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ON SPECTRAL ASYMPTOTICS AND BIFURCATION FOR SOME ELLIPTIC EQUATIONS OF KIRCHHOFF-TYPE WITH ODD SUPERLINEAR TERM*

Baoqiang Yan[1], Donal O'Regan[2] and Ravi P. Agarwal[3,†]

Abstract In this paper, from estimating the eigenvalues for Kirchhoff elliptic equations, we obtain spectral asymptotics and bifurcation concerning the eigenvalues of some related elliptic linear problem.

Keywords Kirchhoff elliptic equations, Liusternik-Schnirelmann (LS) theory, eigenvalue, eigenfunction.

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1. Introduction

In this paper, we consider the following nonlocal elliptic problem

$$\begin{cases} -(a+b\int_\Omega |\nabla u(x)|^2dx)\Delta u + f(x,u) = \lambda u, & x \text{ in } \Omega, \\ u(x) = 0, & x \text{ on } \partial\Omega, \end{cases} \tag{1.1$_\lambda$}$$

where $\Omega \subseteq R^N$ $(N \geq 1)$ is a smooth and bounded domain, and $a > 0$, $b > 0$. This problem is a special case of the following problem

$$\begin{cases} -(a+b\int_\Omega |\nabla u(x)|^2dx)\Delta u = g(x,u), & x \text{ in } \Omega, \\ u(x) = 0, & x \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

which is is related to the stationary analogue of the equation

$$u_{tt} - (a+b\int_\Omega |\nabla u|^2dx)\Delta u = g(x,u)$$

[†]the corresponding author. Email address:Ravi.Agarwal@tamuk.edu(Ravi. P. Agarwal)
[1]School of Mathematical Sciences, Shandong Normal University, Jinan, 250014, China
[2]School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland
[3]Department of Mathematics, Texas A and M University-Kingsville, Texas 78363, USA
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proposed by Kirchhoff [14] as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings. We note that (1.2) has been studied extensively in the literature and some results on the existence of the solutions can be found in [1, 2, 8, 11–13, 17–20, 28] and the references therein. Recently, there are some interesting results on the existence of solutions for nonlocal fractional elliptic equations also (see [4, 5, 22, 23, 30]). Papers in the literature on eigenvalues of nonlinear problems can be found in [3, 6, 7, 21, 25, 26]. In particular, some authors considered the spectral asymptotics and bifurcation for the uniformly elliptic equation

$$\begin{cases} \sum_{i,j=1}^{N} D_i(a_{ij}D_j u) + a_0(x)u + f(x,u) = \lambda u, & x \text{ in } \Omega, \\ u(x) = 0, & x \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq R^N$ $(N \geq 1)$ is a smooth and bounded domain; see [7–10, 24]. There are only a few results on comparing the eigenvalues of $(1.1)_\lambda$ with those of the relative linear problem. Our aim is to obtain spectral asymptotics and bifurcation for $(1.1)_\lambda$.

This paper is organized as follows. In Section 2, using the Liusternik-Schnirelmann (LS) theory, we establish, given any $r > 0$, the existence of infinitely many eigenvalues $\mu_{n,r}(n = 1, 2, \cdots)$ for $(1.1)_\lambda$ associated with eigenfunctions $u_{n,r}$ satisfying $\int_\Omega u_{n,r}^2(x)dx = r^2$. In Section 3, we obtain bifurcation and comparison results concerning the eigenvalues of some related linear problems $(2.1)_\lambda$. In Section 4, we consider the asymptotic laws of the eigenvalues $\mu_{n,r}$ of $(1.1)_\lambda$ as $n \to +\infty$ when f is superlinear at $+\infty$. Our paper was motivated in part by the papers [9, 10, 15, 16, 24, 27, 29].

2. Existence of the eigenvalues of $(1.1)_\lambda$

Let $W_0^{1,2}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the usual Sobolev space $W^{1,2}(\Omega)$ with the scalar product $(u, u) = \int_\Omega \nabla u \cdot \nabla u dx$ and the corresponding norm $\|u\| = \|\nabla u\|_2 = (\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}}$, while $\|u\|_p$, will denote the norm of $u \in L^p(\Omega)$. Problem $(1.1)_\lambda$ is equivalent to its weak formulation, namely that of finding $u \in W_0^{1,2}(\Omega)$ and $\lambda \in R$ such that

$$(a + b\|u\|^2) \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega f(x, u)v dx = \lambda \int_\Omega uv dx$$

for all $v \in W_0^{1,2}(\Omega)$. Let

$$\Phi(u) = \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4, \quad \Psi(u) = \int_\Omega F(x, u(x))dx$$

and

$$I(u) = \Phi(u) + \Psi(u),$$

where

$$F(x, u(x)) = \int_0^{u(x)} f(x, s)ds.$$

For $r > 0$, let

$$M_r := \{u \in W_0^{1,2}(\Omega) | \int_\Omega u^2 dx = r^2\}$$

and for each $n = 1, 2, \cdots$, set

$$K_{n,r} = \{K \subseteq M_r : K \text{ compact, symmetric}, \gamma(K) = n\}$$

where $\gamma(K)$ denotes the genus of K.

