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GENERAL ENERGY DECAY FOR A DEGENERATE VISCOELASTIC PETROVSKY-TYPE PLATE EQUATION WITH BOUNDARY FEEDBACK*

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Abstract In this paper, we consider a degenerate viscoelastic Petrovsky-type plate equation

$$K(\boldsymbol{x})u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)ds + f(u) = 0$$

with boundary feedback. Under the weaker assumption on the relaxation function, the general energy decay is proved by priori estimates and analysis of Lyapunov-like functional. The exponential decay result and polynomial decay result in some literature are special cases of this paper.

Keywords General energy decay, degenerate Petrovsky plate equation, boundary feedback, function approximation.

MSC(2010) 35L05, 35L15, 35L70.

1. Introduction

In this paper, we are concerned with the following initial-boundary value problem of a degenerate viscoelastic Petrovsky-type plate equation

$$K(\boldsymbol{x})u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)ds + f(u) = 0 \ \text{ in } \Omega \times (0,\infty), \tag{1.1}$$

$$u = \frac{\partial u}{\partial \boldsymbol{\nu}} = 0 \ \text{ on } \Gamma_0 \times (0,\infty), \tag{1.2}$$

$$\mathcal{B}_1 u - \mathcal{B}_1 \left\{ \int_0^t g(t-s)u(s)ds \right\} = 0 \ \text{ on } \Gamma_1 \times (0,\infty), \tag{1.3}$$

$$\mathcal{B}_2 u - \mathcal{B}_2 \left\{ \int_0^t g(t-s)u(s)ds \right\} = 0 \ \text{ on } \Gamma_1 \times (0,\infty), \tag{1.4}$$

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \ \ u_t(\boldsymbol{x},0) = u_1(\boldsymbol{x}) \ \text{ in } \Omega, \tag{1.5}$$

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where Ω is a bounded domain of $\mathbb{R}^2$ with a smooth boundary $\Gamma := \partial\Omega$ such that $\Gamma = \Gamma_0 \cup \Gamma_1$ with meas $\left(\overline{\Gamma}_0 \cap \overline{\Gamma}_1\right) = 0$ and Γ_0, Γ_1 have positive measures and $\boldsymbol{x} = (x, y)$, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The vector $\boldsymbol{\nu} = (\nu_1, \nu_2)$ is the unit exterior normal and $\boldsymbol{\eta} = (-\nu_2, \nu_1)$ represents the corresponding unit tangent vector on Γ. We will assume in the sequel that $K \in C^1(\overline{\Omega})$ and $K(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \Omega$. Here, the relaxation function g is positive and nonincreasing function, f is a countinuous function and

$$\mathcal{B}_1 u = \Delta u + (1-\mu) B_1 u, \qquad \mathcal{B}_2 u = \frac{\partial \Delta u}{\partial \nu} + (1-\mu) B_2 u,$$

where the constant $\mu \, (0 < \mu < \frac{1}{2})$ is the Poisson's ratio and the boundary operators B_1 and B_2 are defined by

$$B_1 u = 2\nu_1\nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx}, \quad B_2 u = \frac{\partial}{\partial \eta}\left[(\nu_1^2 - \nu_2^2) u_{xy} + \nu_1\nu_2(u_{yy} - u_{xx})\right].$$

This problem appears in the mathematical description of viscoelastic materials. It is well known that viscoelastic materials exhibit nature damping, which is due to the special property of these materials to keep memory of their past history. From the mathematical point of view, these damping effects are modeled by integro-differential operators. Therefore, dynamics of viscoelastic materials are great important and interesting as they have wide applications in natural sciences. Problems related to (1.1)-(1.5) are interesting not only from the point of view of PDE theory, but also due to its applications in mechanics.

