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TRAVELING WAVES OF A NONLOCAL DIFFUSION SIRS EPIDEMIC MODEL WITH A CLASS OF NONLINEAR INCIDENCE RATES AND TIME DELAY*

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Abstract In this paper, we study the traveling waves of a delayed SIRS epidemic model with nonlocal diffusion and a class of nonlinear incidence rates. When the basic reproduction ratio $\mathscr{R}_0 > 1$, by using the Schauder's fixed point theorem associated with upper-lower solutions, we reduce the existence of traveling waves to the existence of a pair of upper-lower solutions. By constructing a pair of upper-lower solutions, we derive the existence of traveling wave solutions connecting the disease-free steady state and the endemic steady state. When $\mathscr{R}_0 < 1$, the nonexistence of traveling waves is obtained by the comparison principle.

Keywords SIRS, traveling waves, nonlocal diffusion, nonlinear incidence rates, upper-lower solutions.

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1. Introduction

Since Kermack and Mckendrick [6] constructed a mathematical model of an ODE to study epidemiology in 1927, various models have been used to describe various kinds of epidemics, and the dynamics of these systems have been investigated. However, on the one kind, in disease progression, the spatial content of the environment plays a crucial role; the spread of germs, bacteria, and pathogen in the area is the main reason which leads to the spread of infectious disease. On the other kind, due to the diseases latency or immunity, the presence of time delays in such models makes them more realistic. In recent years, the dynamics of delayed diffusion epidemic models have been widely studied by researchers [1, 3, 7, 12–14, 20]. For example, considering the spatial effects and time delay, Gan etc [5] concerned the following delayed SIRS epidemic model with spatial diffusion

$$\begin{cases} \frac{\partial S}{\partial t} = D_S \frac{\partial^2 S}{\partial x^2} + A - dS(x,t) - \beta S(x,t) I(x,t-\tau) + \delta R(x,t), \\ \frac{\partial I}{\partial t} = D_I \frac{\partial^2 I}{\partial x^2} + \beta S(x,t) I(x,t-\tau) - (\gamma+\alpha+d) I(x,t), \\ \frac{\partial R}{\partial t} = D_R \frac{\partial^2 R}{\partial x^2} + \gamma I(x,t) - (\delta+d) R(x,t), \end{cases} \tag{1.1}$$

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where $S(x,t)$, $I(x,t)$ and $R(x,t)$ represent the sizes of the susceptible, infective and recovered individuals at location $x \in \mathbb{R}$ and time $t > 0$, respectively. D_S, D_I and D_R denote the corresponding diffusion rates. The parameters $A, d, \beta, \delta, \gamma, \alpha, \tau$ are positive constants in which A is the recruitment rate of the population, d is the natural death rate of the population, β is the transmission coefficient, δ is the rate at which recovered individuals lose immunity and return to the susceptible class, γ is the recovery rate of the infective individuals, α is the death rate due to disease, and τ is a fixed time during which the infectious agents develop in the vector and it is only after that time that the infected vector can infect a susceptible individual. In [5], by constructing a pair of upper-lower solutions, the existence of traveling wave solutions connecting the disease-free steady state and the endemic steady state was given.

Note that in system (1.1), Gan etc used the bilinear incidence rate βSI based on the law of mass action. However, as the number of susceptible individuals is large, it is reasonable to consider the saturation incidence rate instead of the bilinear incidence rate [4, 16]. Recently, Yang etc [17] considered the following delayed SIR epidemic model with saturation incidence rate and spatial diffusion

$$\begin{cases} \frac{\partial S}{\partial t} = D_S \frac{\partial^2 S}{\partial x^2} + A - dS(x,t) - \frac{\beta S(x,t)I(x,t-\tau)}{1+\alpha I(x,t-\tau)}, \\ \frac{\partial I}{\partial t} = D_I \frac{\partial^2 I}{\partial x^2} + \frac{\beta S(x,t)I(x,t-\tau)}{1+\alpha I(x,t-\tau)} - (\gamma+\mu)I(x,t), \\ \frac{\partial R}{\partial t} = D_R \frac{\partial^2 R}{\partial x^2} + \gamma I(x,t) - \mu_1 R(x,t). \end{cases} \tag{1.2}$$

By using the cross iteration method and the Schauder's fixed point theorem, they investigated the existence of traveling waves of system (1.2). Li etc [8] further considered the minimal wave speed by presenting the existence and nonexistence of traveling wave solutions of system (1.2). Zhou and Wang [21] concerned the following delayed SIRS epidemic model with saturation incidence rate and spatial diffusion

$$\begin{cases} \frac{\partial S}{\partial t} = D_S \frac{\partial^2 S}{\partial x^2} + A - dS(x,t) - \frac{\beta S(x,t)I(x,t-\tau)}{1+\alpha I(x,t-\tau)} + \delta R(x,t), \\ \frac{\partial I}{\partial t} = D_I \frac{\partial^2 I}{\partial x^2} + \frac{\beta S(x,t)I(x,t-\tau)}{1+\alpha I(x,t-\tau)} - (\gamma+\mu+d)I(x,t), \\ \frac{\partial R}{\partial t} = D_R \frac{\partial^2 R}{\partial x^2} + \gamma I(x,t) - (\delta+d)R(x,t), \end{cases} \tag{1.3}$$

and obtained the existence of traveling wave solutions.

Although the Laplacian operator $\Delta := \frac{\partial^2}{\partial x^2}$ can be used to model the diffusion of the species, it is a local operator which suggests that the population at the location x can only be influenced by the variation of the population near the location x. However, in dynamics of infectious diseases, dispersal is better described as a long range process rather that as a local one. Since the long range effect is taken into account, nonlocal diffusion equations have received great interest [9, 10, 15, 18]. Yu etc [19] considered the following delayed SIRS epidemic model with nonlocal spatial diffusion

$$\begin{cases} \frac{\partial S}{\partial t} = [(J*S)(x,t) - S(x,t)] + A - dS(x,t) - \beta S(x,t)I(x,t-\tau) \\ \qquad + \delta R(x,t), \\ \frac{\partial I}{\partial t} = [(J*I)(x,t) - I(x,t)] + \beta S(x,t)I(x,t-\tau) - (\gamma+\alpha+d)I(x,t), \\ \frac{\partial R}{\partial t} = [(J*R)(x,t) - R(x,t)] + \gamma I(x,t) - (\delta+d)R(x,t), \end{cases} \tag{1.4}$$

and obtained the existence of traveling wave solutions, where $J*S$, $J*I$ and $J*R$ are the standard convolutions with the space invariable x. $J*S-S$, $J*I-I$ and

$J * R - R$ indicate that the diffusion of all the susceptible, infective and recovered individuals is free and long range. Tian and Xu [11] investigated the following delayed SIRS epidemic model with saturation incidence rate and nonlocal spatial diffusion

$$\begin{cases} \frac{\partial S}{\partial t} = D[(J * S)(x,t) - S(x,t)] + A - dS(x,t) - \frac{\beta S(x,t)I(x,t-\tau)}{1+\alpha I(x,t-\tau)} \\ \qquad +\delta R(x,t), \\ \frac{\partial I}{\partial t} = D[(J * I)(x,t) - I(x,t)] + \frac{\beta S(x,t)I(x,t-\tau)}{1+\alpha I(x,t-\tau)} - (\gamma + \mu + d)I(x,t), \\ \frac{\partial R}{\partial t} = D[(J * R)(x,t) - R(x,t)] + \gamma I(x,t) - (\delta + d)R(x,t), \end{cases} \tag{1.5}$$

and obtained the existence of traveling wave solutions.

Motivated by the work of [11, 19], in this paper, we study the following delayed SIRS epidemic model with a class of nonlinear incidence rates and nonlocal spatial diffusion

$$\begin{cases} \frac{\partial S}{\partial t} = D[(J * S)(x,t) - S(x,t)] + A - dS(x,t) \\ \qquad -\beta S(x,t) \int_0^h f(\tau) g(I(x,t-\tau)) d\tau + \delta R(x,t), \\ \frac{\partial I}{\partial t} = D[(J * I)(x,t) - I(x,t)] + \beta S(x,t) \int_0^h f(\tau) g(I(x,t-\tau)) d\tau \\ \qquad -(\gamma + \alpha + d)I(x,t), \\ \frac{\partial R}{\partial t} = D[(J * R)(x,t) - R(x,t)] + \gamma I(x,t) - (\delta + d)R(x,t), \end{cases} \tag{1.6}$$

where h is a maximum time taken to become infectious and $f(\tau)$ denotes the fraction of vector population in which the time taken to become infectious is τ [2].

Meanwhile, throughout this paper, we give the following assumptions:

(A1) $J(y) = J(-y) \geq 0$, $\int_{-\infty}^{+\infty} J(y)dy = 1$. For any fixed $\mu > 0$,

$$J_\mu := \int_{-\infty}^{+\infty} J(y) e^{\mu|y|} dy < \infty.$$

(A2) $f(\tau)$ is nonnegative and continuous on $[0, h]$, $f(0) = 0$. Moreover,

$$\int_0^h f(\tau) d\tau = 1.$$

(A3) $g(I)$ is continuous differentiable, monotone increasing on $[0, +\infty)$ with

$$g(0) = 0.$$

(A4) $I/g(I)$ is monotone increasing on $(0, +\infty)$ with

$$\lim_{I \to 0^+} \frac{I}{g(I)} = 1.$$

It is easy to see that $g(I)$ is Lipschitz continuous on $[0, +\infty)$ and $0 < g(I) \leq I$ holds for $I > 0$.

The rest of this paper is organized as follows. In Section 2, by constructing a pair of upper-lower solutions and using the Schauder's fixed point theorem, the existence of traveling wave solutions connecting the disease-free steady state and the endemic steady state of system (1.6) is established. In Section 3, the nonexistence of traveling waves is considered.