From [9], the linear elliptic problem

$$\begin{cases} -\Delta u = \lambda u, & x \text{ in } \Omega, \\ u(x) = 0, & x \text{ on } \partial\Omega, \end{cases} \tag{2.1$_\lambda$}$$

has eigenvalues $\lambda_1 < \lambda_2 \cdots \leq \lambda_n \leq \cdots$ and the corresponding eigenfunction to λ_n is u_n with $u_n \in M_r$ and

$$r^2\lambda_n = \int_\Omega |\nabla u_n|^2 dx. \tag{2.2}$$

Since the set of all eigenfunctions corresponding to λ_n is a linear space, if we choose v_n is a eigenfunction of λ_n with $\int_\Omega |v_n|^2 dx = 1$, then the eigenfunction u_n of λ_n with $u_n \in M_r$ can be written as $u_n = l_n v_n$. From

$$r^2 = \int_\Omega |u_n|^2 dx = \int_\Omega |l_n v_n|^2 dx = l_n^2 \int_\Omega |v_n|^2 dx,$$

we get $l_n = \pm r$, i.e.,

$$u_n = \pm r v_n, \quad n = 1, 2, \cdots, \tag{2.3}$$

which together with (2.2) gives

$$r^2\lambda_n = \int_\Omega |\nabla u_n|^2 dx = \int_\Omega |\nabla(\pm r v_n)|^2 dx = r^2 \int_\Omega |\nabla v_n|^2 dx,$$

and so

$$\lambda_n = \int_\Omega |\nabla v_n|^2 dx.$$

Finally, introduce the "LS critical levels"

$$c_{n,r} = \inf_{K_{n,r}} \sup_K 2I. \tag{2.4}$$

Lemma 2.1 (Lemma 2.1, [10]). *Let $p : 1 \leq p \leq p_0 = (N+2)/(N-2)$ (so that $2 \leq p+1 \leq 2^*$) and let $\beta = (2^*/N)(2^* - (p+1))$. Then, for each $\gamma : 0 \leq \gamma \leq \beta$, there exists $c > 0$ such that*

$$\|u\|_{p+1}^{p+1} \leq c\|\nabla u\|_2^{p+1-\gamma}\|u\|_2^\gamma \tag{2.5}$$

for all $u \in W_0^{1,2}(\Omega)$. (Here and henceforth $\|u\|_p$ denotes the norm of u in $L^p(\Omega)$.)

We will consider the following condition:

(A_1) $f : \Omega \times R \to R$ is continuous, $f(x, -u) = -f(x, u)$ and satisfies

$$|f(x, u)| \leq c|u|^p + d$$

for some c, $d \geq 0$ and some $0 \leq p < \overline{p} = \min\{2^*, 1 + 8/N\}$.

From the LS theory, we have the following existence result.

Theorem 2.1. *Assume (A_1) holds. Then, for given $r > 0$, there exists a sequence $\{u_{n,r}\}$ of (weak) eigenfunctions of $(1.1)_\lambda$ belonging to M_r, and such that*

$$2I(u_{n,r}) = c_{n,r},$$

where $c_{n,r}$ is as in (2.4); the eigenvalue $\mu_{n,r}$ corresponding to $u_{n,r}$ satisfies

$$r^2\mu_{n,r} = a\|\nabla u_{n,r}\|_2^2 + b\|\nabla u_{n,r}\|_2^4 + \int_\Omega f(x, u_{n,r})u_{n,r}(x)dx.$$

Proof. (1) We first show that I is bounded below on M_r (for each r). Note from (A_1) that

$$I(u) \geq \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \int_\Omega |F(x, u(x))|dx.$$

Moreover, (A_1) and Schwarz's inequality imply that

$$\int_\Omega |F(x, u(x))|dx \leq c\int_\Omega |u|^{p+1}dx + d(\int_\Omega |u|^2 dx)^{\frac{1}{2}}$$

for some new constants c, $d > 0$. Next, we use the inequality (2.5) with $\gamma = \beta$: on setting $2\alpha = p + 1 - \beta = (p-1)N/2$, this becomes

$$\int_\Omega |u|^{p+1}dx \leq c'\|\nabla u\|_2^{2\alpha}(\int_\Omega u^2 dx)^{\frac{\beta}{2}},$$

and we conclude that, on M_r,

$$I(u) \geq \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - cc'r^\beta\|\nabla u\|_2^{2\alpha} - dr.$$

The assumption $p < \min\{2^* - 1, 1 + 8/N\}$ is equivalent to $2\alpha < 4$, which implies that I is bounded below on M_r (for each r).

Then

$$-\infty < c_{n,r} = \inf_{K_{n,r}} \sup_K 2I < +\infty.$$

(2) We show that I satisfies the Palais-Smale condition (PS) on M_r, i.e., for $c \neq 0$, $\varepsilon > 0$ small enough, $u_n \in I^{-1}[c-\varepsilon, c+\varepsilon] \cap M_r$ and $\|I'_{M_r}(u_n)\| \to 0$, then there is a $u \in M_r$ and a subsequence $\{u_{n_j}\}$ such that

$$\|\nabla(u_{n_j} - u)\|_2 \to 0.$$

We know that $\{u_n\}$ is bounded, which implies that there exists a $u^* \in W_0^{1,2}(\Omega)$ and subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $u_{n_j} \rightharpoonup u^*$, as $j \to +\infty$. Since

