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AN EFFICIENT STEP METHOD FOR A SYSTEM OF DIFFERENTIAL EQUATIONS WITH DELAY

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Abstract Using the step method, we study a system of delay differential equations and we prove the existence and uniqueness of the solution and the convergence of the successive approximation sequence using the Perov's contraction principle and the step method. Also, we propose a new algorithm of successive approximation sequence generated by the step method and, as an example, we consider some second order delay differential equations with initial conditions.

Keywords System of delay differential equations, step method, Picard operators, generalized fibre contraction principle.

MSC(2010) 47H10, 47N20, 45G10, 45G15.

1. Introduction

We consider the system of delay differential equations

$$\begin{cases} x_1'(t) = f_1(t, x_1(t), x_2(t), x_1(t-h), x_2(t-h)),\ t \in [a, b] \\ x_2'(t) = f_2(t, x_1(t), x_2(t), x_1(t-h), x_2(t-h)) \end{cases} \tag{1.1}$$

with initial conditions

$$\begin{cases} x_1(t) = \varphi_1(t),\ t \in [a-h, a] \\ x_2(t) = \varphi_2(t), \end{cases} \tag{1.2}$$

where $f_1, f_2 \in C([a, b] \times \mathbb{R}^4, \mathbb{R}),\ \varphi_1, \varphi_2 \in C([a-h, a], \mathbb{R})$ and $h > 0$ is a parameter. We denote by $\mathbf{x} = (x_1, x_2),\ \mathbf{f} = (f_1, f_2)$ and $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$. By a solution of the problem (1.1)-(1.2) we mean a function $\mathbf{x} \in C([a-h, b], \mathbb{R}^2) \cap C^1([a, b], \mathbb{R}^2)$ which satisfies the system (1.1) and the conditions (1.2).

In this paper we study this problem using the ideas of I. A. Rus [14] to obtain existence, uniqueness theorems and the convergence of an iterative algorithm using Perov's theorem, fibre contraction principle and step method. As an application, we

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consider a second order functional differential equation with delay and we approximate the solution using the Chebyshev spectral method (see [18–21]). We compare the obtained results with Matlab dde23 procedure.

Such kind of results have been proved in [15] and [4] in the case of integro-differential equations with lags and in [2] in the case of an integral equation from biomathematics. Other results regarding efficient and rapidly convergent algorithms for solving Volterra differential and integral equations can be found in [5, 6, 8].

Let (X, d) be a metric space and $A : X \to X$ an operator. In this paper we use the terminologies and notations from [13]. For the convenience of the reader we shall recall some of them.

Denote by $A^0 := 1_X,\ A^1 := A,\ A^{n+1} := A \circ A^n,\ n \in \mathbb{N}$, the iterate operators of the operator A and by $F_A := \{x \in X |\ A(x) = x\}$ the fixed point set of A.

Definition 1.1. $A : X \to X$ is called a Picard operator (briefly PO) if: $F_A = \{x^*\}$ and $A^n(x) \to x^*$ as $n \to \infty$, for all $x \in X$.

Definition 1.2. $A : X \to X$ is said to be a weakly Picard operator (briefly WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of A.

Definition 1.3. A matrix $Q \in \mathbb{R}_+^{2\times 2}$ is called a matrix convergent to zero iff $Q^k \to 0$ as $k \to \infty$.

As concerns matrices which are convergent to zero, we mention the following equivalent characterizations:

Theorem 1.1. (see [10]) *Let $Q \in \mathbb{R}_+^{2\times 2}$. The following statements are equivalent:*

- *(i) Q is a matrix convergent to zero;*
- *(ii) $Q^k x \to 0$ as $k \to \infty,\ \forall x \in \mathbb{R}^2$;*
- *(iii) $I_2 - Q$ is non-singular and $(I_2 - Q)^{-1} = I_2 + Q + Q^2 + \ldots$;*
- *(iv) $I_2 - Q$ is non-singular and $(I_2 - Q)^{-1}$ has nonnegative elements;*
- *(v) $\lambda \in \mathbb{C},\ \det(Q - \lambda I_2) = 0$ imply $|\lambda| < 1$;*
- *(v) there exits at least one subordinate matrix norm such that $\|Q\| < 1$.*

The matrices convergent to zero were used by Perov [9] to generalize the contraction principle in the case of generalized metric spaces with the metric taking values in the positive cone of $\mathbb{R}^2$.