2. Existence of traveling waves for $\mathscr{R}_0 > 1$

In this section, we apply the Schauder's fixed point theorem associated with upper-lower solutions to study the existence of traveling wave solutions of system (1.6) connecting the disease-free steady state and the endemic steady state. Denote

$$\mathscr{R}_0 = \frac{A\beta}{d(\gamma + \alpha + d)}. \tag{2.1}$$

$\mathscr{R}_0$ is called the basic reproduction ratio of system (1.6), which describes the average number of newly infected cells generated from one infected cell at the beginning of the infectious process. This quantity determines the thresholds for disease transmissions. It is easy to show that system (1.6) always has a disease-free steady state $(A/d, 0, 0)$. From

$$\begin{cases} A - dS - \beta S g(I) + \delta R = 0, \\ \beta S g(I) - (\gamma + \alpha + d) I = 0, \\ \gamma I - (\delta + d) R = 0, \end{cases} \tag{2.2}$$

we obtain

$$A\beta = d(\gamma + \alpha + d)\frac{I}{g(I)} + \beta\left(\frac{\gamma d}{\delta + d} + \alpha + d\right) I.$$

Note that $g(I)$ satisfies (A4). If $\mathscr{R}_0 > 1$, then (2.2) has a unique positive solution (S^*, I^*, R^*), where

$$A\beta = d(\gamma + \alpha + d)\frac{I^*}{g(I^*)} + \beta\left(\frac{\gamma d}{\delta + d} + \alpha + d\right) I^*,$$

$$S^* = \frac{A(\delta + d) + \gamma\delta I^*}{(\delta + d)(d + \beta g(I^*))}, \quad R^* = \frac{\gamma}{\delta + d} I^*.$$

That is, system (1.6) has a unique endemic steady state (S^*, I^*, R^*). Denoting $N = S + I + R$, then system (1.6) is equivalent to the following system

$$\begin{cases} \frac{\partial N}{\partial t} = D[(J * N)(x,t) - N(x,t)] + A - dN(x,t) - \alpha I(x,t), \\ \frac{\partial I}{\partial t} = D[(J * I)(x,t) - I(x,t)] - (\gamma + \alpha + d) I(x,t) \\ \qquad + \beta[N(x,t) - I(x,t) - R(x,t)] \int_0^h f(\tau) g(I(x, t - \tau)) d\tau, \\ \frac{\partial R}{\partial t} = D[(J * R)(x,t) - R(x,t)] + \gamma I(x,t) - (\delta + d) R(x,t). \end{cases} \tag{2.3}$$

By making a change of variables $\tilde{N} = A/d - N, \tilde{I} = I, \tilde{R} = R$ and dropping the tildes, system (2.3) becomes

$$\begin{cases} \frac{\partial N}{\partial t} = D[(J * N)(x,t) - N(x,t)] - dN(x,t) + \alpha I(x,t), \\ \frac{\partial I}{\partial t} = D[(J * I)(x,t) - I(x,t)] - (\gamma + \alpha + d) I(x,t) \\ \qquad + \beta\left[\frac{A}{d} - N(x,t) - I(x,t) - R(x,t)\right] \int_0^h f(\tau) g(I(x, t - \tau)) d\tau, \\ \frac{\partial R}{\partial t} = D[(J * R)(x,t) - R(x,t)] + \gamma I(x,t) - (\delta + d) R(x,t). \end{cases} \tag{2.4}$$

It is easy to show that if $\mathscr{R}_0 > 1$, then system (2.4) has two steady states $(0,0,0)$ and (k_1, k_2, k_3), where

$$k_1 = A/d - S^* - I^* - R^*, \quad k_2 = I^*, \quad k_3 = R^*.$$

Obviously,

$$k_1 = \frac{\alpha}{d} k_2 > 0, \quad k_3 = \frac{\gamma}{\delta + d} k_2, \quad k_1 + k_2 + k_3 = \frac{A}{d} - S^* < \frac{A}{d}.$$

A traveling wave solution of (2.4) is a special translation invariant solution of the form $(N(x,t), I(x,t), R(x,t)) = (\phi(x+ct), \varphi(x+ct), \psi(x+ct))$, where $(\phi, \varphi, \psi) \in C(\mathbb{R}, \mathbb{R}^3)$ is the profile of the wave that propagates through one-dimensional spatial domain at a constant speed $c > 0$. On substituting $(N(x,t), I(x,t), R(x,t)) = (\phi(x+ct), \varphi(x+ct), \psi(x+ct))$ into (2.4) and denoting the traveling wave coordinate $x + ct$ by ξ, we derive that

$$\begin{cases} c\phi'(\xi) = D \int_{-\infty}^{+\infty} J(\xi - y)[\phi(y) - \phi(\xi)]dy + f_1(\phi, \varphi, \psi)(\xi), \\ c\varphi'(\xi) = D \int_{-\infty}^{+\infty} J(\xi - y)[\varphi(y) - \varphi(\xi)]dy + f_2(\phi, \varphi, \psi)(\xi), \\ c\psi'(\xi) = D \int_{-\infty}^{+\infty} J(\xi - y)[\psi(y) - \psi(\xi)]dy + f_3(\phi, \varphi, \psi)(\xi), \end{cases} \tag{2.5}$$

where

$$\begin{cases} f_1(\phi, \varphi, \psi)(\xi) = -d\phi(\xi) + \alpha\varphi(\xi), \\ f_2(\phi, \varphi, \psi)(\xi) = \beta \left[\frac{A}{d} - \phi(\xi) - \varphi(\xi) - \psi(\xi)\right] \int_0^h f(\tau) g(\varphi(\xi - c\tau)) d\tau \\ \qquad\qquad\qquad -(\gamma + \alpha + d)\varphi(\xi), \\ f_3(\phi, \varphi, \psi)(\xi) = \gamma\varphi(\xi) - (\delta + d)\psi(\xi). \end{cases} \tag{2.6}$$

System (2.5) will be solved subject to the following asymptotic boundary conditions

$$\lim_{\xi \to -\infty} (\phi(\xi), \varphi(\xi), \psi(\xi)) = (0, 0, 0), \quad \lim_{\xi \to +\infty} (\phi(\xi), \varphi(\xi), \psi(\xi)) = (k_1, k_2, k_3). \tag{2.7}$$

Note that $\mathscr{R}_0 > 1$ can be rewritten in the form $\frac{A\beta}{d} - (\gamma + \alpha) > d$. On the other hand, we have

$$\alpha \frac{k_2}{k_1} = d, \quad \gamma \frac{k_2}{k_3} - \delta = d.$$

We can select suitable values of M_1, M_2, M_3 such that $M_1 > k_1, M_2 > k_2, M_3 > k_3$ and satisfying

$$\begin{cases} \frac{A\beta}{d} - (\gamma + \alpha) > \alpha \frac{M_2}{M_1} > d, \\ \frac{A\beta}{d} - (\gamma + \alpha) > \gamma \frac{M_2}{M_3} - \delta > d. \end{cases} \tag{2.8}$$

Furthermore, because of $\frac{A}{d} > k_1 + k_2 + k_3$, we can also let

$$\frac{A}{d} > M_1 + M_2 + M_3 \quad \text{and} \quad k_1 + k_2 + k_3 > M_1 + M_3. \tag{2.9}$$

In fact, let

$$M_1 = \left(1 + \frac{\epsilon}{2}\right) k_1, \quad M_2 = (1 + \epsilon) k_2, \quad M_3 = \left(1 + \frac{\epsilon}{2}\right) k_3.$$

By choosing $\epsilon > 0$, we can obtain (2.8) and (2.9).

Let

$$C_{[0,M]}(\mathbb{R}, \mathbb{R}^3) = \{(\phi, \varphi, \psi) \in C(\mathbb{R}, \mathbb{R}^3) : \\ (0,0,0) \le (\phi(\xi), \varphi(\xi), \psi(\xi)) \le (M_1, M_2, M_3) \quad \text{for } \xi \in \mathbb{R}\}.$$

For $\Phi = (\phi_1, \varphi_1, \psi_1), \Psi = (\phi_2, \varphi_2, \psi_2) \in C_{[0,M]}(\mathbb{R}, \mathbb{R}^3)$, we can easily get that

$$|f_1(\phi_1, \varphi_1, \psi_1)(\xi) - f_1(\phi_2, \varphi_2, \psi_2)(\xi)| \leq (\alpha + d)|\Phi - \Psi|_{\mathbb{R}^3},$$

$$|f_2(\phi_1, \varphi_1, \psi_1)(\xi) - f_2(\phi_2, \varphi_2, \psi_2)(\xi)| \leq L|\Phi - \Psi|_{\mathbb{R}^3},$$

$$|f_3(\phi_1, \varphi_1, \psi_1)(\xi) - f_3(\phi_2, \varphi_2, \psi_2)(\xi)| \leq (\gamma + \delta + d)|\Phi - \Psi|_{\mathbb{R}^3},$$

where $L = \frac{A\beta}{d} + \beta[(M_1 + M_2 + M_3)L_g + 3g(M_2)] + (\gamma + \alpha + d)$, L_g is the Lipschitz constant of g and $|\cdot|_{\mathbb{R}^3}$ is the Euclidean norm in $\mathbb{R}^3$.

Define $H = (H_1, H_2, H_3) : C_{[0,M]}(\mathbb{R}, \mathbb{R}^3) \to C(\mathbb{R}, \mathbb{R}^3)$ by

$$H_1(\phi, \varphi, \psi)(\xi) = D \int_{-\infty}^{+\infty} J(\xi - y)\phi(y)dy + \alpha\varphi(\xi),$$

$$\begin{aligned} H_2(\phi, \varphi, \psi)(\xi) =& D \int_{-\infty}^{+\infty} J(\xi - y)\varphi(y)dy + \beta M_2\varphi(\xi) \\ &+ \beta \left[\frac{A}{d} - \phi(\xi) - \varphi(\xi) - \psi(\xi)\right] \int_0^h f(\tau)g(\varphi(\xi - c\tau))d\tau, \end{aligned}$$

$$H_3(\phi, \varphi, \psi)(\xi) = D \int_{-\infty}^{+\infty} J(\xi - y)\psi(y)dy + \gamma\varphi(\xi).$$

The operators H_1, H_2 and H_3 admit the following properties.

Lemma 2.1. *We have*
(i)

$$H_1(\phi, \varphi, \psi)(\xi) \geq 0, \quad H_3(\phi, \varphi, \psi)(\xi) \geq 0$$

for any $(\phi, \varphi, \psi) \in C_{[0,M]}(\mathbb{R}, \mathbb{R}^3)$;
(ii)

$$H_1(\phi_2, \varphi_2, \psi_2)(\xi) \leq H_1(\phi_1, \varphi_1, \psi_1)(\xi),$$

$$H_3(\phi_2, \varphi_2, \psi_2)(\xi) \leq H_3(\phi_1, \varphi_1, \psi_1)(\xi);$$

(iii)

$$H_2(\phi_1, \varphi_1, \psi_1)(\xi) \leq H_2(\phi_2, \varphi_1, \psi_1)(\xi),$$

$$H_2(\phi_1, \varphi_2, \psi_1)(\xi) \leq H_2(\phi_1, \varphi_1, \psi_1)(\xi),$$

$$H_2(\phi_1, \varphi_1, \psi_1)(\xi) \leq H_2(\phi_1, \varphi_1, \psi_2)(\xi) \quad \textit{for } \xi \in \mathbb{R} \quad \textit{with}$$

$$(0, 0, 0) \leq (\phi_2(\xi), \varphi_2(\xi), \psi_2(\xi)) \leq (\phi_1(\xi), \varphi_1(\xi), \psi_1(\xi)) \leq (M_1, M_2, M_3).$$

Proof. Assertion (i) can be easily get from (A1). For (ii), we have

$$\begin{aligned} &H_1(\phi_2, \varphi_2, \psi_2)(\xi) - H_1(\phi_1, \varphi_1, \psi_1)(\xi) \\ &= D \int_{-\infty}^{+\infty} J(\xi - y)[\phi_2(y) - \phi_1(y)]dy + \alpha[\varphi_2(\xi) - \varphi_1(\xi)] \leq 0, \end{aligned}$$

$$\begin{aligned} &H_3(\phi_2, \varphi_2, \psi_2)(\xi) - H_3(\phi_1, \varphi_1, \psi_1)(\xi) \\ &= D \int_{-\infty}^{+\infty} J(\xi - y)[\psi_2(y) - \psi_1(y)]dy + \gamma[\varphi_2(\xi) - \varphi_1(\xi)] \leq 0. \end{aligned}$$

For (iii), we have

$$\begin{aligned}
&H_2(\phi_1,\varphi_1,\psi_1)(\xi)-H_2(\phi_2,\varphi_1,\psi_1)(\xi)\\
&=\beta[\phi_2(\xi)-\phi_1(\xi)]\int_0^h f(\tau)g(\varphi_1(\xi-c\tau))d\tau\le 0,\\
&H_2(\phi_1,\varphi_1,\psi_1)(\xi)-H_2(\phi_1,\varphi_1,\psi_2)(\xi)\\
&=\beta[\psi_2(\xi)-\psi_1(\xi)]\int_0^h f(\tau)g(\varphi_1(\xi-c\tau))d\tau\le 0,
\end{aligned}$$

$$\begin{aligned}
&H_2(\phi_1,\varphi_2,\psi_1)(\xi)-H_2(\phi_1,\varphi_1,\psi_1)(\xi)\\
&=D\int_{-\infty}^{+\infty}J(\xi-y)[\varphi_2(y)-\varphi_1(y)]dy+\beta\left[\frac{A}{d}-\phi_1(\xi)-\varphi_2(\xi)-\psi_1(\xi)\right]\\
&\quad\times\int_0^h f(\tau)[g(\varphi_2(\xi-c\tau))-g(\varphi_1(\xi-c\tau))]d\tau\\
&\quad+\beta[\varphi_2(\xi)-\varphi_1(\xi)]\left[M_2-\int_0^h f(\tau)g(\varphi_1(\xi-c\tau))d\tau\right]\le 0.
\end{aligned}$$

Here we used the face $\frac{A}{d}>M_1+M_2+M_3$ and $M_2-g(M_2)\ge 0$. The proof is complete. □

In terms of H_1,H_2 and H_3, system (2.5) can be rewritten as

$$\begin{cases}c\phi'(\xi)=-b_1\phi(\xi)+H_1(\phi,\varphi,\psi)(\xi),\\ c\varphi'(\xi)=-b_2\varphi(\xi)+H_2(\phi,\varphi,\psi)(\xi),\\ c\psi'(\xi)=-b_3\psi(\xi)+H_3(\phi,\varphi,\psi)(\xi),\end{cases}\tag{2.10}$$

where $b_1=D+d, b_2=D+\beta M_2+\gamma+\alpha+d, b_3=D+\delta+d$.

