$. In the following, we will prove that $U(t,x) \equiv 0$.

As

$$F_t = -F_x X - \bar{X},$$

so

$$\begin{aligned}
F_{tx} &= -\frac{\partial}{\partial x}(F_x X) - \dot{\bar{\Delta}}_x F_x,\\
\frac{\partial U}{\partial t} &= F_{tx}\Delta + F_x\Delta_t - \dot{\bar{\Delta}}_t - \dot{\bar{\Delta}}_x F_t = F_{tx}\Delta + F_x\Delta_t - \dot{\bar{\Delta}}_t + \dot{\bar{\Delta}}_x(F_x X + \bar{X})\\
&= -\frac{\partial}{\partial x}(F_x X)\Delta - \dot{\bar{X}}_x F_x\Delta + F_x\Delta_t - \dot{\bar{\Delta}}_t + \dot{\bar{\Delta}}_x F_x X + \dot{\bar{\Delta}}_x \bar{X}.
\end{aligned}$$

Since

$$\frac{\partial U}{\partial x} = \frac{\partial}{\partial x}(F_x\Delta) - \dot{\bar{\Delta}}_x F_x,$$

thus

$$\frac{\partial U}{\partial t}+\frac{\partial U}{\partial x}X=\frac{\partial}{\partial x}(F_x\Delta)X-\frac{\partial}{\partial x}(F_xX)\Delta-\dot{\bar{\Delta}}_xF_xX-\dot{\bar{X}}_xF_x\Delta+F_x\Delta_t-\dot{\bar{\Delta}}_t+\dot{\bar{\Delta}}_xF_xX+\dot{\bar{\Delta}}_x\bar{X},$$

by the Lemma of [6] we get

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x}X = F_x(\Delta_x X - X_x\Delta) + F_x\Delta_t - \dot{\bar{X}}_x(F_x\Delta - \bar{\Delta}) - \dot{\bar{X}}_x\bar{\Delta} - \dot{\bar{\Delta}}_t + \dot{\bar{\Delta}}_x\bar{X},$$

using this relation and Δ being the μ-integral of (1.1), we get

$$\begin{aligned}
\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x}X + \dot{\bar{X}}_x U &= F_x(\Delta_t + \Delta_x X - X_x\Delta) + \overline{\Delta_t + \Delta_x X - X_x\Delta}\\
&= F_x\mu\Delta - \mu\bar{\Delta} = \mu U, \qquad (2.1)
\end{aligned}$$

so, U is a solution of the Cauchy problem:

$$U_t + U_x X + (\bar{X}_x - \mu E)U = 0, \quad U(0,x) = 0.$$

By the uniqueness of solution of the initial problem of the linear partial differential equation implies that $U(t,x) \equiv 0$. Therefore, the proof is finished. □

Remark 2.1. If $\mu \equiv \mu(t)$ is a continuous odd function, it is not difficult to check that Δ is the reflecting integral of the system (1.1) if and only if $\tilde{\Delta} = e^{\int \mu(t)dt}\Delta(t,x)$ is the μ-integral of the system (1.1).

Remark 2.2. Taking $\mu = 0$ in the above Theorem 2.1, we get the Theorem 1 of [6]. That is said that my theorem is more general than Mironenko's result.

Theorem 2.2. *The differential equations* (1.1) *and* (1.3) ($n = 1$) *are equivalent with the reflecting function* $F(t,x)$, *if and only if, there is a function* μ *such that* $\mu(t,x) + \mu(-t,F) = 0$, *and* Δ *is the* μ*-integral of* (1.1).

Proof. By Theorem 2.1, the sufficiency of the present theorem is correct. Now, we will prove the necessity is correct too. As the equations (1.1) and (1.3) are equivalent,

$$U = F_x\Delta - \bar{\Delta} \equiv 0.$$

Let denote $G = \Delta_t + \Delta_x X - X_x\Delta$, then by (2.1) we get

$$U_t + U_x X + \bar{X}_x U = F_x G + \bar{G} \equiv 0.$$

If tanking $\mu = \frac{G}{\Delta}$, then from the above relations we obtain $\frac{G}{\Delta} = -\frac{\bar{G}}{\bar{\Delta}}$, so $\mu + \bar{\mu} = 0$. Thus,

$$G = \Delta_t + \Delta_x X - X_x\Delta = \mu\Delta,$$

i.e., Δ is the μ-integral of (1.1). □

Example 2.1. For the first order differential equation

$$x' = t + t^3x^3, \tag{2.2}$$

$F = x$ is its reflecting function. It is not difficult to check that for this equation, there is no any reflecting integral Δ in the form of

