Journal of Applied Analysis and Computation
Volume 8, Number 4, August 2018, 1085–1107

Website:http://jaac-online.com/
DOI:10.11948/2018.1085

NONLINEAR PERTURBATIONS FOR LINEAR NONAUTONOMOUS IMPULSIVE DIFFERENTIAL EQUATIONS AND NONUNIFORM (H, K, μ, ν)-DICHOTOMY*

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Abstract We explore nonlinear perturbations of a flow generated by a linear nonautonomous impulsive differential equation $x' = A(t)x, t \neq \tau_i, \Delta x|_{t=\tau_i} = B_i x(\tau_i),\ i \in \mathbb{Z}$ in Banach spaces. Here we assume that the linear nonautonomous impulsive equation admits a more general dichotomy on $\mathbb{R}$ called the nonuniform (h, k, μ, ν)-dichotomy, which extends the existing uniform or nonuniform dichotomies and is related to the theory of nonuniform hyperbolicity. Under nonlinear perturbations, we establish a new version of the Grobman-Hartman theorem and construct stable and unstable invariant manifolds for nonlinear nonautonomous impulsive differential equations $x' = A(t)x+f(t, x)$, $t \neq \tau_i, \Delta x|_{t=\tau_i} = B_i x(\tau_i) + g_i(x(\tau_i)), i \in \mathbb{Z}$ with the help of nonuniform (h, k, μ, ν)-dichotomies.

Keywords Nonautonomous impulsive differential equations, topological equivalence, nonuniform (h, k, μ, ν)-dichotomy, invariant manifolds.

MSC(2010) 34D09, 37D10, 34A37.

1. Introduction

The theory of impulsive differential equations describes a smooth evolution of a dynamics that at certain times changes instantaneously and has been becoming an important field of investigation because of its wide applicability in physics, chemistry, biology, control theory, robotics and so on. For more details on this theory and on its applications, we refer the reader to the references [1, 17, 27].

As one of the most important and useful properties in differential equations, the

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*JZ and LY are supported by Heilong Jiang Postdoctoral Funds for scientific research initiation (Q17148), MF and MC are supported by NSFC(11671072) and RFPD(20130043110001).

nonlinear perturbation theory has been extensively studied in mathematics and frequently used in applied science. In the nonlinear perturbation theory, the Grobman-Hartman theorem and invariant manifolds theory are the most important two aspects. The classical Grobman-Hartman theorem essentially introduced in seminal work of Grobman [12] and Hartman [13] states that the topological equivalence between the nonlinear perturbed system and its corresponding linearization is established and has long been studied such as differential equations [3,6,21,23,29,35], difference equations [6,16,22], impulsive differential equations [2,10,25,26,30], dynamical systems on measure chains [31,32]. The study of invariant manifolds, which is important in the geometric study of global dynamical systems, is another research direction in the nonlinear perturbation theory and has seen much progress in the past decades for uniformly hyperbolic systems [11,14,15] and nonuniformly hyperbolic systems [4–6,32,33,35].

It is well known that the notion of uniform dichotomies and nonuniform dichotomies is an important method and tool in the study of the qualitative and stability problems for nonautonomous dynamical systems. In the past studies of the Grobman-Hartman type theorems and invariant manifolds theory, the (uniform or nonuniform) dichotomy together with some of its variants and generalizations is a key and general assumption in the nonautonomous case for the corresponding linearized systems. However, there is increasing recognition that nonautonomoous dynamical systems can exhibit various different kinds of dichotomic behavior and the growing interest is to look for more general types of dichotomic behavior [6–10,18–20,24,28,36].

Motivated by the existing nice studies and the above considerations, we consider the following linear nonautonomous impulsive differential equation

$$x' = A(t)x,\ t \neq \tau_i,\quad \Delta x|_{t=\tau_i} = B_i x(\tau_i),\ i \in \mathbb{Z} \tag{1.1}$$

and its nonlinear perturbed system

$$\begin{aligned} x' &= A(t)x + f(t,x),\ t \neq \tau_i, \\ \Delta x|_{t=\tau_i} &= B_i x(\tau_i) + g_i(x(\tau_i)),\ i \in \mathbb{Z}, \end{aligned} \tag{1.2}$$

where $\mathbb{I} = \{\tau_i\}_{i=-\infty}^{\infty}$ is a sequence of numbers

$$\cdots < \tau_j < \cdots < \tau_{-2} < \tau_{-1} < \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_i < \cdots,$$
$$\lim_{i\to\infty} \tau_i = \infty \text{ and } \lim_{j\to-\infty} \tau_j = -\infty.$$

In this paper, we first introduce a more general dichotomy on $\mathbb{R}$ called the nonuniform (h,k,μ,ν)-dichotomy for the linear nonautonomous impulsive equation (1.1). The new dichotomy is not only more general and includes the existing dichotomy as special cases in the literatures, but also exhibits more rich and widely dichotomic behavior for nonautonomous impulsive equations. Specially, it has been proved that any linear nonautonomous differential or impulsive equation in a finite-dimensional space has a nonuniform (h,k,μ,ν)-dichotomy in terms of appropriate Lyapunov exponents or Lyapunov functions (see [34, 35]). This means that the nonuniform (h,k,μ,ν)-dichotomy widely exists and arises naturally in nonautonomous equations. We also establish a new version of the Grobman-Hartman theorem and construct invariant manifolds for the nonlinear perturbed system (1.2) if (1.1) admits a nonuniform (h,k,μ,ν)-dichotomy on $\mathbb{R}$.

The content of this paper is as follows. In the next section, we introduce the nonuniform (h,k,μ,ν)-dichotomy on $\mathbb{R}$ for the linear nonautonomous impulsive equation (1.1) and state our main results. The rigorous proofs of the main results are given in Section 3.

2. Nonuniform (h,k,μ,ν)-dichotomy and main results

In this section, we introduce the notion of the nonuniform (h,k,μ,ν)-dichotomy for the linear nonautonomous impulsive differential equation (1.1) and establish our main results.

2.1. Nonuniform (h,k,μ,ν)-dichotomy on $\mathbb{R}$

We let $T(t,s)$ be the evolution operator associated with equation (1.1) satisfying $T(t,s)x(s)=x(t)$ for $t,s\in\mathbb{R}$ and any solution $x(t)$ of equation (1.1) and assume that $T(t,s)$ is invertible for all $t,s\in\mathbb{R}$. We define

$$\Delta := \left\{ m \,\middle|\, \begin{array}{l} m:\mathbb{R}\to(0,+\infty) \text{ is an increasing function with} \\ \lim\limits_{t\to\infty} u(t)=\infty \text{ and } \lim\limits_{t\to-\infty} u(t)=0 \end{array} \right\}.$$

Equation (1.1) is said to admit a nonuniform (h,k,μ,ν)-dichotomy on $\mathbb{R}$ if there exist projections $P(t)$ such that $P(t)T(t,s)=T(t,s)P(s), t,s\in\mathbb{R}$ and there exist constants $a<0\le b$, $\varepsilon\ge 0$ and $K\ge 0$ such that

$$\begin{aligned} \|T(t,s)P(s)\| &\le K\left(\frac{h(t)}{h(s)}\right)^{a}\mu(|s|)^{\varepsilon},\ t\ge s, \\ \|T(t,s)Q(s)\| &\le K\left(\frac{k(s)}{k(t)}\right)^{-b}\nu(|s|)^{\varepsilon},\ s\ge t, \end{aligned} \tag{2.1}$$

where $Q(t)=I-P(t)$ are the complementary projections of $P(t)$ and $h,k,\mu,\nu\in\Delta$. It is shown that any linear nonautonomous impulsive differential equation as in (1.1) admits a nonuniform (h,k,μ,ν)-dichotomy if (1.1) has at least one negative (h,k) Lyapunov exponent (see [34]). The nonuniform (h,k,μ,ν)-dichotomy includes the existing dichotomy in impulsive differential equations as follows:

- exponential dichotomy [1,2] for $h(t)=k(t)=e^t$ and $\varepsilon=0$;
- (h,k)-dichotomy [10,30] for $\varepsilon=0$;
- nonuniform exponential dichotomy [5] for $h(t)=k(t)=\mu(t)=\nu(t)=e^t$;
- nonuniform polynomial dichotomy [4] for $h(t)=k(t)=\mu(t)=\nu(t)=t+1$, $t\in\mathbb{R}_+$;
- ρ-nonuniform exponential dichotomy [4] for $h(t)=k(t)=\mu(t)=\nu(t)=e^{\rho(t)}$, $t\in\mathbb{R}_+$;
- nonuniform (μ,ν)-dichotomy [33] for $h(t)=k(t)$ and $\mu(t)=\nu(t)$, $t\in\mathbb{R}_+$.

It is worth noting that the nonuniform (h,k,μ,ν)-dichotomy exhibits more rich and widely dichotomic behavior which can not be characterized by the existing dichotomies for linear nonautonomous impulsive equations.

Example 2.1. Consider the linear nonautonomous impulsive differential equations in $\mathbb{R}^2$

$$\begin{aligned}
z_1' &= \left[-\delta_1 + (\delta_2/\sqrt{t^2+1})\left(\ln(\sqrt{t^2+1}+t)\cos\ln(\sqrt{t^2+1}+t)-1\right)\right] z_1,\ t\neq\tau_i,\\
\Delta z_1|_{t=\tau_i} &= \left(e^{\bar{\delta}_1(\tau_i-\tau_{i-1})}-1\right) z_1,\\
z_2' &= \left(\delta_3/\sqrt{t^2+1}+3\delta_2 t^2(t^3\cos t^3-1)\right) z_2,\ t\neq\tau_i,\\
\Delta z_2|_{t=\tau_i} &= \left[\left((\sqrt{\tau_{i+1}^2+1}+\tau_{i+1})/(\sqrt{\tau_i^2+1}+\tau_i)\right)^{-\bar{\delta}_2}-1\right] z_2,
\end{aligned} \tag{2.2}$$

where δ_i, $i=1,2,3$ and $\bar{\delta}_j$, $j=1,2$ are positive constants.