$$\begin{aligned} I'_{M_r}(u)(v) &= I'(u)(v) - r^{-2}I'(u)(u)\int_\Omega uvdx \\ &= a\int_\Omega \nabla u\nabla vdx + b\int_\Omega |\nabla u|^2dx\int_\Omega \nabla u\nabla vdx + \int_\Omega f(x,u)vdx \\ &\quad -r^{-2}(a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 + \int_\Omega f(x,u)udx)\int_\Omega uvdx, \quad u, v \in W_0^{1,2}(\Omega), \end{aligned}$$

we have

$$\begin{aligned}&(a+b\|\nabla u_{n_j}\|_2^2)\int_\Omega \nabla u_{n_j}\nabla(u_{n_j}-u^*)dx\\ =\ & I'(u_{n_j})(u_{n_j}-u^*)-\textstyle\int_\Omega f(x,u_{n_j})(u_{n_j}-u^*)dx\\ &+r^{-2}(a\|\nabla u_{n_j}\|_2^2+b\|\nabla u_{n_j}\|_2^4+\int_\Omega f(x,u_{n_j})u_{n_j}dx)\int_\Omega u_{n_j}(u_{n_j}-u^*)dx\\ \to\ & 0,\end{aligned}$$

which together with $p<2^*-1$ implies that

$$\|\nabla u_{n_j}-u^*\|_2\to 0,\ \text{ as } j\to+\infty.$$

(3) We show that $c_{n,r}$ is a critical value of $I(u)$ in M_r, i.e., there exists a $u_{n,r}\in M_r$ such that $c_{n,r}=2I(u_{n,r})$ and $I|'_{M_r}(u_{n,r})=0$.

On the contrary, suppose that there is a $\varepsilon_0>0$ such that $2I^{-1}[c_{n,r}-\varepsilon_0,c_{n,r}-\varepsilon_0]\cap K=\emptyset$, where $K=\{u\in M_r|I|'_{M_r}(u)=0\}$. Let $A_c=\{u|2I(u)\le c\}$ and $K_c=\{u|2I(u)=c,I|'_{M_r}(u)=\theta\}$. From [21], let N be a neighourhood of K_c, there exists a $\eta(t,x)\in C([0,1],W_0^{1,2}(\Omega))$ and $\varepsilon_0>\varepsilon>0$ such that

(1) $\eta(0,u)=u$ for all $u\in W_0^{1,2}(\Omega)$;
(2) $\eta(t,u)=u$ for all $u\in 2I^{-1}[c_{n,r}-\varepsilon_0,c_{n,r}+\varepsilon_0]$ and for all $t\in[0,1]$;
(3) $\eta_t(u)$ is a homeomorphism from $W_0^{1,2}(\Omega)$ onto $W_0^{1,2}(\Omega)$ for all $t\in[0,1]$;
(4) $I(\eta(t,u))\le I(u)$ for all $u\in W_0^{1,2}(\Omega)$, for all $t\in[0,1]$;
(5) $\eta_1(A_{c_{n,r}+\varepsilon})\subseteq A_{c_{n,r}-\varepsilon}$;

Since $c_{n,r}=\inf_{K_{n,r}}\sup_K 2I<+\infty$, there is a $A_n\subseteq M_r$ such that $c_{n,r}\le\sup_{u\in A_n}2I(u)\le c_{n,r}+\varepsilon$. Then $\gamma(A_n)\ge n$ and $\gamma(\eta_1(A_n))=\gamma(A_n)\ge n$ and $\eta_1(A_n)\subseteq A_{c_{n,r}-\varepsilon}$ and so

$$c_{n,r}=\inf_{K_{n,r}}\sup_K 2I\le\sup_{u\in\eta_1(A_n)}2I(u)\le c_{n,r}-\varepsilon.$$

This is contradiction.

Consequently, there exists a $u_{n,r}\in M_r$ such that

$$c_{n,r}=2I(u_{n,r})$$

and

$$I'(u_{n,r})(v)=r^{-2}I'(u_{n,r})(u_{n,r})\cdot u_{n,r}(v),\forall v\in W_0^{1,2}(\Omega).$$

Let $\mu_{n,r}=r^{-2}I'(u_{n,r})(u_{n,r})$. Note one has

$$(a+b\|\nabla u_{n,r}\|_2^2)\int_\Omega\nabla u_{n,r}\nabla v dx+\int_\Omega f(x,u_{n,r})v(x)dx=\mu_{n,r}\int_\Omega u_{n,r}v dx,\forall v\in W_0^{1,2}. \tag{2.6}$$

Let $v=u_{n,r}$. Then (2.6) becomes

$$r^2\mu_{n,r}=(a+b\|\nabla u_{n,r}\|_2^2)\|\nabla u_{n,r}\|_2^2+\int_\Omega f(x,u_{n,r})u_{n,r}dx.$$

The proof is complete. □

Corollary 2.1. *Let $f \equiv 0$ and equation $(1.1)_\lambda$ becomes*

$$\begin{cases} -(a+b\|\nabla u\|_2^2)\Delta u = \lambda u, & x \text{ in } \Omega, \\ u(x) = 0, & x \text{ on } \partial\Omega. \end{cases} \tag{$(2.7)_\lambda$}$$

Then, $(2.7)_\lambda$ has branches

$$C_n = \{(a\lambda_n + br^2\lambda_n^2, \pm r v_n)|, r > 0\}, \quad n = 1, 2, \cdots.$$