$$\Delta = r_0(t) + r_1(t)x + r_2(t)x^2 + r_3(t)x^3.$$

But, for this equation there is a cubic μ-integral $\tilde{\Delta} = 1 + x^3$, where $\mu = \frac{3x^2(t-t^3)}{1+x^3}$. Thus the equation (2.2) is equivalent to the equation

$$x' = t + t^3x^3 + \alpha(t)(1 + x^3),$$

where $\alpha(t)$ is an arbitrary continuously differentiable odd function.

Theorem 2.3. *If $n = 1$ in (1.1), Δ_j $(j = 1, 2, ..., m)$ are the μ_j-integrals of the first-order differential equation (1.1), then*

(A). *If $\sum_{j=1}^m k_j = 1$, then $\Delta = \Delta_1^{k_1}\Delta_2^{k_2}\cdots\Delta_m^{k_m}$ is the $\mu = \sum_{j=1}^m k_j\mu_j$-integral of (1.1), and the equation (1.1) is equivalent to the equation*

$$x' = X(t, x) + \sum_{j=0}^{m} \alpha_j(t)\Delta_j, \tag{2.3}$$

where $\alpha_j(t)(j = 0, 1, 2, ..., m)$ are arbitrary continuous odd functions.

(B). *If $\sum_{j=1}^m k_j = 0$ and $\sum_{j=1}^m k_j\mu_j = 0$, then*

$$\Delta = \Delta_1^{k_1}\Delta_2^{k_2}\cdots\Delta_m^{k_m} = c$$

is the first-integral of (1.1), where c is an arbitrary constant.

(C). *If Δ_1 is a μ-integral of (1.1), then Δ also is the μ-integral of (1.1), if and only if, $\Delta = \Delta_1\phi(u)$, where $u = u(t, x)$ is the first integral of (1.1), ϕ is a continuously differentiable function.*

Proof. **(A).** As Δ_j $(j = 1, 2, ..., m)$ are the μ_j-integral of the first-order equation (1.1) and $\sum_{j=1}^m k_j = 1$, then

$$\Delta_{jt} + \Delta_{jx}X = (X_x - \mu_j)\Delta_j (j = 1, 2, ..., m),$$

so,

$$\Delta_t + \Delta_x X = \sum_{j=1}^{m} k_j \Delta_1^{k_1} \cdots \Delta_j^{k_j - 1} \cdots \Delta_m^{k_m} (\Delta_{jt} + \Delta_{jx} X)$$
$$= (\sum_{j=1}^{m} k_j X_x + \sum_{j=1}^{m} k_j \mu_j)\Delta = (X_x + \mu E)\Delta, \tag{2.4}$$

i.e., Δ is the $\mu = \sum_{j=1}^{m} k_j \mu_j$-integral of (1.1). By Theorem 2.1, the equations (1.1) and (2.3) are equivalent.

(B). Since $\sum_{j=1}^{m} k_j = 0$ and $\sum_{j=1}^{m} k_j \mu_j = 0$, using (2.4), we get

$$\Delta_t + \Delta_x X = 0.$$

It implies that $\Delta = c$ is the first-integral of (1.1).

(C). Necessity: If Δ_1 and Δ are the μ-integrals of (1.1), then

$$\Delta_{1t} + \Delta_{1x} X - X_x \Delta_1 = \mu \Delta_1,$$

$$\Delta_t + \Delta_x X - X_x \Delta = \mu \Delta.$$

From these relations we get

$$(\frac{\Delta}{\Delta_1})_t + (\frac{\Delta}{\Delta_1})_x X = 0,$$

it implies that there exits a continuously differentiable function ϕ such that $\frac{\Delta}{\Delta_1} = \phi(u)$, i.e., $\Delta = \Delta_1 \phi(u)$, where $u = u(t, x)$ is the first integral of (1.1).