We define an operator $Q=(Q_1,Q_2,Q_3):C_{[0,M]}(\mathbb{R},\mathbb{R}^3)\to C(\mathbb{R},\mathbb{R}^3)$ by

$$Q_i[\Phi](\xi)=Q_i(\phi,\varphi,\psi)(\xi)=\frac{1}{c}e^{-\frac{b_i}{c}\xi}\int_{-\infty}^{\xi}e^{\frac{b_i}{c}s}H_i(\phi,\varphi,\psi)(s)ds,\tag{2.11}$$

where $\Phi=(\phi,\varphi,\psi)$. It is easy to see that $Q_1(\phi,\varphi,\psi),Q_2(\phi,\varphi,\psi)$ and $Q_3(\phi,\varphi,\psi)$ satisfy

$$\begin{cases}cQ_1'(\phi,\varphi,\psi)(\xi)=-b_1Q_1(\phi,\varphi,\psi)(\xi)+H_1(\phi,\varphi,\psi)(\xi),\\ cQ_2'(\phi,\varphi,\psi)(\xi)=-b_2Q_2(\phi,\varphi,\psi)(\xi)+H_2(\phi,\varphi,\psi)(\xi),\\ cQ_3'(\phi,\varphi,\psi)(\xi)=-b_3Q_3(\phi,\varphi,\psi)(\xi)+H_3(\phi,\varphi,\psi)(\xi).\end{cases}\tag{2.12}$$

With the properties of H_i, we have the following result for Q.

Lemma 2.2. *For any* $(0,0,0)\le(\phi_2(\xi),\varphi_2(\xi),\psi_2(\xi))\le(\phi_1(\xi),\varphi_1(\xi),\psi_1(\xi))\le(M_1,M_2,M_3)$*, we have*

$$\begin{aligned}
Q_1(\phi_2,\varphi_2,\psi_2)(\xi)&\le Q_1(\phi_1,\varphi_1,\psi_1)(\xi),\\
Q_3(\phi_2,\varphi_2,\psi_2)(\xi)&\le Q_3(\phi_1,\varphi_1,\psi_1)(\xi),\\
Q_2(\phi_1,\varphi_1,\psi_1)(\xi)&\le Q_2(\phi_2,\varphi_1,\psi_1)(\xi),\\
Q_2(\phi_1,\varphi_2,\psi_1)(\xi)&\le Q_2(\phi_1,\varphi_1,\psi_1)(\xi),\\
Q_2(\phi_1,\varphi_1,\psi_1)(\xi)&\le Q_2(\phi_1,\varphi_1,\psi_2)(\xi)\quad \textit{for } \xi\in\mathbb{R}.
\end{aligned}$$

We next verify the continuity of Q. For $\mu\in\big(0,\min\{\frac{b_1}{c},\frac{b_2}{c},\frac{b_3}{c}\}\big)$, define $B_\mu(\mathbb{R},\mathbb{R}^3)=\{\Phi\in C(\mathbb{R},\mathbb{R}^3):|\Phi|_\mu<\infty\}$, where $|\Phi|_\mu=\sup_{\xi\in\mathbb{R}}|\Phi(\xi)|e^{-\mu|\xi|}$. Then it is easy to check that $(B_\mu(\mathbb{R},\mathbb{R}^3),|\cdot|_\mu)$ is a Banach space.

Lemma 2.3. *$Q=(Q_1,Q_2,Q_3)$ is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R}^3)$.*

Proof. Let $\Phi=(\phi_1,\varphi_1,\psi_1)$, $\Psi=(\phi_2,\varphi_2,\psi_2)$. Then we have

$$
\begin{aligned}
&|Q_1[\Phi](\xi)-Q_1[\Psi](\xi)|e^{-\mu|\xi|}\\
&\le \frac{e^{-\mu|\xi|}}{c}\int_{-\infty}^{\xi}e^{-\frac{b_1}{c}(\xi-s)}|H_1(\phi_1,\varphi_1,\psi_1)(s)-H_1(\phi_2,\varphi_2,\psi_2)(s)|ds\\
&\le \frac{e^{-\mu|\xi|}}{c}\int_{-\infty}^{\xi}De^{-\frac{b_1}{c}(\xi-s)}\int_{-\infty}^{+\infty}J(s-y)|\phi_1(y)-\phi_2(y)|dyds\\
&\quad+\frac{e^{-\mu|\xi|}}{c}\int_{-\infty}^{\xi}\alpha e^{-\frac{b_1}{c}(\xi-s)}|\varphi_1(s)-\varphi_2(s)|ds\\
&=\frac{De^{-\mu|\xi|}}{c}\int_{-\infty}^{\xi}e^{-\frac{b_1}{c}(\xi-s)}\int_{-\infty}^{+\infty}J(s-y)|\phi_1(y)-\phi_2(y)|e^{-\mu|y|}e^{\mu|y|}dyds\\
&\quad+\frac{\alpha e^{-\mu|\xi|}}{c}\int_{-\infty}^{\xi}e^{-\frac{b_1}{c}(\xi-s)}|\varphi_1(s)-\varphi_2(s)|e^{-\mu|s|}e^{\mu|s|}ds\\
&\le DJ_\mu|\Phi-\Psi|_\mu\int_{-\infty}^{\xi}\frac{e^{-\mu|\xi|}e^{-\frac{b_1}{c}(\xi-s)}}{c}e^{\mu|s|}ds\\
&\quad+\alpha|\Phi-\Psi|_\mu\int_{-\infty}^{\xi}\frac{e^{-\mu|\xi|}e^{-\frac{b_1}{c}(\xi-s)}}{c}e^{\mu|s|}ds.
\end{aligned}
$$

For $\xi\ge 0$, we have

$$
\begin{aligned}
&\int_{-\infty}^{\xi}\frac{e^{-\mu|\xi|}e^{-\frac{b_1}{c}(\xi-s)}}{c}e^{\mu|s|}ds\\
&=\frac{1}{c}\left[\int_{-\infty}^{0}e^{-\mu\xi}e^{-\frac{b_1}{c}(\xi-s)}e^{-\mu s}ds+\int_{0}^{\xi}e^{-\mu\xi}e^{-\frac{b_1}{c}(\xi-s)}e^{\mu s}ds\right]\\
&=\frac{1}{c}\left[e^{-\frac{b_1+c\mu}{c}\xi}\int_{-\infty}^{0}e^{\frac{b_1-c\mu}{c}s}ds+e^{-\frac{b_1+c\mu}{c}\xi}\int_{0}^{\xi}e^{\frac{b_1+c\mu}{c}s}ds\right]\\
&=\frac{1}{c}\left[\frac{c}{b_1-c\mu}e^{-\frac{b_1+c\mu}{c}\xi}+\frac{c}{b_1+c\mu}\left(1-e^{-\frac{b_1+c\mu}{c}\xi}\right)\right]\\
&\le\frac{1}{b_1-c\mu}+\frac{1}{b_1+c\mu}.
\end{aligned}
$$

For $\xi<0$, we have

$$
\begin{aligned}
&\int_{-\infty}^{\xi}\frac{e^{-\mu|\xi|}e^{-\frac{b_1}{c}(\xi-s)}}{c}e^{\mu|s|}ds\\
&=\frac{1}{c}\int_{-\infty}^{\xi}e^{\mu\xi}e^{-\frac{b_1}{c}(\xi-s)}e^{-\mu s}ds
\end{aligned}
$$

$$= \frac{1}{c} e^{-\frac{b_1 - c\mu}{c}\xi} \int_{-\infty}^{\xi} e^{\frac{b_1 - c\mu}{c} s} ds$$
$$= \frac{1}{b_1 - c\mu}.$$

Thus, we obtain

$$|Q_1[\Phi](\xi) - Q_1[\Psi](\xi)|e^{-\mu|\xi|} \le \left(\frac{1}{b_1 - c\mu} + \frac{1}{b_1 + c\mu} \right) (DJ_\mu + \alpha)|\Phi - \Psi|_\mu.$$

Define

$$E_1 = \left(\frac{1}{b_1 - c\mu} + \frac{1}{b_1 + c\mu} \right) (DJ_\mu + \alpha).$$

By the same argument, we can define E_3 as

$$E_3 = \left(\frac{1}{b_3 - c\mu} + \frac{1}{b_3 + c\mu} \right) (DJ_\mu + \gamma).$$

Note that

$$\begin{aligned} H_2(\phi, \varphi, \psi)(\xi) =& D \int_{-\infty}^{+\infty} J(\xi - y)\varphi(y)dy + (\beta M_2 + \gamma + \alpha + d)\varphi(\xi) \\ &+ f_2(\phi, \varphi, \psi)(\xi). \end{aligned}$$

Therefore, we have

$$\begin{aligned}
&|Q_2[\Phi](\xi) - Q_2[\Psi](\xi)|e^{-\mu|\xi|} \\
&\le \frac{e^{-\mu|\xi|}}{c} \int_{-\infty}^{\xi} e^{-\frac{b_2}{c}(\xi - s)} |H_2(\phi_1, \varphi_1, \psi_1)(s) - H_2(\phi_2, \varphi_2, \psi_2)(s)| ds \\
&\le \frac{e^{-\mu|\xi|}}{c} \int_{-\infty}^{\xi} D e^{-\frac{b_2}{c}(\xi - s)} \int_{-\infty}^{+\infty} J(s - y)|\varphi_1(y) - \varphi_2(y)| dy ds \\
&\quad + \frac{e^{-\mu|\xi|}}{c} \int_{-\infty}^{\xi} (\beta M_2 + \gamma + \alpha + d) e^{-\frac{b_2}{c}(\xi - s)} |\varphi_1(s) - \varphi_2(s)| ds \\
&\quad + \frac{e^{-\mu|\xi|}}{c} \int_{-\infty}^{\xi} e^{-\frac{b_2}{c}(\xi - s)} |f_2(\phi_1, \varphi_1, \psi_1)(s) - f_2(\phi_2, \varphi_2, \psi_2)(s)| ds \\
&\le \frac{e^{-\mu|\xi|}}{c} \int_{-\infty}^{\xi} D e^{-\frac{b_2}{c}(\xi - s)} \int_{-\infty}^{+\infty} J(s - y)|\varphi_1(y) - \varphi_2(y)| dy ds \\
&\quad + \frac{e^{-\mu|\xi|}}{c} \int_{-\infty}^{\xi} (\beta M_2 + \gamma + \alpha + d) e^{-\frac{b_2}{c}(\xi - s)} |\varphi_1(s) - \varphi_2(s)| ds \\
&\quad + \frac{e^{-\mu|\xi|}}{c} \int_{-\infty}^{\xi} e^{-\frac{b_2}{c}(\xi - s)} L|\Phi - \Psi| e^{-\mu|s|} e^{\mu|s|} ds \\
&\le \left(\frac{1}{b_2 - c\mu} + \frac{1}{b_2 + c\mu} \right) (DJ_\mu + \beta M_2 + \gamma + \alpha + d + L)|\Phi - \Psi|_\mu.
\end{aligned}$$

Define $E_2 = \left(\frac{1}{b_2 - c\mu} + \frac{1}{b_2 + c\mu} \right) (DJ_\mu + \beta M_2 + \gamma + \alpha + d + L)$. Let $E = E_1 + E_2 + E_3$. Then we have $|Q[\Phi] - Q[\Psi]|_\mu \le E|\Phi - \Psi|_\mu$. This completes the proof. □

Now, we give the definition of upper and lower solutions of system (2.5) as follows.

