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UNIQUENESS OF SOLUTIONS FOR AN INTEGRAL BOUNDARY VALUE PROBLEM WITH FRACTIONAL Q-DIFFERENCES*

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Abstract This paper deals with uniqueness of solutions for integral boundary value problem $\begin{cases} (D_q^\alpha u)(t) + f(t,u(t)) = 0, & t \in (0,1), \\ u(0) = D_q u(0) = 0, & u(1) = \lambda \int_0^1 u(s) \mathrm{d}_q s, \end{cases}$ where $\alpha \in (2,3]$, $\lambda \in (0, [\alpha]_q)$, D_q^α denotes the q-fractional differential operator of order α. By using the iterative method and one new fixed point theorem, we obtain that there exist a unique nontrivial solution and a unique positive solution.

Keywords Fractional q-difference equation, integral boundary value condition, the first eigenvalue, uniqueness of solutions.

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1. Introduction

Recently, great attention has been devoted to the existence and multiplicity of positive solutions for fractional difference boundary value problems by applying the cone fixed point theory, see [1,3,4,10,11,13]. And we notice that Cabada et al. [3,4] obtained the existence of at least one positive solution with integral boundary value conditions and the authors [10,11] dealt with the existence of positive solutions to fractional q-difference equations. At the same time, the existence and uniqueness of solutions also has been studied extensively, see, for example, [2,5,8]. Especially, under the condition that the Lipschitz constant of the nonlinear term is related to the first eigenvalues corresponding to the relevant u_0-positive operator, Cui [2] obtained the uniqueness of solutions by using the iterative methods.

Inspired by the papers mentioned above, we are interested in studying the uniqueness results for the integral boundary value problem (BVP)

$$\begin{cases} (D_q^\alpha u)(t) + f(t,u(t)) = 0, & t \in (0,1), \\ u(0) = D_q u(0) = 0, & u(1) = \lambda \int_0^1 u(s) \mathrm{d}_q s, \end{cases} \tag{1.1}$$

where $\alpha \in (2,3]$, $\lambda > 0$, D_q^α denotes the q-fractional differential operator of order α. By using the iterative method which resembles [2] and a new fixed point theorem in

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a normal cone, we obtain that there exist a unique nontrivial solution and a unique positive solution.

2. Preliminary and Green's function

We first present some formulas and definitions on fractional q-derivative and fractional q-integral that will be used in what follows.

Let $q \in (0,1)$, $\alpha \in \mathbb{R}$ and ${}_tD_q$ denotes the derivative with respect to variable t. Then

$$\begin{aligned} &{}_tD_q(t-s)^{(\alpha)} = [\alpha]_q(t-s)^{(\alpha-1)};\\ &\left({}_tD_q \int_0^t f(t,s)\mathrm{d}_q s\right)(t) = \int_0^t {}_tD_q f(t,s)\mathrm{d}_q s + f(qt,t). \end{aligned}$$

The q-integral of a function f defined on $[0,b]$ is defined by

$$(I_q f)(t) = \int_0^t f(s)\mathrm{d}_q s = t(1-q)\sum_{k=0}^{\infty} f(tq^k)q^k, \quad t \in [0,b].$$

Definition 2.1 ([7])**.** Let f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is given by

$$(I_q^\alpha f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} f(s)\mathrm{d}_q s, \quad \alpha > 0,\ t \in [0,1].$$

Lemma 2.1 ([7])**.** *Let n be a positive integer. Then*

$$(I_q^\alpha D_q^n f)(t) = (D_q^n I_q^\alpha f)(t) - \sum_{k=0}^{n-1} \frac{t^{\alpha-n+k}}{\Gamma_q(\alpha+k-n+1)}(D_q^k f)(0), \quad \alpha > 0.$$

Let E be a Banach space and $P \subset E$ be a cone.

Definition 2.2 ([9])**.** A bounded linear operator $T : E \to E$ is u_0-positive on cone P if there exists $u_0 \in P\backslash\{\theta\}$ such that for each $u \in P\backslash\{\theta\}$ there exist $n \in \mathbb{N}$ and positive constants $\alpha(u)$, $\beta(u)$ such that

$$\alpha(u)u_0 \leqslant T^n u \leqslant \beta(u)u_0.$$

Recall that φ^* is called a positive eigenfunction of a linear operator T if $\varphi^* \in P\backslash\{\theta\}$ and there exists $\lambda_0 > 0$ such that $\lambda_0 T\varphi^* = \varphi^*$.