The *non-degenerate* Petrovsky plate model can be found in [6]. Indeed, starting with [6] and followed by papers [3,4], uniform decay properties for the energy of the *modified* von Kármán system with boundary dissipation were established. We recall that in the case of the *modified* von Kármán system, the in-plane displacements are not accounted for and the system can be decoupled via the Airy stress function (thus it is reduced to a scalar equation). In [20], the author consider the problem

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2} + a(x)\dfrac{\partial w}{\partial t} + \Delta^2 w = 0, & x \in \Omega, t \geq 0, \\ w = \dfrac{\partial w}{\partial n} = 0, & x \in \partial\Omega, t \geq 0, \\ w|_t = 0 = w_0(x),\, w_t|_{t=0} = v_0(x), & x \in \Omega, \end{cases}$$

where $a(x) \geq a_0 > 0$ a.e. in Ω, and $a(\cdot) \in L^\infty(\Omega)$, and proved the exponential energy decay. The first author of this paper showed the energy functional associated with the viscoelastic Petrovsky decays exponentially or polynomially to zero as time goes to infinity in [8]. In [18], Santos and Junior considered a plate model

$$u_{tt} + \Delta^2 u = 0$$

with a memory boundary condition, and proved that such dissipation is strong enough to produce exponential decay to the solution, provided the relaxation functions also decays exponentially. When the relaxation functions decays polynomially, they proved that the solution decays polynomially and with the same rate. Raposo and Santos in [17] considered the *non-degenerate modified* von Kármán plate model with memory and showed the general decay of the solution as time goes to infinity, under the conditions for the kernel $g \in C^2$ and

$$g, g', g'' \in L^1(0, +\infty),\ g(0) > 0,\ g' < 0,\ \alpha := 1 - \int_0^\infty g(s)ds > 0,$$

and there exists a differential function ξ satisfying

$$g'(t) \leq -\xi(t)g(t),\ \xi(t) > 0,\ \xi'(t) < 0,\ t \geq 0.$$

Kang [5] studied the *non-degenerate modified* von Kármán plate model with memory and boundary nonlinear feedback and established an explicit and general decay rate result, using some properties of the convex functions. Shin an Kang in [19] considered a *degenerate* plate model

$$K(\boldsymbol{x})u_{tt} + \Delta^2 u + f(u) = 0 \text{ in } \Omega \times (0, \infty),$$

with a memory condition at the boundary, under some geometrical assumption on domain and twice continuous differentiable assumption resolvent kernels, they establish a more general decay result.

In [9–11], Li et al. proved the existence uniqueness, uniform energy decay rates, and limit behavior of the solution to nonlinear viscoelastic Marguerre-von *Kármán* shallow shells system, respectively. Li et al. in [12–14], shown the global existence uniqueness and decay estimates for nonlinear viscoelastic equation with boundary dissipation. The authors in [2, 7, 15, 16] studied the blow-up phenomenon for some evolution equations.

Motivated by the above work, we intend to study the global existence and the energy decay for problem (1.1)-(1.5). By using the perturbed energy method and differential inequality, we will prove that under some conditions on g , the solution of the problem exists globally and the general decay rate is obtained. The main contribution of this paper are: (a)this degenerate Petrovsky plate with classical boundary feedback possess physical significance and the possibility of wide application in the future; (b) the detail construction process of the energy functional and auxiliary functionals are given by multiplicator method; (c) we waken the assumptions for $\xi(t)$ in (A_1), only need $\xi(t)$ is locally integrable in $(0, \infty)$, instead of being differentiable, the hypothesis on g are weaker and the estimates are precise for the degenerate characteristic and weak assumptions; (d)the general decay result of the memory-type Petrovsky plate model is proved, the exponential decay result and polynomial decay result in some literature are the special cases of this paper.

The present work is organized as follows. In section 2, we present some assumptions and notations for our work and state the existence result and the general energy decay result to problem (1.1)-(1.5). In section 3, some lemmas and the proof of our main result will be given.

2. Preliminaries and main results

Throughout this paper, we define

$$H^1_{\Gamma_0} = \left\{u \in H^1 : u = 0 \quad \text{on} \quad \Gamma_0\right\},\quad H^2_{\Gamma_0} = \left\{v \in H^2 : v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_0\right\},$$

and the following norms

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx \right)^{\frac{1}{p}}.$$

To simplify the notations, we denote $\|u\|_{L^2(\Omega)}$ by $\|u\|$, and for $\mu\,(0 < \mu < \frac{1}{2})$, we define the bilinear form $a(u, v)$ as follows