Set $P(t)(z_1,z_2)^T=(z_1,0)^T$ and $Q(t)(z_1,z_2)^T=(0,z_2)^T$ for $t\in\mathbb{R}$. Then we obtain

$$T(t,s)P(s)=\begin{pmatrix} e^{-\delta_1(t-s)+\bar{\delta}_1(\tau_i-\tau_{j-1})+\delta_2 c_1(t)} & 0\\ 0 & 0\end{pmatrix}$$

and

$$T(t,s)Q(s)=\begin{pmatrix} 0 & \left(\dfrac{\sqrt{t^2+1}+t}{\sqrt{s^2+1}+s}\right)^{\delta_3}\left(\dfrac{\sqrt{\tau_{i+1}^2+1}+\tau_{i+1}}{\sqrt{\tau_j^2+1}+\tau_j}\right)^{-\bar{\delta}_2} e^{\delta_2 c_2(t)}\\ 0 & 0\end{pmatrix},$$

where

$$\begin{aligned}
c_1(t) &= \ln(\sqrt{t^2+1}+t)(\sin\ln(\sqrt{t^2+1}+t)-1)+\cos\ln(\sqrt{t^2+1}+t)\\
&\quad -\cos\ln(\sqrt{s^2+1}+s)-\ln(\sqrt{s^2+1}+s)(\sin\ln(\sqrt{s^2+1}+s)-1),\\
c_2(t) &= t^3(\sin t^3-1)+\cos t^3-\cos s^3-s^3(\sin s^3-1).
\end{aligned}$$

A direct calculation gives

$$\begin{aligned}
\|T(t,s)P(s)\| &\le e^{2\delta_2}e^{-\delta_1(t-s)}e^{\bar{\delta}_1 t}(\sqrt{s^2+1}+s)^{2\delta_2}\\
&\le e^{2\delta_2}e^{(-\delta_1+\bar{\delta}_1)(t-s)}(e^{s}(\sqrt{s^2+1}+s))^{\bar{\delta}_1+2\delta_2}\\
&\le e^{2\delta_2}e^{(-\delta_1+\bar{\delta}_1)(t-s)}(e^{|s|}(\sqrt{|s|^2+1}+|s|))^{\bar{\delta}_1+2\delta_2},\ t\ge s
\end{aligned}$$

and

$$\begin{aligned}
\|T(t,s)Q(s)\| &\le e^{2\delta_2}\left(\frac{\sqrt{s^2+1}+s}{\sqrt{t^2+1}+t}\right)^{-\delta_3}\left(\frac{\sqrt{\tau_j^2+1}+\tau_j}{\sqrt{\tau_{i+1}^2+1}+\tau_{i+1}}\right)^{\bar{\delta}_2} e^{2\delta_2 s^3}\\
&\le e^{2\delta_2}\left(\frac{\sqrt{s^2+1}+s}{\sqrt{t^2+1}+t}\right)^{-\delta_3+\bar{\delta}_2} e^{2\delta_2|s|^3},\ s\ge t.
\end{aligned}$$

It is clear that the above dichotomic behavior exhibited by (2.2) can not be characterized by the existing dichotomies in the literatures. Let

$$h(t) = e^t,\ k(t) = \sqrt{t^2+1}+t,\ \mu(t) = e^t(\sqrt{t^2+1}+t),\ \nu(t) = e^{t^3},$$

$-\delta_1 + \bar{\delta}_1 < 0$ and $-\delta_3 + \bar{\delta}_2 < 0$. Then (2.2) admits a nonuniform (h, k, μ, ν)-dichotomy on $\mathbb{R}$.

2.2. Topological equivalence

Let $\mathcal{B}(X)$ be the space of bounded linear operators on a Banach space X. We first assume that

- (H$_1$) $A(t) \in \mathcal{B}(X)$ for each $t \in \mathbb{R}$ with $t \mapsto A(t)$ at most with discontinuities of the first kind at the points τ_i and $I + B_i \in \mathcal{B}(X)$ with $(I+B_i)^{-1} \in \mathcal{B}(X)$ for $i \in \mathbb{Z}$;
- (H$_2$) $f\colon \mathbb{R} \times X \to X$ with $f(t,0) = 0$ for every $t \in \mathbb{R}$ such that $t \mapsto f(t,x)$ has at most discontinuities of the first kind at the points τ_i, and $g_i\colon X \to X$ with $g_i(0) = 0$ for every $i \in \mathbb{Z}$.

To facilitate the discussion below, we define

$$\Delta_1 := \left\{ m \in \Delta \;\middle|\; \begin{array}{l} \text{there exist positive constants } l_1 \text{ and } \omega_1 \in \mathbb{N} \\ \text{such that any interval of length } l_1 \text{ of } \mathbb{R} \\ \text{contains at most } \omega_1 \text{ elements of } \{1/m(\tau_i)\}_{i\in\mathbb{Z}} \end{array} \right\},$$

$$\Delta_2 := \left\{ m \in \Delta \;\middle|\; \begin{array}{l} \text{there exist positive constants } l_2 \text{ and } \omega_2 \in \mathbb{N} \\ \text{such that any interval of length } l_2 \text{ of } \mathbb{R} \\ \text{contains at most } \omega_2 \text{ elements of } \{m(\tau_i)\}_{i\in\mathbb{Z}} \end{array} \right\}.$$

For any constant $\tilde{l} < -1$, $t, s \in \mathbb{R}$, without loss of generality, we can choose $l_1 = 1$ and $l_2 = m(s)$, then

$$\sum_{s \le \tau_i} m^{\tilde{l}}(\tau_i) \le \omega_2 m^{\tilde{l}}(s) + \omega_2(2m(s))^{\tilde{l}} + \cdots = \omega_2 m(s)^{\tilde{l}} \zeta_{\tilde{l}}$$

for $m \in \Delta_2$ and

$$\sum_{\tau_i < t} (m(t)/m(\tau_i))^{\tilde{l}} \le \omega_1 1^{\tilde{l}} + \omega_1 2^{\tilde{l}} + \cdots = \omega_1 \zeta_{\tilde{l}}$$

for $m \in \Delta_1$, where $\zeta_{\tilde{l}} := \sum_{i=1}^{\infty} i^{\tilde{l}}$.

Definition 2.1. (1.1) and (1.2) are said to be topologically equivalent if there exists a function $H : \mathbb{R} \times X \to X$ having the following properties:

(i) $H(t,x) - x$ is bounded uniformly with respect to $t \in \mathbb{R}$;

(ii) for each fixed t, $H(t, \cdot)$ is a homeomorphism of X into X;

(iii) $L(t, \cdot) = H^{-1}(t, \cdot)$ also has property (i);

(iv) if $x(t)$ is a solution of (1.2), then $H(t, x(t))$ is a solution of (1.1).

The function H satisfying the above four properties is said to be *the equivalent function* of (1.1) and (1.2).

Now we state the first main finding of this paper, i.e., a new version of the Grobman-Hartman theorem.

Theorem 2.1. *Assume that* (1.1) *admits a nonuniform* (h, k, μ, ν)*-dichotomy on* $\mathbb{R}$*,* h, k *is differentiable for each* $t \neq \tau_i$ *and* $h \in \Delta_2, k \in \Delta_1$*. If* $|a|, b > 1$ *and there exist constants* $\alpha_i, \gamma_i > 0$ $(i = 1, 2)$ *such that for any* $x, x_1, x_2 \in X$

(a_1)

$$\begin{aligned}\|f(t,x)\| &\leq \alpha_1 \min\{h'(t)h(t)^{-1}\mu(|t|)^{-\varepsilon}, k'(t)k(t)^{-1}\nu(|t|)^{-\varepsilon}\},\\ \|f(t,x_1) - f(t,x_2)\| &\leq \gamma_1 \min\left\{\frac{h'(t)}{h(t)}\mu(|t|)^{-\varepsilon}, \frac{k'(t)}{k(t)}\nu(|t|)^{-\varepsilon}\right\}\|x_1 - x_2\|,\end{aligned}$$

for $t \neq \tau_i$*;*

(a_2)

$$\left.\begin{aligned}\|g_i(x)\| &\leq \alpha_2 \min\{\mu(|\tau_i|)^{-\varepsilon}, \nu(|\tau_i|)^{-\varepsilon}\},\\ \|g_i(x_1) - g_i(x_2)\| &\leq \gamma_2 \min\{\mu(|\tau_i|)^{-\varepsilon}, \nu(|\tau_i|)^{-\varepsilon}\}\|x_1 - x_2\|,\end{aligned}\right. \quad i \in \mathbb{Z};$$

(a_3) $K\gamma_1(1/|a| + 1/b) + K\gamma_2(\omega_1\zeta_a + \omega_2\zeta_{-b}) < 1$,

then (1.2) *is topologically equivalent to* (1.1) *and the equivalent function* $H(t, x)$ *satisfies*

$$\|H(t,x) - x\| \leq K\alpha_1(1/|a| + 1/b) + K\alpha_2(\omega_1\zeta_a + \omega_2\zeta_{-b})$$

for each $t \in \mathbb{R}, x \in X$.

In the above theorem, the assumed conditions seem more restrictive in contrast with previous studies such as IS condition (see [10, 30]). This is due to the fact that the linear system (1.1) with the nonuniform (h, k, μ, ν)-dichotomy is a nonuniformly hyperbolic impulsive system, which means that IS condition does not hold when $\varepsilon \neq 0$ in the nonuniform part of nonuniform (h, k, μ, ν)-dichotomies. Therefore, our results enrich and improve the classical Palmer's linearization theorem for nonautonomous impulsive differential equations [2, 10, 30]. Specially, here we also point out the size of the nonlinear term in the linearization theorem of nonuniformly hyperbolic impulsive systems may depend on the specific forms of dichotomy (see conditions (a_1) and (a_2)).

2.3. Invariant manifolds

We describe the construction of stable invariant manifolds on $\mathbb{R}^+$ and the construction of unstable invariant manifolds on $\mathbb{R}^-$ for the nonlinear nonautonomous impulsive differential equation (1.2). We define the *stable* and *unstable subspaces* for each $t \in \mathbb{R}$ by $E(t) = P(t)(X)$ and $F(t) = Q(t)(X)$. We assume that there exist positive constants c and q such that

$$\|f(t,x_1) - f(t,x_2)\| \leq c\|x_1 - x_2\|(\|x_1\|^q + \|x_2\|^q) \tag{2.3}$$

and

$$\|g_i(x_1) - g_i(x_2)\| \leq c\|x_1 - x_2\|(\|x_1\|^q + \|x_2\|^q) \tag{2.4}$$

for any $t \in \mathbb{R}$, $i \in \mathbb{Z}$ and $x_1, x_2 \in X$.