Lemma 2.2 ([9])**.** *Let $T : E \to E$ be linear completely continuous and $T(P) \subset P$. If there exist $\psi \in E\backslash(-P)$ and a constant $c > 0$ such that $cT\psi \geqslant \psi$, then the spectral radius $r(T) \neq 0$ and T has a positive eigenfunction φ corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$, i.e. $\varphi = \lambda_1 T\varphi$.*

The following lemma will be used to prove one of our results.

Lemma 2.3 ([6])**.** *Let $P \subset E$ be a normal cone, $A : P \to P$ be a completely continuous and decreasing operator satisfying the following two conditions:*

(i) $A\theta > \theta, A^2\theta \geqslant \varepsilon_0 A\theta$, where $\varepsilon_0 \in (0,1)$;

(ii) For each $0 < u \leqslant A\theta$ and $k \in (0,1)$, there exists a constant $\eta > 0$ such that

$$A(ku) \leqslant [k(1+\eta)]^{-1} Au.$$

Then A has a single fixed point $u^* \in P$.

Lemma 2.4. *Let $\alpha \in (2,3]$, $\lambda \neq [\alpha]_q$ and $h \in C[0,1]$. Then the boundary value problem (BVP)*

$$\begin{cases} (D_q^\alpha u)(t) + h(t) = 0, \quad t \in (0,1), \\ u(0) = D_q u(0) = 0, \quad u(1) = \lambda \int_0^1 u(s) d_q s, \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^1 G(t, qs) h(s) d_q s, \quad t \in [0,1],$$

where Green's function is represented in the form

$$G(t,qs) = \begin{cases} \dfrac{t^{\alpha-1}(1-qs)^{(\alpha-1)}([\alpha]_q - \lambda + \lambda s q^\alpha) - ([\alpha]_q - \lambda)(t-qs)^{(\alpha-1)}}{([\alpha]_q - \lambda)\Gamma_q(\alpha)}, & 0 \leqslant qs \leqslant t \leqslant 1; \\ \dfrac{t^{\alpha-1}(1-qs)^{(\alpha-1)}([\alpha]_q - \lambda + \lambda s q^\alpha)}{([\alpha]_q - \lambda)\Gamma_q(\alpha)}, & 0 \leqslant t \leqslant qs \leqslant 1. \end{cases}$$

Proof. From **le2.1**, the equation $(D_q^\alpha u)(t) = -h(t)$ can be change into

$$u(t) = -\frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s) \mathrm{d}_q s + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}, \tag{2.2}$$

where c_1, c_2, c_3 are constants. Condition $u(0) = 0$ shows that $c_3 = 0$. Compute the q-derivative both sides of (2.2),

$$(D_q u)(t) = -\frac{1}{\Gamma_q(\alpha)} \int_0^t [\alpha-1]_q (t-qs)^{(\alpha-2)} h(s) \mathrm{d}_q s + c_1 [\alpha-1]_q t^{\alpha-2} + c_2 [\alpha-2]_q t^{\alpha-3}.$$

Condition $D_q u(0) = 0$ implies $c_2 = 0$. Using condition $u(1) = \lambda \int_0^1 u(s) \mathrm{d}_q s$, we have from (2.2) that

$$c_1 = \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} h(s) \mathrm{d}_q s + \lambda \int_0^1 u(s) \mathrm{d}_q s.$$

Thus, $u(t)$ becomes

$$\begin{aligned} u(t) = &-\frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s) \mathrm{d}_q s + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} h(s) \mathrm{d}_q s \\ &+ \lambda t^{\alpha-1} \int_0^1 u(s) \mathrm{d}_q s. \end{aligned} \tag{2.3}$$

Set $C = \int_0^1 u(s)\mathrm{d}_q s$. Applying q-integral to (2.3) on $[0,1]$, we get

$$\begin{aligned} C = &-\frac{1}{\Gamma_q(\alpha)}\int_0^1 \mathrm{d}_q t\int_0^t (t-qs)^{(\alpha-1)}h(s)\mathrm{d}_q s \\ &+\frac{1}{\Gamma_q(\alpha)}\int_0^1 t^{\alpha-1}\mathrm{d}_q t\int_0^1 (1-qs)^{(\alpha-1)}h(s)\mathrm{d}_q s + \lambda C\int_0^1 t^{\alpha-1}\mathrm{d}_q t. \end{aligned}$$

Noticing that $\int_0^1 t^{\alpha-1}\mathrm{d}_q t = \frac{1}{[\alpha]_q}$ and

$$\int_0^1 \mathrm{d}_q t\int_0^t (t-qs)^{(\alpha-1)}h(s)\mathrm{d}_q s = \frac{1}{[\alpha]_q}\int_0^1 (1-qs)^{(\alpha)}h(s)\mathrm{d}_q s,$$

thus

$$\begin{aligned} C = &-\frac{1}{([\alpha]_q-\lambda)\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha)}h(s)\mathrm{d}_q s \\ &+\frac{1}{([\alpha]_q-\lambda)\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-1)}h(s)\mathrm{d}_q s. \end{aligned}$$