Definition 1.4 ([9])**.** Let (X, d) be a complete generalized metric space with $d : X \times X \to \mathbb{R}_+^2$ and $A : X \to X$. The operator A is called a Q-contraction if there exists a matrix $Q \in \mathbb{R}_+^{2\times 2}$ such that:

- (i) Q is a matrix convergent to zero;
- (ii) $d(A(x), A(y)) \le Q d(x, y),\ \forall x, y \in X$.

Theorem 1.2 (Perov, [2, 12])**.** *Let (X, d) be a complete generalized metric space with $d : X \times X \to \mathbb{R}_+^2$ and $A : X \to X$ be a Q-contraction. Then*

- *(i) A is a Picard operator, $F_A = F_{A^n} = \{x^*\},\ \forall n \in \mathbb{N}^*$;*
- *(ii) $d(A^n(x), x^*) \le (I_2 - Q)^{-1} Q^n d(x, A(x)), \forall x \in X$.*

Finally, we recall the following result that is a generalization of the fibre contraction theorem (see I. A. Rus [12], [2]):

Theorem 1.3 (Theorem 9.1., [11]). *Let (X_i, d_i), $i = \overline{0,m}$, $m \geq 1$, be some generalized metric spaces. Let $A_i : X_0 \times \cdots \times X_i \to X_i$, $i = \overline{0,m}$, be some operators. We suppose that:*

(i) (X_i, d_i), $i = \overline{1,m}$, *are generalized complete metric spaces;*

(ii) *the operator A_0 is a weakly Picard operator;*

(iii) *there exist the matrices $Q_i \in \mathbb{R}_+^{2\times 2}$ which converge to zero, such that the operators $A_i(x^0, \ldots, x^{i-1}, \cdot) : X_i \to X_i$, $i = \overline{1,m}$ are Q_i-generalized contractions, for all $x^i \in X_i, i = \overline{1,m}$;*

(iv) *the operators A_i, $i = \overline{1,m}$, are continuous.*

Then the operator $A : X_0 \times \cdots \times X_m \to X_0 \times \cdots \times X_m$,

$$A(x^0, \ldots, x^m) = (A_0(x^0), A_1(x^0, x^1), \ldots, A_m(x^0, \ldots, x^m))$$

is a weakly Picard operator. Moreover, if A_0 is a Picard operator, then A is a Picard operator.

2. Main result

We begin this section with an existence theorem for the solution of the problem (1.1)-(1.2). We denote by $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}_+^2$ the vectorial norm

$$\|\mathbf{u}\| := \begin{pmatrix} |u_1| \\ |u_2| \end{pmatrix}, \ \mathbf{u} = (u_1, u_2) \in \mathbb{R}^2.$$

Relative to the problem (1.1)-(1.2) we consider the following conditions:

(H_1) $\mathbf{f} \in C([a,b] \times \mathbb{R}^4, \mathbb{R}^2)$, $\boldsymbol{\varphi} \in C([a-h,a], \mathbb{R}^2)$, $h \in \mathbb{R}_+^*$, $a, b \in \mathbb{R}$, $a < b$;

(H_2) there exists $L \in \mathbb{R}_+^{2\times 2}$ such that

$$\left\|\mathbf{f}(t, \mathbf{u}^1, \mathbf{v}^1) - \mathbf{f}(t, \mathbf{u}^2, \mathbf{v}^2)\right\| \leq L(\left\|\mathbf{u}^1 - \mathbf{u}^2\right\| + \left\|\mathbf{v}^1 - \mathbf{v}^2\right\|),$$

$t \in [a,b], \mathbf{u}^1, \mathbf{u}^2, \mathbf{v}^1, \mathbf{v}^2 \in \mathbb{R}^2$;

(H_2') there exists $L' \in \mathbb{R}_+^{2\times 2}$ such that

$$\left\|\mathbf{f}(t, \mathbf{u}^1, \mathbf{v}) - \mathbf{f}(t, \mathbf{u}^2, \mathbf{v})\right\| \leq L'(\left\|\mathbf{u}^1 - \mathbf{u}^2\right\|),$$

$t \in [a,b], \mathbf{u}^1, \mathbf{u}^2, \mathbf{v} \in \mathbb{R}^2$.

