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APPROXIMATE CONTROLLABILITY OF RIEMANN-LIOUVILLE FRACTIONAL EVOLUTION EQUATIONS WITH INTEGRAL CONTRACTOR ASSUMPTION*

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Abstract We propose and investigate an evolution system with a Riemann-Liouville fractional derivative. With the aid of a resolvent method, we formulate a suitable notion of solutions to this system and demonstrate the corresponding existence and uniqueness of solutions under a regular integral contractor condition. Furthermore, by applying a space decomposition technique, we exhibit the approximate controllability result of the system. This paper closes with a simple example, which confirms our analytical findings.

Keywords Approximate controllability, Riemann-Liouville fractional derivatives, resolvent, regular integral contractor.

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1. Introduction

In the past two decades, differential equations with fractional derivatives have attracted increasing research attention because they play a central role in describing many physical phenomena. Many researchers have studied them extensively and much significant literature about them has been displayed [5, 7, 13, 15, 25, 26, 30, 31]. In [31], Laplace transformations and probability densities were applied to formulate a suitable concept of solutions to a Riemann-Liouville fractional system when A generates a C_0-semigroup. In 2014, Fan [7] analyzed a Riemann-Liouville fractional linear inhomogeneous system by using a resolvent method introduced by Li and Peng [15]. With the help of the uniform integrability assumption on the resolvent, Fan presented an appropriate notion of solutions to this system. Considering that the resolvent method, a generalization of semigroup approach, is convenient and efficient, we will also utilize this technique to investigate solutions of Riemann-Liouville fractional semilinear systems when A generates a resolvent.

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On the other hand, there are widespread applications of controllability in many fields, such as control theory, engineering practice, technical science, etc. Thereby, we need pay attention to various control problems. Especially, many researchers have shown an intense interest in analyzing approximate controllability problems (e.g., [4,9,12,14,16–19,22–24]). In addition, many authors have explored these problems by applying a space decomposition method (refer to [12,18,19,22,23]). However, few results on approximate controllability for Riemann-Liouville fractional systems have been displayed by employing this technique. Therefore, the objective of the present paper is to address the approximate controllability of Riemann-Liouville fractional semilinear systems by utilizing this method.

Here, under suitable conditions of the operator $t^{1-\alpha}T_\alpha(t)$, we first formulate a concept of solutions to system (2.1) by a resolvent approach. Emphasis here is that this is a nontrivial extension of Fan [7] since the assumptions on the operator $T_\alpha(t)$ is different from Fan's. Then, we display the existence and uniqueness of solutions by utilizing an integral contractor assumption. It is worth mentioning here that this assumption is weaken than a Lipschitz condition. Next, we analyze the approximate controllability problems by applying a technique of space decomposition. Finally, we present a simple example to illustrate that the hypotheses on $t^{1-\alpha}T_\alpha(t)$ are suitable. We emphasize that the technique combining integral contractor, resolvent and space decomposition can enable us to treat all previous works on approximate controllability problems in Hilbert spaces.

The work is structured as follows. Section 2 contains some preliminaries, including symbols, definitions and lemmas. In Section 3, we introduce a suitable concept of solutions and exhibit the existence and uniqueness result. Section 4 is devoted to the approximate controllability problem. Finally, a simple application is proposed in Section 5 to demonstrate the validity of our analytical findings.

2. Problem statement and preliminaries

We explore the following Riemann-Liouville fractional evolution system:

$$\begin{cases} D^\alpha x(t) = Ax(t) + (Bu)(t) + f(t, x(t)), \ \frac{1}{2} < \alpha < 1, \ t \in J' = (0, b], \\ \lim\limits_{t\to 0^+} \Gamma(\alpha)t^{1-\alpha}x(t) = x_0, \end{cases} \tag{2.1}$$

where $A : D(A) \subset H \to H$ generates an α-order resolvent $\{T_\alpha(t)\}_{t>0}$ on a Hilbert space H and $f : J \times H \to H$ is a nonlinear function. Furthermore, $u \in L^2(J;U)$ and $B : L^2(J;U) \to L^2(J;H)$ is a bounded and linear operator, where $J = [0, b]$ and U is a Hilbert space.

To establish our main results, we first summarize some preliminaries, including notations, definitions and lemmas. We employ the symbol $*$ to denote the convolution of functions, i.e., $(f * h)(s) = \int_0^s f(s-\tau)h(\tau)\mathrm{d}\tau, \ s > 0$. Additionally, the notation $L(H)$ represents the class of all bounded and linear operators from H to itself. Moreover, we denote the closure of the set D by $\overline{D}$. Let

$$C_{1-\alpha}(J;H) = \{x \in C(J';H) : \widetilde{x}(s) = s^{1-\alpha}x(s), \ \widetilde{x}(0) = \lim_{s\to 0^+} \widetilde{x}(s), \ \widetilde{x} \in C(J;H)\}$$

be normed by $\|x\|_{C_{1-\alpha}} = \sup\limits_{s\in J} \|\widetilde{x}(s)\|$. Then $C_{1-\alpha}(J;H)$ is a Banach space.

Definition 2.1 ([21]). Let $\alpha > 0$. For any $f \in L^1(J;H)$, the α-order fractional integral is

$$J_s^\alpha f(s) = (g_\alpha * f)(s),\ \ s > 0,$$

where

$$g_\alpha(s) = \frac{s^{\alpha-1}}{\Gamma(\alpha)},\ \ s > 0.$$

Definition 2.2 ([21]). Let $0 < \alpha < 1$ and $f \in L^1(J;H)$. The α-order fractional derivative of f, in the Riemann-Liouville sense, can be expressed by

$$D^\alpha f(s) = \frac{\mathrm{d}}{\mathrm{d}s}(g_{1-\alpha} * f)(s),\ \ s > 0.$$

Definition 2.3 ([15]). Let $0 < \alpha < 1$. By an α-order resolvent, we mean a strongly continuous family $\{T_\alpha(s)\}_{s>0} \subseteq L(H)$ satisfying
(a) $\lim_{s\to 0^+} \Gamma(\alpha)s^{1-\alpha}T_\alpha(s)x = x$, $x \in H$;
(b) $T_\alpha(\tau)T_\alpha(s) = T_\alpha(s)T_\alpha(\tau)$, $s,\ \tau > 0$;
(c) $T_\alpha(\tau)J_s^\alpha T_\alpha(s) - J_\tau^\alpha T_\alpha(\tau)T_\alpha(s) = g_\alpha(\tau)J_s^\alpha T_\alpha(s) - g_\alpha(s)J_\tau^\alpha T_\alpha(\tau)$, $s, \tau > 0$.