In view of $(a-b)^{(\alpha)} = (a-bq^{\alpha-1})(a-b)^{(\alpha-1)}$, replacing the value of C in (2.3), we deduce that

$$\begin{aligned} u(t) = &-\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}h(s)\mathrm{d}_q s + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-1)}h(s)\mathrm{d}_q s \\ &-\frac{\lambda t^{\alpha-1}}{([\alpha]_q-\lambda)\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha)}h(s)\mathrm{d}_q s \\ &+\frac{\lambda t^{\alpha-1}}{([\alpha]_q-\lambda)\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-1)}h(s)\mathrm{d}_q s \\ = &-\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}h(s)\mathrm{d}_q s \\ &+\frac{t^{\alpha-1}}{([\alpha]_q-\lambda)\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-1)}([\alpha]_q-\lambda+\lambda sq^{\alpha})h(s)\mathrm{d}_q s \\ = &\int_0^1 G(t,qs)h(s)\mathrm{d}_q s. \end{aligned}$$

□

Lemma 2.5. *Let $\alpha \in (2,3]$ and $\lambda \in (0,[\alpha]_q)$. The function G given above satisfies*

(i) $G(0,qs) = G(t,1) = 0, \;\; t,s \in [0,1]$;

(ii) $G(t,qs) > 0, \;\; t,s \in (0,1)$;

(iii) $t^{\alpha-1}(1-qs)^{(\alpha-1)}\lambda sq^{\alpha} \leqslant ([\alpha]_q-\lambda)\Gamma_q(\alpha)G(t,qs) \leqslant [\alpha]_q t^{\alpha-1}(1-qs)^{(\alpha-1)}$;

(iv) $G : [0,1]\times[0,1] \to [0,+\infty)$ *is continuous.*

Proof. Results (i) and (iv) are obvious. Now we prove properties (ii) and (iii). In the case when $0 < qs \leqslant t < 1$, the function

$$\begin{aligned}g_1(t,s) =& t^{\alpha-1}(1-qs)^{(\alpha-1)}([\alpha]_q - \lambda + \lambda s q^\alpha) - ([\alpha]_q - \lambda)(t-qs)^{(\alpha-1)}\\ =& t^{\alpha-1}(1-qs)^{(\alpha-1)}([\alpha]_q - \lambda + \lambda s q^\alpha) - t^{\alpha-1}([\alpha]_q - \lambda)(1-\frac{qs}{t})^{(\alpha-1)}\\ \geqslant& t^{\alpha-1}(1-qs)^{(\alpha-1)}([\alpha]_q - \lambda + \lambda s q^\alpha) - t^{\alpha-1}([\alpha]_q - \lambda)(1-qs)^{(\alpha-1)}\\ =& \lambda s q^\alpha t^{\alpha-1}(1-qs)^{(\alpha-1)} > 0.\end{aligned}$$

In the case when $0 < t \leqslant qs < 1$, notice that $[\alpha] - \lambda > 0$, then the function

$$g_2(t,s) = t^{\alpha-1}(1-qs)^{(\alpha-1)}([\alpha]_q - \lambda + \lambda s q^\alpha) \geqslant \lambda s q^\alpha t^{\alpha-1}(1-qs)^{(\alpha-1)} > 0$$

and

$$g_2(t,s) = t^{\alpha-1}(1-qs)^{(\alpha-1)}([\alpha]_q - \lambda(1-sq^\alpha)) \leqslant [\alpha] t^{\alpha-1}(1-qs)^{(\alpha-1)}.$$

Furthermore, it is clear that $g_1(t,s) \leqslant g_2(t,s)$ for all $t,s \in [0,1]$. Thus, we finish the proof. □

From now on, we let $E = C[0,1]$ be endowed with the norm $\|u\| = \sup_{t\in[0,1]} |u(t)|$ and cone $P = \{u \in E : u(t) \geqslant 0,\ t \in [0,1]\}$, it is well known that P is a normal cone.

Define operators $T,\ A : E \to E$ respectively by

$$Tu(t) = \int_0^1 G(t,qs)u(s)\mathrm{d}_q s, \quad t \in [0,1], \tag{2.4}$$

$$Au(t) = \int_0^1 G(t,qs)f(s,u(s))\mathrm{d}_q s, \quad t \in [0,1]. \tag{2.5}$$

Obviously, $T : E \to E(P \to P)$ is linear completely continuous.