We consider the space $X := C([a-h,b], \mathbb{R}^2)$ endowed with the generalized norm $\|\cdot\|_B$ where $\|\mathbf{x}\|_B := \binom{|x_1|_B}{|x_2|_B}$, $\mathbf{x} = (x_1, x_2)$ and

$$|x_i|_B := \max_{a-h \leq t \leq b} (|x_i(t)|\, e^{-\tau(t-a+h)}), \ \tau > 0, i = 1, 2.$$

It is clear that the space $(X, \|\cdot\|_B)$ is a generalized Banach space. Any solution of the problem (1.1)-(1.2) is a fixed point of the operator $A_f : X \to X$, defined by

$$A_f(\mathbf{x})(t) = \begin{cases} \boldsymbol{\varphi}(t), \ t \in [a-h, a] \\ \boldsymbol{\varphi}(a) + \int\limits_{t_0}^{t} \mathbf{f}(s, \mathbf{x}(s), \mathbf{x}(s-h))ds, t \in [a, b]. \end{cases} \tag{2.1}$$

Let $m \in \mathbb{N}^*$ be such that:

$$a + (m-1)h < b \text{ and } a + mh \geq b.$$

We denote by $t_{-1} := a - h,\ t_0 := a,\ t_i := a + ih,\ i = \overline{1, m-1},\ t_m := b.$

The following result is well known (see [7]).

Theorem 2.1. *We suppose that the conditions (H_1) and (H_2) hold. Then:*

(i) the problem (1.1)-(1.2) has a unique solution $\mathbf{x}^* \in C([t_{-1}, t_m], \mathbb{R}^2) \cap C^1([t_0, t_m], \mathbb{R}^2)$;

(ii) the successive approximations sequence $(\mathbf{x}^n)_{n \in \mathbb{N}^*}$, defined by

$$\mathbf{x}^{n+1}(t) := \begin{cases} \boldsymbol{\varphi}(t), \ t \in [t_{-1}, t_0] \\ \boldsymbol{\varphi}(t_0) + \int\limits_{t_0}^{t} \mathbf{f}(s, \mathbf{x}^n(s), \mathbf{x}^n(s-h))ds, t \in [t_0, t_m] \end{cases}$$

converges to $\mathbf{x}^*$, $\forall \mathbf{x}^0 \in C([t_{-1}, t_m], \mathbb{R}^2)$;

(iii) the operator A_f is a Picard operator.

Proof. In a standard way we obtain

$$\|A_f(\mathbf{x}) - A_f(\mathbf{y})\|_B \leq \frac{1}{\tau} L \|\mathbf{x} - \mathbf{y}\|_B, \forall \mathbf{x}, \mathbf{y} \in X.$$

We can choose τ sufficiently large such that A_f is Q-contraction with $Q := \frac{1}{\tau}L$. So we can apply the Perov's Theorem (Theorem 1.2) for $A_f : X \to X$. □

Delay differential equations may be solved as ordinary differential equations over successive intervals $[t_m, t_{m+1}]$ by the step method (see, for example [3] or [1]).

Under the condition (H_1), the step method for the problem (1.1)-(1.2) consists of the following equations:

(p^0) $\mathbf{x}^0(t) = \boldsymbol{\varphi}(t), t \in [t_{-1}, t_0];$

(p^1) $\mathbf{x}^1(t) = \boldsymbol{\varphi}(t_0) + \int_{t_0}^{t} \mathbf{f}(s, \mathbf{x}^1(s), \boldsymbol{\varphi}(s-h))ds,\ t \in [t_0, t_1];$

(p^2) $\mathbf{x}^2(t) = \mathbf{x}^{1,*}(t_1) + \int_{t_1}^{t} \mathbf{f}(s, \mathbf{x}^2(s), \mathbf{x}^{1,*}(s-h))ds,\ t \in [t_1, t_2];$

...

(p^{m-1}) $\mathbf{x}^{m-1}(t) = \mathbf{x}^{m-2,*}(t_{m-2}) + \int_{t_{m-2}}^{t} \mathbf{f}(s, \mathbf{x}^{m-1}(s), \mathbf{x}^{m-2,*}(s-h))ds, t \in [t_{m-2}, t_{m-1}];$

(p^m) $\mathbf{x}^m(t) = \mathbf{x}^{m-1,*}(t_{m-1}) + \int_{t_{m-1}}^{t} \mathbf{f}(s, \mathbf{x}^m(s), \mathbf{x}^{m-1,*}(s-h))ds,\ t \in [t_{m-1}, t_m];$

where $\mathbf{x}^{i,*} = (x_1^{i,*}, x_2^{i,*}) \in C([t_{i-1}, t_i], \mathbb{R}^2)$ is the unique solution of the equation $(p^i),\ i = \overline{1, m}.$

So, by using the step method and an idea from [14], we obtain:

Theorem 2.2. *We suppose that the conditions (H_1) and (H_2') hold. Then:*

(i) the problem (1.1)-(1.2) has a unique solution $\mathbf{x}^*$ *in* $C([t_{-1}, t_m], \mathbb{R}^2)$, *where*

$$\mathbf{x}^*(t) = \begin{cases} \boldsymbol{\varphi}(t), t \in [t_{-1}, t_0] \\ \mathbf{x}^{1,*}(t), t \in [t_0, t_1] \\ \dots \\ \mathbf{x}^{m,*}(t), t \in [t_{m-1}, t_m] \end{cases}$$

(ii) for each $\mathbf{x}^{i,0} = (x_1^{i,0}, x_2^{i,0}) \in C([t_{i-1}, t_i], \mathbb{R}^2), i = \overline{1, m}$, *the sequence defined by:*

$$\mathbf{x}^{i,n+1}(t) = \mathbf{x}^{i-1,*}(t_{i-1}) + \int_{t_{i-1}}^{t} \mathbf{f}(s, \mathbf{x}^{i,n}(s), \mathbf{x}^{i-1,*}(s-h))ds,$$

for $t \in [t_{i-1}, t_i]$, *(with* $\mathbf{x}^{0,*}(t_0) := \boldsymbol{\varphi}(t_0)$*), converges and* $\lim_{n\to\infty} \mathbf{x}^{i,n} = \mathbf{x}^{i,*}$, $i = \overline{1, m}$.

Proof. In order to prove this theorem we apply Perov's theorem for each step $[t_{i-1}, t_i]$, $i = \overline{1, m}$.

For the first step, we consider the Banach space $X_1 := (C([t_0, t_1], \mathbb{R}^2), \|\cdot\|_{1B})$, where

$$\|\cdot\|_{1B} := \max_{t_0 \le t \le t_1} (\|\mathbf{x}(t)\| \, e^{-\tau(t-t_0)}), \ \tau > 0$$

and the operator $A_1 : X_1 \to X_1$ defined by

$$A_1(\mathbf{x})(t) = \boldsymbol{\varphi}(t_0) + \int_{t_0}^{t} \mathbf{f}(s, \mathbf{x}(s), \boldsymbol{\varphi}(s-h))ds.$$

For $\mathbf{x}, \mathbf{y} \in X_1$, we obtain

$$\|A_1(\mathbf{x}) - A_1(\mathbf{y})\|_{1B} \le \frac{1}{\tau} L' \, \|\mathbf{x} - \mathbf{y}\|_{1B} \, .$$

We can choose τ sufficiently large such that A_1 is $Q_1 := \frac{1}{\tau} L'$-contraction, therefore $F_{A_1} := \{\mathbf{x}_1^*\}$.

For the next steps, we consider the Banach spaces $X_i := (C([t_{i-1}, t_i], \mathbb{R}^2), \|\cdot\|_{iB}), i = \overline{2, m}$, where

$$\|\mathbf{x}\|_{iB} := \max_{t_{i-1} \le t \le t_i} (\|\mathbf{x}(t)\| \, e^{-\tau(t-t_{i-1})}), \ \tau > 0,$$

and the operators $A_i : X_i \to X_i$, defined by

$$A_i(\mathbf{x})(t) = \mathbf{x}^{i-1,*}(t_{i-1}) + \int_{t_{i-1}}^{t} \mathbf{f}(s, \mathbf{x}(s), \mathbf{x}^{i-1,*}(s-h))ds.$$

For $\mathbf{x}, \mathbf{y} \in X_i$, we obtain

$$\|A_i(\mathbf{x}) - A_i(\mathbf{y})\|_{iB} \le \frac{1}{\tau} L' \, \|\mathbf{x} - \mathbf{y}\|_{iB} \, .$$

We can choose τ sufficiently large such that A_i is $Q_i := \frac{1}{\tau} L'$-contraction, therefore $F_{A_i} := \{\mathbf{x}^{i,*}\}$, $i = \overline{2, m}$.

