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EXISTENCE OF DISSIPATIVE-AFFINE-PERIODIC SOLUTIONS FOR DISSIPATIVE-AFFINE-PERIODIC SYSTEMS*

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Abstract In this paper, we prove that every first order dissipative-(T, a)-affine-periodic system admits a dissipative-(T, a)-affine-periodic solution in $[0, \infty)$ via the Leray-Schauder degree theory and the lower and upper solutions method.

Keywords Dissipative-affine-periodic solution, dissipative-affine-periodic system, existence.

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1. Introduction

As is well known, the qualitative theory of dynamical systems was established by the French mathematician Poincaré and the Russian mathematician Lyapunov in the nineteenth century from regular solutions to irregular solutions, such as stationary, periodic, anti-periodic, quasi-periodic, almost periodic, almost automorphic, Birkhoff recurrent, almost recurrent, non-wondering, chain recurrent, chaos etc, and more details can be found in Liu etc [9]. An essential problem on these kinds of solutions is the "well-posed problem", which was introduced by the French mathematician Hadamard, and he believed that the mathematical model of physical phenomena should have the following properties:

Existence a solution exists;

Uniqueness the solution is unique;

Continuity the solution's behaviour depends continuously on the data (initial values and boundary values) and parameters.

If a problem satisfies all above properties, then the problem is called well-posed, otherwise ill-posed. The meaning of existence is clear, that is, there exists a solution satisfying the mathematical model. The uniqueness of the solution means that the

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solution is unique within a certain class of functions. For example, if a problem has many solutions but only one of them is bounded, then we would say that the solution is unique in the space of bounded functions. The continuity of the solution means that if the small change in the data and parameters brings about small change in the solution, i.e., the change of the solution can be controlled by the change of the data and parameters continuously.

In this paper, we consider the well-posedness of a new kind of solutions to ordinary differential equations (for short ODEs), which are called "affine-periodic" solutions. This new type solution was introduced by Li etc [20] to describe some physical phenomena which exhibit a certain symmetry as the evolution of time and it depicts how to interact between time and space. More details on affine-periodic solutions can be found in [1, 2, 8, 10, 13–15, 19].

Now, let us consider the following first order ODE

$$x^{'} = f(t,x), \quad {}^{'} = \frac{\mathrm{d}}{\mathrm{d}t}, \tag{1.1}$$

where the nonlinearity term f is continuous and ensures the existence and uniqueness of the solution with respect to initial value, furthermore f is an affine-periodic function, i.e., there is some $T > 0$ and $a \in \mathbb{R} \setminus \{0\}$ such that

$$f(t+T,x) = af(t,a^{-1}x), \quad \forall t \in \mathbb{R}, \forall x \in \mathbb{R}. \tag{1.2}$$

According to the size of a, system (1.1)-(1.2) can be roughly classified into several cases:

periodic system if $a = 1$, then (1.1)-(1.2) is called a T-periodic system;

antiperiodic system if $a = -1$, then (1.1)-(1.2) is called a T-antiperiodic system;

repulsive system if $|a| > 1$, then (1.1)-(1.2) is called a repulsive-(T,a)-affine-periodic system;

dissipative system if $0 < |a| < 1$, then (1.1)-(1.2) is called a dissipative-(T,a)-affine-periodic system.

For a given (T,a)-affine-periodic system, our aim is to seek for the (T,a)-affine-periodic solution, i.e., there exists a mapping x such that $x(t)$ satisfies equation (1.1) and the following equality

$$x(t+T) = ax(t), \quad \forall t \in \mathbb{R}. \tag{1.3}$$

Similarly, the solution $x(t)$ of (1.1)-(1.2) with the equality (1.3) can also be roughly classified into several cases:

periodic solution if $a = 1$, then the solution x is called a T-periodic solution;

antiperiodic solution if $a = -1$, then the solution x is called a T-antiperiodic solution;

repulsive solution if $|a| > 1$, then the solution x is called a repulsive-(T,a)-affine-periodic solution;

dissipative solution if $0 < |a| < 1$, then the solution x is called a dissipative-(T,a)-affine-periodic solution.