Proof. From the L-S procedure in Theorem 2.1, $(2.7)_\lambda$ has exactly the eigenvalues $\mu_{n,r}^0$ with the corresponding eigenfunction $u_{n,r}^0(\|u_{n,r}^0\|_2 = r)$ which satisfies

$$\begin{cases} -\Delta u_{n,r}^0 = \mu_{n,r}^0 \frac{1}{a+b\|\nabla u_{n,r}^0\|_2^2} u_{n,r}^0, & x \text{ in } \Omega, \\ u_{n,r}^0(x) = 0, & x \text{ on } \partial\Omega. \end{cases}$$

Comparing $(2.7)_\lambda$ with $(2.1)_\lambda$, we get

$$\mu_{n,r}^0 \frac{1}{a + b\|\nabla u_{n,r}^0\|_2^2} = \lambda_n$$

and $u_{n,r}^0 = k_n u_n$, where u_n is the corresponding eigenvalue function to λ_n of $(2.1)_\lambda$ with $\|u_n\|_2 = r$. Moreover,

$$c_{n,r}^0 = 2\Phi(u_{n,r}^0) = a\|\nabla u_{n,r}^0\|_2^2 + \frac{b}{2}\|\nabla u_{n,r}^0\|_2^4, \quad \mu_{n,r}^0 = a\lambda_n + b\|\nabla u_{n,r}^0\|_2^2\lambda_n. \tag{2.8}$$

Since $u_{n,r}^0 = k_n u_n$, one has

$$r = \|u_{n,r}^0\|_2 = \|k_n u_n\|_2 = |k_n| r,$$

which implies $k_n = \pm 1$ and $u_{n,r}^0 = \pm u_n$. Hence, (2.8) becomes

$$c_{n,r}^0 = 2\Phi(u_{n,r}^0) = a\|\nabla u_n\|_2^2 + \frac{b}{2}\|\nabla u_n\|_2^4, \quad \mu_{n,r}^0 = a\lambda_n + b\|\nabla u_n\|_2^2\lambda_n.$$

From (2.2), we have

$$r^2\lambda_n = \int_\Omega |\nabla u_n|^2 dx = \|\nabla u_n\|_2^2 = \|\nabla u_{n,r}^0\|_2^2,$$

and so

$$c_{n,r}^0 = 2\Phi(u_{n,r}^0) = ar^2\lambda_n + \frac{b}{2}r^4\lambda_n^2, \quad \mu_{n,r}^0 = a\lambda_n + br^2\lambda_n^2, \tag{2.9}$$

which together with (2.3) implies that $(2.7)_\lambda$ has branches

$$C_n = \{(a\lambda_n + br^2\lambda_n^2, \pm r v_n)|, r > 0\}, \quad n = 1, 2, \cdots.$$

The proof is complete. □

3. Bifurcation and comparison results concerning the eigenvalues of some related linear problem to $(1.1)_\lambda$

In the last section, we obtained the branches of solutions of $(1.1)_\lambda$ when $f \equiv 0$. Now we consider the case $f \not\equiv 0$.

Theorem 3.1. *Let the assumptions of Theorem 2.1 be satisfied with $p > 1$ and $d = 0$ in the growth assumption (A_1). Then each $a\lambda_n$ is a bifurcation point (in $W_0^{1,2}(\Omega)$) for $(1.1)_\lambda$; more precisely, for each $n = 1, 2, \cdots$, the eigenvalue-eigenfunction pairs $(\mu_{n,r}, u_{n,r})$ given by Theorem 2.1 satisfy $\mu_{n,r} = a\lambda_n + br^2\lambda_n^2 + O(r^{\min\{2,p-1\}})$.*

Proof.

There are two cases to be considered, namely $p < 1 + 4/N$ and $1 + 4/N \le p < 1 + 8/N$.

(a) The case $p < 1 + 4/N$. Let $\gamma = p - 1$. Then (see Lemma 2.1) we have

$$\|u\|_{p+1}^{p+1} \le c\|\nabla u\|_2^2 \|u\|_2^{p-1}, u \in W_0^{1,2}(\Omega). \tag{3.1}$$

Note

$$|I(u) - \Phi(u)| = |\int_\Omega F(x,u)dx| \le c\int_\Omega |u|^{p+1}dx. \tag{3.2}$$

Since

$$\Phi(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4,$$

from (3.1), we have

$$\int_\Omega |u|^{p+1}dx \le c\Phi(u)\|u\|_2^{p-1}.$$

Hence, for all $u \in M_r$, one has

$$\int_\Omega |u|^{p+1}dx \le c\Phi(u)r^{p-1}.$$

Therefore from (3.2), we have

$$(1 - cr^{p-1})\Phi(u) \le I(u) \le (1 + cr^{p-1})\Phi(u),$$

and so

$$(1 - cr^{p-1}) \inf_{K_{n,r}} \sup_K 2\Phi(u) \le \inf_{K_{n,r}} \sup_K 2I(u) \le (1 + cr^{p-1}) \inf_{K_{n,r}} \sup_K 2\Phi(u),$$