We have that $\boldsymbol{\varphi}(t_0) = \mathbf{x}^{1,*}(t_0)$ and from definition of $A_i, i = \overline{2, m}$, we obtain

$$\mathbf{x}^{i-1,*}(t_{i-1}) = \mathbf{x}^{i,*}(t_{i-1}), \ i = \overline{2, m},$$

therefore

$$\mathbf{x}^*(t) = \begin{cases} \boldsymbol{\varphi}(t), t \in [t_{-1}, t_0] \\ \mathbf{x}^{1,*}(t), t \in [t_0, t_1] \\ \dots \\ \mathbf{x}^{m,*}(t), t \in [t_{m-1}, t_m] \end{cases}$$

is the unique solution in $C([t_{-1}, t_m], \mathbb{R}^2)$. □

Next, we will study if it is possible to replace $\mathbf{x}^{i,*}$ by the approximation $\mathbf{x}^{i,n}$, $i = \overline{1,m}$ in the conclusion (ii) of the Theorem 2.2. Applying the results from [14] we have

Theorem 2.3. *In the condition of Theorem 2.2, for each* $\mathbf{x}^{i,0} \in C([t_{i-1}, t_i], \mathbb{R}^2), i = \overline{1,m}$, *the sequences defined by:*

$$\begin{aligned} &\mathbf{x}^{1,n+1}(t) = \boldsymbol{\varphi}(t_0) + \int_{t_0}^{t} \mathbf{f}(s, \mathbf{x}^{1,n}(s), \boldsymbol{\varphi}(s-h))ds, \ \textit{for } t \in [t_0, t_1] \\ &\mathbf{x}^{2,n+1}(t) = \mathbf{x}^{1,n}(t_1) + \int_{t_1}^{t} \mathbf{f}(s, \mathbf{x}^{2,n}(s), \mathbf{x}^{1,n}(s-h))ds, \ \textit{for } t \in [t_1, t_2] \\ &\qquad \dots \\ &\mathbf{x}^{m,n+1}(t) = \mathbf{x}^{m-1,n}(t_{m-1}) + \int_{t_{m-1}}^{t} \mathbf{f}(s, \mathbf{x}^{m,n}(s), \mathbf{x}^{m-1,n}(s-h))ds, \ \textit{for } t \in [t_{m-1}, t_m] \end{aligned} \tag{2.2}$$

converge and $\lim\limits_{n\to\infty} \mathbf{x}^{i,n} = \mathbf{x}^{i,*}, \ i = \overline{1,m}$.

Proof. We consider the following Banach spaces $X_0 := (C([t_{-1}, t_0], \mathbb{R}^2), \|\cdot\|_{0B})$, where

$$\|\cdot\|_{0B} := \max_{t_{-1} \le t \le t_0} (\|\mathbf{x}(t)\| \, e^{-\tau(t-t_{-1})}), \ \tau > 0$$

and $X_i := (C([t_{i-1}, t_i], \mathbb{R}^2), \|\cdot\|_{iB}), \ i = \overline{1,m}$ (as in the proof of Theorem 2.2) and the operators

$$\begin{aligned} &A_0 : X_0 \to X_0, \ A_0(\mathbf{x}^0)(t) = \boldsymbol{\varphi}(t), \ t \in [t_{-1}, t_0], \\ &A_i : X_{i-1} \times X_i \to X_i, i = \overline{1,m} \\ &A_i(\mathbf{x}^{i-1}, \mathbf{x}^i)(t) = \mathbf{x}^{i-1}(t_{i-1}) + \int_{t_{i-1}}^{t} \mathbf{f}(s, \mathbf{x}^i(s), \mathbf{x}^{i-1}(s-h))ds, t \in [t_{i-1}, t_i], \end{aligned}$$

and let A be the operator $A : X_0 \times X_1 \times \cdots \times X_m \to X_0 \times X_1 \times \cdots \times X_m$ defined by

$$A(\mathbf{x}^0, \mathbf{x}^1, \dots, \mathbf{x}^m) = (A_0(\mathbf{x}^0), A_1(\mathbf{x}^0, \mathbf{x}^1), \dots, A_m(\mathbf{x}^{m-1}, \mathbf{x}^m)).$$

It is easy to see that for fixed $(\mathbf{x}^0, \mathbf{x}^1, \dots, \mathbf{x}^m) \in X_0 \times X_1 \times \cdots \times X_m$ the sequence defined by (2.2) means $(\mathbf{x}^{0,n}, \mathbf{x}^{1,n}, \dots, \mathbf{x}^{m,n}) = A^n(\mathbf{x}^0, \mathbf{x}^1, \dots, \mathbf{x}^m)$. To prove the conclusion we need to prove that the operator A is a Picard operator and for this we apply Theorem 1.3.