The generator $A : D(A) \subseteq H \to H$ of the resolvent $\{T_\alpha(s)\}_{s>0}$ is defined by

$$Ax = \Gamma(2\alpha) \lim_{s\to 0^+} \frac{s^{1-\alpha}T_\alpha(s)x - \frac{x}{\Gamma(\alpha)}}{s^\alpha},$$

where

$$D(A) = \left\{ x \in H : \lim_{s\to 0^+} \frac{s^{1-\alpha}T_\alpha(s)x - \frac{x}{\Gamma(\alpha)}}{s^\alpha} \text{ exists} \right\}.$$

Remark 2.1. It is easily seen that the operator $s^{1-\alpha}T_\alpha(s)$ is bounded on J, where $s^{1-\alpha}T_\alpha(s)|_{s=0} = \lim\limits_{s\to 0^+} s^{1-\alpha}T_\alpha(s)$. For brevity, set $M = \sup\limits_{s\in J} \|s^{1-\alpha}T_\alpha(s)\|$.

Lemma 2.1 ([15]). *Let A generate a resolvent $\{T_\alpha(s)\}_{s>0}$. Then*

(i) $T_\alpha(s)D(A) \subseteq D(A)$, $s > 0$;

(ii) $\overline{D(A)} = H$.

Hereafter, we always assume that

(HA) $\{s^{1-\alpha}T_\alpha(s)\}_{s>0}$ is compact and there exists a positive constant C such that $\left\| \frac{\mathrm{d}(s^{1-\alpha}T_\alpha(s))}{\mathrm{d}s} \right\| \le \frac{C}{s}$, $s \in J'$.

Remark 2.2. In view of Lemma 3.8 in [8], if $\{s^{1-\alpha}T_\alpha(s)\}_{s>0}$ is an analytic compact operator family of analyticity type (ω_0, θ_0), then (HA) is fulfilled.

Following the proofs in Lemmas 3.4, 3.5 and 3.8 of [8], we can derive the following properties of the operator $s^{1-\alpha}T_\alpha(s)$.

Lemma 2.2. *Let condition (HA) hold. Then for $s \in J'$, we have*

(a) $\lim\limits_{\tau\to 0} \left\| (s+\tau)^{1-\alpha}T_\alpha(s+\tau) - s^{1-\alpha}T_\alpha(s) \right\| = 0$;

(b) $\lim_{\tau\to 0^+} \left\|(s+\tau)^{1-\alpha}T_\alpha(s+\tau) - (\Gamma(\alpha)\tau^{1-\alpha}T_\alpha(\tau))(s^{1-\alpha}T_\alpha(s))\right\| = 0;$

(c) $\lim_{\tau\to 0^+} \left\|s^{1-\alpha}T_\alpha(s) - (\Gamma(\alpha)\tau^{1-\alpha}T_\alpha(\tau))((s-\tau)^{1-\alpha}T_\alpha(s-\tau))\right\| = 0.$

Lemma 2.3 ([28]). *Let $1 \le p < \infty$. If $\psi \in L^p(J;H)$, then*

$$\lim_{s\to 0}\int_0^b \|\psi(s+\tau)-\psi(\tau)\|^p \mathrm{d}\tau = 0,$$

where $\psi(t) = 0$ for $t \notin J$.

We now introduce the concept of a regular integral contractor associated with the resolvent $\{T_\alpha(s)\}_{s>0}$ (refer to [1,9–11]).

Definition 2.4. Let $\gamma : J \times H \to L(C_{1-\alpha}(J;H))$. γ is called a bounded integral contractor of f, depending on the resolvent $\{T_\alpha(s)\}_{s>0}$, if

$$\begin{aligned}\Bigg\| f\Bigg(s, x(s)+y(s)+\int_0^s T_\alpha(s-\tau)\tau^{1-\alpha}(\gamma(\tau,x(\tau))y)(\tau)\mathrm{d}\tau\Bigg)\\ -f(s,x(s)) - s^{1-\alpha}(\gamma(s,x(s))y)(s)\Bigg\| \le \lambda s^{1-\alpha}\|y(s)\|\end{aligned} \tag{2.2}$$

with $\lambda > 0$ holds for all $s \in J$ and $x,\ y \in C_{1-\alpha}(J;H)$. In addition, if for any $x,\ z \in C_{1-\alpha}(J;H)$, the following equation

$$y(s) + \int_0^s T_\alpha(s-\tau)\tau^{1-\alpha}(\gamma(\tau,x(\tau))y)(\tau)\mathrm{d}\tau = z(s)$$

possesses a solution $y \in C_{1-\alpha}(J;H)$, then γ is called a regular integral contractor.

For convenience, we set $\beta = \sup\{\|\gamma(s,x(s))\| : s \in J,\ x \in C_{1-\alpha}(J;H)\}$.

Remark 2.3. If for $x,\ y \in H$ and $s \in J$, $\|f(s,x)-f(s,y)\| \le \lambda s^{1-\alpha}\|x-y\|$ with $\lambda > 0$, then f has a regular integral contractor $\gamma = 0$.

3. Existence results

The task of this section is to introduce a suitable concept of solutions to system (2.1) and exhibit the corresponding existence and uniqueness of solutions by employing a resolvent method. To achieve our goal, we suppose that $f : J \times H \to H$ satisfies

(Hf) (i) for a.e. $s \in J$, $y \to f(s,y)$ is continuous;

(ii) for all $y \in H$, $s \to f(s,y)$ is measurable;

(iii) for a.e. $s \in J$ and all $y \in H$, $\|f(s,y)\| \le k + \varrho s^{1-\alpha}\|y\|$ with $\varrho,\ k > 0$;

(iv) $f : J \times H \to H$ has a regular integral contractor γ.

Definition 3.1. For fixed $u \in L^2(J;U)$, by a solution to (2.1) depending on u, we mean a function $x \in C_{1-\alpha}(J;H)$ satisfying

$$x(t) = g_\alpha(t)x_0 + AJ_t^\alpha x(t) + J_t^\alpha((Bu)(t) + f(t,x(t))), \tag{3.1}$$

for $t \in J'$.

In order to propose an equivalent concept of solutions to (2.1) by employing the resolvent method, for $h \in L^2(J;H)$, we first analyze the continuity of $T_\alpha * h$ in $C(J;H)$. We emphasize that it is essential to study the continuity, since $T_\alpha(t)$ has singularity at zero.

Lemma 3.1. *Assume that condition* (HA) *is satisfied and* $h \in L^2(J;H)$. *Then* $T_\alpha * h \in C(J;H)$.