$$a(u, v) = \int_\Omega \{u_{xx}v_{xx} + u_{yy}v_{yy} + \mu(u_{xx}v_{yy} + u_{yy}v_{xx}) + 2(1-\mu)u_{xy}v_{xy}\} d\Omega,$$

where $d\Omega = dxdy$. since meas$(\Gamma_0) \neq \emptyset$ we know that $\sqrt{a(u,u)}$ is a norm equivalent to the usual *Sobolev* norm on $H^2(\Omega)$, that is,

$$c_0\|u\|^2_{H^2(\Omega)} \leq a(u, u) \leq C_0\|u\|^2_{H^2(\Omega)}, \tag{2.1}$$

where c_0 and C_0 are generic positive constants.

Assumptions on the functions g and f:

(A_1) $g(t)$: $\mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 function such that

$$g(0) > 0, \quad 1 - \int_0^\infty g(s)ds = 1 - l > 0,$$

and there exists a nonincreasing function ξ with $\xi \in L^1_{loc}(0, +\infty)$ satisfying

$$g'(t) \leq -\xi(t)g(t). \tag{2.2}$$

(A_2) Let f be a Lipschitz continuous function with $f(0) = 0$ and satisfy

$$f(s)s \geq 0,\ \forall s \in \mathbb{R},\ \|f(u)\|^2 \leq Ca(u, u).$$

Additionally, we suppose that f is superlinear, that is,

$$f(s)s \geq (2+\alpha)F(s) \geq 0, \quad F(z) = \int_0^z f(s)ds,\ \forall s \in \mathbb{R},$$

for some $\alpha > 0$.

To simplify calculation in our analysis we introduce the following notations

$$(g * u)(t) := \int_0^t g(t-s)u(s)ds,\ g \diamond u := \int_0^t g(t-s)\|u(\cdot,t) - u(\cdot,s)\|^2 ds,$$

$$g \diamond \partial^2 u := \int_0^t g(t-s)a\left(u(\cdot,t) - u(\cdot,s), u(\cdot,t) - u(\cdot,s)\right) ds.$$

Our results is based on the following existence and regularity theorem of the solution to the problem (1.1)-(1.5).

Theorem 2.1. *If* $(u_0, u_1) \in (H^4(\Omega) \cap H^2_{\Gamma_0}) \times H^2_{\Gamma_0}$, *then there exists a unique solution of system* (1.1)-(1.5) *satisfying*

$$u \in L^\infty_{loc}\left(0, \infty; H^4(\Omega) \cap H^2_{\Gamma_0}\right), u_t \in L^\infty_{loc}\left(0, \infty; H^2(\Omega) \cap H^1_{\Gamma_0}\right), u_{tt} \in L^\infty_{loc}(0, \infty; L^2(\Omega)).$$

Proof. The proof can be obtained by Faedo-Galerkin method and calculus theorem in [1,9].

Theorem 2.2. *Let u be the global solution of the problem* (1.1)-(1.5) *with the conditions* (A_1)-(A_2). *We define the energy functional as*

$$E(t) = \frac{1}{2}\int_\Omega K(\boldsymbol{x})|u_t|^2 d\Omega + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right) a(u,u) + \frac{1}{2} g \diamond \partial^2 u + \int_\Omega F(u) d\Omega.$$

Then, for some $t_0 > 0$ there exist positive constants C_0 and k such that

$$E(t) \le C_0 e^{-k\int_{t_0}^t \xi(s)ds}.$$

3. The proof of main result Theorem 2.2

To demonstrate the stability of the system (1.1)-(1.5) the lemmas below are essential.

Lemma 3.1. *For any $v \in C^1\left([0,T]; H^2(\Omega)\right)$, we have*

$$a(g * v, v_t) = -\frac{1}{2} g(t) a(v,v) + \frac{1}{2} g' \diamond \partial^2 v - \frac{1}{2}\frac{d}{dt}\left\{ g \diamond \partial^2 v - \left(\int_0^t g(s)ds\right) a(v,v)\right\}.$$

Proof. Differenting $-\frac{1}{2}\left\{g \diamond \partial^2 v - \left(\int_0^t g(s)ds\right) a(v,v)\right\}$, we have