i.e.,

$$(1 - cr^{p-1})c_{n,r}^0 \le c_{n,r} \le (1 + cr^{p-1})c_{n,r}^0,$$

which implies that

$$|c_{n,r} - c_{n,r}^0| \le cc_{n,r}^0 r^{p-1}.$$

Now (2.9) guarantees that

$$|c_{n,r} - c_{n,r}^0| \le cr^{p+1}$$

and

$$|c_{n,r}| \le cr^2. \tag{3.3}$$

From Theorem 2.1 and (3.2), we have

$$\begin{aligned} c_{n,r} &= a\|\nabla u_{n,r}\|_2^2 + \frac{b}{2}\|\nabla u_{n,r}\|_2^4 + 2\int_\Omega F(x,u_{n,r})dx \\ &\geq a\|\nabla u_{n,r}\|_2^2 + \frac{b}{2}\|\nabla u_{n.r}\|_2^4 - 2cr^{p-1}\|\nabla u_{n,r}\|_2^2 \\ &= (a-2cr^{p-1})\|\nabla u_{n,r}\|_2^2 + \frac{b}{2}\|\nabla u_{n,r}\|_2^4, \end{aligned}$$

which together with (3.3) implies that

$$\begin{aligned} \|\nabla u_{n,r}\|_2^2 &\leq \frac{\sqrt{(a-2cr^{p-1})^2+2bc_{n,r}}-(a-2cr^{p-1})}{b} \\ &= \frac{2c_{n,r}}{\sqrt{(a-2cr^{p-1})^2+2bc_{n,r}}+(a-2cr^{p-1})} \\ &\leq cr^2 \end{aligned}$$

and

$$\|\nabla u_{n,r}\|_2^4 \leq cr^4.$$

From Theorem 2.1 and (3.2), one has

$$\begin{aligned} |c_{n,r}-r^2\mu_n| &\leq \frac{b}{2}\|\nabla u_{n,r}\|_2^4 + 2|\int_\Omega F(x,u_{n,r})dx| + |\int_\Omega f(x,u_{n,r})u_{n,r}dx| \\ &\leq \frac{b}{2}\|\nabla u_{n,r}\|_2^4 + c|\int_\Omega |u_{n,r}(x)|^{p+1}dx| \\ &\leq \frac{b}{2}\|\nabla u_{n,r}\|_2^4 + c\|\nabla u_{n,r}\|_2^2\|u\|_2^{p-1} \\ &\leq cr^{\min\{4,p+1\}}. \end{aligned}$$

Then

$$\begin{aligned} |r^2\mu_{n,r}-r^2\mu^0_{n,r}(r)| &= |r^2\mu_{n,r}-c_{n,r}+c_{n,r}-c^0_{n,r}+c^0_{n,r}-r^2\mu^0_{n,r}(r)| \\ &\leq |r^2\mu_{n,r}-c_{n,r}|+|c_{n,r}-c^0_{n,r}|+|c^0_{n,r}-r^2\mu^0_{n,r}(r)| \\ &\leq c_1r^{\min\{4,p+1\}}+c_2r^{p+1}+c_3r^4. \\ &\leq cr^{\min\{4,p+1\}}, \end{aligned}$$

which implies that

$$|\mu_{n,r}-\mu^0_{n,r}| \leq cr^{\min\{2,p-1\}}.$$

Consequently,

$$\mu_{n,r} = a\lambda_n + br^2\lambda_n^2 + O(r^{\min\{2,p-1\}}).$$

(b) The case $1+4/N \leq p < 1+8/N$. Let $\gamma = \beta > 0$, which guarantees that $4 \geq p+1-\beta \geq 2$. Then (2.5) becomes

$$\|u\|_{p+1}^{p+1} \leq c\|\nabla u\|_2^{p+1-\beta}\|u\|_2^\beta, u \in W_0^{1,2}(\Omega). \tag{3.4}$$

Since

$$\Phi(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4, \quad \text{and } 4 \geq p+1-\beta \geq 2,$$

(3.4) guarantees that

$$\int_\Omega |u|^{p+1}dx \le c\Phi(u)\|u\|_2^\beta,$$

and so,

$$\int_\Omega |u|^{p+1}dx \le c\Phi(u)r^\beta, \quad \forall u \in M_r,$$

Therefore from (3.2), we have

$$(1-cr^\beta)\Phi(u) \le I(u) \le (1+cr^\beta)\Phi(u),$$

and so

$$(1-cr^\beta)\inf_{K_{n,r}}\sup_K 2\Phi(u) \le \inf_{K_{n,r}}\sup_K 2I(u) \le (1+cr^\beta)\inf_{K_{n,r}}\sup_K 2\Phi(u),$$

i.e.,

$$(1-cr^\beta)c_{n,r}^0 \le c_{n,r} \le (1+cr^\beta)c_{n,r}^0,$$

which implies that

$$|c_{n,r}-c_{n,r}^0| \le cc_{n,r}^0 r^\beta.$$

Now (2.9) guarantees that

$$|c_{n,r}-c_{n,r}^0| \le cr^{\beta+2}.$$

Since $\beta+2 \ge p+1$, we have

$$|c_{n,r}-c_{n,r}^0| \le cr^{p+1} \text{ and } c_{n,r} \le c_{n,r}^0 + cr^{p+1} \le cr^2 \tag{3.5}$$

for some new constant c. In the following we show that

$$\|\nabla u_{n,r}\|_2^2 \le cr^2 \text{ and } \|\nabla u_{n,r}\|_2^4 \le cr^4. \tag{3.6}$$