Proof. We begin by showing the existence of $(T_\alpha * h)(s)$, $s \in J$. By a similar proof as employed in Proposition 1.3.4 of [2], we can check the measurability of $T_\alpha(s-\cdot)h(\cdot)$ on $(0,s)$, $s \in J'$. Furthermore, we have

$$\left\|\int_0^s T_\alpha(s-\tau)h(\tau)\mathrm{d}\tau\right\| = \left\|\int_0^s ((s-\tau)^{1-\alpha}T_\alpha(s-\tau))(s-\tau)^{\alpha-1}h(\tau)\mathrm{d}\tau\right\| \le M\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\|h\|_{L^2}.$$

Thus, $(T_\alpha * h)(s)$ exists.

We now turn to prove the continuity of $T_\alpha * h$ on J. It suffices to show that $T_\alpha * h \in C(J';H)$, since it is easily seen that $T_\alpha * h$ is continuous at $s=0$. Let $0<\varepsilon<t_1<t_2\le b$. We have

$$\begin{aligned}
&\|(T_\alpha * h)(t_2)-(T_\alpha * h)(t_1)\| \\
\le& \left\|\int_0^{t_1-\varepsilon}((t_2-\tau)^{1-\alpha}T_\alpha(t_2-\tau)-(t_1-\tau)^{1-\alpha}T_\alpha(t_1-\tau))(t_2-\tau)^{\alpha-1}h(\tau)\mathrm{d}\tau\right\| \\
&+\left\|\int_{t_1-\varepsilon}^{t_1}((t_2-\tau)^{1-\alpha}T_\alpha(t_2-\tau)-(t_1-\tau)^{1-\alpha}T_\alpha(t_1-\tau))(t_2-\tau)^{\alpha-1}h(\tau)\mathrm{d}\tau\right\| \\
&+\left\|\int_0^{t_1}(t_1-\tau)^{1-\alpha}T_\alpha(t_1-\tau)((t_2-\tau)^{\alpha-1}-(t_1-\tau)^{\alpha-1})h(\tau)\mathrm{d}\tau\right\| \\
&+\left\|\int_{t_1}^{t_2}(t_2-\tau)^{1-\alpha}T_\alpha(t_2-\tau)(t_2-\tau)^{\alpha-1}h(\tau)\mathrm{d}\tau\right\| \\
\le& \sup_{\tau\in[0,t_1-\varepsilon]}\|(t_2-\tau)^{1-\alpha}T_\alpha(t_2-\tau)-(t_1-\tau)^{1-\alpha}T_\alpha(t_1-\tau)\|\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\|h\|_{L^2} \\
&+2M\int_{t_1-\varepsilon}^{t_1}(t_2-\tau)^{\alpha-1}\|h(\tau)\|\mathrm{d}\tau \\
&+M\|h\|_{L^2}\left(\int_0^{t_1}[(t_2-\tau)^{\alpha-1}-(t_1-\tau)^{\alpha-1}]^2\mathrm{d}\tau\right)^{\frac{1}{2}} \\
&+M\int_{t_1}^{t_2}(t_2-\tau)^{\alpha-1}\|h(\tau)\|\mathrm{d}\tau.
\end{aligned}$$

Thus, it follows from Lemmas 2.2 and 2.3, the absolute continuity of integration of $(t_2-\cdot)^{\alpha-1}\|h(\cdot)\|$ and the arbitrariness of ε, we can infer that

$$\|(T_\alpha * h)(t_2)-(T_\alpha * h)(t_1)\| \to 0,\ t_2\to t_1.$$

Hence, $T_\alpha * h \in C(J;H)$. □

With the help of Lemma 3.1, we give the following equivalent concept of solutions to (2.1).

Lemma 3.2. *For each $u \in L^2(J;U)$, if hypotheses (HA) and (Hf) hold, then $x \in C_{1-\alpha}(J;H)$ is a solution depending on u to* (2.1) *if and only if x satisfies*

$$x(t) = T_\alpha(t)x_0 + \int_0^t T_\alpha(t-\tau)((Bu)(\tau) + f(\tau, x(\tau)))\mathrm{d}\tau, \ t \in J'. \tag{3.2}$$

Proof. Due to Lemma 3.1, we can infer that the expression (3.2) is well defined. Similar to the proofs of Theorems 3.2 and 3.3 in [7], we can easily verify the assertion of the lemma. □

We are now in a position to establish the existence and uniqueness result.

Theorem 3.1. *Under conditions (HA) and (Hf), for fixed $u \in L^2(J;U)$, problem* (2.1) *possesses a unique solution.*

Proof. To begin with, we show the existence of solutions. Define the sequences $\{y_n\}$ and $\{z_n\}$ in $C_{1-\alpha}(J;H)$ by using the following iteration procedures:

$$y_0(t) = T_\alpha(t)x_0 + \int_0^t T_\alpha(t-\tau)(Bu)(\tau)\mathrm{d}\tau,$$

$$z_n(t) = y_n(t) - y_0(t) - \int_0^t T_\alpha(t-\tau) f(\tau, y_n(\tau))\mathrm{d}\tau, \tag{3.3}$$

$$y_{n+1}(t) = y_n(t) - \left(\int_0^t T_\alpha(t-\tau)\tau^{1-\alpha}(\gamma(\tau, y_n(\tau))z_n)(\tau)\mathrm{d}\tau + z_n(t)\right). \tag{3.4}$$

Based on Lemma 3.1, we can conclude that the sequences $\{y_n\}$ and $\{z_n\}$ are well defined in $C_{1-\alpha}(J;H)$. Plugging (3.3) into (3.4) yields

$$\begin{aligned} y_{n+1}(t) = & \int_0^t T_\alpha(t-\tau) f(\tau, y_n(\tau))\mathrm{d}\tau + y_0(t) \\ & - \int_0^t T_\alpha(t-\tau)\tau^{1-\alpha}(\gamma(\tau, y_n(\tau))z_n)(\tau)\mathrm{d}\tau. \end{aligned} \tag{3.5}$$

In view of (3.3)-(3.5),

$$\begin{aligned} z_{n+1}(t) = & \int_0^t T_\alpha(t-\tau) f(\tau, y_n(\tau))\mathrm{d}\tau - \int_0^t T_\alpha(t-\tau)\tau^{1-\alpha}(\gamma(\tau, y_n(\tau))z_n)(\tau)\mathrm{d}\tau \\ & - \int_0^t T_\alpha(t-\tau) f\Bigg(\tau, y_n(\tau) - z_n(\tau) \\ & - \int_0^\tau T_\alpha(\tau-\theta)\theta^{1-\alpha}(\gamma(\theta, y_n(\theta))z_n)(\theta)\mathrm{d}\theta\Bigg)\mathrm{d}\tau. \end{aligned}$$