To establish the existence of stable invariant manifolds on $\mathbb{R}^+$, we rewrite (2.1) of the nonuniform (h, k, μ, ν)-dichotomy as the following equivalent way:

$$\begin{aligned} \|T(t,s)P(s)\| &\leq K\left(\frac{h(t)}{h(s)}\right)^a \mu(s)^\varepsilon, \\ \|T(t,s)^{-1}Q(t)\| &\leq K\left(\frac{k(t)}{k(s)}\right)^{-b} \nu(t)^\varepsilon \end{aligned} \tag{2.5}$$

for $t \geq s \geq 0$. Let

$$\int_0^\infty h(\tau)^{aq} \max\{\mu(\tau)^\varepsilon, \nu(\tau)^\varepsilon\} d\tau + \sum_{0 \leq \tau_i} h(\tau_i)^{aq} \max\{\mu(\tau_i)^\varepsilon, \nu(\tau_i)^\varepsilon\}$$

be convergent and

$$\beta(t) = k(t)^{b/(\varepsilon q)} h(t)^{-a(q+1)/(\varepsilon q)} \mu(t)^{1+1/q} C(t)^{1/\varepsilon q}, \tag{2.6}$$

where

$$C(t) = \int_t^\infty h(\tau)^{aq} \max\{\mu(\tau)^\varepsilon, \nu(\tau)^\varepsilon\} d\tau + \sum_{t \leq \tau_i} h(\tau_i)^{aq} \max\{\mu(\tau_i)^\varepsilon, \nu(\tau_i)^\varepsilon\}.$$

Consider the set of initial conditions

$$Z_\beta(\eta) = \{(s, \xi) : s \geq 0,\ \xi \in B_s(\beta(s)^{-\varepsilon}/\eta)\},$$

where $B_s(\beta(s)^{-\varepsilon}/\eta) \subset E(s)$ is the open ball of radius $\beta(s)^{-\varepsilon}/\eta$ centered at zero. Let $Z_\beta(1) = Z_\beta$. Denote by $\mathcal{X}$ the space of functions $\Phi\colon Z_\beta \to X$ that are left-continuous in s, at most with discontinuities of the first kind at the points τ_i, such that $\Phi(s, 0) = 0$, $\Phi(s, B_s(\beta(s)^{-\varepsilon})) \subset F(s)$ and

$$\|\Phi(s, \xi_1) - \Phi(s, \xi_2)\| \leq \|\xi_1 - \xi_2\| \tag{2.7}$$

for every $s \geq 0$ and $\xi_1, \xi_2 \in B_s(\beta(s)^{-\varepsilon})$. It is not difficult to show that $\mathcal{X}$ is a complete metric space with the norm

$$|\Phi|' = \sup\left\{\frac{\|\Phi(s,\xi)\|}{\|\xi\|} : s \geq 0 \text{ and } \xi \in B_s(\beta(s)^{-\varepsilon}) \setminus \{0\}\right\}.$$

Given $\Phi \in \mathcal{X}$, consider the graph

$$\mathcal{W} = \{(s, \xi, \Phi(s, \xi)) : (s, \xi) \in Z_\beta\} \tag{2.8}$$

and, for each $(s, u(s), v(s)) \in \mathbb{R}^+ \times E(s) \times F(s)$, the semiflow

$$\Psi_\kappa(s, u(s), v(s)) = (t, u(t), v(t)), \quad \kappa = t - s \geq 0 \tag{2.9}$$

generated by (1.2), where

$$\begin{aligned} u(t) = T(t,s)u(s) &+ \int_s^t T(t,\tau)P(\tau)f(\tau, u(\tau), v(\tau))d\tau \\ &+ \sum_{s \leq \tau_i < t} T(t, \tau_i^+)P(\tau_i^+)g_i(u(\tau_i), v(\tau_i)), \end{aligned} \tag{2.10}$$

$$v(t) = T(t,s)v(s) + \int_s^t T(t,\tau)Q(\tau)f(\tau,u(\tau),v(\tau))d\tau + \sum_{s\le\tau_i<t} T(t,\tau_i^+)Q(\tau_i^+)g_i(u(\tau_i),v(\tau_i)).$$

Theorem 2.2. *Assume that*

(b_1) (1.1) *admits a nonuniform* (h,k,μ,ν)*-dichotomy on* $\mathbb{R}^+$*;*

(b_2) $\lim_{t\to\infty} k(t)^{-b}h(t)^a\nu(t)^\varepsilon = 0$;

(b_3) $h(t)^a\beta(t)^\varepsilon$ *is decreasing.*

If c is sufficiently small in (2.3) *and* (2.4)*, then*

(c_1) *there exists a unique function* $\Phi \in \mathcal{X}$ *such that* $\mathcal{W}$ *is forward invariant with respect to* Ψ_κ *in the sense that*

$$\Psi_\kappa(s,\xi,\Phi(s,\xi)) \in \mathcal{W} \quad \textit{for any} \quad (s,\xi)\in Z_{\beta\cdot\mu}(2K),\ \kappa = t-s\ge 0; \tag{2.11}$$

(c_2) *there exists a constant* $d>0$ *such that*

$$\|\Psi_\kappa(s,\xi_1,\Phi(s,\xi_1)) - \Psi_\kappa(s,\xi_2,\Phi(s,\xi_2))\| \le d(h(t)/h(s))^a\mu(s)^\varepsilon\|\xi_1-\xi_2\| \tag{2.12}$$

for any $\kappa = t-s\ge 0$ *and* $(s,\xi_1),(s,\xi_2)\in Z_{\beta\cdot\mu}(2K)$.

Remark 2.1. Theorem 2.2 generalizes and extends some previous works, such as, Theorem 1 in [4], Theorem 1 in [5]. Here a new discovery is that the different forms of functions h,k,μ,ν influence the size of stable invariant manifold (see (2.6) and (2.8)). This implies that the types of dichotomies may play an important role in the construction of invariant manifolds.

We now establish the existence of unstable invariant manifolds of (1.2) on $\mathbb{R}^-$. In this case we say that (1.1) admits a nonuniform (h,k,μ,ν)-dichotomy on $\mathbb{R}^-$ if there exist constants $a\le 0<b$, $\varepsilon\ge 0$ and $K\ge 0$ such that

$$\|T(t,s)^{-1}P(t)\| \le K\left(\frac{h(s)}{h(t)}\right)^a \mu(|t|)^\varepsilon,$$
$$\|T(t,s)Q(s)\| \le K\left(\frac{k(s)}{k(t)}\right)^{-b} \nu(|s|)^\varepsilon$$

for $0\ge s\ge t$. We let

$$\int_{-\infty}^0 k(\tau)^{bq}\max\{\mu(|\tau|)^\varepsilon,\nu(|\tau|)^\varepsilon\}d\tau + \sum_{\tau_i\le 0} k(\tau_i)^{bq}\max\{\mu(|\tau_i|)^\varepsilon,\nu(|\tau_i|)^\varepsilon\}$$

be convergent and

$$\beta^u(t) = h(t)^{a/(\varepsilon q)}k(t)^{-b(q+1)/(\varepsilon q)}\nu(|t|)^{1+1/q}C^u(t)^{1/\varepsilon q}$$

for $t\le 0$, where

$$C^u(t) = \int_{-\infty}^t k(\tau)^{bq}\max\{\mu(|\tau|)^\varepsilon,\nu(|\tau|)^\varepsilon\}d\tau + \sum_{\tau_i\le t} k(\tau_i)^{bq}\max\{\mu(|\tau_i|)^\varepsilon,\nu(|\tau_i|)^\varepsilon\}.$$

Let $B_s^u(\beta^u(s)^{-\varepsilon}) \subset F(s)$ for $s \le 0$ be the open ball of radius $\beta^u(s)^{-\varepsilon}$ centered at zero. We consider the set of initial conditions $Z_{\beta^u}^u = \{(s,\xi) : s \le 0, \ \xi \in B_s^u(\beta^u(s)^{-\varepsilon})\}$ and denote by $\mathcal{X}^u$ the space of functions $\Phi^u : Z_\beta^u \to X$ that are left-continuous in s, at most with discontinuities of the first kind at the points τ_i, such that $\Phi^u(s,0) = 0$, $\Phi^u(s, B_s^u(\beta^u(s)^{-\varepsilon})) \subset E(s)$ and satisfying (2.7) for every $s \le 0$ and $\xi_1, \xi_2 \in B_s^u(\beta^u(s)^{-\varepsilon})$. For each $\Phi^u \in \mathcal{X}^u$, we consider the graph $\mathcal{W}^u = \left\{(s, \Phi^u(s,\xi), \xi) : (s,\xi) \in Z_{\beta^u}^u\right\}$ and the semiflow Ψ_κ^u defined by $\Psi_\kappa(s, u(s), v(s)) = (t, u(t), v(t))$, $\kappa = t - s \le 0$, where

$$\begin{aligned}
u(t) &= T(t,s)u(s) - \int_t^s T(t,\tau)P(\tau)f(\tau,u(\tau),v(\tau))d\tau \\
&\quad - \sum_{t \le \tau_i < s} T(t,\tau_i^+)P(\tau_i^+)g_i(u(\tau_i),v(\tau_i)), \\
v(t) &= T(t,s)v(s) - \int_t^s T(t,\tau)Q(\tau)f(\tau,u(\tau),v(\tau))d\tau \\
&\quad - \sum_{t \le \tau_i < s} T(t,\tau_i^+)Q(\tau_i^+)g_i(u(\tau_i),v(\tau_i)).
\end{aligned}$$

Theorem 2.3. *Assume that* (1.1) *admits a nonuniform* (h,k,μ,ν)*-dichotomy on* $\mathbb{R}^-$ *with* $\lim_{t\to-\infty} k(t)^b h(t)^{-a} \mu(|t|)^\varepsilon = 0$. *If* $k(t)^{-b}\beta(t)^\varepsilon$ *is decreasing and* c *is sufficiently small in* (2.3) *and* (2.4), *then there exists a unique function* $\Phi^u \in \mathcal{X}^u$ *such that*

$$\Psi_\kappa^u(s, \Phi^u(s,\xi), \xi) \in \mathcal{W}^u \quad \text{for any} \quad (s,\xi) \in Z_{\beta\cdot\nu}^u(2K), \ \kappa = t - s \le 0.$$

Moreover, there exists a constant $d^u > 0$ *such that*

$$\|\Psi_\kappa^u(s,\Phi^u(s,\xi_1),\xi_1) - \Psi_\kappa^u(s,\Phi^u(s,\xi_2),\xi_2)\| \le d^u (k(s)/k(t))^{-b} \nu(|s|)^\varepsilon \|\xi_1 - \xi_2\|$$

for any $\kappa = t - s \le 0$ *and* $(s,\xi_1), (s,\xi_2) \in Z_{\beta\cdot\nu}(2K)$.