Since $A_0 : X_0 \to X_0$ is a constant operator then A_0 is Q_0-contraction where Q_0 is the null matrix, so A_0 is a Picard operator and $\mathbf{x}^{0,*} = \boldsymbol{\varphi}$. For $i = \overline{1,m}$, we have the inequalities:

$$\left\|A_i(\mathbf{x}^{i-1}, \mathbf{x}^i) - A_i(\mathbf{x}^{i-1}, \mathbf{y}^i)\right\|_{iB} \le \tfrac{1}{\tau} L' \left\|\mathbf{x}^i - \mathbf{y}^i\right\|_{iB}$$

for all $\mathbf{x}^{i-1} \in X_{i-1}$ and $\mathbf{x}^i, \mathbf{y}^i \in X_i$. For τ sufficiently large we get that $A_i(\mathbf{x}^{i-1}, \cdot) : X_i \to X_i$ are Q_i-contractions with $Q_i = \frac{1}{\tau} L'$, so we are in the conditions of Theorem

1.3, therefore A is a Picard operator and $F_A = \{(\mathbf{x}^{0,*}, \dots, \mathbf{x}^{m,*})\}$, thus

$$(\mathbf{x}^{0,n}, \mathbf{x}^{1,n}, \dots, \mathbf{x}^{m,n}) = A^n(\mathbf{x}^0, \mathbf{x}^1, \dots, \mathbf{x}^m) \to (\mathbf{x}^{0,*}, \dots, \mathbf{x}^{m,*}),$$

with $\mathbf{x}^{0,n} = \boldsymbol{\varphi}$ and $\mathbf{x}^{1,n}, \dots, \mathbf{x}^{m,n}$ are defined by (2.2), for all $n \in \mathbb{N}$. From the definitions of $A_i, i = \overline{1,m}$, we have

$$\mathbf{x}^{i-1,*}(t_{i-1}) = \mathbf{x}^{i,*}(t_{i-1}),\ i = \overline{1,m}$$

and therefore

$$\mathbf{x}^*(t) = \begin{cases} \boldsymbol{\varphi}(t), t \in [t_{-1}, t_0] \\ \mathbf{x}^{1,*}(t), t \in [t_0, t_1] \\ \dots \\ \mathbf{x}^{m,*}(t), t \in [t_{m-1}, t_m] \end{cases}$$

is the unique solution in $C([t_{-1}, t_m], \mathbb{R}^2)$. □

3. Application

We consider the following second order delay differential equation

$$-x''(t) = f(t, x(t), x(t-h)),\ t \in [a,b] \tag{3.1}$$

with initial conditions

$$\begin{cases} x(t) = \varphi(t),\ t \in [a-h, a] \\ x'(t) = \varphi'(t),\ t \in [a-h, a], \end{cases} \tag{3.2}$$

where $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ h \in \mathbb{R}_+^*,\ a, b \in \mathbb{R},\ a < b,\ \varphi, \varphi' : [a-h, a] \to \mathbb{R}$.

The problem (3.1)-(3.2) can be written in the following form

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} y_2(t) \\ -f(t, y_1(t), y_1(t-h)) \end{pmatrix},\ t \in [a,b] \tag{3.3}$$

with initial conditions

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \varphi(t) \\ \varphi'(t) \end{pmatrix},\ t \in [a-h, a] \tag{3.4}$$

where $\mathbf{y} := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}$, $\mathbf{F} \in C([a,b] \times \mathbb{R}^4, \mathbb{R}^2)$, $\mathbf{F}\,(t, \mathbf{y}, \mathbf{v}) = \mathbf{F}\,(t, y_1, y_2, v_1, v_2) :=$ $\begin{pmatrix} f_1\,(t, \mathbf{y}, \mathbf{v}) \\ f_2\,(t, \mathbf{y}, \mathbf{v}) \end{pmatrix} = \begin{pmatrix} y_2 \\ -f\,(t, y_1, v_1) \end{pmatrix}$, $\boldsymbol{\varphi} \in C([a-h, a], \mathbb{R}^2)$, $\boldsymbol{\varphi} := \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}$ and $h > 0$ is a parameter.

Relative to the problem (3.3)-(3.4) we consider the following conditions:

(C_1) $f \in C([a,b]\times\mathbb{R}^2,\mathbb{R}),\ \varphi \in C^1([a-h,a],\mathbb{R})$;

(C_2) there exists $l_f \in \mathbb{R}_+$ such that

$$|f(t,u_1,v)-f(t,u_2,v)| \le l_f\,|u_1-u_2|\,,$$

$t\in[a,b], u_1,u_2,v\in\mathbb{R}$.

Let $m\in\mathbb{N}^*$ be such that:

$$a+(m-1)h<b \text{ and } a+mh\ge b.$$

We denote by $t_{-1}:=a-h,\ \ t_0:=a,\ \ t_i:=a+ih,\ \ i=\overline{1,m-1},\ t_m:=b$.