Exploiting (2.2) with $x = y_n$ and $y = -z_n$ gives

$$t^{1-\alpha}\|z_{n+1}(t)\| \le b^{1-\alpha} M\lambda \int_0^t (t-\tau)^{\alpha-1}\tau^{1-\alpha}\|z_n(\tau)\|\mathrm{d}\tau.$$

By induction, we can easily obtain

$$\|z_n\|_{C_{1-\alpha}} \le \frac{(bM\lambda\Gamma(\alpha))^n}{\Gamma(n\alpha+1)}\|z_0\|_{C_{1-\alpha}}. \tag{3.6}$$

Moreover, by virtue of the definition of $\{y_n\}$ and $\{z_n\}$, straightforward calculations tell us that

$$\|z_0\|_{C_{1-\alpha}} \leq \frac{bM}{\alpha}\left(k + M\varrho\|x_0\| + M\varrho\sqrt{\frac{b}{2\alpha-1}}\|Bu\|_{L^2}\right).$$

Thus, it follows from the convergence of $E_\alpha(bM\lambda\Gamma(\alpha)) = \sum_{n=0}^{\infty}\frac{(bM\lambda\Gamma(\alpha))^n}{\Gamma(n\alpha+1)}$ that $\lim_{n\to\infty} z_n = 0$ in $C_{1-\alpha}(J;H)$.

On the other hand, by means of (3.4) and (3.6), we get

$$\begin{aligned}
& t^{1-\alpha}\|y_{n+1}(t) - y_n(t)\| \\
\leq{}& t^{1-\alpha}\|z_n(t)\| + t^{1-\alpha}\beta M\int_0^t (t-\tau)^{\alpha-1}\|z_n\|_{C_{1-\alpha}}\mathrm{d}\tau \\
\leq{}& \left(1 + \frac{b\beta M}{\alpha}\right)\|z_n\|_{C_{1-\alpha}} \\
\leq{}& \left(1 + \frac{b\beta M}{\alpha}\right)\frac{(bM\lambda\Gamma(\alpha))^n}{\Gamma(n\alpha+1)}\|z_0\|_{C_{1-\alpha}}.
\end{aligned}$$

As such, we have

$$\begin{aligned}
\|y_n - y_m\|_{C_{1-\alpha}} &\leq \sum_{k=m}^{n-1}\|y_{k+1} - y_k\|_{C_{1-\alpha}} \\
&\leq \left(1 + \frac{b\beta M}{\alpha}\right)\sum_{k=m}^{n-1}\frac{(bM\lambda\Gamma(\alpha))^k}{\Gamma(k\alpha+1)}\|z_0\|_{C_{1-\alpha}},
\end{aligned}$$

for $n > m \geq 0$, which indicates that $\{y_n\}$ is a Cauchy sequence in $C_{1-\alpha}(J;H)$. Thus, we can assume that $y_n \to y^*$ in $C_{1-\alpha}(J;H)$ for some y^* in $C_{1-\alpha}(J;H)$, as $n \to \infty$. Hence, we arrive at $y_n(t) \to y^*(t)$, $t \in J'$. Therefore, according to the expression (3.3), condition (Hf) and the dominated convergence theorem, we can deduce that

$$y^*(t) = T_\alpha(t)x_0 + \int_0^t T_\alpha(t-\tau)(Bu)(\tau)\mathrm{d}\tau + \int_0^t T_\alpha(t-\tau)f(\tau, y^*(\tau))\mathrm{d}\tau,\ t \in J',$$

which means that y^* is a solution of (2.1).

We proceed to verify the uniqueness of solutions. For fixed $u \in L^2(J;U)$, we assume that y_1 and y_2 are two solutions depending on u to (2.1). Then one has

$$y_2(t) - y_1(t) = \int_0^t T_\alpha(t-\tau)[f(\tau, y_2(\tau)) - f(\tau, y_1(\tau))]\mathrm{d}\tau. \tag{3.7}$$

Moreover, according to Definition 2.4, the following equation

$$z(t) + \int_0^t T_\alpha(t-\tau)\tau^{1-\alpha}(\gamma(\tau, y_1(\tau))z)(\tau)\mathrm{d}\tau = y_2(t) - y_1(t) \tag{3.8}$$

admits a solution $z \in C_{1-\alpha}(J;H)$. Combining (3.7) with (3.8) yields

$$\begin{aligned}
z(t) = \int_0^t T_\alpha(t-\tau)\Big[f\Big(\tau, y_1(\tau) + z(\tau) + \int_0^\tau T_\alpha(\tau-\theta)\theta^{1-\alpha}(\gamma(\theta, y_1(\theta))z)(\theta)\mathrm{d}\theta\Big) \\
- f(\tau, y_1(\tau)) - \tau^{1-\alpha}(\gamma(\tau, y_1(\tau))z)(\tau)\Big]\mathrm{d}\tau.
\end{aligned}$$

Applying (2.2) gives

$$
\begin{aligned}
t^{1-\alpha}\|z(t)\| &\le t^{1-\alpha}\int_0^t \|(t-\tau)^{1-\alpha}T_\alpha(t-\tau)\|(t-\tau)^{\alpha-1}\lambda\tau^{1-\alpha}\|z(\tau)\|\mathrm{d}\tau \\
&\le M\lambda b^{1-\alpha}\int_0^t (t-\tau)^{\alpha-1}\tau^{1-\alpha}\|z(\tau)\|\mathrm{d}\tau.
\end{aligned}
$$

Thus, by Gronwall inequality of singular version [27], we get $t^{1-\alpha}\|z(t)\| = 0$, for any $t \in J$, which implies that $\|z\|_{C_{1-\alpha}} = 0$. Furthermore, based upon (3.8), we get

$$
\begin{aligned}
&t^{1-\alpha}\|y_2(t) - y_1(t)\| \\
&\le t^{1-\alpha}\|z(t)\| + Mt^{1-\alpha}\int_0^t \tau^{1-\alpha}\|\gamma((\tau, y_1(\tau))z)(\tau)\|(t-\tau)^{\alpha-1}\mathrm{d}\tau \\
&\le \left(1 + \frac{Mb\beta}{\alpha}\right)\|z\|_{C_{1-\alpha}}.
\end{aligned}
$$

Hence, we have $\|y_2 - y_1\|_{C_{1-\alpha}} = 0$, which indicates that $y_1 = y_2$. Therefore, we acquire the uniqueness of solutions. □