3. Proofs of main results

3.1. Proofs of Theorem 2.1

Let $X(t,t_0,x_0)$ be the solution of (1.2) with the initial value $X(t_0) = x_0$ and $Y(t,t_0,y_0)$ be the solution of (1.1) with the initial value $Y(t_0) = y_0$. We first prove some auxiliary results.

Step 1. Construction of bounded solutions.

Lemma 3.1. *For any fixed* $(\bar{t},\xi) \in \mathbb{R} \times X$, *the system*

$$\begin{aligned}
z' &= A(t)z - f(t, X(t,\bar{t},\xi)), \quad t \ne \tau_i, \\
\Delta z|_{t=\tau_i} &= B_i z(\tau_i) - g_i(X(\tau_i,\bar{t},\xi)), \quad i \in \mathbb{Z},
\end{aligned} \tag{3.1}$$

has a unique bounded solution $h(t,(\bar{t},\xi))$ *and*

$$\|h(t,(\bar{t},\xi))\| \le K\alpha_1(1/|a| + 1/b) + K\alpha_2(\omega_1\zeta_a + \omega_2\zeta_{-b}), \quad t \in \mathbb{R}.$$

Proof. Let

$$h(t,(\bar{t},\xi)) = -\int_{-\infty}^{t} T(t,\tau)P(\tau)f(\tau,X(\tau,\bar{t},\xi))d\tau + \int_{t}^{\infty} T(t,\tau)Q(\tau)f(\tau,X(\tau,\bar{t},\xi))d\tau$$
$$- \sum_{\tau_i<t} T(t,\tau_i^+)P(\tau_i^+)g_i(x(\tau_i)) + \sum_{t\le\tau_i} T(t,\tau_i^+)Q(\tau_i^+)g_i(x(\tau_i)).$$

It follows from direct calculation that $h(t,(\bar{t},\xi))$ is a solution of (3.1). By (2.1), (a_1) and (a_2), we have

$$\begin{aligned} A_1 :&= \int_{-\infty}^{t} \|T(t,\tau)P(\tau)\|\|f(\tau,X(\tau,\bar{t},\xi))\|d\tau \\ &\quad + \int_{t}^{\infty} \|T(t,\tau)Q(\tau)\|\|f(\tau,X(\tau,\bar{t},\xi))\|d\tau \\ &\le K\alpha_1 h(t)^a \int_{-\infty}^{t} h(\tau)^{-a-1}h'(\tau)d\tau + K\alpha_1 k(t)^b \int_{t}^{\infty} k(\tau)^{-b-1}k'(\tau)d\tau \\ &\le K\alpha_1(1/|a|+1/b) \end{aligned}$$

and

$$\begin{aligned} B_1 :&= \sum_{\tau_i<t} \|T(t,\tau_i^+)P(\tau_i^+)\|\|g_i(x(\tau_i))\| + \sum_{t\le\tau_i} \|T(t,\tau_i^+)Q(\tau_i^+)\|\|g_i(x(\tau_i))\| \\ &\le K\alpha_2 \sum_{\tau_i<t}[h(t)/h(\tau_i)]^a + K\alpha_2 k(t)^b \sum_{t\le\tau_i} k(\tau_i)^{-b} \\ &\le K\alpha_2(\omega_1\zeta_a+\omega_2\zeta_{-b}). \end{aligned}$$

Therefore,

$$\|h(t,(\bar{t},\xi))\| \le A_1 + B_1 \le K\alpha_1(1/|a|+1/b) + K\alpha_2(\omega_1\zeta_a+\omega_2\zeta_{-b})$$

for any $t\in\mathbb{R}$. This means that $h(t,(\bar{t},\xi))$ is the unique bounded solution of (3.1) since

$$\begin{aligned} z' &= A(t)z, \quad t\ne\tau_i, \\ \Delta z|_{t=\tau_i} &= B_i z(\tau_i), \quad i\in\mathbb{Z}, \end{aligned}$$

admits a nonuniform (h,k,μ,ν)-dichotomy. □

Lemma 3.2. *For any fixed* $(\bar{t},\xi)\in\mathbb{R}\times X$, *the system*

$$\begin{aligned} z' &= A(t)z + f(t,Y(t,\bar{t},\xi)+z) \quad t\ne\tau_i, \\ \Delta z|_{t=\tau_i} &= B_i z(\tau_i) + g_i(Y(\tau_i,\bar{t},\xi)+z(\tau_i)), \quad i\in\mathbb{Z}, \end{aligned} \tag{3.2}$$

has a unique bounded solution $l(t,(\bar{t},\xi))$ *and*

$$\|l(t,(\bar{t},\xi))\| \le K\alpha_1(1/|a|+1/b) + K\alpha_2(\omega_1\zeta_a+\omega_2\zeta_{-b}), \quad t\in\mathbb{R}.$$

Proof. We denote by Ω_1 the space of functions $z:\mathbb{R}\to X$ that are left-continuous in t, at most with discontinuities of the first kind at the points τ_i, such that

$$\|z\| \le K\alpha_1(1/|a|+1/b) + K\alpha_2(\omega_1\zeta_a+\omega_2\zeta_{-b}),$$

where $\|z\| := \sup_{t\in\mathbb{R}} \|z(t)\|$. It is not difficult to show that $(\Omega_1, \|\cdot\|)$ is a Banach space. Define the mapping T on Ω_1 by

$$\begin{aligned}(Tz)(t) = &\int_{-\infty}^{t} T(t,\tau)P(\tau)f(\tau, Y(\tau,\bar{t},\xi) + z(\tau))d\tau \\ &- \int_{t}^{\infty} T(t,\tau)Q(\tau)f(\tau, Y(\tau,\bar{t},\xi) + z(\tau))d\tau \\ &+ \sum_{\tau_i<t} T(t,\tau_i^+)P(\tau_i^+)g_i(Y(\tau_i,\bar{t},\xi) + z(\tau_i)) \\ &- \sum_{t\le\tau_i} T(t,\tau_i^+)Q(\tau_i^+)g_i(Y(\tau_i,\bar{t},\xi) + z(\tau_i)).\end{aligned}$$

It is clear that Tz is left-continuous in t, at most with discontinuities of the first kind at the points τ_i and it follows from (2.1), (a_1) and (a_2) that

$$\|Tz\| \le K\alpha_1(1/|a| + 1/b) + K\alpha_2(\omega_1\zeta_a + \omega_2\zeta_{-b}).$$

Therefore, $T(\Omega_1) \subset \Omega_1$. In addition, for any $z_1(t), z_2(t) \in \Omega_1$, we have

$$\begin{aligned}A_2 := &\int_{-\infty}^{t} \|T(t,\tau)P(\tau)\|\|f(\tau, Y(\tau,\bar{t},\xi) + z_1(\tau)) - f(\tau, Y(\tau,\bar{t},\xi) + z_2(\tau))\|d\tau \\ &+ \int_{t}^{\infty} \|T(t,\tau)Q(\tau)\|\|f(\tau, Y(\tau,\bar{t},\xi) + z_1(\tau)) - f(\tau, Y(\tau,\bar{t},\xi) + z_2(\tau))\|d\tau \\ \le\ & K\gamma_1(1/|a| + 1/b)\|z_1 - z_2\|\end{aligned}$$

and

$$\begin{aligned}B_2 := &\sum_{\tau_i<t} \|T(t,\tau_i^+)P(\tau_i^+)\|\|g_i(Y(\tau_i,\bar{t},\xi) + z_1(\tau_i)) - g_i(Y(\tau_i,\bar{t},\xi) + z_2(\tau_i))\| \\ &+ \sum_{t\le\tau_i} \|T(t,\tau_i^+)Q(\tau_i^+)\|\|g_i(Y(\tau_i,\bar{t},\xi) + z_1(\tau_i)) - g_i(Y(\tau_i,\bar{t},\xi) + z_2(\tau_i))\| \\ \le\ & K\gamma_2(\omega_1\zeta_a + \omega_2\zeta_{-b})\|z_1 - z_2\|.\end{aligned}$$

Therefore,

$$\|Tz_1 - Tz_2\| \le [K\gamma_1(1/|a| + 1/b) + K\gamma_2(\omega_1\zeta_a + \omega_2\zeta_{-b})]\|z_1 - z_2\|.$$

Then by (a_3), $T : \Omega_1 \to \Omega_1$ is a contraction mapping. By the Banach fixed point theorem (also known as the contraction mapping theorem or contraction mapping principle), we conclude that T has a unique fixed point $l(t,(\bar{t},\xi))$, i.e.,

$$\begin{aligned}l(t,(\bar{t},\xi)) = &\int_{-\infty}^{t} T(t,\tau)P(\tau)f(\tau, Y(\tau,\bar{t},\xi) + l(\tau))d\tau \\ &- \int_{t}^{\infty} T(t,\tau)Q(\tau)f(\tau, Y(\tau,\bar{t},\xi) + l(\tau))d\tau \\ &+ \sum_{\tau_i<t} T(t,\tau_i^+)P(\tau_i^+)g_i(Y(\tau_i,\bar{t},\xi) + l(\tau_i)) \\ &- \sum_{t\le\tau_i} T(t,\tau_i^+)Q(\tau_i^+)g_i(Y(\tau_i,\bar{t},\xi) + l(\tau_i)).\end{aligned}$$