From Theorem 2.1, we have

$$\begin{aligned}
c_{n,r} &= 2I(u_{n,r})\\
&= a\|\nabla u_{n,r}\|_2^2 + \frac{b}{2}\|\nabla u_{n.r}\|_2^4 + 2\int_\Omega F(x,u_{n,r})dx\\
&\ge a\|\nabla u_{n,r}\|_2^2 + \frac{b}{2}\|\nabla u_{n,r}\|_2^4 - 2c\|\nabla u_{n,r}\|_2^{p+1-\beta}r^\beta\\
&\ge (a-2cr^\beta)\|\nabla u_{n,r}\|_2^2 + \frac{b}{2}\|\nabla u_{n,r}\|_2^4, \text{ if } \|\nabla u_{n,r}\|_2 \le 1,
\end{aligned}$$

which together with (3.5) implies that

$$\begin{aligned}
\|\nabla u_{n,r}\|_2^2 &\le \frac{\sqrt{(a-2cr^\beta)^2+2bc_{n,r}}-(a-2cr^\beta)}{b}\\
&= \frac{2c_{n,r}}{\sqrt{(a-2cr^\beta)^2+2bc_{n,r}}+(a-2cr^\beta)}\\
&\le cr^2
\end{aligned}$$

and

$$\|\nabla u_{n,r}\|_2^4 \le cr^4.$$

Similarly, from

$$\begin{aligned}c_{n,r} &= 2I(u_{n,r})\\ &= a\|\nabla u_{n,r}\|_2^2 + \frac{b}{2}\|\nabla u_{n.r}\|_2^4 + 2\int_\Omega F(x,u_{n,r})dx\\ &\geq a\|\nabla u_{n,r}\|_2^2 + \frac{b}{2}\|\nabla u_{n,r}\|_2^4 - 2c\|\nabla u_{n,r}\|_2^{p+1-\beta} r^\beta\\ &\geq a\|\nabla u_{n,r}\|_2^2 + (\frac{b}{2} - 2cr^\beta)\|\nabla u_{n,r}\|_2^4,\ \text{if } \|\nabla u_{n,r}\|_2 > 1,\end{aligned}$$

we get

$$\begin{aligned}\|\nabla u_{n,r}\|_2^2 &\leq \frac{\sqrt{a^2 + 4c_{n,r}(\frac{b}{2} - 2cr^\beta)} - a}{2(\frac{b}{2} - 2cr^\beta)}\\ &= \frac{2c_{n,r}}{\sqrt{a^2 + 4c_{n,r}(\frac{b}{2} - 2cr^\beta)} + a}\\ &\leq cr^2\end{aligned}$$

and

$$\|\nabla u_{n,r}\|_2^4 \leq cr^4.$$

Hence, (3.6) is true.

From the definitions of $c_{n,r}$ and $\mu_{n,r}$, we have, from (3.6),

$$\begin{aligned}|c_{n,r} - r^2\mu_n| &\leq \frac{b}{2}\|\nabla u_{n,r}\|_2^4 + |2\int_\Omega F(x,u_{n,r})dx - \int_\Omega f(x,u_{n,r})u_{n,r}dx|\\ &\leq \frac{b}{2}\|\nabla u_{n,r}\|_2^4 + c|\int_\Omega |u_{n,r}(x)|^{p+1}dx|\\ &\leq \frac{b}{2}\|\nabla u_{n,r}\|_2^4 + c\|\nabla u_{n,r}\|_2^{p+1-\beta}\|u\|_2^\beta\\ &\leq cr^{\min\{4,p+1\}},\end{aligned}$$

and so

$$\begin{aligned}|r^2\mu_{n,r} - r^2\mu_{n,r}^0(r)| &= |r^2\mu_{n,r} - c_{n,r} + c_{n,r} - c_{n,r}^0 + c_{n,r}^0 - r^2\mu_{n,r}^0(r)|\\ &\leq |r^2\mu_{n,r} - c_{n,r}| + |c_{n,r} - c_{n,r}^0| + |c_{n,r}^0 - r^2\mu_{n,r}^0(r)|\\ &\leq c_1 r^{\min\{4,p+1\}} + c_2 r^{p+1} + c_3 r^4.\\ &\leq cr^{\min\{4,p+1\}},\end{aligned}$$

which means that

$$|\mu_{n,r} - \mu_{n,r}^0| \leq cr^{\min\{2,p-1\}}.$$

Consequently,

$$\mu_{n,r} = a\lambda_n + br^2\lambda_n^2 + O(r^{\min\{2,p-1\}}).$$

The proof is complete. □

4. The asymptotic distribution of the eigenvalue $\mu_{n,r}$ of $(1.1)_\lambda$

In this section, we consider the asymptotic laws of the eigenvalue $\mu_{n,r}$ of $(1.1)_\lambda$.