4. Approximate controllability results

In this section, we treat the approximate controllability of (2.1) by employing a technique combining integral contractor, resolvent and space decomposition. For convenience of later analysis. Let $N = \{h \in L^2(J; H) : Gh = 0\}$, where $G : L^2(J; H) \to H$ is a linear map given by

$$
Gh = \int_0^b T_\alpha(b-\tau)h(\tau)\mathrm{d}\tau,\ h \in L^2(J; H).
$$

The symbol $N^\perp$ means the orthogonal complement of N. Moreover, let $\mathscr{F} : C_{1-\alpha}(J; H) \to L^2(J; H)$ be a map defined by $(\mathscr{F}x)(\tau) = f(\tau, x(\tau))$. We also introduce a set

$$
M_0 = \{m \in C_{1-\alpha}(J; H) : m = \phi n,\ n \in N\},
$$

where $\phi : L^2(J; H) \to C_{1-\alpha}(J; H)$ is a linear map defined as

$$
(\phi z)(t) = \int_0^t T_\alpha(t-\tau)z(\tau)\mathrm{d}\tau,\ z \in L^2(J; H). \tag{4.1}
$$

Definition 4.1. By the reachable set of system (2.1), we mean the set $K_b(f) = \{x(b; u) \in H : x(b; u)$ is the state of problem (2.1) at b depending on a control $u \in L^2(J; U)\}$. Moreover, if $\overline{K_b(f)} = H$, then system (2.1) is said to be approximately controllable on J.

To investigate the approximate controllability of (2.1), we need the additional assumption:

(Hc) for any $h \in L^2(J; H)$, there exists $h^* \in \overline{R(B)}$ satisfying $Gh = Gh^*$.

In view of (Hc), we can conclude that for any $h \in N^{\perp}$, $\overline{R(B)} \cap \{h+N\} \neq \emptyset$. Thus, we can define a mapping $Q : N^{\perp} \to \overline{R(B)}$ by

$$Qh = \{h^* : h^* \in \overline{R(B)} \cap \{h+N\},\ \|h^*\|_{L^2} = \min\{\|q\|_{L^2} : q \in \overline{R(B)} \cap \{h+N\}\}\}.$$

According to Naito [19], the mapping Q is well defined, linear and bounded. For convenience, we assume that $\|Q\| \leq c$, $c > 0$.

For $h \in N^{\perp}$, we have $Qh = h + n_0 \in \overline{R(B)}$, $n_0 \in N$. Moreover, for any $z \in L^2(J;H)$, z possesses a unique decomposition $z = n_1 + h$, $n_1 \in N$, $h \in N^{\perp}$. Thus, one has $z = n_1 - n_0 + (n_0 + h) = n_1 - n_0 + Qh$. Hence, z admits a unique decomposition

$$z = n + h^*,\ h^* = Qh \in \overline{R(B)},\ n = n_1 - n_0 \in N,\ h \in N^{\perp}. \tag{4.2}$$

Furthermore, by means of [23], we have

$$\|n\|_{L^2} \leq (1+c)\|z\|_{L^2}. \tag{4.3}$$

We first deal with the approximate controllability for the linear system of (2.1).

Lemma 4.1. *Suppose that conditions (HA) and (Hc) hold. Then $\overline{K_b(0)} = H$.*

Proof. For arbitrary $\varepsilon > 0$, due to $\overline{D(A)} = H$, we can choose $\eta \in D(A)$ satisfying $\|\eta - x_0\| < \frac{b^{1-\alpha}\varepsilon}{3M}$, where x_0 is the initial value of (2.1). Let $\xi \in D(A)$. According to Lemma 2.1, one has $\xi - T_\alpha(b)\eta \in D(A)$.

On the other hand, owing to Remark 2.1 and (a) of Definition 2.3, we can pick $\bar{b} \in J'$ such that

$$\begin{aligned}
&\left\| \left(\Gamma(\alpha)\bar{b}^{1-\alpha}T_\alpha(\bar{b})\right)^2 (\xi - T_\alpha(b)\eta) - (\xi - T_\alpha(b)\eta) \right\| \\
&\leq \left\| \Gamma(\alpha)\bar{b}^{1-\alpha}T_\alpha(\bar{b}) + I \right\| \left\| \left(\Gamma(\alpha)\bar{b}^{1-\alpha}T_\alpha(\bar{b}) - I\right)(\xi - T_\alpha(b)\eta) \right\| \\
&< \frac{\varepsilon}{3}.
\end{aligned}$$

Now, we set

$$h(\tau) = \begin{cases} \frac{-(b-\tau)^{1-\alpha}\Gamma^2(\alpha)}{\bar{b}} \Big[-(b-\tau)^{1-\alpha}T_\alpha(b-\tau) \\ \qquad + 2(b-\tau)\frac{\mathrm{d}\left((b-\tau)^{1-\alpha}T_\alpha(b-\tau)\right)}{\mathrm{d}\tau} \Big] (\xi - T_\alpha(b)\eta), & t \in [b-\bar{b}, b], \\ 0, & t \in [0, b-\bar{b}]. \end{cases}$$

Then $Gh = \left(\Gamma(\alpha)\bar{b}^{1-\alpha}T_\alpha(\bar{b})\right)^2 (\xi - T_\alpha(b)\eta)$. Moreover, thanks to (HA) and Lemma 2.2, we have $h \in L^2(J;H)$. Hence, on account of (Hc), we can choose $q \in \overline{R(B)}$ obeying

$$Gq = Gh = \left(\Gamma(\alpha)\bar{b}^{1-\alpha}T_\alpha(\bar{b})\right)^2 (\xi - T_\alpha(b)\eta).$$

In addition, by means of $q \in \overline{R(B)}$, for the above-mentioned $\varepsilon > 0$, one can find $u \in L^2(J;U)$ satisfying

$$\|Bu - q\|_{L^2} < \sqrt{\frac{2\alpha - 1}{b^{2\alpha-1}}}\frac{\varepsilon}{3M}.$$

Thus, we get

$$\begin{aligned}
&\|Gq - GBu\| \\
\leq &\int_0^b \|(b-\tau)^{1-\alpha}T_\alpha(b-\tau)\|(b-\tau)^{\alpha-1}\|q(\tau)-(Bu)(\tau)\|\mathrm{d}\tau \\
\leq &M\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\|Bu-q\|_{L^2} \\
< &\frac{\varepsilon}{3}.
\end{aligned}$$

Hence, we derive

$$\begin{aligned}
&\|\xi - (GBu + T_\alpha(b)x_0)\| \\
\leq &\|\xi - T_\alpha(b)\eta - Gq\| + \|Gq - GBu\| + \|T_\alpha(b)\eta - T_\alpha(b)x_0\| \\
< &\frac{\varepsilon}{3} + \frac{\varepsilon}{3} + b^{\alpha-1}\|(b^{1-\alpha}T_\alpha(b))(x_0-\eta)\|. \\
\leq &\frac{2\varepsilon}{3} + b^{\alpha-1}M\|x_0-\eta\| \\
< &\varepsilon,
\end{aligned}$$

which implies that $D(A) \subset \overline{K_b(0)}$. Therefore, $\overline{K_b(0)} = H$. □

We then study the compactness of the map ϕ given by (4.1).