In the following, we prove that $l(t,(\bar{t},\xi))$ is unique in the whole space. Otherwise, assume that there is another bounded solution $l^0(t,(\bar{t},\xi))$ of (3.2), which can be written as

$$\begin{aligned} l^0(t,(\bar{t},\xi)) = &\int_{-\infty}^{t} T(t,\tau)P(\tau)f(\tau, Y(\tau,\bar{t},\xi)+l^0(\tau))d\tau \\ &- \int_{t}^{\infty} T(t,\tau)Q(\tau)f(\tau, Y(\tau,\bar{t},\xi)+l^0(\tau))d\tau \\ &+ \sum_{\tau_i<t} T(t,\tau_i^+)P(\tau_i^+)g_i(Y(\tau_i,\bar{t},\xi)+l^0(\tau_i)) \\ &- \sum_{t\le\tau_i} T(t,\tau_i^+)Q(\tau_i^+)g_i(Y(\tau_i,\bar{t},\xi)+l^0(\tau_i)). \end{aligned}$$

Proceeding in a similar manner to the above arguments, we have

$$\|l-l^0\| \le [K\gamma_1(1/|a|+1/b)+K\gamma_2(\omega_1\zeta_a+\omega_2\zeta_{-b})]\|l-l^0\|.$$

Then, by (a_3), one has $l\equiv l^0$. Therefore, $l(t,(\bar{t},\xi))$ is a unique bounded solution of (3.2) with

$$\|l(t,(\bar{t},\xi)\| \le K\alpha_1(1/|a|+1/b)+K\alpha_2(\omega_1\zeta_a+\omega_2\zeta_{-b}), \quad t\in\mathbb{R}.$$

□

Lemma 3.3. *If $x(t)$ is any solution of* (1.2), *then*

$$\begin{aligned} z' &= A(t)z + f(t,x(t)+z) - f(t,x(t)) \quad t\ne\tau_i, \\ \Delta z|_{t=\tau_i} &= B_i z(\tau_i) + g_i(x(\tau_i)+z(\tau_i)) - g_i(x(\tau_i)), \quad i\in\mathbb{Z}, \end{aligned} \tag{3.3}$$

has a unique bounded solution $z(t)\equiv 0$.

Proof. Obviously, $z(t)\equiv 0$ is a bounded solution of (3.3). Next we show that $z(t)\equiv 0$ is the unique bounded solution. Assume that $z^0(t)$ is any bounded solution of (3.3), then $z^0(t)$ can be written in the form

$$\begin{aligned} z^0(t) = &\int_{-\infty}^{t} T(t,\tau)P(\tau)[f(\tau,x(\tau)+z(\tau))-f(\tau,x(\tau))]d\tau \\ &- \int_{t}^{\infty} T(t,\tau)Q(\tau)[f(\tau,x(\tau)+z(\tau))-f(\tau,x(\tau))]d\tau \\ &+ \sum_{\tau_i<t} T(t,\tau_i^+)P(\tau_i^+)[g_i(x(\tau_i)+z(\tau_i))-g_i(x(\tau_i))] \\ &- \sum_{t\le\tau_i} T(t,\tau_i^+)Q(\tau_i^+)[g_i(x(\tau_i)+z(\tau_i))-g_i(x(\tau_i))]. \end{aligned}$$

It is easy to show that

$$\|z^0-0\| \le [K\gamma_1(1/|a|+1/b)+K\gamma_2(\omega_1\zeta_a+\omega_2\zeta_{-b})]\|z^0-0\|,$$

which implies that $z(t)\equiv 0$. □

Step 2. Construction of the topologically equivalent function.

Define

$$H(t,x) = x + h(t,(t,x)), \quad L(t,y) = y + l(t,(t,y)), \quad x,y\in X. \tag{3.4}$$

Lemma 3.4. *For any fixed* $(\bar{t}, x(\bar{t})) \in \mathbb{R} \times X$, $H(t, X(t, \bar{t}, x(\bar{t})))$ *is a solution of* (1.1).

Proof. It follows from Lemma 3.1 that

$$h(t,(t,X(t,\bar{t},x(\bar{t})))) = h(t,(\bar{t},x(\bar{t})))$$

and

$$H(t,X(t,\bar{t},x(\bar{t}))) = X(t,\bar{t},x(\bar{t}))+h(t,(t,X(t,\bar{t},x(\bar{t})))) = X(t,\bar{t},x(\bar{t}))+h(t,(\bar{t},x(\bar{t}))).$$

Since $X(t,\bar{t},x(\bar{t}))$ and $h(t,(\bar{t},x(\bar{t})))$ are solutions of (1.2) and (3.1), respectively, for $t \neq \tau_i$, we have

$$\begin{aligned} H'(t,X(t,\bar{t},x(\bar{t}))) &= X'(t,\bar{t},x(\bar{t})) + h'(t,(\bar{t},x(\bar{t}))) \\ &= A(t)X(t,\bar{t},x(\bar{t})) + f(t,X(t,\bar{t},x(\bar{t}))) \\ &\quad + A(t)h(t,(\bar{t},x(\bar{t}))) - f(t,X(t,\bar{t},x(\bar{t}))) \\ &= A(t)H(t,X(t,\bar{t},x(\bar{t}))), \end{aligned}$$

and, for $t = \tau_i$, $i \in \mathbb{Z}$, we have

$$\begin{aligned} H(\tau_i^+,X(\tau_i^+,\bar{t},x(\bar{t}))) &= X(\tau_i^+,\bar{t},x(\bar{t})) + h(\tau_i^+,(\bar{t},x(\bar{t}))) \\ &= B_iX(\tau_i,\bar{t},x(\bar{t})) + g_i(X(\tau_i,\bar{t},x(\bar{t}))) \\ &\quad + B_ih(\tau_i,(\bar{t},x(\bar{t}))) - g_i(X(\tau_i,\bar{t},x(\bar{t}))) \\ &= B_iH(\tau_i,X(\tau_i,\bar{t},x(\bar{t}))). \end{aligned}$$

This implies that $H(t,X(t,\bar{t},x(\bar{t})))$ is a solution of (1.1). □

A similar argument to the proof of Lemma 3.4, we have

Lemma 3.5. *For any fixed* $(\bar{t},y(\bar{t}))\in\mathbb{R}\times X$, $L(t,Y(t,\bar{t},y(\bar{t})))$ *is a solution of* (1.2).

Lemma 3.6. *For any fixed* $t \in \mathbb{R}$ *and* $y \in X$, $H(t,L(t,y)) = y$ *holds.*

Proof. Let $y(t)$ be any solution of (1.1). It follows from Lemma 3.4 and Lemma 3.5 that $L(t,y(t))$ is a solution of (1.2) and $H(t,L(t,y(t)))$ is a solution of (1.1). Moreover, one has

$$\begin{aligned} H'(t,L(t,y(t))) - y'(t) &= A(t)H(t,L(t,y(t))) - A(t)y(t) \\ &= A(t)(H(t,L(t,y(t))) - y(t)),\ t \neq \tau_i \end{aligned}$$

and

$$\begin{aligned} H(\tau_i^+,L(\tau_i^+,y(\tau_i^+))) - y(\tau_i^+) &= B_iH(\tau_i,L(\tau_i,y(\tau_i))) - B_iy(\tau_i) \\ &= B_i(H(\tau_i,L(\tau_i,y(\tau_i))) - y(\tau_i)),\ i \in \mathbb{Z}. \end{aligned}$$

In addition,

$$\begin{aligned} \|H(t,L(t,y(t))) - y(t)\| &\leq \|H(t,L(t,y(t))) - L(t,y(t))\| + \|L(t,y(t)) - y(t)\| \\ &\leq 2[K\alpha_1(1/|a| + 1/b) + K\alpha_2(\omega_1\zeta_a + \omega_2\zeta_{-b})] \end{aligned}$$

for any $t \in \mathbb{R}$. This shows that $H(t,L(t,y(t))) - y(t)$ is a bounded solution of (1.1), and hence

$$H(t,L(t,y(t))) - y(t) \equiv 0.$$

For any fixed $t \in \mathbb{R}, y \in X$, there exists a solution of (1.1) with the initial value $y(t) = y$. Then $H(t,L(t,y)) = y$ holds. □

Similarly, we have

Lemma 3.7. *For any fixed* $t \in \mathbb{R}, x \in X$, $L(t, H(t, x)) = x$ *holds.*

We are now at the right position to establish Theorem 2.1. We are going to prove topological equivalence between (1.1) and (1.2), that is, to verify that $H(t, x)$ is a topologically equivalent function.

Proof of condition (i). It follows from (3.4) and Lemma 3.1 that for any $t \in \mathbb{R}$

$$\|H(t,x) - x\| = \|h(t,x)\| \leq K\alpha_1(1/|a| + 1/b) + K\alpha_2(\omega_1\zeta_a + \omega_2\zeta_{-b}),\ x \in X.$$

Proof of condition (ii). By Lemma 3.6 and Lemma 3.7, for each fixed $t \in \mathbb{R}$, $L(t, \cdot) = H^{-1}(t, \cdot)$ is homeomorphism.

Proof of condition (iii). From (3.4) and Lemma 3.2, for any $t \in \mathbb{R}$, we have

$$\|L(t,y) - y\| = \|l(t,,y)\| \leq K\alpha_1(1/|a| + 1/b) + K\alpha_2(\omega_1\zeta_a + \omega_2\zeta_{-b}),\ y \in X.$$

Proof of condition (iv). By using Lemma 3.4 and Lemma 3.5, the condition (iv) holds.

The proof of Theorem 2.1 is complete.

3.2. Proofs of Theorems 2.2 and 2.3

The proof of Theorem 2.3 is the similar arguments to the proof of Theorem 2.2 and can be obtained by reversing the time. Therefore, here it is omitted.

To obtain the stable manifolds, we first introduce an auxiliary space. Let $\mathcal{X}^*$ be the space of functions $\Phi\colon \mathbb{R}^+ \times X \to X$ that are left-continuous in s, at most with discontinuities of the first kind at the points τ_i, such that $\Phi|_{Z_\beta} \in \mathcal{X}$ and

$$\Phi(s,\xi) = \Phi\big(s, \beta(s)^{-\varepsilon}\xi/\|\xi\|\big) \quad \text{for every} \quad (s,\xi) \notin Z_\beta.$$

We note that there is a one-to-one correspondence between functions in $\mathcal{X}$ and functions in $\mathcal{X}^*$. Moreover, $\mathcal{X}^*$ is a Banach space with the norm $\mathcal{X}^* \ni \Phi \mapsto |\Phi|Z_\beta|'$. It is not difficult to show that for each $\Phi \in \mathcal{X}^*$ we have

$$\|\Phi(s,\xi_1) - \Phi(s,\xi_2)\| \leq 2\|\xi_1 - \xi_2\| \tag{3.5}$$

for every $s \geq 0$ and $\xi_1, \xi_2 \in E(s)$.