Lemma 2.6. *T is a u_0-positive operator with $u_0 = t^{\alpha-1}$.*

Proof. For each $u \in P\backslash\{\theta\}$, by (iii) of **le2.5**, we have

$$Tu(t) = \int_0^1 G(t,qs)u(s)\mathrm{d}_q s \leqslant \frac{[\alpha]_q t^{\alpha-1}}{([\alpha]_q - \lambda)\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} u(s)\mathrm{d}_q s$$

and

$$Tu(t) = \int_0^1 G(t,qs)u(s)\mathrm{d}_q s \geqslant \frac{\lambda q^\alpha t^{\alpha-1}}{([\alpha]_q - \lambda)\Gamma_q(\alpha)} \int_0^1 s(1-qs)^{(\alpha-1)} u(s)\mathrm{d}_q s.$$

The two inequalities above show that T is a u_0-positive operator with $u_0 = t^{\alpha-1}$. □

Remark 2.1. Taking $\psi(t) = t^{\alpha-1}$, by **le2.6**, one may choose a number

$$c = \frac{([\alpha]_q - \lambda)\Gamma_q(\alpha)}{\lambda q^\alpha} \left(\int_0^1 s^\alpha (1-qs)^{(\alpha-1)} \mathrm{d}_q s\right)^{-1} > 0$$

such that $cT\psi \geqslant \psi$. According to **le2.2**, we see that the spectral radius $r(T) \neq 0$ and T has a positive eigenfunction corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$.

Lemma 2.7. *Assume φ^* be the positive eigenfunction corresponding to λ_1, namely, $\lambda_1 T\varphi^* = \varphi^*$. Then T is a φ^*-positive operator.*

3. Main results

In this section, based on the iterative method and a fixed point **le2.3**, we give the existence of a unique nontrivial solution and a unique positive solution to BVP(1.1), respectively.

Theorem 3.1. *Suppose that $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$ and $f(t,0) \not\equiv 0$ for all $t \in [0,1]$. If there exists $K \in [0,1)$ such that*

$$|f(t,x_1) - f(t,x_2)| \leqslant K\lambda_1 |x_1 - x_2|, \quad t \in [0,1],\ x_1, x_2 \in \mathbb{R}, \tag{3.1}$$

where λ_1 is the first eigenvalue of T. Then BVP(1.1) has a unique nontrivial solution u^* in E, and for each $u_0 \in E$, the iterative sequence $u_n = Au_{n-1}(n = 1, 2, \ldots)$ converges to u^*.

Proof. One important fact is that $A : E \to E$ is completely continuous and a nontrivial solution of BVP(1.1) in E is equivalent to a nonzero fixed point of A in E.

First, we prove existence. Following the approach in [1], we need to construct an iterative sequence. For any given $u_0 \in E$, set $u_n = Au_{n-1}(n = 1, 2, \cdots,)$. By **le2.6** and **le2.7**, there exists $a(|u_1 - u_0|) > 0$ such that

$$T(|u_1 - u_0|)(t) \leqslant a\varphi^*(t), \quad t \in [0,1].$$

For any $n \in \mathbb{N}$ and $t \in [0,1]$, it follows from (2.4), (2.5) and (3.1) that

$$\begin{aligned}
|u_{n+1}(t) - u_n(t)| &= \left| \int_0^1 G(t,qs) f(s, u_n(s)) \mathrm{d}_q s - \int_0^1 G(t,qs) f(s, u_{n-1}(s)) \mathrm{d}_q s \right| \\
&\leqslant \int_0^1 G(t,qs) |f(s,u_n(s)) - f(s,u_{n-1}(s))| \mathrm{d}_q s \\
&\leqslant K\lambda_1 T(|u_n - u_{n-1}|)(t) \\
&\leqslant \cdots \leqslant K^n \lambda_1^n T^n(|u_1 - u_0|)(t) \\
&\leqslant K^n \lambda_1^n T^{n-1}(a\varphi^*(t)) \\
&= K^n a \lambda_1 \varphi^*(t).
\end{aligned}$$

So for any given $n, p \in \mathbb{N}$, we have

$$\begin{aligned}
&|u_{n+p}(t) - u_n(t)| \\
\leqslant &|u_{n+p}(t) - u_{n+p-1}(t)| + |u_{n+p-1}(t) - u_{n+p-2}(t)| + \cdots + |u_{n+1}(t) - u_n(t)| \\
\leqslant &a\lambda_1 (K^{n+p-1} + \cdots + K^n) \varphi^*(t) \\
= &a\lambda_1 \frac{K^n(1-K^p)}{1-K} \varphi^*(t).
\end{aligned}$$

Since $K \in [0,1)$, then

$$\|u_{n+p} - u_n\| < \frac{a\lambda_1 K^n}{1-K} \|\varphi^*\| \to 0, \quad n \to \infty.$$

According to the completeness of E, there exists $u^* \in E$ such that $\lim_{n\to\infty} u_n = u^*$. Since A is continuous and $u_n = Au_{n-1}$, we get that u^* is a fixed point of A in E.