Lemma 4.2. *Under condition (HA), the map $\phi : L^2(J;H) \to C_{1-\alpha}(J;H)$ is compact.*

Proof. Assume that $\{z_n\}_{n\geq 1} \subseteq L^2(J;H)$ with $\|z_n\|_{L^2} \leq r$ for some $r > 0$. We are reduced to verifying the relative compactness of $\{\phi z_n\}_{n\geq 1}$ in $C_{1-\alpha}(J;H)$. Put $\widetilde{z_n}(\cdot) = (\cdot)^{1-\alpha}(\phi z_n)(\cdot)$. Then, we need only check the relative compactness of $\{\widetilde{z_n}\}_{n\geq 1}$ in $C(J;H)$.

We begin by examining the uniform boundedness of $\{\widetilde{z_n}\}_{n\geq 1}$ in $C(J;H)$. For any $t \in J$ and positive integer n, one has

$$\begin{aligned}
\|\widetilde{z_n}(t)\| &\leq Mb^{1-\alpha}\int_0^t (t-\tau)^{\alpha-1}\|z_n(\tau)\|\mathrm{d}\tau \\
&\leq Mr\sqrt{\frac{b}{2\alpha-1}},
\end{aligned}$$

which ensures the uniform boundedness of $\{\widetilde{z_n}\}_{n\geq 1}$.

We then analyze the equicontinuity of $\{\widetilde{z_n}\}_{n\geq 1}$ in $C(J;H)$. Let $t_1,\ t_2 \in J$ and $t_1 < t_2$, one has

$$\begin{aligned}
&\|\widetilde{z_n}(t_2) - \widetilde{z_n}(t_1)\| \\
\leq &\|t_2^{1-\alpha}(\phi z_n)(t_2) - t_1^{1-\alpha}(\phi z_n)(t_1)\| \\
\leq &(t_2^{1-\alpha} - t_1^{1-\alpha})\|(\phi z_n)(t_2)\| + b^{1-\alpha}\|(\phi z_n)(t_2) - (\phi z_n)(t_1)\| \\
\leq &(t_2^{1-\alpha} - t_1^{1-\alpha})Mr\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}} + b^{1-\alpha}\|(\phi z_n)(t_2) - (\phi z_n)(t_1)\|.
\end{aligned}$$

Thus, similar to the proof of Lemma 3.1, we can easily obtain the equicontinuity of $\{\widetilde{z_n}\}_{n\geq 1}$ in $C(J;H)$.

Finally, we address the relative compactness of $\{\widetilde{z_n}(t)\}_{n\geq 1}$ in H for $t\in J$. Since the compactness of $\{\widetilde{z_n}(0)\}_{n\geq 1}=\{0\}$ is obvious, it suffices to prove the case of $t\in J'$. For arbitrary $\varepsilon\in(0,t)$, $t\in J'$, in view of the compactness of $\varepsilon^{1-\alpha}T_\alpha(\varepsilon)$, it follows that $\{\widetilde{z_n}^{\varepsilon}(t)\}_{n\geq 1}$ is relatively compact, where

$$\widetilde{z_n}^{\varepsilon}(t)=t^{1-\alpha}(\Gamma(\alpha)\varepsilon^{1-\alpha}T_\alpha(\varepsilon))\int_0^{t-\varepsilon}T_\alpha(t-\tau-\varepsilon)z_n(\tau)\mathrm{d}\tau.$$

Now, for simplicity of notation, we set

$$\Psi_t(\varepsilon,\tau)=((t-\tau-\varepsilon)^{1-\alpha}T_\alpha(t-\tau-\varepsilon))(\Gamma(\alpha)\varepsilon^{1-\alpha}T_\alpha(\varepsilon))$$

and

$$\Phi_t(\varepsilon,\tau)=(t-\tau)^{1-\alpha}T_\alpha(t-\tau)-\Psi_t(\varepsilon,\tau).$$

Let $\delta\in(\varepsilon,t),\ t\in J'$. Then

$$\begin{aligned}
&\|\widetilde{z_n}(t)-\widetilde{z_n}^{\varepsilon}(t)\|\\
\leq\ & b^{1-\alpha}\int_0^{t-\delta}\|\Phi_t(\varepsilon,\tau)\|(t-\tau)^{\alpha-1}\|z_n(\tau)\|\mathrm{d}\tau\\
&+b^{1-\alpha}\int_{t-\delta}^{t-\varepsilon}\|\Phi_t(\varepsilon,\tau)\|(t-\tau)^{\alpha-1}\|z_n(\tau)\|\mathrm{d}\tau\\
&+b^{1-\alpha}\int_0^{t-\varepsilon}\|\Psi_t(\varepsilon,\tau)\|((t-\tau-\varepsilon)^{\alpha-1}-(t-\tau)^{\alpha-1})\|z_n(\tau)\|\mathrm{d}\tau\\
&+b^{1-\alpha}\int_{t-\varepsilon}^{t}\|(t-\tau)^{1-\alpha}T_\alpha(t-\tau)\|(t-\tau)^{\alpha-1}\|z_n(\tau)\|\mathrm{d}\tau\\
\leq\ & b^{1-\alpha}\int_0^{t-\delta}\|\Phi_t(\varepsilon,\tau)\|(t-\tau)^{\alpha-1}\|z_n(\tau)\|\mathrm{d}\tau\\
&+(M+\Gamma(\alpha)M^2)b^{1-\alpha}\int_{t-\delta}^{t-\varepsilon}(t-\tau)^{\alpha-1}\|z_n(\tau)\|\mathrm{d}\tau\\
&+\Gamma(\alpha)M^2b^{1-\alpha}r\left(\int_0^{t-\varepsilon}((t-\tau-\varepsilon)^{\alpha-1}-(t-\tau)^{\alpha-1})^2\mathrm{d}\tau\right)^{\frac{1}{2}}\\
&+Mb^{1-\alpha}\int_{t-\varepsilon}^{t}(t-\tau)^{\alpha-1}\|z_n(\tau)\|\mathrm{d}\tau\\
:=\ & I_1+I_2+I_3+I_4.
\end{aligned}$$