The proof of Theorem 2.2 is obtained in several steps. We first prove that, for each $\Phi \in \mathcal{X}^*$, there exists a unique function u satisfying (2.10).

Lemma 3.8. *Let c be sufficiently small. Then, for each $\Phi \in \mathcal{X}^*$ and $(s,\xi) \in Z_\beta$, there exists a unique function $u : \mathbb{R}^+ \to X$ with $u(s) = \xi$ such that, for any $t \geq s$,* (2.10) *holds and*

$$\|u(t)\| \leq 2K(h(t)/h(s))^a \mu(s)^\varepsilon \|\xi\|. \tag{3.6}$$

Proof. Let Ω_2 be the space of left-continuous functions $x\colon [s,\infty) \to X$ at most with discontinuities of the first kind at the points τ_i with the initial value $x(s)$ and $\|x\|_* \leq \beta(s)^{-\varepsilon}$, where

$$\|x\|_* = \frac{1}{2K} \sup\left\{ \frac{\|x(t)\|}{(h(t)/h(s))^a \mu(s)^\varepsilon} : t \geq s \right\}. \tag{3.7}$$

It is not difficult to show that $(\Omega_2, \|\cdot\|_*)$ is a Banach space.

Given $(s,\xi)\in Z_\beta$ and $\Phi\in\mathcal{X}^*$, define an operator L in Ω_2 by

$$(Lu)(t)=T(t,s)\xi+\int_s^t T(t,\tau)P(\tau)f(\tau,u(\tau),\Phi(\tau,u(\tau))d\tau$$
$$+\sum_{s\le\tau_i<t}T(t,\tau_i^+)P(\tau_i^+)g_i(u(\tau_i),\Phi(\tau_i,u(\tau_i)))$$

for each $t\ge s$. It is easy to show that Lu is left-continuous in $[s,\infty)$ at most with discontinuities of the first kind at the points τ_i, and that $(Lu)(s)=\xi$ and $(Lu)(t)\in E(t)$ for every $t\ge s$. It follows from (2.3), (2.4) and (2.5) that

$$\begin{aligned}C_1^1(\tau):&=\|f(\tau,u(\tau),\Phi(\tau,u(\tau)))\|\\&\le c\left(\|u(\tau)\|+\|\Phi(\tau,u(\tau))\|\right)\left(\|u(\tau)\|+\|\Phi(\tau,u(\tau))\|\right)^q\\&\le 3^{q+1}c\|u(\tau)\|^{q+1}\\&\le 6^{q+1}cK^{q+1}\left(\frac{h(\tau)}{h(s)}\right)^{a(q+1)}\mu(s)^{\varepsilon(q+1)}(\|u\|_*)^{q+1},\ \tau\ne\tau_i\end{aligned}$$

and

$$\begin{aligned}C_1^2(\tau_i):&=\|g_i(u(\tau_i),\Phi(\tau_i,u(\tau_i)))\|\\&\le c\left(\|u(\tau_i)\|+\|\Phi(\tau_i,u(\tau_i))\|\right)\left(\|u(\tau_i)\|+\|\Phi(\tau_i,u(\tau_i))\|\right)^q\\&\le 3^{q+1}c\|u(\tau_i)\|^{q+1}\\&\le 6^{q+1}cK^{q+1}\left(\frac{h(\tau_i)}{h(s)}\right)^{a(q+1)}\mu(s)^{\varepsilon(q+1)}(\|u\|_*)^{q+1},\ i\in\mathbb{N}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|(Lu)(t)\|&\le\|T(t,s)\|\|\xi\|+\int_s^t\|T(t,\tau)P(\tau)\|C_1^1(\tau)d\tau\\&\quad+\sum_{s\le\tau_i<t}\|T(t,\tau_i^+)P(\tau_i^+)\|C_1^2(\tau_i)\\&\le K\left(\frac{h(t)}{h(s)}\right)^a\mu(s)^\varepsilon\|\xi\|\\&\quad+6^{q+1}cK^{q+2}\left(\frac{h(t)}{h(s)}\right)^a h(s)^{-aq}\mu(s)^{\varepsilon(q+1)}(\|u\|_*)^{q+1}C(s).\end{aligned}$$

By (2.6), we have

$$\begin{aligned}\|Lu\|_*&\le\frac12\left(\|\xi\|+6^{q+1}cK^{q+1}h(s)^{-aq}\mu(s)^{\varepsilon q}(\|u\|_*)^{q+1}C(s)\right)\\&\le\frac12\left(1+6^{q+1}cK^{q+1}h(s)^{-aq}\mu(s)^{\varepsilon q}\beta(s)^{-\varepsilon q}C(s)\right)\beta(s)^{-\varepsilon}\\&\le\frac12(1+6^{q+1}cK^{q+1})\beta(s)^{-\varepsilon}.\end{aligned}$$

If c is sufficiently small so that $6^{q+1}cK^{q+1}<1$, then we have $L(\Omega_2)\subset\Omega_2$. On the other hand, for each $u_1,u_2\in\Omega_2$, we get

$$\begin{aligned}C_2^1(\tau):&=\|f(\tau,u_1(\tau),\Phi(\tau,u_1(\tau)))-f(\tau,u_2(\tau),\Phi(\tau,u_2(\tau)))\|\\&\le 2\cdot 6^{q+1}cK^{q+1}\|u_1-u_2\|_*\beta(s)^{-\varepsilon q}(h(\tau)/h(s))^{a(q+1)}\mu(s)^{\varepsilon(q+1)},\ \tau\ne\tau_i,\end{aligned}$$

and

$$\begin{aligned} C_2^2(\tau_i) :&= \|g_i(u_1(\tau_i), \Phi(\tau_i, u_1(\tau_i))) - g_i(u_2(\tau_i), \Phi(\tau_i, u_2(\tau_i)))\| \\ &\le 2 \cdot 6^{q+1} c K^{q+1} \|u_1 - u_2\|_* \beta(s)^{-\varepsilon q} (h(\tau_i)/h(s))^{a(q+1)} \mu(s)^{\varepsilon(q+1)},\ i \in \mathbb{N}. \end{aligned}$$

Then

$$\begin{aligned} \|Lu_1(t) - Lu_2(t)\| &\le \int_s^t \|T(t,\tau)P(\tau)\| C_2^1(\tau) d\tau + \sum_{s \le \tau_i < t} \|T(t, \tau_i^+) P(\tau_i^+)\| C_2^2(\tau_i) \\ &\le 2 \cdot 6^{q+1} c K^{q+2} \|u_1 - u_2\|_* \left(\frac{h(t)}{h(s)}\right)^a \mu(s)^\varepsilon. \end{aligned}$$

Taking c sufficiently small such that $6^{q+1} c K^{q+1} < 1$, we have

$$\|Lu_1 - Lu_2\|_* \le 6^{q+1} c K^{q+1} \|u_1 - u_2\|_*.$$

Then L is a contraction in Ω_2, and there exists a unique function $u \in \Omega_2$ such that $Lu = u$. Moveover, it is easy to show that

$$\|u\|^* \le \frac{1}{2}\|\xi\| + \frac{1}{2} 6^{q+1} c K^{q+1} \|u\|_*,$$

and

$$\|u(t)\| \le 2K (h(t)/h(s))^a \mu(s)^\varepsilon \|\xi\| \quad \text{for any} \quad t \ge s,$$

since $K/(1 - (1/2) 6^{q+1} c K^{q+1}) < 2K$. □

Let $u = u_\xi^\Phi$ be the unique function given by Lemma 3.8, that is,

$$\begin{aligned} u(t) = T(t,s)\xi &+ \int_s^t T(t,\tau) P(\tau) f(\tau, u(\tau), \Phi(\tau, u(\tau)) d\tau \\ &+ \sum_{s \le \tau_i < t} T(t, \tau_i^+) P(\tau_i^+) g_i(u(\tau_i), \Phi(\tau_i, u(\tau_i))) \end{aligned} \tag{3.8}$$

for each $t \ge s$.

Lemma 3.9. *Given c sufficiently small and $\Phi \in \mathcal{X}^*$, the following properties hold:*

1. *for each $(s, \xi) \in Z_\beta$ and $t \ge s$, if*

$$\begin{aligned} \Phi(t, u(t)) = T(t,s)\Phi(s,\xi) &+ \int_s^t T(t,\tau) Q(\tau) f(\tau, u(\tau), \Phi(\tau, u(\tau))) d\tau \\ &+ \sum_{s \le \tau_i < t} T(t, \tau_i^+) Q(\tau_i^+) g_i(u(\tau_i), \Phi(\tau_i, u(\tau_i))), \end{aligned} \tag{3.9}$$

then

$$\begin{aligned} \Phi(s,\xi) = &- \int_s^\infty T(\tau, s)^{-1} Q(\tau) f(\tau, u(\tau), \Phi(\tau, u(\tau))) d\tau \\ &- \sum_{s \le \tau_i} T(\tau_i^+, s)^{-1} Q(\tau_i^+) g_i(u(\tau_i), \Phi(\tau_i, u(\tau_i))). \end{aligned} \tag{3.10}$$

2. *if identity (3.10) holds for each $s \ge 0$ and $\xi \in B_s(\beta(s)^{-\varepsilon})$, then (3.9) holds for each $(s, \xi) \in Z_{\beta\cdot\mu}(2K)$.*