In fact, a natural concept of dissipation is to assume that there is a bounded set such that every orbit eventually enters into the set and remains. In 1944, Levinson [7] posed a conjecture on the existence of periodic solutions for dissipative systems. He claimed that such a periodic system admits a periodic solution. In the theory of dynamical systems, Levinson's conjecture is a basic result which was completely solved by Küpper etc [6]. Furthermore, Li etc proved this classical results is true in dissipative-affine-periodic systems:

- Zhang etc [20] proved that every dissipative affine-periodic system admits an affine-periodic solution via Horn's fixed point theorem.
- Li etc [8] considered the existence of dissipative-repulsive-affine-periodic solutions to dissipative-repulsive-affine-periodic systems using the method of topological degree.
- Chang etc [1] proved the existence of affine-periodic solutions to a class of second order dissipative dynamical systems by the method of topological degree.

Motivated by these fruitful results and ideas, we are concerned with the existence of dissipative-affine-periodic solutions for first order dissipative-affine-periodic systems in $\mathbb{R}$ in the present paper.

The main tool we used is the lower and upper solutions (for short LUS) method which is known to be an easy and elementary method for differential equations, and it replaces a difficulty problem "*How to find a solution of a boundary value problem?*" by an easier problem "*How to find lower and upper solutions?*". This method can be dated back to the use of successive approximation by Picard [12] in 1893. With the development of the LUS method, monotone method, Nagumo condition, maximal and minimal solutions, non well-ordered upper and lower solutions and so on appeared in the stage of mathematics. Furthermore, LUS method was connected with other theories including degree theory, variational method and so on. Above all, LUS can be used to deal with many problems, such as resonant and nonresonant problems, singular perturbation problems and so on. More details about the LUS method can be found in the book of Coster etc [3]. Here we only list some main results using the LUS method connected with our work.

- Fabry etc [4] obtained the existence theorems for the Picard boundary value problem via the LUS method and the homotophy invariance of Leray-Schauder degree. Here we list this essential theorem about the homotopy invariance of Leray-Schauder degree.

  Lemma 1.1 (Theorem 1, [4]). *Let X be a Banach space, and $A : X \to X$ be a compact mapping such that $(I - A)$ is one to one , and Ω be an open bounded set such that $0 \in (I - A)(\Omega)$. Then the compact mapping $H : \overline{\Omega} \to X$ has a fixed point in $\overline{\Omega}$ if for any $\lambda \in (0, 1)$, the equation*

  $$x = \lambda Hx + (1 - \lambda)Ax$$

  has no solution on the boundary $\partial\Omega$.

- Wu [16] gave the existence result of anti-periodic solution to first order ODEs using the LUS method.
- Wu etc [17,18] obtained the existence result of antiperiodic solution to second order ODEs via the LUS method.

- Nieto [11] showed that the LUS method is valid for finding periodic solutions to third order ODEs.
- Howes [5] considered boundary value problems for higher order equations using the LUS method.

Outline of this paper: In Section 2, we state our main results Theorem 2.1, as applications: Theorems 2.2 and 2.3. In Section 3, we give the proof of our main results via the Leray-Schauder degree theory and the LUS method. In Section 4, we provide some examples to illustrate our results.

2. Main Results

In this section, we list our main results as an abstract Theorem 2.1, as applications: Theorems 2.2 and 2.3.

At first, we introduce Theorem 2.1.

Theorem 2.1. *Assume that the dissipative-(T,a)-affine-periodic system* (1.1)-(1.2) *has a solution $x_0(t)$ in $[0,\infty)$, and there exists a function $\mathcal{O}(t)$ such that*

$$|x(t)-y(t)| \le \mathcal{O}(t)|x(0)-y(0)|, \quad \forall t \in [0,\infty)$$

for any possible solutions $x(t)$ and $y(t)$ to (1.1)-(1.2)*, where $\mathcal{O}(t)$ satisfies dissipative condition, i.e.,*

$$\mathcal{O}(t) \le |a|^{m+\varepsilon(m)}, \quad mT \le t < (m+1)T, m \in \mathbb{N}, \quad \varepsilon(m) = \varepsilon\chi_{\mathbb{N}^*}(m), \varepsilon > 0,$$

where χ is the indicator function. Then there admits a unique dissipative-(T,a)-affine-periodic solution for (1.1)-(1.2)*.*

Remark 2.1. In Theorem 2.1, we give a specific condition on $\mathcal{O}(t)$ according to the dissipative coefficient a, which satisfies the ordinary dissipative condition in Wu [16]

$$\mathcal{O}(t) \ge 0 \quad \text{and} \quad \lim_{t\to\infty} \mathcal{O}(t) = 0.$$

Then we introduce Theorem 2.2 for $0 < a < 1$ and Theorem 2.3 for $-1 < a < 0$.