Due to Lemma 2.2, we have

$$\lim_{\varepsilon\to 0}\|\Phi_t(\varepsilon,\tau)\|=0,\ \tau\in[0,t-\delta].$$

Combing this and the dominated convergence theorem leads to

$$\lim_{\varepsilon\to 0^+}I_1=0.$$

Moreover, based on Lemma 2.3, the arbitrariness of δ and the absolute continuity of integration of $(t-\cdot)^{\alpha-1}\|z_n(\cdot)\|$, we have

$$\lim_{\varepsilon\to 0^+}(I_2+I_3+I_4)=0.$$

Hence,

$$\lim_{\varepsilon\to 0^+} \|\widetilde{z_n}(t) - \widetilde{z_n}^{\varepsilon}(t)\| = 0.$$

Thus, for every $t \in J'$, we derive the relative compactness of $\{\widetilde{z_n}(t)\}_{n\geq 1}$.

Therefore, the compactness of Φ is derived by the Arzela-Ascoli theorem. □

For convenience of subsequent analysis, with the help of Theorem 3.1, we introduce the solution map $W : L^2(J;H) \to C_{1-\alpha}(J;H)$ defined by $(Wz)(\cdot) = y(\cdot;z)$, $z \in L^2(J;H)$, where $y(\cdot;z)$ is the unique solution of the equation:

$$y(t) = T_\alpha(t)x_0 + \int_0^t T_\alpha(t-\tau)(z(\tau) + f(\tau, y(\tau)))\mathrm{d}\tau,\ t \in J'.$$

Lemma 4.3. *Let conditions* (HA) *and* (Hf) *hold. Then for* $z_1,\ z_2 \in L^2(J;H)$*, we have*

$$\|Wz_1 - Wz_2\|_{C_{1-\alpha}} \leq \left(1 + \frac{Mb\beta}{\alpha}\right) M\sqrt{\frac{b}{2\alpha-1}} E_\alpha(M\lambda b\Gamma(\alpha))\|z_1 - z_2\|_{L^2}.$$

Proof. The proof of this lemma is similar to that of the uniqueness of solutions in Theorem 3.1, so we omit it here. □

With the aid of above lemmas, we are now in a position to exhibit our main result.

Theorem 4.1. *Let conditions* (HA), (Hf) *and* (Hc) *be fulfilled. Then system* (2.1) *is approximately controllable if*

$$\frac{Mb\varrho(1+c)}{\sqrt{2\alpha-1}} < 1. \tag{4.4}$$

Proof. Owing to Lemma 4.1, it suffices to check that $K_b(0) \subset \overline{K_b(f)}$.

For $u \in L^2(J;U)$, let $x^0 \in C_{1-\alpha}(J;H)$ be a solution to the linear system of (2.1) given by

$$x^0(t) = T_\alpha(t)x_0 + \int_0^t T_\alpha(t-\tau)(Bu)(\tau)\mathrm{d}\tau.$$

We can define a map $f_{x^0} : \overline{M_0} \to M_0$ by $f_{x^0}(m) = \phi n$, where n is chosen to satisfy the following unique decomposition (in the sense of expression (4.2)):

$$\mathscr{F}(x^0 + m) = n + q,\ n \in N,\ q \in \overline{R(B)}. \tag{4.5}$$

Firstly, Lemma 4.2 yields the compactness of the map f_{x^0}. Moreover, by virtue of (4.3) and (4.5), we have

$$\begin{aligned}
& t^{2-2\alpha}\|f_{x^0}(m)(t)\|^2 \\
\leq\ & b^{2-2\alpha}\left\|\int_0^t T_\alpha(t-\tau)n(\tau)\mathrm{d}\tau\right\|^2 \\
\leq\ & M^2 b^{2-2\alpha}\left(\int_0^t (t-\tau)^{\alpha-1}\|n(\tau)\|\mathrm{d}\tau\right)^2 \\
\leq\ & \frac{M^2 b}{2\alpha-1}\|n\|_{L^2}^2
\end{aligned}$$

$$\begin{aligned}&\le \frac{M^2(1+c)^2b}{2\alpha-1}\|\mathscr{F}(x^0+m)\|_{L^2}^2\\&\le \frac{M^2(1+c)^2b}{2\alpha-1}\int_0^b \|f(t,x^0(t)+m(t))\|^2\mathrm{d}t\\&\le \frac{M^2(1+c)^2b}{2\alpha-1}\int_0^b (k+t^{1-\alpha}\varrho\|x^0(t)\|+t^{1-\alpha}\varrho\|m(t)\|)^2\mathrm{d}t\\&\le \frac{M^2(1+c)^2b^2}{2\alpha-1}(k+\varrho\|x^0\|_{C_{1-\alpha}}+\varrho\|m\|_{C_{1-\alpha}})^2,\end{aligned}$$

which implies that

$$\limsup_{\|m\|_{C_{1-\alpha}}\to\infty}\frac{\|f_{x^0}(m)\|_{C_{1-\alpha}}}{\|m\|_{C_{1-\alpha}}}\le \frac{Mb\varrho(1+c)}{\sqrt{2\alpha-1}}<1.$$

Thus, we can find $r>0$ such that $f_{x^0}B_r\subset B_r$, where

$$B_r=\{m\in M_0:\|m\|_{C_{1-\alpha}}\le r\}.$$

Hence, thanks to Schauder's fixed point theorem, the operator f_{x^0} possesses a fixed point m^*, i.e.,

$$m^*=f_{x^0}(m^*)=\phi n. \tag{4.6}$$

Therefore, by means of (4.5) and (4.6), we obtain

$$x^0+\phi\mathscr{F}(x^0+m^*)=x^0+\phi n+\phi q=x^0+m^*+\phi q.$$

Let $y=x^0+m^*$. Then one has

$$x^0+\phi\mathscr{F}(y)=y+\phi q,$$

that is,

$$y(t)=T_\alpha(t)x_0+\int_0^t T_\alpha(t-s)((Bu)(\tau)-q(\tau))\mathrm{d}\tau+\int_0^t T_\alpha(t-\tau)f(\tau,y(\tau))\mathrm{d}\tau.$$