Proof. By (2.5), (2.3), (2.4), (3.5) and (3.6), for each $\tau \geq s$ we have

$$\begin{aligned}
C_3^1(\tau) :&= \|T(\tau,s)^{-1}Q(\tau)\| \cdot \|f(\tau,u(\tau),\Phi(\tau,u(\tau)))\| \\
&\leq 3^{q+1}cK\left(\frac{k(\tau)}{k(s)}\right)^{-b}\nu(\tau)^{\varepsilon}\|u(\tau)\|^{q+1} \\
&\leq 6^{q+1}cK^{q+2}\left(\frac{k(\tau)}{k(s)}\right)^{-b}\nu(\tau)^{\varepsilon}\left(\frac{h(\tau)}{h(s)}\right)^{a(q+1)}\mu(s)^{\varepsilon(q+1)}\|\xi\|^{q+1} \\
&\leq 6^{q+1}cK^{q+2}\left(\frac{k(\tau)}{k(s)}\right)^{-b}\nu(\tau)^{\varepsilon}\left(\frac{h(\tau)}{h(s)}\right)^{a(q+1)}\mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon(q+1)}
\end{aligned}$$

and similarly, for each $\tau_i \geq s$ we have

$$\begin{aligned}
C_3^2(\tau_i) :&= \|T(\tau_i^+,s)^{-1}Q(\tau_i^+)\| \cdot \|g_i(u(\tau_i),\Phi(\tau_i,u(\tau_i)))\| \\
&\leq 3^{q+1}cK\left(\frac{k(\tau_i)}{k(s)}\right)^{-b}\nu(\tau_i)^{\varepsilon}\|u(\tau_i)\|^{q+1} \\
&\leq 6^{q+1}cK^{q+2}\left(\frac{k(\tau_i)}{k(s)}\right)^{-b}\nu(\tau_i)^{\varepsilon}\left(\frac{h(\tau_i)}{h(s)}\right)^{a(q+1)}\mu(s)^{\varepsilon(q+1)}\|\xi\|^{q+1} \\
&\leq 6^{q+1}cK^{q+2}\left(\frac{k(\tau_i)}{k(s)}\right)^{-b}\nu(\tau_i)^{\varepsilon}\left(\frac{h(\tau_i)}{h(s)}\right)^{a(q+1)}\mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon(q+1)}.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_s^{\infty} C_3^1(\tau)d\tau + \sum_{s\leq\tau_i} C_3^2(\tau_i) \\
&\leq 6^{q+1}cK^{q+2}k(s)^b h(s)^{-a(q+1)}\mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon(q+1)} \\
&\quad\times\left(\int_s^{\infty} k(\tau)^{-b}h(\tau)^{a(q+1)}\nu(\tau)^{\varepsilon}d\tau + \sum_{s\leq\tau_i} k(\tau_i)^{-b}h(\tau_i)^{a(q+1)}\nu(\tau_i)^{\varepsilon}\right) \\
&\leq 6^{q+1}cK^{q+2}k(s)^b h(s)^{-a(q+1)}\mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon q}C(s) < \infty.
\end{aligned}$$

This implies that the right-hand side of (3.10) is always well-defined.

Now we assume that (3.9) holds for each $(s,\xi) \in Z_\beta$ and $t \geq s$. Identity (3.9) can be written in the form

$$\begin{aligned}
\Phi(s,\xi) = T(t,s)^{-1}\Phi(t,u(t)) &- \int_s^t T(\tau,s)^{-1}Q(\tau)f(\tau,u(\tau),\Phi(\tau,u(\tau)))d\tau \\
&- \sum_{s\leq\tau_i<t} T(\tau_i^+,s)^{-1}Q(\tau_i^+)g_i(u(\tau_i),\Phi(\tau_i,u(\tau_i))).
\end{aligned} \tag{3.11}$$

By (2.5), (3.5) and (3.6), we obtain

$$\begin{aligned}
\|T(t,s)^{-1}\Phi(t,u(t))\| &\leq 4K^2\left(\frac{k(t)}{k(s)}\right)^{-b}\nu(t)^{\varepsilon}\left(\frac{h(t)}{h(s)}\right)^{a}\mu(s)^{\varepsilon}\beta(s)^{-\varepsilon} \\
&\leq 4K^2k(t)^{-b}h(t)^a\nu(t)^{\varepsilon}k(s)^b h(s)^{-a}\mu(s)^{\varepsilon}\beta(s)^{-\varepsilon}.
\end{aligned}$$

Therefore, letting $t \to \infty$ in (3.11) yields identity (3.10).

Assume that (3.10) holds for any $(s,\xi) \in Z_\beta$. For each $(s,\xi) \in Z_{\beta\cdot\mu}(2K)$, by (3.6), we have

$$\|u(t)\| \le 2K\left(\frac{h(t)}{h(s)}\right)^a \mu(s)^\varepsilon \|\xi\| \le \beta(t)^{-\varepsilon}\frac{h(t)^a\beta(t)^\varepsilon}{h(s)^a\beta(s)^\varepsilon} \le \beta(t)^{-\varepsilon},$$

and hence, $(t,u(t)) \in Z_\beta$ for any $t \ge s$. It follows from (3.10) that

$$\begin{aligned}
T(t,s)\Phi(s,\xi) = &-\int_s^t T(t,\tau)Q(\tau)f(\tau,u(\tau),\Phi(\tau,u(\tau)))d\tau \\
&-\int_t^\infty T(t,\tau)Q(\tau)f(\tau,u(\tau),\Phi(\tau,u(\tau)))d\tau \\
&-\sum_{s\le\tau_i<t} T(t,\tau_i^+)Q(\tau_i^+)g_i(u(\tau_i),\Phi(\tau_i,u(\tau_i))) \\
&-\sum_{t\le\tau_i} T(t,\tau_i^+)Q(\tau_i^+)g_i(u(\tau_i),\Phi(\tau_i,u(\tau_i))) \\
= &-\int_s^t T(t,\tau)Q(\tau)f(\tau,u(\tau),\Phi(\tau,u(\tau)))d\tau \\
&-\sum_{s\le\tau_i<t} T(t,\tau_i^+)Q(\tau_i^+)g_i(u(\tau_i),\Phi(\tau_i,u(\tau_i))) + \Phi(t,u(t)),
\end{aligned}$$

using (3.10) in the last identity with (s,ξ) replaced by $(t,u(t))$. □

Lemma 3.10. *If c is sufficiently small, then there exists a $K_1 > 0$ such that*

$$\|u_1(t) - u_2(t)\| \le K_1(h(t)/h(s))^a\mu(s)^\varepsilon\|\xi_1 - \xi_2\| \tag{3.12}$$

for any $\Phi \in \mathcal{X}^*$, $(s,\xi_1),(s,\xi_2) \in Z_\beta$ and $t \ge s$.

Proof. Write $u_i = u_{\xi_i}^\Phi$. By (2.5), (2.3), (2.4), (3.5) and (3.6), we have

$$\begin{aligned}
C_4^1 &:= \int_s^t \|T(t,\tau)P(\tau)\|\|f(\tau,u_1(\tau),\Phi(\tau,u_1(\tau))) - f(\tau,u_2(\tau),\Phi(\tau,u_2(\tau)))\|d\tau \\
&\le 2\cdot 6^{q+1}cK^{q+2}\|u_1-u_2\|_*\left(\frac{h(t)}{h(s)}\right)^a\mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon q}\int_s^t h(\tau)^{aq}\mu(\tau)^\varepsilon d\tau
\end{aligned}$$

and

$$\begin{aligned}
C_4^2 &:= \sum_{s\le\tau_i<t}\|T(t,\tau_i^+)P(\tau_i^+)\|\|g_i(u_1(\tau_i),\Phi(\tau_i,u_1(\tau_i))) - g_i(u_2(\tau_i),\Phi(\tau_i,u_2(\tau_i)))\| \\
&\le 2\cdot 6^{q+1}cK^{q+2}\|u_1-u_2\|_*\left(\frac{h(t)}{h(s)}\right)^a\mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon q}\sum_{s\le\tau_i<t} h(\tau_i)^{aq}\mu(\tau_i)^\varepsilon.
\end{aligned}$$

Then

$$\begin{aligned}
\|u_1(t)-u_2(t)\| &\le \|T(t,s)(\xi_1-\xi_2)\| + C_4^1 + C_4^2 \\
&\le K\left(\frac{h(t)}{h(s)}\right)^a\mu(s)^\varepsilon(\|\xi_1-\xi_2\| + 2\cdot 6^{q+1}cK^{q+1}\|u_1-u_2\|_*).
\end{aligned}$$

This implies that

$$\|u_1 - u_2\|^* \leq \frac{1}{2}\|\xi_1 - \xi_2\| + 6^{q+1}cK^{q+1}\|u_1 - u_2\|^*,$$

which yields (3.12) with $K_1 = K/(1 - 6^{q+1}cK^{q+1})$. $\square$

Lemma 3.11. *If c is sufficiently small, then there exists a $K_2 > 0$ such that*

$$\|u^{\Phi_1}(t) - u^{\Phi_2}(t)\| \leq K_2(h(t)/h(s))^a\|\xi\| \cdot |\Phi_1 - \Phi_2|' \tag{3.13}$$

for any $\Phi_1, \Phi_2 \in \mathcal{X}^*$, $(s,\xi) \in Z_\beta$ and $t \geq s$.

Proof. For simplicity, write $u_i = u_\xi^{\Phi_i}$ for $i = 1, 2$. Note that

$$\begin{aligned}
&\|\Phi_1(\tau, u_1(\tau)) - \Phi_2(\tau, u_2(\tau))\| \\
&\leq \|\Phi_1(\tau, u_1(\tau)) - \Phi_2(\tau, u_1(\tau))\| + \|\Phi_2(\tau, u_1(\tau)) - \Phi_2(\tau, u_2(\tau))\| \\
&\leq \|u_1(\tau)\| \cdot |\Phi_1 - \Phi_2|' + 2\|u_1(\tau) - u_2(\tau)\|.
\end{aligned}$$

With similar arguments to those in Lemmas 3.8 and 3.10, we obtain

$$\begin{aligned}
C_5^1(\tau) :&= \|f(\tau, u_1(\tau), \Phi_1(\tau, u_1(\tau))) - f(\tau, u_2(\tau), \Phi_2(\tau, u_2(\tau)))\| \\
&\leq 3^q c\,[3(\|u_1(\tau) - u_2(\tau)\|)(\|u_1(\tau)\|^q + \|u_2(\tau)\|^q) \\
&\quad + (\|u_1(\tau)\| \cdot |\Phi_1 - \Phi_2|')(\|u_1(\tau)\|^q + \|u_2(\tau)\|^q)] \\
&\leq [2 \cdot 6^{q+1}cK^{q+1}\|u_1 - u_2\|_* + 4 \cdot 6^q cK^{q+1}\|\xi\| \cdot |\Phi_1 - \Phi_2|'] \\
&\quad \times (h(\tau)/h(s))^{a(q+1)}\mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon q},\ \tau \geq s
\end{aligned}$$

and

$$\begin{aligned}
C_5^2(\tau_i) :&= \|g_i(u_1(\tau_i), \Phi_1(\tau_i, u_1(\tau_i))) - g_i(u_2(\tau_i), \Phi_2(\tau_i, u_2(\tau_i)))\| \\
&\leq 3^q c\,[3(\|u_1(\tau_i) - u_2(\tau_i)\|)(\|u_1(\tau_i)\|^q + \|u_2(\tau_i)\|^q) \\
&\quad + (\|u_1(\tau_i)\| \cdot |\Phi_1 - \Phi_2|')(\|u_1(\tau_i)\|^q + \|u_2(\tau_i)\|^q)] \\
&\leq [2 \cdot 6^{q+1}cK^{q+1}\|u_1 - u_2\|_* + 4 \cdot 6^q cK^{q+1}\|\xi\| \cdot |\Phi_1 - \Phi_2|'] \\
&\quad \times (h(\tau_i)/h(s))^{a(q+1)}\mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon q},\ \tau_i \geq s.
\end{aligned}$$