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\left\{ g \diamond \partial^2 v - \left(\int_0^t g(s)ds\right) a(v,v)\right\} \\
&= -\frac{1}{2} g' \diamond \partial^2 v - \frac{1}{2}\int_0^t g(t-s)\left[a(v(t)-v(s), v(t)-v(s))\right]_t ds \\
&\quad + \frac{1}{2} g(t) a(v,v) + \frac{1}{2}\int_0^t g(s)ds \left[a(v(t),v(t))\right]_t \\
&= -\frac{1}{2} g' \diamond \partial^2 v + \int_0^t g(t-s) a(v_t(t), v(s)) ds + \frac{1}{2} g(t) a(v,v) \\
&= -\frac{1}{2} g' \diamond \partial^2 v + a(g * v, v_t) + \frac{1}{2} g(t) a(v,v).
\end{aligned}$$

This completes the proof of Lemma 3.1 □

Lemma 3.2. *For any $u \in H^4(\Omega)$ and $v \in H^2(\Omega)$, we have*

$$\int_\Omega (\Delta^2 u) v d\Omega = a(u,v) + \int_\Gamma \left\{(\mathcal{B}_2 u) v - (\mathcal{B}_1 u)\frac{\partial v}{\partial \nu}\right\} d\Gamma. \tag{3.1}$$

Proof. The definition of $a(u,v)$ gives

$$\int_\Omega \Delta u \Delta v d\Omega = a(u,v) + \int_\Omega \left\{(1-\mu)(u_{xx}v_{yy} + u_{yy}v_{xx}) - 2(1-\mu)u_{xy}v_{xy}\right\} d\Omega.$$

Using Green's formula, we see

$$\begin{aligned}
\int_\Omega (\Delta^2 u) v d\Omega &= \int_\Gamma \left(\frac{\partial \Delta u}{\partial \nu}\right) v d\Gamma - \int_\Gamma \Delta u \frac{\partial v}{\partial \nu} d\Gamma + \int_\Omega \Delta u \Delta v d\Omega \\
&= \int_\Gamma \left(\frac{\partial \Delta u}{\partial \nu}\right) v d\Gamma - \int_\Gamma \Delta u \frac{\partial v}{\partial \nu} d\Gamma + a(u,v)
\end{aligned}$$

$$+\int_\Omega \{(1-\mu)(u_{xx}v_{yy}+u_{yy}v_{xx})-2(1-\mu)u_{xy}v_{xy}\}\,d\Omega.$$

Recalling the definition of B_1 and B_2 and using the fact that

$$\int_\Omega (u_{xx}v_{yy}+u_{yy}v_{xx}-2u_{xy}v_{xy})d\Omega=\int_\Gamma \left\{(B_2u)v-(B_1u)\frac{\partial v}{\partial \nu}\right\}d\Gamma,$$

our conclusion follows. □

In order to define the energy function $E(t)$ of the problem (1.1)-(1.5), we give the following computation. Multiplying equation (1.1) by u_t, integrating the result over Ω and employing Green's formula and boundary conditions, we get from Lemma 3.1 and Lemma 3.2 that

$$\begin{aligned}0&=\int_\Omega \left(K(\boldsymbol{x})u_{tt}+\Delta^2u-\int_0^t g(t-s)\Delta^2u(s)ds+f(u)\right)u_tdx\\&=\frac{1}{2}\frac{d}{dt}\left(\int_\Omega K(\boldsymbol{x})|u_t|^2d\Omega+a(u,u)\right)+\frac{d}{dt}\int_\Omega F(u)d\Omega-a(g*u,u_t)\\&=\frac{1}{2}\frac{d}{dt}\left(\int_\Omega K(\boldsymbol{x})|u_t|^2d\Omega+\left(1-\int_0^t g(s)ds\right)a(u,u)+g\diamond\partial^2u\right)+\frac{d}{dt}\int_\Omega F(u)d\Omega\\&\quad+\frac{1}{2}g(t)a(u,u)-\frac{1}{2}g'\diamond\partial^2u.\end{aligned}$$

The above computation inspires us to define energy functional as follows

$$E(t)=\frac{1}{2}\int_\Omega K(\boldsymbol{x})|u_t|^2d\Omega+\frac{1}{2}\left(1-\int_0^t g(s)ds\right)a(u,u)+\frac{1}{2}g\diamond\partial^2u+\int_\Omega F(u)d\Omega.$$