Definition 2.1. A pair of continuous functions $\overline{\Phi}(\xi) = (\overline{\phi}(\xi), \overline{\varphi}(\xi), \overline{\psi}(\xi))$ and $\underline{\Phi}(\xi) = (\underline{\phi}(\xi), \underline{\varphi}(\xi), \underline{\psi}(\xi))$ are called a pair of upper-lower solutions of system (2.5), respectively, if there exists a set $\mathcal{S} = \{\xi_i \in \mathbb{R}, i = 1, 2, \dots, m\}$ with finite points such that $\overline{\Phi}'(\xi)$ and $\underline{\Phi}'(\xi)$ exist and are bounded for $\xi \in \mathbb{R}\backslash\mathcal{S}$, and $\overline{\Phi}(\xi), \underline{\Phi}(\xi)$ satisfy

$$\begin{cases} D\int_{-\infty}^{+\infty} J(\xi-y)[\overline{\phi}(y)-\overline{\phi}(\xi)]dy - c\overline{\phi}'(\xi) + f_1(\overline{\phi},\overline{\varphi},\overline{\psi})(\xi) \le 0, \\ D\int_{-\infty}^{+\infty} J(\xi-y)[\overline{\varphi}(y)-\overline{\varphi}(\xi)]dy - c\overline{\varphi}'(\xi) + f_2(\underline{\phi},\overline{\varphi},\underline{\psi})(\xi) \le 0, \\ D\int_{-\infty}^{+\infty} J(\xi-y)[\overline{\psi}(y)-\overline{\psi}(\xi)]dy - c\overline{\psi}'(\xi) + f_3(\overline{\phi},\overline{\varphi},\overline{\psi})(\xi) \le 0, \end{cases}$$

$$\begin{cases} D\int_{-\infty}^{+\infty} J(\xi-y)[\underline{\phi}(y)-\underline{\phi}(\xi)]dy - c\underline{\phi}'(\xi) + f_1(\underline{\phi},\underline{\varphi},\underline{\psi})(\xi) \ge 0, \\ D\int_{-\infty}^{+\infty} J(\xi-y)[\underline{\varphi}(y)-\underline{\varphi}(\xi)]dy - c\underline{\varphi}'(\xi) + f_2(\overline{\phi},\underline{\varphi},\overline{\psi})(\xi) \ge 0, \\ D\int_{-\infty}^{+\infty} J(\xi-y)[\underline{\psi}(y)-\underline{\psi}(\xi)]dy - c\underline{\psi}'(\xi) + f_3(\underline{\phi},\underline{\varphi},\underline{\psi})(\xi) \ge 0 \end{cases}$$

for $\xi \in \mathbb{R}\backslash\mathcal{S}$, respectively.

We assume that a pair of upper-lower solutions $\overline{\Phi}(\xi) = (\overline{\phi}(\xi), \overline{\varphi}(\xi), \overline{\psi}(\xi))$ and $\underline{\Phi}(\xi) = (\underline{\phi}(\xi), \underline{\varphi}(\xi), \underline{\psi}(\xi))$ are given such that

(P1) $(0,0,0) \le (\underline{\phi}(\xi), \underline{\varphi}(\xi), \underline{\psi}(\xi)) \le (\overline{\phi}(\xi), \overline{\varphi}(\xi), \overline{\psi}(\xi)) \le (M_1, M_2, M_3)$ for $\xi \in \mathbb{R}$;

(P2) $\lim_{\xi\to-\infty}(\overline{\phi}(\xi), \overline{\varphi}(\xi), \overline{\psi}(\xi)) = (0,0,0)$,
$\lim_{\xi\to+\infty}(\underline{\phi}(\xi), \underline{\varphi}(\xi), \underline{\psi}(\xi)) = \lim_{\xi\to+\infty}(\overline{\phi}(\xi), \overline{\varphi}(\xi), \overline{\psi}(\xi)) = (k_1, k_2, k_3)$;

(P3) $\overline{\Phi}'(\xi+) \le \overline{\Phi}'(\xi-)$, $\underline{\Phi}'(\xi+) \ge \underline{\Phi}'(\xi-)$ for $\xi \in \mathbb{R}$.

Define a profile set

$$\begin{aligned} \Gamma &:= \Gamma\left((\underline{\phi},\underline{\varphi},\underline{\psi}),(\overline{\phi},\overline{\varphi},\overline{\psi})\right) \\ &= \left\{\left(\underline{\phi}(\xi),\underline{\varphi}(\xi),\underline{\psi}(\xi)\right) \le (\phi(\xi),\varphi(\xi),\psi(\xi)) \le \left(\overline{\phi}(\xi),\overline{\varphi}(\xi),\overline{\psi}(\xi)\right)\right\}. \end{aligned}$$

Obviously, Γ is nonempty, convex, closed and bounded. We have the following two lemmas.

Lemma 2.4. $Q(\Gamma) \subset \Gamma$.

Proof. From Lemma 2.2, for any (ϕ, φ, ψ) satisfying

$$(\underline{\phi}(\xi),\underline{\varphi}(\xi),\underline{\psi}(\xi)) \le (\phi(\xi),\varphi(\xi),\psi(\xi)) \le (\overline{\phi}(\xi),\overline{\varphi}(\xi),\overline{\psi}(\xi)) \quad \text{for } \xi \in \mathbb{R},$$

we have

$$\begin{aligned} Q_1(\underline{\phi},\underline{\varphi},\underline{\psi})(\xi) &\le Q_1(\phi,\varphi,\psi)(\xi) \le Q_1(\overline{\phi},\overline{\varphi},\overline{\psi})(\xi), \\ Q_2(\overline{\phi},\underline{\varphi},\overline{\psi})(\xi) &\le Q_2(\phi,\varphi,\psi)(\xi) \le Q_2(\underline{\phi},\overline{\varphi},\underline{\psi})(\xi), \\ Q_3(\underline{\phi},\underline{\varphi},\underline{\psi})(\xi) &\le Q_3(\phi,\varphi,\psi)(\xi) \le Q_3(\overline{\phi},\overline{\varphi},\overline{\psi})(\xi) \quad \text{for} \quad \xi \in \mathbb{R}. \end{aligned}$$

By the definition of upper-lower solutions, we have

$$H_1(\overline{\phi},\overline{\varphi},\overline{\psi})(\xi) \le c\overline{\phi}'(\xi) + b_1\overline{\phi}(\xi) \quad \text{for} \quad \xi \in \mathbb{R}\backslash\mathcal{S}.$$

Let $\xi_0 = -\infty$ and $\xi_{m+1} = +\infty$. Then, for $\xi_{i-1} < \xi < \xi_i$ with $i = 1, 2, \dots, m+1$, we have

$$\begin{aligned} &Q_1(\overline{\phi},\overline{\varphi},\overline{\psi})(\xi) \\ &= \frac{1}{c}e^{-\frac{b_1}{c}\xi}\int_{-\infty}^{\xi} e^{\frac{b_1}{c}s}H_1(\overline{\phi},\overline{\varphi},\overline{\psi})(s)ds \end{aligned}$$

$$\leq \frac{1}{c}e^{-\frac{b_1}{c}\xi}\left\{\left(\sum_{j=1}^{i-1}\int_{\xi_{j-1}}^{\xi_j}+\int_{\xi_{i-1}}^{\xi}\right)e^{\frac{b_1}{c}s}[c\overline{\phi}'(s)+b_1\overline{\phi}(s)]ds\right\}$$
$$=\overline{\phi}(\xi) \quad \text{for} \quad \xi\in\mathbb{R}\backslash\mathcal{S}.$$

By the continuity of $Q_1(\overline{\phi},\overline{\varphi},\overline{\psi})(\xi)$ and $\overline{\phi}(\xi)$, we get that $Q_1(\overline{\phi},\overline{\varphi},\overline{\psi})(\xi)\leq\overline{\phi}(\xi)$ for $\xi\in\mathbb{R}$.

By a similar argument, we can get

$$Q_1(\underline{\phi},\underline{\varphi},\underline{\psi})(\xi)\geq\underline{\phi}(\xi),$$
$$Q_2(\overline{\phi},\underline{\varphi},\overline{\psi})(\xi)\geq\underline{\varphi}(\xi),\quad Q_2(\underline{\phi},\overline{\varphi},\underline{\psi})(\xi)\leq\overline{\varphi}(\xi),$$
$$Q_3(\underline{\phi},\underline{\varphi},\underline{\psi})(\xi)\geq\underline{\psi}(\xi),\quad Q_3(\overline{\phi},\overline{\varphi},\overline{\psi})(\xi)\leq\overline{\psi}(\xi)\quad \text{for}\quad \xi\in\mathbb{R}.$$

Then $Q(\Gamma)\subset\Gamma$. The proof is complete. □

Lemma 2.5. *$Q:\Gamma\to\Gamma$ is compact with respect to the decay norm $|\cdot|_\mu$.*

Proof. Noting that

$$Q_1'(\phi,\varphi,\psi)(\xi)=\frac{1}{c}H_1(\phi,\varphi,\psi)(\xi)-\frac{b_1}{c^2}e^{-\frac{b_1}{c}\xi}\int_{-\infty}^{\xi}e^{\frac{b_1}{c}s}H_1(\phi,\varphi,\psi)(s)ds,$$

it follows from Lemma 2.2 that $Q_1'(\phi,\varphi,\psi)(\xi)\geq 0$. By (ii) in Lemma 2.1, we have

$$0\leq Q_1'(\phi,\varphi,\psi)(\xi)\leq\frac{1}{c}H_1(\phi,\varphi,\psi)(\xi)\leq\frac{1}{c}H_1(\overline{\phi},\overline{\varphi},\overline{\psi})(\xi)\quad \text{for } \xi\in\mathbb{R}.$$

Hence, (P1) implies that there exists a positive constant N_1 such that $|Q_1'(\phi,\varphi,\psi)|_\mu\leq N_1$.