Applying the results from Section 2 we have the following theorems.

Theorem 3.1. *We suppose that the conditions (C_1) and (C_2) hold. Then:*

(i) the problem (3.3)-(3.4) has a unique solution $\mathbf{y}^*$ *in* $C([t_{-1},t_m],\mathbb{R}^2)$ *where*

$$\mathbf{y}^*(t)=\begin{cases}\boldsymbol{\varphi}(t_0), t\in[t_{-1},t_0]\\ \mathbf{y}^{1,*}(t), t\in[t_0,t_1]\\ \dots\\ \mathbf{y}^{m,*}(t), t\in[t_{m-1},t_m]\end{cases}$$

(ii) for each $\mathbf{y}^{i,0}\in C([t_{i-1},t_i],\mathbb{R}^2), i=\overline{1,m-1},\ \ \mathbf{y}^{m,0}\in C([t_{m-1},t_m],\mathbb{R}^2)$*, the sequence defined by:*

$$\mathbf{y}^{i,n+1}(t)=\mathbf{y}^{i-1,*}(t_{i-1})+\int_{t_{i-1}}^{t}\mathbf{F}(s,\mathbf{y}^{i,n}(s),\mathbf{y}^{i-1,*}(s-h))ds,$$

for $t\in[t_{i-1},t_i]$*, converges and* $\lim\limits_{n\to\infty}\mathbf{y}^{i,n}=\mathbf{y}^{i,*},\ \ i=\overline{1,m}$.

Proof. From condition (C_1) we have that $\mathbf{F}\in C([a,b]\times\mathbb{R}^4,\mathbb{R}^2),\ \boldsymbol{\varphi}\in C([a-h,a],\mathbb{R}^2)$.

From (C_2) we have

$$\begin{aligned}\left\|\mathbf{F}(t,\mathbf{u}^1,\mathbf{v})-\mathbf{F}(t,\mathbf{u}^2,\mathbf{v})\right\| &= \left\|\begin{pmatrix}u_2^1\\ -f(t,u_1^1,v_1)\end{pmatrix}-\begin{pmatrix}u_2^2\\ -f(t,u_1^2,v_1)\end{pmatrix}\right\|\\ &\le \begin{pmatrix}0&1\\ l_f&0\end{pmatrix}\begin{pmatrix}\left|u_1^1-u_1^2\right|\\ \left|u_2^1-u_2^2\right|\end{pmatrix},\end{aligned}$$

for all $\mathbf{u}^1=(u_1^1,u_2^1),\ \mathbf{u}^2=(u_1^2,u_2^2),\ \mathbf{v}=(v_1,v_2)\in\mathbb{R}^2$. So the problem (3.3)-(3.4) verifies the conditions of the Theorem 2.2. □

Theorem 3.2. *In the condition of Theorem 3.1, for each* $\mathbf{y}^{i,0}\in C([t_{i-1},t_i],\mathbb{R}^2), i=\overline{1,m}$*, the sequences defined by:*

$$\mathbf{y}^{1,n+1}(t)=\boldsymbol{\varphi}(t_0)+\int_{t_0}^{t}\mathbf{F}(s,\mathbf{y}^{1,n}(s),\boldsymbol{\varphi}(s-h))ds,\ \textit{for } t\in[t_0,t_1] \tag{3.5}$$

$$\mathbf{y}^{2,n+1}(t)=\mathbf{y}^{1,n}(t_1)+\int_{t_1}^{t}\mathbf{F}(s,\mathbf{y}^{2,n}(s),\mathbf{y}^{1,n}(s-h))ds,\ \textit{for } t\in[t_1,t_2]$$

$$\dots$$

$$\mathbf{y}^{m,n+1}(t)=\mathbf{y}^{m-1,n}(t_{m-1})+\int_{t_{m-1}}^{t}\mathbf{F}(s,\mathbf{y}^{m,n}(s),\mathbf{y}^{m-1,n}(s-h))ds,\ \textit{for } t\in[t_{m-1},t_m]$$

converge and $\lim\limits_{n\to\infty}\mathbf{y}^{i,n}=\mathbf{y}^{i,*},\ \ i=\overline{1,m}$.

4. Numerical method

In this section we test some second order initial value problems to show the efficiency and accuracy of the proposed method. We follow the technique from D. Trif [20] where the approximating solution is given by a finite sum of the Chebyshev series. The same technique was used in [2, 4, 15] for integro-differential equations with delays.