Lemma 3.3. *The energy function $E(t)$ of the problem* (1.1)-(1.5) *satisfies*

$$E'(t)=-\frac{1}{2}g(t)a(u,u)+\frac{1}{2}g'\diamond\partial^2u\le 0.$$

Proof. By the above computation and from assumptions (A_1), (A_2), it is easy to see that

$$E'(t)=-\frac{1}{2}g(t)a(u,u)+\frac{1}{2}g'\diamond\partial^2u\le 0.$$

□

From Lemma 3.3, we only can conclude the energy is decreasing. To get our desired energy decay rates, we give the following Lemmas.

Lemma 3.4. *Let (A_1)-(A_2) hold. If u is the solution of* (1.1)-(1.5)*, then*

$$\Phi(t)=\int_\Omega K(\boldsymbol{x})u_tud\Omega,$$

satisfies

$$\begin{aligned}\Phi'(t)\le&\int_\Omega K(\boldsymbol{x})|u_t|^2d\Omega-\left(1-(\delta+1)\int_0^t g(s)ds\right)a(u,u)+\frac{1}{4\delta}g\diamond\partial^2u\\&-(2+\alpha)\int_\Omega F(u)d\Omega,\end{aligned}$$

for all $\delta>0$.

Proof. Multiplying (1.1) by u and performing an integration on Ω, we get

$$\begin{aligned}\int_\Omega K(\boldsymbol{x})u_{tt}ud\Omega &= -a(u,u) - \int_{\Gamma_1}\left[(\mathcal{B}_2 u)\,u - (\mathcal{B}_1 u)\,\frac{\partial u}{\partial \nu}\right]d\Gamma + a(g*u,u)\\ &\quad + \int_{\Gamma_1}\left[(\mathcal{B}_2(g*u))\,u - (\mathcal{B}_1(g*u))\,\frac{\partial u}{\partial \nu}\right]d\Gamma - \int_\Omega f(u)ud\Omega\\ &= -a(u,u) + a(g*u,u) - \int_\Omega f(u)ud\Omega.\end{aligned}$$

So, we have

$$\begin{aligned}\Phi'(t) &= \int_\Omega K(\boldsymbol{x})u_{tt}ud\Omega + \int_\Omega K(\boldsymbol{x})|u_t|^2 d\Omega\\ &= -a(u,u) + a(g*u,u) + \int_\Omega K(\boldsymbol{x})|u_t|^2 d\Omega - \int_\Omega f(u)ud\Omega.\end{aligned}$$

Using Young's inequality and the assumption (A_2), we obtain

$$a(g*u,u) \le (\delta+1)\left(\int_0^t g(s)ds\right)a(u,u) + \frac{1}{4\delta}g\diamond\partial^2 u,$$

and

$$-\int_\Omega f(u)ud\Omega \le -(2+\alpha)\int_\Omega F(u)d\Omega.$$

From these estimates, we have

$$\begin{aligned}\Phi'(t) \le& \int_\Omega K(x)|u_t|^2 d\Omega - \left(1-(\delta+1)\int_0^t g(s)ds\right)a(u,u) + \frac{1}{4\delta}g\diamond\partial^2 u\\ &- (2+\alpha)\int_\Omega F(u)d\Omega,\end{aligned}$$

for all $\delta > 0$. □

Lemma 3.5. *Let* (A_1)*-*(A_2) *hold. If* u *is the solution of* (1.1)-(1.5), *then there exist positive constants* C, c_0 *and* $\lambda > 0$ *such that*

$$\Psi(t) = -\int_\Omega K(\boldsymbol{x})u_t\int_0^t g(t-s)(u(t)-u(s))dsd\Omega,$$

satisfies

$$\begin{aligned}\Psi'(t) \le& \lambda(l+2l^2+C)a(u,u) + \left(\frac{1+(8\lambda l+1+c_0^{-1})l}{4\lambda}\right)g\diamond\partial^2 u - \frac{g(0)}{4\lambda c_0}g'\diamond\partial^2 u\\ &+ \left(\lambda\max_{\boldsymbol{x}\in\overline{\Omega}}K(\boldsymbol{x}) - \int_0^t g(s)ds\right)\int_\Omega K(x)|u_t|^2 d\Omega.\end{aligned}$$