Lemma 4.1. *Assume (A_1) holds. For $r > 0$ and $n = 1, 2, ...,$ let $\mu_{n,r}$, $c_{n,r}$ be as in Theorem 2.1 , and let λ_n be the eigenvalues of the linear problem $(2.1)_\lambda$. Then*

$$|c_{n,r} - c^0_{n,r}| \le cr^\beta (c^0_{n,r})^\alpha + dr \tag{4.1}$$

and

$$|c_{n,r} - \frac{1}{2} r^2 \mu_{n,r}| \le \frac{a}{2}\sqrt{2bc_{n,r}} + O(c_{n,r}^{-\frac{1}{2}}) + cr^\beta(\sqrt{2bc_{n,r}})^\alpha (1 + O(c_{n,r}^{\frac{\alpha}{2}-1})), \tag{4.2}$$

where $\alpha = (p-1)N/4$ and $\beta = (p+1)-(p-1)N/2$; here and henceforth c, d denote some, but not always the same, positive constants.

Proof. First notice that the growth assumption (A_1) implies

$$|\int_\Omega F(x,u)dx| \le c\int_\Omega |u|^{p+1}dx + d\int_\Omega |u|dx,$$

and similarly

$$|\int_\Omega f(x,u)udx| \le c\int_\Omega |u|^{p+1}dx + d\int_\Omega |u|dx.$$

Next, as $1 \le p < p_0$, from Lemma 2.1, if $\int_\Omega u^2dx = r^2$, we have

$$|\int_\Omega F(x,u)dx| \le c\|\nabla u\|_2^{2\alpha} r^\beta + dr, \tag{4.3}$$

and similarly

$$|\int_\Omega f(x,u)udx| \le c\|\nabla u\|_2^{2\alpha} r^\beta + dr, \tag{4.4}$$

with α and β as in the statement of Lemma 4.1.

To prove (4.1), observe that (4.3) implies

$$\begin{aligned} I(u) &= \Phi(u) + \int_\Omega F(x,u)dx \\ &\le \Phi(u) + r^\beta(\Phi(u))^\alpha + dr \end{aligned}$$

holds. In other words, we have

$$I(u) \le g(\Phi(u))$$

where $g : R^+ \to R^+$ is defined by

$$g(t) = t + cr^\beta t^\alpha + dr.$$

As g is continuous and nondecreasing, we get

$$\inf_{K_{n,r}} \sup_{K\in K_{n,r}} I(u) \le \inf_{K_{n,r}} \sup_{K\in K_{n,r}} g(\Phi(u)) = g(\inf_{K_{n,r}} \sup_{K\in K_{n,r}} \Phi(u)).$$

Now Theorem 2.1 implies that

$$c_{n,r} \leq 2g(c^0_{n,r}) = c^0_{n,r} + cr^\beta (c^0_{n,r})^\alpha + dr$$

for some new constants c and $d > 0$. Therefore,

$$|c_{n,r} - c^0_{n,r}| \leq cr^\beta (c^0_{n,r})^\alpha + dr,$$

which shows (4.1) is true.

Since

$$c_{n,r} = a\|u_{n,r}\|_2^2 + \frac{b}{2}\|u_{n,r}\|_2^4 + 2\int_\Omega F(x,u)dx,$$

we have

$$\|\nabla u_{n,r}\|_2^2 = \frac{\sqrt{a^2 + 2b(c_{n,r} - 2\int_\Omega F(x,u_{n,r})dx)} - a}{b} = \sqrt{2bc_{n,r}} + O(c_{n,r}^{-\frac{1}{2}}).$$

From Theorem 2.1 and (4.3)-(4.4), we have

$$\begin{aligned}|c_{n,r} - \tfrac{1}{2}r^2\mu_{n,r}| &= |\frac{a}{2}\|\nabla u_{n,r}\|_2^2 + 2\int_\Omega F(x,u_{n,r})dx - \frac{1}{2}\int_\Omega f(x,u_{n,r})u_{n,r}dx| \\ &\leq \frac{a}{2}(\sqrt{2bc_{n,r}} + O(c_{n,r}^{-\frac{1}{2}})) + cr^\beta(\sqrt{2bc_{n,r}} + O(c_{n,r}^{-\frac{1}{2}}))^\alpha \\ &= \frac{a}{2}\sqrt{2bc_{n,r}} + O(c_{n,r}^{-\frac{1}{2}}) + cr^\beta(\sqrt{2bc_{n,r}})^\alpha(1 + O(c_{n,r}^{\frac{\alpha}{2}-1})),\end{aligned}$$

which completes the proof of the lemma.

□

Lemma 4.2 (Theorem 2, [7]). *The eigenvalues λ_n of $(2.1)_\lambda$ satisfy, as $n \to +\infty$*

$$\lambda_n = kn^{2/N} + O(n^{1/N}\log n), \quad n = 1, 2, \cdots, \tag{4.5}$$

where

$$k = (2\pi)^2(V)^{-2/N} \tag{4.6}$$

and V is the value of $B(\theta, 1)$.

Theorem 4.1. *Assume that (A_1) holds. Then given any $r > 0$, $(1.1)_\lambda$ has infinitely many eigenfunctions $u_{n,r}(n = 1, 2, ...)$ with $\int_\Omega u_{n,r}^2 dx = r^2$, whose corresponding eigenvalues $\mu_{n,r}$ satisfy, as $n \to +\infty$ and with k as in (4.6),*

$$\mu_{n,r} = \begin{cases} br^2kn^{4/N} + O(n^{3/N}\log n)), & \text{if } 0 < p \leq (N+4)/N; \\ O(kn^{4\alpha/N}), & \text{if } (N+4)/N < p < \overline{p}, \end{cases}$$

where $\overline{p}$ is defined in (A_1).