According to $m^*=\phi n$ and $y=x^0+m^*$, it follows that

$$x^0(b)=y(b)=(W(Bu-q))(b).$$

In addition, in view of $q\in\overline{R(B)}$, we can choose a sequence $\{u_n\}_{n\ge1}\in L^2(J;U)$ obeying $Bu_n\to q$ in $L^2(J;H)$ as $n\to\infty$. As such, it follows from Lemma 4.3 that $W(Bu-Bu_n)\to W(Bu-q)$ in $C_{1-\alpha}(J;H)$ as $n\to\infty$. Thus, we obtain $(W(Bu-Bu_n))(b)\to(W(Bu-q))(b)=x^0(b)\in K_b(0)$ as $n\to\infty$. Additionally, we can observe that $(W(Bu-Bu_n))(b)=x(b;u-u_n)\in K_b(f)$. Hence, we have $K_b(0)\subseteq\overline{K_b(f)}$. Therefore, system (2.1) is approximately controllable. □

Remark 4.1. By utilizing the technique combining integral contractor, resolvent and space decomposition, we have displayed the approximate controllability result for system (2.1). This method can enable us to treat the approximate controllability problems of integer-order evolution systems and fractional evolution equations with Riemann-Liouville type derivatives or Caputo type. However, the question whether the results obtained in this article hold for evolution equations with Caputo-Fabrizio type [6] or Atangana-Baleanu type [3] fractional derivative is at present far from being solved, since there is no corresponding resolvent theory.

Owing to Remark 2.3 and Theorem 4.1, we can obtain the following result:

Corollary 4.1. *Let assumptions (HA), $(Hf)(ii)$ and (Hc) hold. Moreover, we suppose that*
(i) *for $s>0$ and $x,\ y\in H$, $\|f(s,x)-f(s,y)\|\le \lambda s^{1-\alpha}\|x-y\|$ with $\lambda>0$;*
(ii) $\sup\limits_{s\in J}\|f(s,0)\|<\infty$;
(iii) $\frac{Mb\lambda(1+c)}{\sqrt{2\alpha-1}}<1$.
Then problem (2.1) is approximately controllable.

5. An application

Let us explore the following system with a Riemann-Liouville type fractional derivative:

$$\begin{cases} D^\alpha z(s,x)=\frac{\partial^2}{\partial x^2}z(s,x)+f(s,z(s,x))+(Bu)(s,x), x\in(0,1), s\in(0,1],\\ z(s,0)=z(s,1)=0,\\ \lim\limits_{s\to 0^+}\Gamma(\alpha)s^{1-\alpha}z(s,x)=g(x)=\sum\limits_{k=1}^{\infty}c_k\sin k\pi x, \end{cases} \tag{5.1}$$

where $\frac{1}{2}<\alpha<1$.

Let $H=U=L^2(0,1)$, $e_k(x)=\sqrt{2}\sin(k\pi x)$, $k=1,2,\cdots$ and $A=\frac{\partial^2}{\partial x^2}$ with domain

$$D(A)=\{\xi\in H:\xi',\ \xi''\in H \text{ and } \xi(0)=\xi(1)=0\}.$$

Then A generates an α-order resolvent $\{T_\alpha(s)\}_{s>0}$ (see [15]):

$$T_\alpha(s)g(x)=\sum_{n=1}^{\infty}s^{\alpha-1}E_{\alpha,\alpha}(-n^2\pi^2s^\alpha)c_n\sin n\pi x.$$

Meanwhile, by means of [20], A also generates an analytic compact semigroup $\{T(s)\}_{s\ge 0}$:

$$T(s)v=\sum_{n=1}^{\infty}e^{-n^2\pi^2 s}\langle v,e_n\rangle e_n,\ v\in H.$$

Thus, one can easily derive $\|T(s)\|\le 1$ and

$$T(s)g(x)=\sum_{n=1}^{\infty}e^{-n^2\pi^2 s}c_n\sin n\pi x.$$

Moreover, due to [20], one has $T'(s)=AT(s)$ and $\|AT(s)\|\le\frac{C_1}{s}$, $0<s\le 1$, where C_1 is a constant. Hence, by utilizing probability densities function and Laplace transformations [16], one can easily infer that for any $g\in H$,

$$s^{1-\alpha}T_\alpha(s)g(x)=\alpha\int_0^\infty\tau\xi_\alpha(\tau)T(s^\alpha\tau)g(x)\mathrm{d}\tau,$$

Therefore,

$$s^{1-\alpha}T_\alpha(s) = \alpha \int_0^\infty \tau\xi_\alpha(\tau)T(s^\alpha\tau)\mathrm{d}\tau,$$

where

$$\xi_\alpha(\tau) = \frac{1}{\alpha}\tau^{-1-\frac{1}{\alpha}}\varpi_\alpha(\tau^{-\frac{1}{\alpha}}),$$

and

$$\varpi_\alpha(\tau) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\tau^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha),\ \tau \in (0,\infty).$$

Thanks to [30], we derive the compactness of $\{s^{1-\alpha}T_\alpha(s)\}_{s>0}$. Furthermore, on account of $T'(s) = AT(s)$, $\|AT(s)\| \le \frac{C_1}{s}$, $0 < s \le 1$ and dominated convergence theorem, it is not difficult to check that

$$\frac{\mathrm{d}(s^{1-\alpha}T_\alpha(s))}{\mathrm{d}s} = \alpha^2 s^{\alpha-1}\int_0^\infty \tau^2\xi_\alpha(\tau)AT(s^\alpha\tau)\mathrm{d}\tau,\ 0 < s \le 1$$

and

$$\left\|\frac{\mathrm{d}(s^{1-\alpha}T_\alpha(s))}{\mathrm{d}s}\right\| \le \frac{\alpha C_1}{s\Gamma(\alpha)},\ 0 < s \le 1.$$

Additionally, it is easily seen that $\|s^{1-\alpha}T_\alpha(s)\| \le \frac{1}{\Gamma(\alpha)}$. Therefore, (HA) is fulfilled.

Now, take $U = \{v : v = \sum_{k=2}^{\infty} v_k e_k$ with $\sum_{k=2}^{\infty} v_k^2 < \infty\}$ and define a mapping $\widetilde{B} : U \to H$ by

$$\widetilde{B}v = 2v_2e_1 + \sum_{k=2}^{\infty} v_k e_k,\ v \in U.$$

Moreover, we also define a mapping $B : L^2(J;U) \to L^2(J;H)$ by $(Bu)(s) = \widetilde{B}u(s)$, $u \in L^2(J;U)$. Then, in view of Naito [19] and Zhou [29], (Hc) holds and B is a bounded linear operator.

In addition, we assume that condition $H(f)$ holds and $\frac{\varrho(1+c)}{\Gamma(\alpha)\sqrt{2\alpha-1}} < 1$. Then by virtue of Theorem 4.1, system (5.1) is approximately controllable.

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