Proof. Differentiating $\Psi(t)$ with respect to t yields

$$\Psi'(t) = -\int_\Omega K(\boldsymbol{x})u_{tt}\int_0^t g(t-s)(u(t)-u(s))dsd\Omega$$

$$-\int_\Omega K(\boldsymbol{x})u_t\int_0^t g'(t-s)(u(t)-u(s))dsd\Omega-\int_0^t g(s)ds\int_\Omega K(\boldsymbol{x})|u_t|^2d\Omega.$$

Using (1.1)-(1.5) and Lemma 3.2, we see

$$\begin{aligned}\Psi'(t)=&\int_0^t g(t-s)a(u(t)-u(s),u(t))ds-\int_0^t g(t-s)a\left(u(t)-u(s),g*u\right)ds\\&-\int_\Omega K(\boldsymbol{x})u_t\int_0^t g'(t-s)(u(t)-u(s))dsd\Omega\\&+\int_\Omega f(u)\int_0^t g(t-s)(u(t)-u(s))dsd\Omega-\left(\int_0^t g(s)ds\right)\int_\Omega K(\boldsymbol{x})|u_t|^2d\Omega\\:=&I_1+I_2+I_3+I_4-\left(\int_0^t g(s)ds\right)\int_\Omega K(\boldsymbol{x})|u_t|^2d\Omega.\end{aligned}\tag{3.2}$$

Now, let us estimate the terms in the right side of (3.2). Using Young's, Hölder's inequality, (2.1) and the assumption (A_2) , we have

$$\begin{aligned}|I_1|\le&\int_0^t g(t-s)[a(u(t)-u(s),u(t)-u(s))]^{\frac12}[a(u(t),u(t))]^{\frac12}ds\\\le&\lambda la(u,u)+\frac{1}{4\lambda}g\diamond\partial^2u,\\|I_2|\le&\frac{l}{4\lambda}\int_0^t g(t-s)a(u(t)-u(s),u(t)-u(s))ds+\lambda l\int_0^t g(t-s)a(g*u,g*u)ds\\\le&\frac{l}{4\lambda}g\diamond\partial^2u+\lambda l^2a(g*u,g*u)\\=&\frac{l}{4\lambda}g\diamond\partial^2u\\&+\lambda l^2a\bigg(\int_0^t g(t-s)(u(t)-u(s))ds-\int_0^t g(s)dsu(t),\\&\qquad\int_0^t g(t-s)(u(t)-u(s))ds-\int_0^t g(s)dsu(t)\bigg)\\\le&\left(2\lambda l^2+\frac{l}{4\lambda}\right)g\diamond\partial^2u+2l^3\lambda a(u,u),\\|I_3|\le&\lambda\max_{\boldsymbol{x}\in\overline{\Omega}}K(\boldsymbol{x})\int_\Omega K(\boldsymbol{x})|u_t|^2d\Omega+\frac{1}{4\lambda}\int_\Omega\left(\int_0^t g'(t-s)|u(t)-u(s)|ds\right)^2d\Omega\\\le&\lambda\max_{\boldsymbol{x}\in\overline{\Omega}}K(\boldsymbol{x})\int_\Omega K(\boldsymbol{x})|u_t|^2d\Omega-\frac{g(0)}{4\lambda c_0}g'\diamond\partial^2u,\\|I_4|\le&\lambda\int_\Omega|f(u)|^2d\Omega+\frac{1}{4\lambda}\int_\Omega\left(\int_0^t g(t-s)|u(t)-u(s)|ds\right)^2d\Omega\\\le&\lambda Ca(u,u)+\frac{l}{4\lambda c_0}g\diamond\partial^2u.\end{aligned}$$