We divide the working interval by the points $P_k = k,\ k = 0, 1, \dots, M$, where $M = 8$ and represents the number of subintervals. On each subinterval $I_k = [P_{k-1}, P_k],\ k = 1, \dots, M$, we find the numerical solution by the following form

$$y_{1,k} = c^1_{0,k}\frac{T_0}{2} + c^1_{1,k}T_1(\xi) + c^1_{2,k}T_2(\xi) + \ldots + c^1_{n-1,k}T_{n-1}(\xi),$$
$$y_{2,k} = c^2_{0,k}\frac{T_0}{2} + c^2_{1,k}T_1(\xi) + c^2_{2,k}T_2(\xi) + \ldots + c^2_{n-1,k}T_{n-1}(\xi),$$

where $T_i(\xi) = \cos(i \arccos(\xi))$ are Chebyshev polynomials of i degree, $i = 0, \dots, n-1$ $(n = 25)$, and $t = \alpha\xi + \beta$ where $\alpha = (P_k - P_{k-1})/2$ and $\beta = (P_k + P_{k-1})/2$ (see [17, 18]).

For the efficiency estimation of this algorithm, the integral equation system is written in the form of delay differential system and we use the Matlab command dde23 (Matlab procedure which solves numerically delay differential equations, for details see Shampine [16]) to solve it and we compare the running times. We impose the relative error to 10^{-8} and the absolute error to 10^{-12} to obtain a accuracy comparable with the step method. We display the graph of solutions.

Example 4.1. Consider the following:

$$\begin{cases} x''(t) = e^{-2t}\frac{x^2(t-\tau)}{x(t)},\ t \in [0, 8], \tau = 1 \\ x(t) = e^{-t},\ x'(t) = -e^{-t},\ t \in [-1, 0]. \end{cases}$$

Exact solution: $x(t) = e^{-t}$.

For this example, the step method obtains the solution in 1377 iterations with an error of 10^{-9} in 0.061830 CPU seconds. The Matlab program dde23 needs 0.737448 CPU seconds for a similar precision.

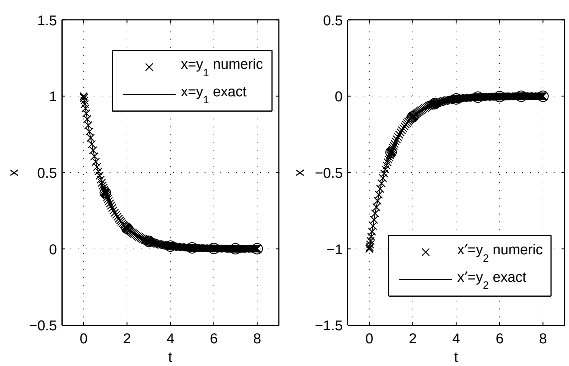

Figure 1. The graphs of the exact and numerical solution for Example 4.1.

Example 4.2. Consider the following:

$$\begin{cases} x''(t) = x^2(t-\tau) - \frac{1}{4\sqrt{(1+t)^3}} - (1+t) + \tau, \ t \in [0,8], \tau = 1 \\ x(t) = \sqrt{1+t}, \ x'(t) = \frac{1}{2\sqrt{1+t}}, \ t \in [-1,0]. \end{cases}$$

Exact solution: $x(t) = \sqrt{1+t}$.

In this case, the step method obtains the solution in 686 iterations with an error of 10^{-8} in 0.017437 CPU seconds. The Matlab program dde23 needs 0.490628 CPU seconds for a similar precision.

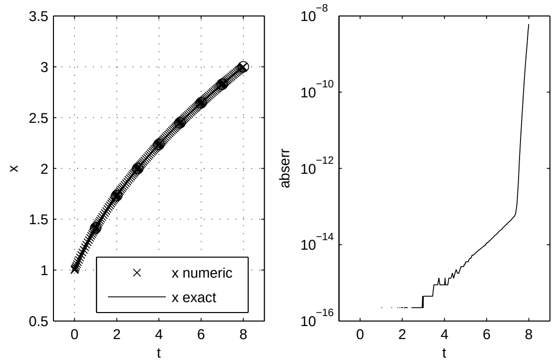

Figure 2. The graphs of the exact and numerical solution and absolute error evolution for Example 4.2.

5. Conclusions

In this paper we introduce a combination of a step method and a Chebyshev spectral method.

For the first example, the running time of the step method is 11 times faster than Matlab dde23 procedure and for the second example is 28 times faster than Matlab dde23 procedure for the similar precision. The above comparisons validate the step method from the accuracy and efficiency point of view.

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