By a same argument to $Q_3(\phi,\varphi,\psi)(\xi)$, we obtain that there exists a positive constant N_3 such that $|Q_3'(\phi,\varphi,\psi)|_\mu\leq N_3$.

For Q_2, we have

$$Q_2'(\phi,\varphi,\psi)(\xi)=\frac{1}{c}H_2(\phi,\varphi,\psi)(\xi)-\frac{b_2}{c^2}e^{-\frac{b_2}{c}\xi}\int_{-\infty}^{\xi}e^{\frac{b_2}{c}s}H_2(\phi,\varphi,\psi)(s)ds.$$

Thus, we obtain

$$\begin{aligned}
&|Q_2'(\phi,\varphi,\psi)|_\mu\\
&=\sup_{\xi\in\mathbb{R}}\left|\frac{1}{c}H_2(\phi,\varphi,\psi)(\xi)-\frac{b_2}{c^2}e^{-\frac{b_2}{c}\xi}\int_{-\infty}^{\xi}e^{\frac{b_2}{c}s}H_2(\phi,\varphi,\psi)(s)ds\right|e^{-\mu|\xi|}\\
&\leq\frac{1}{c}\sup_{\xi\in\mathbb{R}}H_2(\phi,\varphi,\psi)(\xi)e^{-\mu|\xi|}\\
&\quad+\frac{b_2}{c^2}\sup_{\xi\in\mathbb{R}}e^{-\frac{b_2}{c}\xi-\mu|\xi|}\int_{-\infty}^{\xi}e^{\frac{b_2}{c}s}e^{\mu|s|}e^{-\mu|s|}H_2(\phi,\varphi,\psi)(s)ds\\
&\leq\frac{1}{c}|H_2(\phi,\varphi,\psi)|_\mu+\frac{b_2}{c^2}|H_2(\phi,\varphi,\psi)|_\mu\sup_{\xi\in\mathbb{R}}\left\{e^{-\frac{b_2}{c}\xi-\mu|\xi|}\int_{-\infty}^{\xi}e^{\frac{b_2}{c}s}e^{\mu|s|}ds\right\}.
\end{aligned}$$

If $\xi > 0$, we get

$$\begin{aligned}
& e^{-\frac{b_2}{c}\xi-\mu|\xi|}\int_{-\infty}^{\xi} e^{\frac{b_2}{c}s}e^{\mu|s|}ds \\
&= e^{-\frac{b_2+c\mu}{c}\xi}\left[\int_{-\infty}^{0} e^{\frac{b_2-c\mu}{c}s}ds + \int_{0}^{\xi} e^{\frac{b_2+c\mu}{c}s}ds\right] \\
&\leq \frac{c}{b_2-c\mu} + \frac{c}{b_2+c\mu}.
\end{aligned}$$

If $\xi < 0$, we have

$$\begin{aligned}
& e^{-\frac{b_2}{c}\xi-\mu|\xi|}\int_{-\infty}^{\xi} e^{\frac{b_2}{c}s}e^{\mu|s|}ds \\
&= e^{-\frac{b_2-c\mu}{c}\xi}\int_{-\infty}^{\xi} e^{\frac{b_2-c\mu}{c}s}ds \\
&\leq \frac{c}{b_2-c\mu}.
\end{aligned}$$

Therefore, we obtain

$$|Q_2'(\phi,\varphi,\psi)|_\mu \leq \left[\frac{1}{c} + \frac{b_2}{c}\left(\frac{1}{b_2-c\mu} + \frac{1}{b_2+c\mu}\right)\right]|H_2(\phi,\varphi,\psi)|_\mu.$$

Noting that $(0,0,0) \leq (\phi(\xi),\varphi(\xi),\psi(\xi)) \leq (M_1,M_2,M_3)$, it follows from Lemma 2.1 that

$$H_2(M_1,0,M_3) \leq H_2(\phi,\varphi,\psi)(\xi) \leq H_2(0,M_2,0) \quad \text{for} \quad \xi \in \mathbb{R}.$$

Hence, there exists a positive constant N_2 such that $|Q_2'(\phi,\varphi,\psi)|_\mu \leq N_2$.

The above estimates for $Q'(\phi,\varphi,\psi)$ shows that $Q(\Gamma)$ is equicontinuous. It follows from the proof of Lemma 2.4 that $Q(\Gamma)$ is uniform bounded.

Next, we define

$$Q^n(\phi,\varphi,\psi)(\xi) = \begin{cases} Q(\phi,\varphi,\psi)(-n), & \xi \in (-\infty,-n), \\ Q(\phi,\varphi,\psi)(\xi), & \xi \in [-n,n], \\ Q(\phi,\varphi,\psi)(n), & \xi \in (n,+\infty). \end{cases}$$

Then, for each $n \geq 1$, $Q^n(\Gamma)$ is also equicontinuous and uniform bounded. Now, in the interval $[-n,n]$, it follows from the Arzela-Ascoli Theorem that Q^n is compact. On the other hand, $Q^n \to Q$ in $B_\mu(\mathbb{R},\mathbb{R}^3)$ as $n \to \infty$, since

$$\begin{aligned}
& \sup_{\xi\in\mathbb{R}} |Q^n(\phi,\varphi,\psi)(\xi) - Q(\phi,\varphi,\psi)(\xi)|e^{-\mu|\xi|} \\
&\leq 2(M_1+M_2+M_3)e^{-\mu n} \to 0, \quad n \to \infty.
\end{aligned}$$

By Proposition 2.12 in [22], we obtain that Q is compact on Γ. The proof is complete. □

Theorem 2.1. *Assume that (A1)-(A4) hold. Suppose that there exists a pair of upper-lower solutions* $\overline{\Phi}(\xi) = (\overline{\phi}(\xi),\overline{\varphi}(\xi),\overline{\psi}(\xi))$ *and* $\underline{\Phi}(\xi) = (\underline{\phi}(\xi),\underline{\varphi}(\xi),\underline{\psi}(\xi))$ *of system (2.5) satisfying (P1), (P2) and (P3), then system (2.4) has a traveling wave solution satisfying the asymptotic boundary conditions (2.7).*

Proof. Combining Lemmas 2.1-2.5 with the Schauder's fixed point theorem, we know that there exist a fixed point $\Phi^* = (\phi^*, \varphi^*, \psi^*) \in \Gamma$ of Q, which gives a traveling wave solution to system (2.4).

By (P2) and the fact that

$$(\underline{\phi}(\xi), \underline{\varphi}(\xi), \underline{\psi}(\xi)) \leq (\phi^*(\xi), \varphi^*(\xi), \psi^*(\xi)) \leq (\overline{\phi}(\xi), \overline{\varphi}(\xi), \overline{\psi}(\xi)),$$

we get that

$$\lim_{\xi\to-\infty}(\phi^*(\xi), \varphi^*(\xi), \psi^*(\xi)) = (0,0,0),$$
$$\lim_{\xi\to+\infty}(\phi^*(\xi), \varphi^*(\xi), \psi^*(\xi)) = (k_1,k_2,k_3).$$

Therefore, the fixed point $(\phi^*, \varphi^*, \psi^*)$ satisfies the asymptotic boundary conditions (2.7). The proof is complete. □

In order to construct appropriate upper-lower solutions for (2.5), we consider the following functions

$$\begin{cases} \Delta_1(\lambda,c) = D\int_{-\infty}^{+\infty} J(y)e^{-\lambda y}dy - D - c\lambda - d + \alpha\frac{M_2}{M_1}, \\ \Delta_2(\lambda,c) = D\int_{-\infty}^{+\infty} J(y)e^{-\lambda y}dy - D - c\lambda + \frac{A\beta}{d} - (\gamma+\alpha+d), \\ \Delta_3(\lambda,c) = D\int_{-\infty}^{+\infty} J(y)e^{-\lambda y}dy - D - c\lambda - (\delta+d) + \gamma\frac{M_2}{M_3}. \end{cases}$$

From the selection of M_1 and M_2, we can obtain that $\alpha\frac{M_2}{M_1} - d > 0$. Furthermore, by a direct calculation, we have

$$\Delta_1(0,c) = \alpha\frac{M_2}{M_1} - d > 0;$$
$$\Delta_1(\lambda,\infty) = -\infty \quad \text{for any given} \quad \lambda > 0;$$
$$\frac{\partial\Delta_1(\lambda,c)}{\partial c} = -\lambda < 0 \quad \text{for any given} \quad \lambda > 0;$$
$$\frac{\partial^2\Delta_1(\lambda,c)}{\partial\lambda^2} = D\int_{-\infty}^{+\infty} y^2J(y)e^{-\lambda y}dy > 0.$$

Thus, we conclude that there is a $c_1^* > 0$ such that $\Delta_1(\lambda,c) = 0$ has two zeros $0 < \lambda_1 < \lambda_2$ for $c > c_1^*$.

By the same arguments, noting the selection of M_2 and M_3, we can also have

$$\Delta_2(\lambda,c) = 0 \quad \text{has two zeros } 0 < \lambda_3 < \lambda_4 \text{ for } c > c_2^*,$$
$$\Delta_3(\lambda,c) = 0 \quad \text{has two zeros } 0 < \lambda_5 < \lambda_6 \text{ for } c > c_3^*.$$

Let $c^* = \max\{c_1^*, c_2^*, c_3^*\}$. We obtain a lemma as follows.

Lemma 2.6. *Assume that* $\mathscr{R}_0 > 1$*, then we have* $\lambda_1 < \lambda_3$ *and* $\lambda_5 < \lambda_3$.

Proof. Define

$$h(\lambda) = D\int_{-\infty}^{+\infty} J(y)e^{-\lambda y}dy,$$
$$g_1(\lambda) = c\lambda + D + d - \alpha\frac{M_2}{M_1},$$
$$g_2(\lambda) = c\lambda + D - \frac{A\beta}{d} + (\gamma+\alpha+d),$$

$$g_3(\lambda) = c\lambda + D + (\delta + d) - \gamma \frac{M_2}{M_3}.$$

Then we have

$$h(\lambda_1) = g_1(\lambda_1), \quad h(\lambda_3) = g_2(\lambda_3), \quad h(\lambda_5) = g_3(\lambda_5).$$

From $\mathscr{R}_0 > 1$ and (2.8), we can get

$$g_2(0) < g_1(0), \quad g_2(0) < g_3(0).$$

Note that $h''(\lambda) > 0$ for all λ; hence, we have $\lambda_1 < \lambda_3$ and $\lambda_5 < \lambda_3$. □

Assuming that $\mathscr{R}_0 > 1$, we say that there exist $\varepsilon_i > 0 (i = 1, 2, \ldots, 6)$, $\varepsilon_2 \in (0, k_1)$, $\varepsilon_4 \in (0, k_2)$, $\varepsilon_6 \in (0, k_3)$ satisfying the system of inequalities

$$\begin{cases} -d(k_1 + \varepsilon_1) + \alpha M_2 < 0, \\ \beta\left(\frac{A}{d} - k_1 + \varepsilon_2 - k_2 - \varepsilon_3 - k_3 + \varepsilon_6\right) g(k_2 + \varepsilon_3) \\ < (\gamma + \alpha + d)(k_2 + \varepsilon_3), \\ \gamma M_2 - (\delta + d)(k_3 + \varepsilon_5) < 0, \\ -d(k_1 - \varepsilon_2) + \alpha(k_2 - \varepsilon_4) > 0, \\ \beta\left(\frac{A}{d} - M_1 - k_2 + \varepsilon_4 - M_3\right) g(k_2 - \varepsilon_4) \\ > (\gamma + \alpha + d)(k_2 - \varepsilon_4), \\ \gamma(k_2 - \varepsilon_4) - (\delta + d)(k_3 - \varepsilon_6) > 0. \end{cases} \tag{2.13}$$