Sufficiency: If Δ_1 is the $\mu-$integral of (1.1) and $\Delta = \Delta_1 \phi(u)$, then

$$\Delta_{1t} + \Delta_{1x} X - X_x \Delta_1 = \mu \Delta_1, \; u_t + u_x X = 0.$$

So,

$$\Delta_t + \Delta_x X - X_x \Delta = (\Delta_{1t} + \Delta_{1x} X - X_x \Delta_1)\phi(u) + \phi'(u)(u_t + u_x X)\Delta_1 = \mu \Delta_1 \phi(u) = \mu \Delta.$$

i.e., Δ is the μ-integral of (1.1). The proof is completed. □

Remark 2.3. By Theorem 2.3 we see that for the first -order differential equation ($n = 1$), if there is a μ-integral Δ_1, then we know its all the μ-integral have the form $\Delta = \Delta_1 \phi(u)$, where u is the first integral.

Theorem 2.4. *Suppose that $u(t, x)$ and Δ are respectively the first integral and the reflecting integral of (1.1)($n \geq 1$). Then $\tilde{\Delta} = \Delta e^{\phi(t, u(t,x))}$ is the μ-integral of (1.1), and the system (1.1) is equivalent to the system*

$$x' = X(t, x) + (\alpha_1(t) + \alpha_2(t) e^{\phi(t, u(t,x))})\Delta, \tag{2.5}$$

where $\alpha_1(t)$ and $\alpha_2(t)$ are the arbitrary continuously differentiable odd functions, $\phi(t, u)$ is a continuously differentiable function such that $\phi(t, u) = \phi(-t, u)$, $\mu = \frac{\partial \phi(t,u)}{\partial t}$.

Proof. By the above assumptions, we get

$$\Delta_t + \Delta_x X - X_x \Delta = 0, \quad u_t + u_x X = 0,$$

so,

$$\tilde{\Delta} + \tilde{\Delta}_x X - X_x \tilde{\Delta} = \phi_t \tilde{\Delta},$$

i.e., $\tilde{\Delta}$ is the $\mu = \phi_t$-integral of (1.1). By Theorem2.1, the present conclusion is correct. □

Remark 2.4. Obviously, from Theorem 2.4 we find some new types of systems, such as system (2.5), which are equivalent to the system (1.1).

Remark 2.5. From Theorem 2.4, we know the Hamilton system

$$\begin{cases} x' = H_y(x,y), \\ y' = -H_x(x,y) \end{cases}$$

is equivalent to system

$$\begin{cases} x' = H_y(x,y)(1 + \sum_{i=1}^n \alpha_i(t) e^{\phi_i(t,H(x,y))}), \\ y' = -H_x(x,y)(1 + \sum_{i=1}^n \alpha_i(t) e^{\phi_i(t,H(x,y))}), \end{cases}$$

where $\alpha_i(t)$ are continuously differentiable scalar odd functions, $\phi_i(t,H)(i = 1,2,...,n)$ are arbitrary continuously differentiable functions such that $\phi_i(t,H) = \phi_i(-t,H)$.

Example 2.2. It is not difficult to check that

$$\Delta_1 = \begin{pmatrix} y \\ -x \end{pmatrix}$$

and

$$\Delta_2 = \begin{pmatrix} x(x^2+y^2) \\ y(x^2+y^2) \end{pmatrix}$$

are the reflecting integrals of the following system

$$\begin{cases} x' = y + x(x^2+y^2)\cos t, \\ y' = -x + y(x^2+y^2)\cos t. \end{cases} \tag{2.6}$$

$u = 2\sin t + (x^2+y^2)^{-1}$ is the first integral of the system (2.6).

Taking

$$\phi_i(t,u) = \frac{\beta_i(t)}{4+u} = \frac{(x^2+y^2)\beta_i(t)}{1+2(2+\sin t)(x^2+y^2)},\ \mu_i = \frac{(x^2+y^2)\beta_i'(t)}{1+2(2+\sin t)(x^2+y^2)}(i=1,2),$$

then $\tilde{\Delta}_i = e^{\phi_i}\Delta_i(i=1,2)$ are the $\mu_i(i=1,2)$-integrals of (2.6), where $\beta_i(t)(i=1,2)$ are the arbitrary continuously differentiable even functions. Thus, the system (2.6) is equivalent to the system

$$\begin{cases} x' = x(x^2+y^2)[\cos t + \alpha_1 + \alpha_2 e^{\phi_2}] + y[1 + \alpha_3 + \alpha_4 e^{\phi_1}], \\ y' = y(x^2+y^2)[\cos t + \alpha_1 + \alpha_2 e^{\phi_2}] - x[1 + \alpha_3 + \alpha_4 e^{\phi_1}], \end{cases} \tag{2.7}$$

where $\alpha_i (i = 1, 2, 3, 4)$ are the continuously differentiable odd scalar functions.