Then

$$\begin{aligned}
\|u_1(t) - u_2(t)\| &\leq \int_s^t \|T(t,\tau)P(\tau)\|C_5^1(\tau)d\tau + \sum_{s \leq \tau_i < t}\|T(t,\tau_i^+)P(\tau_i^+)\|C_5^2(\tau_i) \\
&\leq [2 \cdot 6^{q+1}cK^{q+1}\|u_1 - u_2\|_* + 4 \cdot 6^q cK^{q+1}\|\xi\| \cdot |\Phi_1 - \Phi_2|'] \\
&\quad \times K\left(\frac{h(t)}{h(s)}\right)^a \mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon q}C(s).
\end{aligned}$$

This implies that

$$\|u_1 - u_2\|_* \leq [6^{q+1}cK^{q+1}\|u_1 - u_2\|_* + 2 \cdot 6^q cK^{q+1}\|\xi\| \cdot |\Phi_1 - \Phi_2|']\mu(s)^{-\varepsilon}.$$

This establishes inequality (3.13). $\square$

Lemma 3.12. *If c is sufficiently small, then there exists a unique function $\Phi \in \mathcal{X}^*$ such that* (3.10) *holds for any $(s,\xi) \in Z_\beta$.*

Proof. For each $\Phi \in \mathcal{X}^*$ and $(s,\xi) \in Z_\beta$, we define an operator J by

$$\begin{aligned}(J\Phi)(s,\xi) = &-\int_s^\infty T(\tau,s)^{-1}Q(\tau)f(\tau,u(\tau),\Phi(\tau,u(\tau)))d\tau \\ &-\sum_{s\le\tau_i} T(\tau_i^+,s)^{-1}Q(\tau_i^+)g_i(u(\tau_i),\Phi(\tau_i,u(\tau_i))),\end{aligned}$$

where $u = u_\xi^\Phi$ is the unique function given by Lemma 3.8. It is easy to show that $J\Phi$ is left-continuous in s at most with discontinuities of the first kind at the points τ_i, and that $(J\Phi)(s,0) = 0$ for $s \ge 0$. Moreover, for any $\xi_1,\xi_2 \in B_s(\beta(s)^{-\varepsilon})$, let $u_i = u_{\xi_i}^\Phi$ for $i = 1,2$, by (2.5), (3.6) and (3.12), we have

$$\begin{aligned}C_6^1(\tau) :&= \|f(\tau,u_1(\tau),\Phi(\tau,u_1(\tau))) - f(\tau,u_2(\tau),\Phi(\tau,u_2(\tau)))\| \\ &\le 6^{q+1}cK^qK_1\left(\frac{h(\tau)}{h(s)}\right)^{a(q+1)}\mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon q}\|\xi_1-\xi_2\|\end{aligned}$$

and

$$\begin{aligned}C_6^2(\tau_i) :&= \|g_i(u_1(\tau_i),\Phi(\tau_i,u_1(\tau_i))) - g_i(u_2(\tau_i),\Phi(\tau_i,u_2(\tau_i)))\| \\ &\le 6^{q+1}cK^qK_1\left(\frac{h(\tau_i)}{h(s)}\right)^{a(q+1)}\mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon q}\|\xi_1-\xi_2\|.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|(J\Phi)(s,\xi_1) - (J\Phi)(s,\xi_2)\| \\ &\le \int_s^\infty \|T(\tau,s)^{-1}Q(\tau)\|C_6^1(\tau)d\tau + \sum_{s\le\tau_i}\|T(\tau_i^+,s)^{-1}Q(\tau_i^+)\|C_6^2(\tau_i) \\ &\le 6^{q+1}cK^{q+1}K_1k(s)^bh(s)^{-a(q+1)}\mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon q}C(s)\|\xi_1-\xi_2\| \\ &\le 6^{q+1}cK^{q+1}K_1\|\xi_1-\xi_2\|.\end{aligned}$$

If c is sufficiently small, then

$$\|(J\Phi)(s,\xi_1) - (J\Phi)(s,\xi_2)\| \le \|\xi_1-\xi_2\|$$

and one can extend $J\Phi$ to $\mathbb{R}^+ \times X$ by $(J\Phi)(s,\xi) = (J\Phi)\big(s,\beta(s)^{-\varepsilon}\xi/\|\xi\|\big)$ for any $(s,\xi) \notin Z_\beta$, and hence, $J(\mathcal{X}^*) \subset \mathcal{X}^*$.

Now we show that J is a contraction. Given $\Phi_1,\Phi_2 \in \mathcal{X}^*$ and writing $u_i = u_\xi^{\Phi_i}$ for $i = 1,2$, by (3.5), (3.6) and (3.13), for each $(s,\xi) \in Z_\beta$ we have

$$\begin{aligned}C_7^1(\tau) :&= \|f(\tau,u_1(\tau),\Phi_1(\tau,u_1(\tau))) - f(\tau,u_2(\tau),\Phi_2(\tau,u_2(\tau)))\| \\ &\le 3^qc\big(3\|u_1(\tau)-u_2(\tau)\| + \|u_1(\tau)\|\cdot|\Phi_1-\Phi_2|'\big)(\|u_1(\tau)\|^q + \|u_2(\tau)\|^q) \\ &\le 2\cdot 6^qcK^q(2K+3K_2)\|\xi\|\cdot|\Phi_1-\Phi_2|'\left(\frac{h(\tau)}{h(s)}\right)^{a(q+1)}\mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon q}\end{aligned}$$

and

$$\begin{aligned}C_7^2(\tau_i) :&= \|g_i(u_1(\tau_i),\Phi_1(\tau_i,u_1(\tau_i))) - g_i(u_2(\tau_i),\Phi_2(\tau_i,u_2(\tau_i)))\| \\ &\le 3^qc\big(3\|u_1(\tau_i)-u_2(\tau_i)\| + \|u_1(\tau_i)\|\cdot|\Phi_1-\Phi_2|'\big)(\|u_1(\tau_i)\|^q + \|u_2(\tau_i)\|^q) \\ &\le 2\cdot 6^qcK^q(2K+3K_2)\|\xi\|\cdot|\Phi_1-\Phi_2|'\left(\frac{h(\tau_i)}{h(s)}\right)^{a(q+1)}\mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon q}.\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|(J\Phi_1)(s,\xi)-(J\Phi_2)(s,\xi)\| \\
&\le \int_s^\infty \|T(\tau,s)^{-1}Q(\tau)\|C_7^1(\tau)d\tau+\sum_{s\le\tau_i}\|T(\tau_i^+,s)^{-1}Q(\tau_i^+)\|C_7^2(\tau_i) \\
&\le 2\cdot 6^q cK^q(2K+3K_2)\|\xi\|\cdot|\Phi_1-\Phi_2|'
\end{aligned}$$

and provided that c is sufficiently small, the operator J is a contraction. Therefore, there exists a unique function $\Phi\in\mathcal{X}^*$ such that (3.10) holds for every $(s,\xi)\in Z_\beta$. □

We are now at the right position to establish Theorem 2.2. It follows from Lemma 3.8 that, for each $(s,\xi)\in Z_\beta$ and $\Phi\in\mathcal{X}^*$, there exists a unique function $u=u_\xi^\Phi\in\Omega_2$. By Lemmas 3.9 , 3.12 and the one-to-one correspondence between $\mathcal{X}$ and $\mathcal{X}^*$, for each $s\ge 0$ and $\xi\in B_s\left((\beta(s)\cdot\mu(s))^{-\varepsilon}/(2K)\right)$, there exists a unique function $\Phi\in\mathcal{X}$ such that (3.9) holds. For each $(s,\xi)\in Z_{\beta(s)\cdot\mu(s)}(2K)$, by (3.6), we have

$$\begin{aligned}
\|u(t)\| &\le 2K(h(t)/h(s))^a\mu(s)^\varepsilon\frac{1}{2K}(\beta(s)\cdot\mu(s))^{-\varepsilon} \\
&\le (h(t)/h(s))^a\beta(s)^{-\varepsilon}\le\beta(s)^{-\varepsilon},
\end{aligned}$$

which implies that $(t,u(t))\in Z_\beta$ for any $t\ge s$. Therefore, (2.11) holds and $\mathcal{W}$ is forward invariant under the semiflow Ψ_κ. For any $(s,\xi_1),(s,\xi_2)\in Z_{\beta(s)\cdot\mu(s)}(2K)$ and $\kappa=t-s\ge 0$, by Lemmas 3.10, we have

$$\begin{aligned}
&\|\Psi_\kappa(s,\xi_1,\Phi(s,\xi_1))-\Psi_\kappa(s,\xi_2,\Phi(s,\xi_2))\| \\
&=\|(t,u^{\xi_1}(t),\Phi(t,u^{\xi_1}(t)))-(t,u^{\xi_2}(t),\Phi(t,u^{\xi_2}(t)))\| \\
&\le 3\|u^{\xi_1}(t)-u^{\xi_2}(t)\|\le 3K_1(h(t)/h(s))^a\mu(s)^\varepsilon\|\xi_1-\xi_2\|.
\end{aligned}$$

The proof of Theorem 2.2 is complete.

Acknowledgements

The authors are grateful to the anonymous referee for carefully reading the manuscript and for important suggestions and comments, which lead to the improvement of their manuscript. The authors would like to thank Prof. Luis Barreira at Departamento de Matemática of Instituto Superior Técnico in Portugal for leading them into the field of the nonuniform dichotomies.

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