Noting that $k_1 = \frac{\alpha}{d} k_2$ and $k_3 = \frac{\gamma}{\delta + d} k_2$, we can find $\varepsilon_1 > 0, \varepsilon_5 > 0$ such that

$$\begin{aligned} &\varepsilon_1 > \frac{\alpha}{d}(M_2 - k_2) = \frac{\alpha}{d} M_2 - k_1 > 0, \\ &\Rightarrow -d(k_1 + \varepsilon_1) + \alpha M_2 < 0, \end{aligned}$$

$$\begin{aligned} &\varepsilon_5 > \frac{\gamma}{\delta + d}(M_2 - k_2) = \frac{\gamma}{\delta + d} M_2 - k_3 > 0, \\ &\Rightarrow \gamma M_2 - (\delta + d)(k_3 + \varepsilon_5) < 0. \end{aligned}$$

By the second equation of system (2.5), we have

$$\beta\left(\frac{A}{d} - k_1 - k_2 - k_3\right) g(k_2) = (\gamma + \alpha + d) k_2.$$

Let $\varepsilon_4 > (M_1 - k_1) + (M_3 - k_3)$. It then follows from (A4) that

$$\begin{aligned} &\beta\left(\frac{A}{d} - M_1 - k_2 + \varepsilon_4 - M_3\right) g(k_2 - \varepsilon_4) \\ &> (\gamma + \alpha + d)(k_2 - \varepsilon_4). \end{aligned}$$

Noting (2.9), for $\varepsilon_4 \in ((M_1 - k_1) + (M_3 - k_3), k_2)$, we can find $\varepsilon_2 \in (0, k_1)$, $\varepsilon_6 \in (0, k_3)$ such that

$$\begin{aligned} &k_1 > \varepsilon_2 > \frac{\alpha}{d} \varepsilon_4 = k_1 - \frac{\alpha}{d}(k_2 - \varepsilon_4) > 0, \\ &\Rightarrow -d(k_1 - \varepsilon_2) + \alpha(k_2 - \varepsilon_4) > 0, \end{aligned}$$

$$k_3 > \varepsilon_6 > \frac{\gamma}{\delta+d}\varepsilon_4 = k_3 - \frac{\gamma}{\delta+d}(k_2-\varepsilon_4) > 0,$$
$$\Rightarrow \gamma(k_2-\varepsilon_4) - (\delta+d)(k_3-\varepsilon_6) > 0.$$

Let $\varepsilon_3 > \varepsilon_2 + \varepsilon_6$. It then follows from (A4) that

$$\beta\left(\frac{A}{d} - k_1 + \varepsilon_2 - k_2 - \varepsilon_3 - k_3 + \varepsilon_6\right) g(k_2+\varepsilon_3)$$
$$< (\gamma+\alpha+d)(k_2+\varepsilon_3).$$

Therefore, we claim that such a group of $\varepsilon_i (i = 1, 2, \ldots, 6)$ could be found.

Let $\varepsilon_i (i = 1, 2, \ldots, 6)$ be defined as in (2.13). We define two continuous functions $\overline{\Phi}(\xi) = (\overline{\phi}(\xi), \overline{\varphi}(\xi), \overline{\psi}(\xi))$ and $\underline{\Phi}(\xi) = (\underline{\phi}(\xi), \underline{\varphi}(\xi), \underline{\psi}(\xi))$ as follows

$$\overline{\phi}(\xi) = \begin{cases} k_1 e^{\lambda_1 \xi}, & \xi \le \xi_1, \\ k_1 + \varepsilon_1 e^{-\lambda\xi}, & \xi > \xi_1, \end{cases} \quad \underline{\phi}(\xi) = \begin{cases} 0, & \xi \le \xi_2, \\ k_1 - \varepsilon_2 e^{-\lambda\xi}, & \xi > \xi_2, \end{cases}$$

$$\overline{\varphi}(\xi) = \begin{cases} \eta k_2 e^{\lambda_3 \xi}, & \xi \le \xi_3, \\ k_2 + \varepsilon_3 e^{-\lambda\xi}, & \xi > \xi_3, \end{cases} \quad \underline{\varphi}(\xi) = \begin{cases} 0, & \xi \le \xi_4, \\ k_2 - \varepsilon_4 e^{-\lambda\xi}, & \xi > \xi_4, \end{cases}$$

$$\overline{\psi}(\xi) = \begin{cases} k_3 e^{\lambda_5 \xi}, & \xi \le \xi_5, \\ k_3 + \varepsilon_5 e^{-\lambda\xi}, & \xi > \xi_5, \end{cases} \quad \underline{\psi}(\xi) = \begin{cases} 0, & \xi \le \xi_6, \\ k_3 - \varepsilon_6 e^{-\lambda\xi}, & \xi > \xi_6, \end{cases}$$

where $\xi_1, \xi_3, \xi_5 > 0$, $\xi_2, \xi_4, \xi_6 < 0$ and $\lambda > 0$ is a small enough constant to be chosen later. Let M_1, M_2, M_3 be chosen such that (2.8) and (2.9) hold. Then we first choose $\lambda > 0$ to be sufficiently small such that $\xi_1 > 0, \xi_3 > 0, \xi_5 > 0$ satisfying

$$k_1 + \varepsilon_1 > M_1 = \sup_{\xi\in\mathbb{R}} \overline{\phi}(\xi) = k_1 e^{\lambda_1\xi_1} > k_1,$$

$$k_2 + \varepsilon_3 > M_2 = \sup_{\xi\in\mathbb{R}} \overline{\varphi}(\xi) = \eta k_2 e^{\lambda_3\xi_3} > k_2,$$

$$k_3 + \varepsilon_5 > M_3 = \sup_{\xi\in\mathbb{R}} \overline{\psi}(\xi) = k_3 e^{\lambda_5\xi_5} > k_3.$$

Furthermore, we can choose $\eta \in (0,1)$ such that $\xi_3 \ge \max\{\xi_1, \xi_5\}$. On the other hand, noting that

$$\xi_2 = \frac{1}{\lambda}\ln\frac{\varepsilon_2}{k_1} = \frac{1}{\lambda}\ln\frac{\varepsilon_2}{\frac{\alpha k_2}{d}}, \quad \xi_4 = \frac{1}{\lambda}\ln\frac{\varepsilon_4}{k_2}, \quad \xi_6 = \frac{1}{\lambda}\ln\frac{\varepsilon_6}{k_3} = \frac{1}{\lambda}\ln\frac{\varepsilon_6}{\frac{\gamma k_2}{\delta+d}},$$

we have from $\varepsilon_2 > \frac{\alpha}{d}\varepsilon_4$ and $\varepsilon_6 > \frac{\gamma}{\delta+d}\varepsilon_4$ that $\xi_4 \le \min\{\xi_2, \xi_6\}$. It is obvious that $\overline{\Phi}(\xi)$ and $\underline{\Phi}(\xi)$ satisfy (P1),(P2) and (P3).

Lemma 2.7. *Let $\mathscr{R}_0 > 1$. Assume that (A1)-(A4) hold, then $\overline{\Phi}(\xi) = (\overline{\phi}(\xi), \overline{\varphi}(\xi), \overline{\psi}(\xi))$ is an upper solution and $\underline{\Phi}(\xi) = (\underline{\phi}(\xi), \underline{\varphi}(\xi), \underline{\psi}(\xi))$ is a lower solution of system (2.5), respectively.*

Proof. Firstly, we have the following facts

$$\begin{aligned} \overline{\phi}(\xi) &\le k_1 e^{\lambda_1\xi}, \quad \overline{\phi}(\xi) \le k_1 + \varepsilon_1 e^{-\lambda\xi} \quad \text{for} \quad \xi\in\mathbb{R}, \\ \overline{\varphi}(\xi) &\le \eta k_2 e^{\lambda_3\xi}, \quad \overline{\varphi}(\xi) \le k_2 + \varepsilon_3 e^{-\lambda\xi} \quad \text{for} \quad \xi\in\mathbb{R}, \\ \overline{\psi}(\xi) &\le k_3 e^{\lambda_5\xi}, \quad \overline{\psi}(\xi) \le k_3 + \varepsilon_5 e^{-\lambda\xi} \quad \text{for} \quad \xi\in\mathbb{R}, \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} &\underline{\phi}(\xi) \geq 0, \quad \underline{\phi}(\xi) \geq k_1 - \varepsilon_2 e^{-\lambda\xi} \quad \text{for} \quad \xi \in \mathbb{R}, \\ &\underline{\varphi}(\xi) \geq 0, \quad \underline{\varphi}(\xi) \geq k_2 - \varepsilon_4 e^{-\lambda\xi} \quad \text{for} \quad \xi \in \mathbb{R}, \\ &\underline{\psi}(\xi) \geq 0, \quad \underline{\psi}(\xi) \geq k_3 - \varepsilon_6 e^{-\lambda\xi} \quad \text{for} \quad \xi \in \mathbb{R}. \end{aligned} \tag{2.15}$$

If $\xi \leq \xi_1$, then $\overline{\phi}(\xi) = k_1 e^{\lambda_1 \xi}$, $\overline{\varphi}(\xi) = \eta k_2 e^{\lambda_3 \xi}$, it follows from (2.14) that

$$\begin{aligned} &D \int_{-\infty}^{+\infty} J(\xi - y)[\overline{\phi}(y) - \overline{\phi}(\xi)]dy - c\overline{\phi}'(\xi) - d\overline{\phi}(\xi) + \alpha\overline{\varphi}(\xi) \\ &\leq Dk_1 \int_{-\infty}^{+\infty} J(\xi - y)\left(e^{\lambda_1 y} - e^{\lambda_1 \xi}\right)dy - c\lambda_1 k_1 e^{\lambda_1 \xi} \\ &\quad - dk_1 e^{\lambda_1 \xi} + \alpha\eta k_2 e^{\lambda_3 \xi} \\ &\leq \left[D \int_{-\infty}^{+\infty} J(y)e^{-\lambda_1 y}dy - D - c\lambda_1 - d + \alpha\eta \frac{k_2}{k_1} e^{(\lambda_3 - \lambda_1)\xi_1}\right] k_1 e^{\lambda_1 \xi} \\ &= \left[D \int_{-\infty}^{+\infty} J(y)e^{-\lambda_1 y}dy - D - c\lambda_1 - d + \alpha\eta \frac{k_2 e^{\lambda_3 \xi_1}}{k_1 e^{\lambda_1 \xi_1}}\right] k_1 e^{\lambda_1 \xi} \\ &= \left[D \int_{-\infty}^{+\infty} J(y)e^{-\lambda_1 y}dy - D - c\lambda_1 - d + \alpha\eta \frac{k_2 e^{\lambda_3 \xi_1}}{M_1}\right] k_1 e^{\lambda_1 \xi} \\ &\leq \left[D \int_{-\infty}^{+\infty} J(y)e^{-\lambda_1 y}dy - D - c\lambda_1 - d + \alpha \frac{M_2}{M_1}\right] k_1 e^{\lambda_1 \xi} \\ &= \Delta_1(\lambda_1, c)k_1 e^{\lambda_1 \xi} = 0. \end{aligned}$$

If $\xi > \xi_1$, then $\overline{\phi}(\xi) = k_1 + \varepsilon_1 e^{-\lambda\xi}$, $\overline{\varphi}(\xi) \leq M_2$, we have from (2.14) that

$$\begin{aligned} &D \int_{-\infty}^{+\infty} J(\xi - y)[\overline{\phi}(y) - \overline{\phi}(\xi)]dy - c\overline{\phi}'(\xi) - d\overline{\phi}(\xi) + \alpha\overline{\varphi}(\xi) \\ &\leq D\varepsilon_1 \int_{-\infty}^{+\infty} J(\xi - y)\left(e^{-\lambda y} - e^{-\lambda\xi}\right)dy + c\lambda\varepsilon_1 e^{-\lambda\xi} \\ &\quad - d\left(k_1 + \varepsilon_1 e^{-\lambda\xi}\right) + \alpha M_2 \\ &\leq \left[D\varepsilon_1 \int_{-\infty}^{+\infty} J(y)e^{\lambda y}dy - D\varepsilon_1 + c\lambda\varepsilon_1\right] e^{-\lambda\xi} \\ &\quad - d\left(k_1 + \varepsilon_1 e^{-\lambda\xi}\right) + \alpha M_2 \\ &=: P_1(\lambda). \end{aligned}$$

From (2.13), we have that $P_1(0) = -d(k_1 + \varepsilon_1) + \alpha M_2 < 0$. Thus there exists $\lambda_1^* > 0$ such that

$$D \int_{-\infty}^{+\infty} J(\xi - y)[\overline{\phi}(y) - \overline{\phi}(\xi)]dy - c\overline{\phi}'(\xi) - d\overline{\phi}(\xi) + \alpha\overline{\varphi}(\xi) \leq P_1(\lambda) < 0$$

for $\lambda \in (0, \lambda_1^*)$.