The reflecting function of the systems (2.6) and (2.7) is

$$F = \frac{1}{\sqrt{1 + 4(x^2 + y^2)\sin t}} \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Thus, all the solutions of the systems (2.6) and (2.7) defined on $[-\pi, \pi]$ are 2π-periodic, when $\alpha_i(t + 2\pi) = \alpha_i(t) (i = 1, 2, 3, 4)$ and $\beta_j(t + 2\pi) = \beta_j(t) (j = 1, 2)$.

Corollary 2.1. *If $x\varphi_x + y\varphi_y = 2\varphi$, then the system*

$$\begin{cases} x' = y + x\varphi(x, y), \\ y' = -x + y\varphi(x, y) \end{cases} \tag{2.8}$$

is equivalent to the system

$$\begin{cases} x' = y + x\varphi(x, y) + (\alpha_1 + \alpha_2 e^{\phi(t,u)})x(x^2 + y^2), \\ y' = -x + y\varphi(x, y) + (\alpha_1 + \alpha_2 e^{\phi(t,u)})y(x^2 + y^2), \end{cases} \tag{2.9}$$

where $u = t - arctan\frac{x}{y}$, $\phi(t, u)$ is a continuously differentiable function and $\phi(t, u) = \phi(-t, u)$, $\alpha_i (i = 1, 2)$ are the arbitrary continuously differentiable odd functions.

In addition, if the system (2.9) is 2π-periodic with respect to t, then the qualitative behavior of the $2\pi-$ periodic solutions of (2.8) and (2.9) with the same initial conditions at $t = -\pi$ are the same.

Proof. It is not difficult to check that $\Delta = \begin{pmatrix} x(x^2 + y^2) \\ y(x^2 + y^2) \end{pmatrix}$ and $u = t - arctan\frac{x}{y}$ are respectively the reflecting integral and the first integral of the system (2.8), by Theorem 2.4 the present corollary is correct. ☐

Example 2.3. The system

$$\begin{cases} x' = y + x(a_1x^2 + a_2xy + a_3y^2) = P(x, y), \\ y' = -x + y(a_1x^2 + a_2xy + a_3y^2) = Q(x, y) \end{cases} \tag{2.10}$$

has the first integral $u = arctan\frac{y}{x} - t$ and the reflecting integral

$$\Delta = \begin{pmatrix} x(x^2 + y^2) \\ y(x^2 + y^2) \end{pmatrix}.$$

If taking

$$\phi_1 = \ln \frac{e^{\cos t}}{1 + \tan^2 u}, \quad \phi_2 = \frac{\gamma(t)}{2 + \cos^2 u},$$

then

$$\tilde{\Delta}_1 = e^{\phi_1}\Delta = e^{\cos t}(x \sin t + y \cos t)^2 \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$\tilde{\Delta}_2 = e^{\phi_2}\Delta = e^{\frac{\gamma(t)(x^2+y^2)}{3(x^2+y^2)+xy\sin 2t}} \begin{pmatrix} x(x^2+y^2) \\ y(x^2+y^2) \end{pmatrix}$$

are respectively the μ_1-integral and μ_2-integral of (2.10), where $\mu_1 = -\sin t$, $\mu_2 = \frac{\gamma'(t)(x^2+y^2)}{3(x^2+y^2)+xy\sin 2t}$. Thus the system (2.10) is equivalent to the system

$$\begin{cases} x' = P(x,y) + \alpha_1 e^{\cos t} x(x\sin t + y\cos t)^2 + (\alpha_2 + \alpha_3 e^{\phi_2})x(x^2+y^2), \\ y' = Q(x,y) + \alpha_1 e^{\cos t} y(x\sin t + y\cos t)^2 + (\alpha_2 + \alpha_3 e^{\phi_2})y(x^2+y^2), \end{cases} \tag{2.11}$$

where $\alpha_i (i = 1,2,3)$ are the arbitrary continuously differentiable odd functions, $\gamma(t)$ is an arbitrary continuously differentiable even function, $a_i (i = 1,2,3)$ are constants. If $\alpha_i(t+2\pi) = \alpha_i(t) (i = 1,2,3)$ and $\gamma(t+2\pi) = \gamma(t)$, then the qualitative behavior of the 2π-periodic solutions of the initial conditions at $t = -\pi$ of the above two equivalent systems are the same.

Remark 2.6. By this example, one can study the qualitative behavior of the solutions of a complicated system (2.11) by using a simple differential system (2.10).

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