Now we prove uniqueness. If there exists the other point $\overline{u} \in E$ also satisfying $A\overline{u} = \overline{u}$. For each $n \in \mathbb{N}$,

$$\|u^* - \overline{u}\| = \|A^n u^* - A^n \overline{u}\| \leqslant K^n b\lambda_1 \|\varphi^*\|, \tag{3.2}$$

where $b(|u^* - \overline{u}|) > 0$. From (3.2), there is $\|u^* - \overline{u}\| \to 0$ as $n \to \infty$. This implies that $u^* = \overline{u}$, namely, A has at most one fixed point in E. The condition $f(t, 0) \not\equiv 0$ for all $t \in [0, 1]$ guarantees that u^* is a nonzero fixed point. Therefore, BVP(1.1) has a unique nontrivial solution in E. □

Theorem 3.2. *Suppose there exist functions $\alpha, \beta \in C([0, 1], (0, +\infty))$ such that*

$$f(t, x) = [\alpha(t) + \beta(t)x]^{-1}, \quad t \in [0, 1], \ x \in (0, +\infty). \tag{3.3}$$

Then BVP(1.1) has a unique positive solution.

Proof. According to the premise of f, it is easy to show that $A : P \to P$ is completely continuous and decreasing and the positive solution of BVP(1.1) is equivalent to the fixed point of A in P.

We verify the all conditions of **le2.3**.

(1) Using the positivity of α, it follows from the condition (3.3) and (ii) of **le2.5** that

$$(A\theta)(t) = \int_0^1 G(t, qs)[\alpha(s)]^{-1} \mathrm{d}_q s > 0, \quad t \in (0, 1)$$

and

$$\begin{aligned}(A^2\theta)(t) &= \int_0^1 G(t, qs)[\alpha(s) + \beta(s)(A\theta)(s)]^{-1} \mathrm{d}_q s \\ &= \int_0^1 G(t, qs)[\alpha(s)]^{-1}[1 + \beta(s)(A\theta)(s)(\alpha(s))^{-1}]^{-1} \mathrm{d}_q s \\ &\geqslant \frac{1}{1+M} A\theta(t), \quad t \in [0, 1],\end{aligned}$$

where $M = \|\beta(A\theta)\alpha^{-1}\|$. Let $\varepsilon_0 = \frac{1}{1+M}$, then $\varepsilon_0 \in (0, 1)$ and $A^2\theta \geqslant \varepsilon_0 A\theta$. Hence, the condition (i) holds.

(2) For each $u \in P$ and $k \in (0, 1)$, we have

$$\begin{aligned}A(ku)(t) &= \int_0^1 G(t, qs)[\alpha(s) + k\beta(s)u(s)]^{-1} \mathrm{d}_q s \\ &= \frac{1}{k} \int_0^1 G(t, qs)[k^{-1}\alpha(s) + \beta(s)u(s)]^{-1} \mathrm{d}_q s.\end{aligned}$$

Noticing $k^{-1}\alpha(s) + \beta(s)u(s) > \alpha(s) + \beta(s)u(s)$ for all $s \in [0, 1]$. Let

$$1 + \eta = \min_{s \in [0,1]} \frac{k^{-1}\alpha(s) + \beta(s)u(s)}{\alpha(s) + \beta(s)u(s)}.$$

Then $\eta > 0$ and we deduce that

$$\begin{aligned}A(ku)(t) &\leqslant \frac{1}{k(1+\eta)} \int_0^1 G(t, qs)[\alpha(s) + \beta(s)u(s)]^{-1} \mathrm{d}_q s \\ &= \frac{1}{k(1+\eta)} Au(t), \quad t \in [0, 1].\end{aligned}$$

Hence, the condition (ii) holds. According to **le2.3**, A only has one fixed point in P, and then BVP(1.1) has a unique positive solution. □

Example 3.1. Let $f(t,x) = [1 + t^2 + x]^{-1}$, $(t,x) \in [0,1] \times [0,+\infty)$. it is obvious that f satisfies all the conditions of **th3.2**. Hence, BVP(1.1) has a unique positive solution.

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