If $\xi \leq \xi_3$, then $\overline{\varphi}(\xi) = \eta k_2 e^{\lambda_3 \xi}$, $\overline{\varphi}(\xi - c\tau) \leq \eta k_2 e^{\lambda_3(\xi - c\tau)}$, we derive from (2.14) that

$$\begin{aligned}
&D\int_{-\infty}^{+\infty} J(\xi-y)[\overline{\varphi}(y)-\overline{\varphi}(\xi)]dy - c\overline{\varphi}'(\xi) + \beta\left[\frac{A}{d} - \underline{\phi}(\xi) - \overline{\varphi}(\xi) - \underline{\psi}(\xi)\right] \\
&\quad \times \int_0^h f(\tau)g(\overline{\varphi}(\xi - c\tau))d\tau - (\gamma+\alpha+d)\overline{\varphi}(\xi) \\
&\le \Bigg[D\int_{-\infty}^{+\infty} J(y)e^{-\lambda_3 y}dy - D - c\lambda_3 + \frac{A\beta}{d}\int_0^h f(\tau)e^{-\lambda_3 c\tau}d\tau \\
&\quad - (\gamma+\alpha+d)\Bigg]\eta k_2 e^{\lambda_3\xi} \\
&\le \left[D\int_{-\infty}^{+\infty} J(y)e^{-\lambda_3 y}dy - D - c\lambda_3 + \frac{A\beta}{d} - (\gamma+\alpha+d)\right]\eta k_2 e^{\lambda_3\xi} \\
&= \Delta_2(\lambda_3, c)\eta k_2 e^{\lambda_3\xi} = 0.
\end{aligned}$$

If $\xi > \xi_3$, then

$$\begin{aligned}
\overline{\varphi}(\xi) &= k_2 + \varepsilon_3 e^{-\lambda\xi}, \\
\underline{\phi}(\xi) &= k_1 - \varepsilon_2 e^{-\lambda\xi}, \\
\underline{\psi}(\xi) &= k_3 - \varepsilon_6 e^{-\lambda\xi}, \\
\overline{\varphi}(\xi - c\tau) &\le k_2 + \varepsilon_3 e^{-\lambda(\xi - c\tau)},
\end{aligned}$$

it follows from (2.14) that

$$\begin{aligned}
&D\int_{-\infty}^{+\infty} J(\xi-y)[\overline{\varphi}(y)-\overline{\varphi}(\xi)]dy - c\overline{\varphi}'(\xi) + \beta\left[\frac{A}{d} - \underline{\phi}(\xi) - \overline{\varphi}(\xi) - \underline{\psi}(\xi)\right] \\
&\quad \times \int_0^h f(\tau)g(\overline{\varphi}(\xi - c\tau))d\tau - (\gamma+\alpha+d)\overline{\varphi}(\xi) \\
&\le D\varepsilon_3\int_{-\infty}^{+\infty} J(\xi-y)\left(e^{-\lambda y} - e^{-\lambda\xi}\right)dy + c\lambda\varepsilon_3 e^{-\lambda\xi} \\
&\quad + \beta\left(\frac{A}{d} - k_1 + \varepsilon_2 e^{-\lambda\xi} - k_2 - \varepsilon_3 e^{-\lambda\xi} - k_3 + \varepsilon_6 e^{-\lambda\xi}\right) \\
&\quad \times \int_0^h f(\tau)g\left(k_2 + \varepsilon_3 e^{-\lambda(\xi - c\tau)}\right)d\tau - (\gamma+\alpha+d)\left(k_2 + \varepsilon_3 e^{-\lambda\xi}\right) \\
&=: P_2(\lambda).
\end{aligned}$$

From (2.13), we have that

$$P_2(0) = -(\gamma+\alpha+d)(k_2+\varepsilon_3) + \beta\left(\frac{A}{d} - k_1 + \varepsilon_2 - k_2 - \varepsilon_3 - k_3 + \varepsilon_6\right)g(k_2+\varepsilon_3) < 0.$$

Thus there exists $\lambda_3^* > 0$ such that

$$\begin{aligned}
&D\int_{-\infty}^{+\infty} J(\xi-y)[\overline{\varphi}(y)-\overline{\varphi}(\xi)]dy - c\overline{\varphi}'(\xi) + \beta\left[\frac{A}{d} - \underline{\phi}(\xi) - \overline{\varphi}(\xi) - \underline{\psi}(\xi)\right] \\
&\times \int_0^h f(\tau)g(\overline{\varphi}(\xi - c\tau))d\tau - (\gamma+\alpha+d)\overline{\varphi}(\xi) \\
&\le P_2(\lambda) < 0
\end{aligned}$$

for $\lambda \in (0, \lambda_3^*)$.

If $\xi \le \xi_5$, then $\overline{\psi}(\xi) = k_3 e^{\lambda_5 \xi}$, $\overline{\varphi}(\xi) = \eta k_2 e^{\lambda_3 \xi}$, we get from (2.14) that

$$\begin{aligned}
& D\int_{-\infty}^{+\infty} J(\xi - y)[\overline{\psi}(y) - \overline{\psi}(\xi)]dy - c\overline{\psi}'(\xi) + \gamma\overline{\varphi}(\xi) - (\delta + d)\overline{\psi}(\xi) \\
& \le \left[D\int_{-\infty}^{+\infty} J(y)e^{-\lambda_5 y}dy - D - c\lambda_5 - (\delta + d) + \gamma\eta\frac{k_2}{k_3}e^{(\lambda_3 - \lambda_5)\xi_5}\right] k_3 e^{\lambda_5 \xi} \\
& \le \left[D\int_{-\infty}^{+\infty} J(y)e^{-\lambda_5 y}dy - D - c\lambda_5 - (\delta + d) + \gamma\frac{M_2}{M_3}\right] k_3 e^{\lambda_5 \xi} \\
& = \Delta_3(\lambda_5, c)k_3 e^{\lambda_5 \xi} = 0.
\end{aligned}$$

If $\xi > \xi_5$, then $\overline{\psi}(\xi) = k_3 + \varepsilon_5 e^{-\lambda\xi}$, $\overline{\varphi}(\xi) \le M_2$, we have from (2.14) that

$$\begin{aligned}
& D\int_{-\infty}^{+\infty} J(\xi - y)[\overline{\psi}(y) - \overline{\psi}(\xi)]dy - c\overline{\psi}'(\xi) + \gamma\overline{\varphi}(\xi) - (\delta + d)\overline{\psi}(\xi) \\
& \le D\varepsilon_5 \int_{-\infty}^{+\infty} J(\xi - y)\left(e^{-\lambda y} - e^{-\lambda\xi}\right)dy + c\lambda\varepsilon_5 e^{-\lambda\xi} \\
& \quad + \gamma M_2 - (\delta + d)\left(k_3 + \varepsilon_5 e^{-\lambda\xi}\right) \\
& \le \left[D\varepsilon_5 \int_{-\infty}^{+\infty} J(y)e^{\lambda y}dy - D\varepsilon_5 + c\lambda\varepsilon_5\right] e^{-\lambda\xi} \\
& \quad + \gamma M_2 - (\delta + d)\left(k_3 + \varepsilon_5 e^{-\lambda\xi}\right) \\
& =: P_3(\lambda).
\end{aligned}$$

From (2.13), we have that $P_3(0) = \gamma M_2 - (\delta + d)(k_3 + \varepsilon_5) < 0$. Thus there exists $\lambda_5^* > 0$ such that

$$D\int_{-\infty}^{+\infty} J(\xi - y)[\overline{\psi}(y) - \overline{\psi}(\xi)]dy - c\overline{\psi}'(\xi) + \gamma\overline{\varphi}(\xi) - (\delta + d)\overline{\psi}(\xi) \le P_3(\lambda) < 0$$

for $\lambda \in (0, \lambda_5^*)$.

If $\xi \le \xi_2$, then $\underline{\phi}(\xi) = 0$, we have from (2.15) that

$$D\int_{-\infty}^{+\infty} J(\xi - y)[\underline{\phi}(y) - \underline{\phi}(\xi)]dy - c\underline{\phi}'(\xi) - d\underline{\phi}(\xi) + \alpha\underline{\varphi}(\xi) \ge \alpha\underline{\varphi}(\xi) \ge 0.$$

If $\xi > \xi_2$, then $\underline{\phi}(\xi) = k_1 - \varepsilon_2 e^{-\lambda\xi}$, $\underline{\varphi}(\xi) = k_2 - \varepsilon_4 e^{-\lambda\xi}$, it follows from (2.15) that

$$\begin{aligned}
& D\int_{-\infty}^{+\infty} J(\xi - y)[\underline{\phi}(y) - \underline{\phi}(\xi)]dy - c\underline{\phi}'(\xi) - d\underline{\phi}(\xi) + \alpha\underline{\varphi}(\xi) \\
& \ge \left[-D\varepsilon_2 \int_{-\infty}^{+\infty} J(y)e^{\lambda y}dy + D\varepsilon_2 - c\lambda\varepsilon_2\right] e^{-\lambda\xi} \\
& \quad - d\left(k_1 - \varepsilon_2 e^{-\lambda\xi}\right) + \alpha\left(k_2 - \varepsilon_4 e^{-\lambda\xi}\right) \\
& =: P_4(\lambda).
\end{aligned}$$

From (2.13), we have that $P_4(0) = -d(k_1 - \varepsilon_2) + \alpha(k_2 - \varepsilon_4) > 0$. Thus there exists $\lambda_2^* > 0$ such that

$$D\int_{-\infty}^{+\infty} J(\xi - y)[\underline{\phi}(y) - \underline{\phi}(\xi)]dy - c\underline{\phi}'(\xi) - d\underline{\phi}(\xi) + \alpha\underline{\varphi}(\xi) \ge P_4(\lambda) > 0$$

for $\lambda \in (0, \lambda_2^*)$.