Substituting these inequalities into (3.2), we finish the proof. □

Lemma 3.6. *Let $L(t)=NE(t)+\varepsilon\Phi(t)+\Psi(t)$, for large enough N, then there exist two positive constants $\alpha_1,\alpha_2>0$ such that*

$$\alpha_1E(t)\le L(t)\le\alpha_2E(t).$$

Proof. From Young's inequality, Hölder's inequality and (2.1), we deduce

$$|\Phi(t)| \le \left|\int_\Omega K(\boldsymbol{x})u_t u d\Omega\right| \le \frac{1}{2}\max_{\boldsymbol{x}\in\overline{\Omega}} K(\boldsymbol{x})\int_\Omega K(\boldsymbol{x})|u_t|^2 d\Omega + \frac{1}{2c_0}a(u,u), \tag{3.3}$$

$$\begin{aligned}|\Psi(t)| &\le \frac{1}{2}\max_{\boldsymbol{x}\in\overline{\Omega}} K(\boldsymbol{x})\int_\Omega K(\boldsymbol{x})|u_t|^2 d\Omega + \frac{1}{2}\int_\Omega\left(\int_0^t g(t-s)|u(t)-u(s)|ds\right)^2 d\Omega \\ &\le \frac{1}{2}\max_{\boldsymbol{x}\in\overline{\Omega}} K(\boldsymbol{x})\int_\Omega K(\boldsymbol{x})|u_t|^2 d\Omega + \frac{l}{2c_0} g\diamond\partial^2 u. \end{aligned}\tag{3.4}$$

Thus, from (3.3) and (3.4), we obtain

$$\begin{aligned}|L(t)-NE(t)| &\le \frac{1}{2}\max_{\boldsymbol{x}\in\overline{\Omega}} K(\boldsymbol{x})(1+\varepsilon)\int_\Omega K(\boldsymbol{x})|u_t|^2 d\Omega + \frac{\varepsilon}{2c_0}a(u,u) + \frac{l}{2c_0}g\diamond\partial^2 u \\ &\le C_0 E(t),\end{aligned}$$

where C_0 is a positive constant depending on ε, l, c_0 and $\max_{\boldsymbol{x}\in\overline{\Omega}} K(\boldsymbol{x})$. Choosing $N > 0$ large enough, we complete the proof of Lemma 3.6. □

Lemma 3.7. *For any fixed $t_0 > 0$ and sufficiently large $N > 0$, there exist positive constants β and γ such that*

$$L'(t) \le -\beta E(t) + \gamma g\diamond\partial^2 u, \quad \forall t \ge t_0. \tag{3.5}$$

Proof. Using Lemma 3.6, we obtain

$$L'(t) = NE'(t) + \varepsilon\Phi'(t) + \Psi'(t). \tag{3.6}$$

Since g is positive, for any fixed $t_0 > 0$, we have $\int_0^t g(s)ds \ge \int_0^{t_0} g(s)ds := g_0 > 0\, \forall t \ge t_0$. Thus, substituting Lemma 3.3-Lemma 3.5 into (3.6), it yields

$$\begin{aligned}L'(t) \le &-\left(g_0 - \varepsilon - \lambda\max_{\boldsymbol{x}\in\overline{\Omega}} K(\boldsymbol{x})\right)\int_\Omega K(\boldsymbol{x})|u_t|^2 d\Omega \\ &- \left[\frac{N}{2}g(t) + \varepsilon(1-(1+\delta)l) - \lambda(l+2l^2+C)\right]a(u,u) + \left(\frac{N}{2} - \frac{g(0)}{4\lambda c_0}\right)g'\diamond\partial^2 u \\ &+ \left(\frac{\varepsilon}{4\delta} + \frac{1}{4\lambda} + \frac{l}{4\lambda} + \frac{l}{4\lambda c_0} + 2l^2\right)g\diamond\partial^2 u - (2+\alpha)\int_\Omega F(u)d\Omega.\end{aligned}$$

We firstly take $\varepsilon > 0$ and $\delta > 0$ small enough such that $g_0 - \varepsilon > 0$ and $1 - (1+\delta)l > 0$ respectively. Then, we choose $\lambda > 0$ sufficiently small such that $g_0 - \varepsilon - \lambda\max_{\boldsymbol{x}\in\overline{\Omega}} K(\boldsymbol{x}) > 0$ and $\varepsilon(1-(1+\delta)l) - \lambda(l+2l^2+C) > 0$. Finally, taking $N > 0$ large enough, this finish the conclusion. □