Proof. (1) We consider the case $\alpha < (p-1)(N/4) \leq 1$, i.e., $p \leq (N+4)/N$, which implies that $0 < 2\alpha \leq 1$. Then, we have

$$\begin{aligned} c_{n,r} &= c^0_{n,r} + O((c^0_{n,r})^\alpha) \\ &= \frac{b}{2}r^4\lambda_n^2 + O(\lambda_n) + O(\frac{b}{2}r^4\lambda_n^2 + O(\lambda_n))^\alpha \\ &= \frac{b}{2}r^4\lambda_n^2 + O(\lambda_n) + O(\frac{b}{2}r^4\lambda_n^2(1 + O(\lambda_n^{-1}))^\alpha \\ &= \frac{b}{2}r^4\lambda_n^2 + O(\lambda_n) + O((\frac{b}{2}r^4)^\alpha\lambda_n^{2\alpha}(1 + O(\lambda_n^{2\alpha-1}))) \\ &= \frac{b}{2}r^4\lambda_n^2 + O(\lambda_n) \end{aligned}$$

and

$$\begin{aligned}(c_{n,r})^{\alpha} &= (\frac{b}{2}r^4\lambda_n^2 + O(\lambda_n))^{\alpha} \\ &= (\frac{b}{2}r^4\lambda_n^2(1+O(\lambda_n^{-1}))^{\alpha} \\ &= (\frac{b}{2}r^4)^{\alpha}\lambda_n^{2\alpha}(1+O(\lambda_n^{-1})) \\ &= O(\lambda_n), \\ (c_{n,r})^{\frac{1}{2}} &= O(\lambda_n),\end{aligned}$$

which together (4.1) and (4.2) imply that

$$\begin{aligned}|\frac{1}{2}r^2\mu_{n,r} - c^0_{n,r}| &= |\frac{1}{2}r^2\mu_{n,r} - c_{n,r} + c_{n,r} - c^0_{n,r}| \\ &\le |\frac{1}{2}r^2\mu_{n,r} - c_{n,r}| + |c_{n,r} - c^0_{n,r}| \\ &\le \frac{a}{2}\sqrt{2bc_{n,r}} + O(c_{n,r}^{-\frac{1}{2}}) + cr^{\beta}(\sqrt{2bc_{n,r}})^{\alpha}(1+O(c_{n,r}^{\frac{\alpha}{2}-1})) + cr^{\beta}(c^0_{n,r})^{\alpha} + dr \\ &= O(\lambda_n),\end{aligned}$$

and so

$$\frac{1}{2}r^2\mu_{n,r} = c^0_{n,r} + O(\lambda_n) = ar^2\lambda_n + \frac{b}{2}r^4\lambda_n^2 + O(\lambda_n).$$

Consequently

$$\mu_{n,r} = 2a\lambda_n + br^2\lambda_n^2 + O(\lambda_n) = br^2\lambda_n^2 + O(\lambda_n).$$

Since

$$\lambda_n = kn^{2/N} + O(n^{1/N}\log n),$$

we have

$$\begin{aligned}\mu_{n,r} &= br^2(kn^{2/N} + O(n^{1/N}\log n))^2 + O(kn^{2/N} + O(n^{1/N}\log n)) \\ &= br^2k^2n^{4/N} + O(n^{3/N}\log n).\end{aligned}$$

(2) We consider the case $(N+4)/N < p < \overline{p}$, i.e., $2 \ge \alpha > 1$. Lemma 4.1 guarantees that

$$c_{n,r} = c^0_{n,r} + O((c^0_{n,r}))^{\alpha} = O(\lambda_n^{2\alpha})$$

and

$$(c_{n,r})^{\alpha} = O(\lambda_n^{2\alpha^2}),$$

which together (4.1) and (4.2) imply that

$$\begin{aligned}|\frac{1}{2}r^2\mu_{n,r} - c^0_{n,r}| &= |\frac{1}{2}r^2\mu_{n,r} - c_{n,r} + c_{n,r} - c^0_{n,r}| \\ &\le |\frac{1}{2}r^2\mu_{n,r} - c_{n,r}| + |c_{n,r} - c^0_{n,r}| \\ &\le \frac{a}{2}\sqrt{2bc_{n,r}} + O(c_{n,r}^{-\frac{1}{2}}) + cr^{\beta}(\sqrt{2bc_{n,r}})^{\alpha}(1+O(c_{n,r}^{\frac{\alpha}{2}-1})) + cr^{\beta}(c^0_{n,r})^{\alpha} + dr \\ &= O(\lambda_n^{2\alpha}) + O(\lambda_n^{\alpha^2}) \\ &= O(\lambda_n^{2\alpha}),\end{aligned}$$

and so

$$\frac{1}{2}r^2\mu_{n,r} = c^0_{n,r} + O(\lambda_n^{2\alpha}) = O(\lambda_n^{2\alpha}).$$

Consequently

$$\mu_{n,r} = O(\lambda_n^{2\alpha}).$$

Since

$$\lambda_n = kn^{2/N} + O(n^{1/N}\log n),$$

then

$$\mu_{n,r} = O(kn^{4\alpha/N}).$$

The proof is complete. □

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