If $\xi \leq \xi_4$, then $\underline{\varphi}(\xi) = 0$. Note that $\overline{\phi}(\xi) + \underline{\varphi}(\xi) + \overline{\psi}(\xi) \leq M_1 + M_2 + M_3 < \frac{A}{d}$, we derive from (2.15) that

$$D\int_{-\infty}^{+\infty} J(\xi - y)[\underline{\varphi}(y) - \underline{\varphi}(\xi)]dy - c\underline{\varphi}'(\xi) + \beta\left[\frac{A}{d} - \overline{\phi}(\xi) - \underline{\varphi}(\xi) - \overline{\psi}(\xi)\right] \times \int_0^h f(\tau)g(\underline{\varphi}(\xi - c\tau))d\tau - (\gamma + \alpha + d)\underline{\varphi}(\xi) \geq 0.$$

If $\xi > \xi_4$, then

$$\underline{\varphi}(\xi) = k_2 - \varepsilon_4 e^{-\lambda\xi},$$
$$\overline{\phi}(\xi) \leq M_1,$$
$$\overline{\psi}(\xi) \leq M_3,$$
$$\underline{\varphi}(\xi - c\tau) \geq k_2 - \varepsilon_4 e^{-\lambda(\xi - c\tau)},$$

it follows from (2.15) that

$$\begin{aligned}
&D\int_{-\infty}^{+\infty} J(\xi - y)[\underline{\varphi}(y) - \underline{\varphi}(\xi)]dy - c\underline{\varphi}'(\xi) + \beta\left[\frac{A}{d} - \overline{\phi}(\xi) - \underline{\varphi}(\xi) - \overline{\psi}(\xi)\right] \\
&\quad \times \int_0^h f(\tau)g(\underline{\varphi}(\xi - c\tau))d\tau - (\gamma + \alpha + d)\underline{\varphi}(\xi) \\
&\geq -D\varepsilon_4 \int_{-\infty}^{+\infty} J(\xi - y)\left(e^{-\lambda y} - e^{-\lambda\xi}\right)dy - c\lambda\varepsilon_4 e^{-\lambda\xi} \\
&\quad + \beta\left(\frac{A}{d} - M_1 - k_2 + \varepsilon_4 e^{-\lambda\xi} - M_3\right) \\
&\quad \times \int_0^h f(\tau)g\left(k_2 - \varepsilon_4 e^{-\lambda(\xi - c\tau)}\right)d\tau - (\gamma + \alpha + d)\left(k_2 - \varepsilon_4 e^{-\lambda\xi}\right) \\
&=: P_5(\lambda).
\end{aligned}$$

From (2.13), we have that

$$P_5(0) = -(\gamma + \alpha + d)(k_2 - \varepsilon_4) + \beta\left(\frac{A}{d} - M_1 - k_2 + \varepsilon_4 - M_3\right)g(k_2 - \varepsilon_4) > 0.$$

Thus there exists $\lambda_4^* > 0$ such that

$$\begin{aligned}
&D\int_{-\infty}^{+\infty} J(\xi - y)[\underline{\varphi}(y) - \underline{\varphi}(\xi)]dy - c\underline{\varphi}'(\xi) + \beta\left[\frac{A}{d} - \overline{\phi}(\xi) - \underline{\varphi}(\xi) - \overline{\psi}(\xi)\right] \\
&\times \int_0^h f(\tau)g(\underline{\varphi}(\xi - c\tau))d\tau - (\gamma + \alpha + d)\underline{\varphi}(\xi) \\
&\geq P_5(\lambda) > 0
\end{aligned}$$

for $\lambda \in (0, \lambda_4^*)$.

If $\xi \leq \xi_6$, then $\underline{\psi}(\xi) = 0$, we get from (2.15) that

$$D\int_{-\infty}^{+\infty} J(\xi - y)[\underline{\psi}(y) - \underline{\psi}(\xi)]dy - c\underline{\psi}'(\xi) + \gamma\underline{\varphi}(\xi) - (\delta + d)\underline{\psi}(\xi) \geq \gamma\underline{\varphi}(\xi) \geq 0.$$

If $\xi > \xi_6$, then $\underline{\psi}(\xi) = k_3 - \varepsilon_6 e^{-\lambda\xi}$, $\underline{\varphi}(\xi) = k_2 - \varepsilon_4 e^{-\lambda\xi}$, we have from (2.15) that

$$\begin{aligned}
&D\int_{-\infty}^{+\infty} J(\xi-y)[\underline{\psi}(y)-\underline{\psi}(\xi)]dy - c\underline{\psi}'(\xi) + \gamma\underline{\varphi}(\xi) - (\delta+d)\underline{\psi}(\xi)\\
&\geq -D\varepsilon_6 \int_{-\infty}^{+\infty} J(\xi-y)\left(e^{-\lambda y} - e^{-\lambda\xi}\right)dy - c\lambda\varepsilon_6 e^{-\lambda\xi}\\
&\quad + \gamma\left(k_2 - \varepsilon_4 e^{-\lambda\xi}\right) - (\delta+d)\left(k_3 - \varepsilon_6 e^{-\lambda\xi}\right)\\
&\geq \left[-D\varepsilon_6 \int_{-\infty}^{+\infty} J(y)e^{\lambda y}dy + D\varepsilon_6 - c\lambda\varepsilon_6\right]e^{-\lambda\xi}\\
&\quad + \gamma\left(k_2 - \varepsilon_4 e^{-\lambda\xi}\right) - (\delta+d)\left(k_3 - \varepsilon_6 e^{-\lambda\xi}\right)\\
&=: P_6(\lambda).
\end{aligned}$$

From (2.13), we have that $P_6(0) = \gamma(k_2 - \varepsilon_4) - (\delta+d)(k_3 - \varepsilon_6) > 0$. Thus there exists $\lambda_6^* > 0$ such that

$$D\int_{-\infty}^{+\infty} J(\xi-y)[\underline{\psi}(y)-\underline{\psi}(\xi)]dy - c\underline{\psi}'(\xi) + \gamma\underline{\varphi}(\xi) - (\delta+d)\underline{\psi}(\xi) \geq P_6(\lambda) > 0$$

for $\lambda \in (0, \lambda_6^*)$.

Taking $\lambda \in (0, \min_{i=1,2,\ldots,6}\{\lambda_i^*\})$, we see that $\overline{\Phi}(\xi)$ is an upper solution and $\underline{\Phi}(\xi)$ is a lower solution of system (2.5), respectively. □

Now we can state our main result in this section.

Theorem 2.2. *Let $\mathscr{R}_0 > 1$. Assume that (A1)-(A4) hold, then for any $c > c^*$, system (2.4) has a traveling wave solution $(\phi(\xi), \varphi(\xi), \psi(\xi))$ with speed c, which connects two equilibria $(0,0,0)$ and (k_1, k_2, k_3). That is to say, system (1.6) has a traveling wave solution with speed c, which connects two equilibria $(A/d, 0, 0)$ and (S^*, I^*, R^*). Furthermore, $(\phi(\xi), \varphi(\xi), \psi(\xi))$ satisfies*

$$0 \leq \phi(\xi) \leq M_1, \quad 0 \leq \varphi(\xi) \leq M_2, \quad 0 \leq \psi(\xi) \leq M_3.$$

Thus,

$$0 \leq \phi(\xi) + \varphi(\xi) + \psi(\xi) \leq \frac{A}{d}. \tag{2.16}$$

3. Nonexistence of traveling waves for $\mathscr{R}_0 < 1$

In this section, we focus on the nonexistence of traveling waves. We have the following result.

Theorem 3.1. *Suppose that $\mathscr{R}_0 < 1$. Then system (2.5) has no nontrivial bounded nonnegative solution satisfying (2.7) for any $c > 0$.*

Proof. Suppose that (2.5) admits a nontrivial bounded nonnegative solution $(\phi(\xi), \varphi(\xi), \psi(\xi))$ satisfying (2.7). We first claim that $\varphi(\xi) > 0$ for any $\xi \in \mathbb{R}$. Indeed, suppose that there exists $\xi_0 \in \mathbb{R}$ such that $\varphi(\xi_0) = 0$, then $\varphi'(\xi_0) = 0$ and

$$(J * \varphi - \varphi)(\xi_0) \geq 0$$

with equality holding if and only if $\varphi \equiv 0$. Since $\varphi(\xi)$ is nontrivial, we have

$$(J * \varphi - \varphi)(\xi_0) > 0.$$

Then by the second equation of (2.5), we have

$$
\begin{aligned}
0 &= c\varphi'(\xi_0) \\
&= D(J*\varphi-\varphi)(\xi_0)+\beta\left[\frac{A}{d}-\phi(\xi_0)-\varphi(\xi_0)-\psi(\xi_0)\right]\int_0^h f(\tau)g(\varphi(\xi_0-c\tau))d\tau \\
&\quad -(\gamma+\alpha+d)\varphi(\xi_0) \\
&= D(J*\varphi-\varphi)(\xi_0)+\beta\left[\frac{A}{d}-\phi(\xi_0)-\varphi(\xi_0)-\psi(\xi_0)\right]\int_0^h f(\tau)g(\varphi(\xi_0-c\tau))d\tau \\
&> 0,
\end{aligned}
$$

which reduces to a contradiction.

On the other hand, we have

$$
\begin{aligned}
&c\varphi'(\xi) \\
&= D(J*\varphi-\varphi)(\xi)+\beta\left[\frac{A}{d}-\phi(\xi)-\varphi(\xi)-\psi(\xi)\right]\int_0^h f(\tau)g(\varphi(\xi-c\tau))d\tau \\
&\quad -(\gamma+\alpha+d)\varphi(\xi) \\
&\le D(J*\varphi-\varphi)(\xi)+\frac{A\beta}{d}\int_0^h f(\tau)\varphi(\xi-c\tau)d\tau-(\gamma+\alpha+d)\varphi(\xi).
\end{aligned}
$$

Thus, $u(x,t)=\varphi(x+ct)$ satisfies

$$
\begin{cases}
\frac{\partial u(x,t)}{\partial t}\le D(J*u-u)(x,t)+\frac{A\beta}{d}\int_0^h f(\tau)u(x,t-\tau)d\tau-(\gamma+\alpha+d)u(x,t),\\
u(x,s)=\varphi(x+cs)>0,\quad s\in[-h,0].
\end{cases}
$$

Let $\omega_0=\sup_{\xi\in\mathbb{R}}\varphi(\xi)$, it is easy to see that $\omega_0>0$. Consider the initial value problem

$$
\begin{cases}
\frac{d\omega(t)}{dt}=\frac{A\beta}{d}\int_0^h f(\tau)\omega(t-\tau)d\tau-(\gamma+\alpha+d)\omega(t),\quad t>0,\\
\omega(0)=2\omega_0>0.
\end{cases}
\tag{3.1}
$$

The comparison principle implies that

$$
0\le u(x,t)\le 2\omega_0 e^{\lambda t},\quad t>0, \tag{3.2}
$$

where λ satisfies

$$
\lambda+(\gamma+\alpha+d)=\frac{A\beta}{d}\int_0^h f(\tau)e^{-\lambda\tau}d\tau.
$$

Since $\frac{A\beta}{d}<\gamma+\alpha+d$, it is not difficult to get that $Re\lambda<0$.

By (3.2) and the invariant form of $\varphi(\xi)$, we get that $\varphi(\xi)\equiv 0$ for $\xi\in\mathbb{R}$, which contradicts that $\varphi(\xi)$ is nontrivial. This completes the proof. □

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