Lemma 3.8. *Define $\eta \in C^\infty(\mathbb{R}^1_+)$ by*

$$\eta(t) = \begin{cases} C\exp(\frac{1}{t^2-1}), & 0 \le t < 1, \\ 0, & t \ge 1, \end{cases}$$

the constant $C > 0$ is selected such that $\int_0^{+\infty}\eta(t)dt = 1$. For each $\varepsilon > 0$, set

$$\eta_\varepsilon(t) = \frac{1}{\varepsilon}\eta\left(\frac{t}{\varepsilon}\right).$$

If $\xi : \mathbb{R}^1_+ \to R$ is locally integrable, we define the mollification

$$\xi^\varepsilon := \int_0^\varepsilon \eta_\varepsilon(s)\xi(t-s)ds,\ t \in (\varepsilon, +\infty).$$

Then the following properties of ξ^ε hold

(i) $\xi^\varepsilon \in C^\infty(\varepsilon, +\infty)$,

(ii) $\xi^\varepsilon \to \xi$ *a.e. as* $\varepsilon \to 0$,

(iii) If $1 \le p < \infty$ *and* $\xi \in L^p_{loc}$, *then* $\xi^\varepsilon \to \xi$ *in* $L^p_{loc}[0, +\infty)$,

(iv) $\frac{d}{dt}\xi^\varepsilon(t) \le 0$.

Proof. The proof of (i)-(iii) is similar to [1, Theorem 6, pp. 629–631].

Now, we prove (iv). The assumption that ξ does not increase gives for $\tau \ge 0$

$$\xi^\varepsilon(t+\tau) - \xi^\varepsilon(t) = \int_0^\varepsilon \eta_\varepsilon(s)\left[\xi(t+\tau-s) - \xi(t-s)\right] ds \le 0,$$

which implies the conclusion. □

Proof of Theorem 2.2. Multiplying (3.5) by $\xi(t)$, noting that $\xi(t)$ is nonincreasing, and using (2.2) and Lemma 3.3, we obtain

$$\begin{aligned} \xi(t)L'(t) &\le -\beta\xi(t)E(t) + \gamma\xi(t)g \diamond \partial^2 u \le -\beta\xi(t)E(t) - \gamma g' \diamond \partial^2 u \\ &\le -\beta\xi(t)E(t) - 2\gamma E'(t),\ \forall t \ge t_0. \end{aligned} \tag{3.7}$$

Let $F^\varepsilon(t) := L(t)\xi^\varepsilon(t) + 2\gamma E(t)$. Using Lemma 3.8, we have

$$F(t) := \lim_{\varepsilon\to 0} F^\varepsilon(t) = L(t)\xi(t) + 2\gamma E(t)$$

and

$$F^{\varepsilon\prime}(t) := L'(t)\xi^\varepsilon(t) + L(t)\xi^{\varepsilon\prime}(t) + 2\gamma E'(t) \le L'(t)\xi^\varepsilon(t) + 2\gamma E'(t).$$

Letting $\varepsilon \to 0$ leads to

$$F'(t) \le L'(t)\xi(t) + 2\gamma E'(t).$$

Thus from (3.7), we have

$$F'(t) \le -\beta\xi(t)E(t).$$

Since $L(t) \sim E(t)$, we have $F(t) \sim E(t)$. Therefore, there exists a positive constant $k > 0$ such that

$$F'(t) \le -k\xi(t)F(t).$$

Integrating by part over (t_0, t), we get

$$F(t) \le ce^{-k\int_{t_0}^t \xi(s)ds}.$$

Owing to $F(t) \sim E(t)$, we conclude that there exists a positive constant C_0 such that

$$E(t) \le C_0 e^{-k\int_{t_0}^t \xi(s)ds}.$$

The proof is completed. □

Remark 3.1. From Theorem 2.2, we obtain exponential decay for $\xi(t) = a$ and polynomial decay for $\xi(t) = a(1+t)^{-1}$, where $a > 0$ is a constant. Hence the exponential decay result and polynomial decay result in some literatures [8,17,18,20] are the special cases of this paper. The assumptions on g, $\xi(t)$ and the domain are weaker than that in [5,19].

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