Theorem 2.2 $(0 < a < 1)$**.** *Assume that the following hold:*

(i) There exist a strict lower solution $\alpha \in C^1([0,\infty);\mathbb{R})$ to (1.1)-(1.2) *and a strict upper solution $\beta \in C^1([0,\infty);\mathbb{R})$, i.e.,*

$$\alpha' < f(t,\alpha), \quad \beta' > f(t,\beta), \quad \forall t \in [0,\infty).$$

(ii) α and β are (T,a)-affine-periodic functions, i.e.,

$$\alpha(t+T) = a\alpha(t), \quad \beta(t+T) = a\beta(t), \quad \forall t \in [0,\infty).$$

(iii) α is a negative function and β is a positive function on $[0,\infty)$, i.e.,

$$\alpha(t) < 0 < \beta(t), \quad \forall t \in [0,\infty).$$

Then the dissipative-(T,a)-affine-periodic system (1.1)-(1.2) *admits a dissipative-(T,a)-affine-periodic solution in the order interval (α,β).*

Remark 2.2. It should be emphasized that the lower solution α and the upper solution β in Theorem 2.2 may not satisfy $\alpha+\beta=0$ for $0<a<1$, which is necessary in Fabry [4] where the solution x holds $|x(t)|<\psi(t)$, that is, if set $\alpha=-\psi, \beta=\psi$, then $\alpha+\beta=0$. In this sense, we weaken the conditions on the lower and upper solutions.

Theorem 2.3 $(-1<a<0)$**.** *Assume that there exists a function* $\psi \in C^1([0,\infty);\mathbb{R})$ *satisfying the following conditions:*

(i) $-\psi$ *is a strict lower solution and* ψ *is a strict upper solution for* (1.1)-(1.2), *i.e.,*

$$-\psi^{'}<f(t,-\psi), \quad \psi^{'}>f(t,\psi), \quad \forall t\in[0,\infty).$$

(ii) ψ *is a* $(T,-a)$*-affine-periodic function, i.e.,*

$$\psi(t+T)=-a\psi(t), \quad \forall t\in[0,\infty).$$

(iii) ψ *is a positive function, i.e.,*

$$\psi(t)>0, \quad \forall t\in[0,\infty).$$

Then dissipative-(T,a)*-affine-periodic system* (1.1)-(1.2) *admits a dissipative-*(T,a)*-affine-periodic solution in the order interval* $(-\psi,\psi)$.

3. Proof of Main Results

In this section, we prove Theorem 2.1, Theorem 2.2 and Theorem 2.3.

3.1. Proof of Theorem 2.1

Proof. Since $x_0(t)$ is a solution to (1.1)-(1.2), then $x_i(t)\triangleq a^{-i}x_0(t+iT)$ is also the solution to (1.1)-(1.2) for all $i\in\mathbb{N}$. In fact,

$$\begin{aligned} x_i^{'}(t) &= a^{-i}x_0^{'}(t+iT) \\ &= a^{-i}f(t+iT,x_0(t+iT)) \\ &= f(t,a^{-i}x_0(t+iT)) \\ &= f(t,x_i(t)). \end{aligned}$$

Furthermore

$$\begin{aligned} \left|a^{-(i+1)}x_0((i+1)T)-a^{-i}x_0(iT)\right| &\le |a|^{-i}\mathcal{O}(iT)\left|a^{-1}x_0(T)-x_0(0)\right| \\ &\le |a|^{\varepsilon(i)}\left|a^{-1}x_0(T)-x_0(0)\right| \end{aligned}$$

and then

$$\begin{aligned} \left|a^{-(m+1)}x_0((m+1)T)\right| &\le \sum_{i=0}^{m}\left|a^{-(i+1)}x_0((i+1)T)-a^{-i}x_0(iT)\right|+|x_0(0)| \\ &\le (m|a|^{\varepsilon}+1)\left|a^{-1}x_0(T)-x_0(0)\right|+|x_0(0)|. \end{aligned}$$

Thus

$$|x_0((m+1)T)| \le |a|^{m+1} \left((m|a|^{\varepsilon}+1) \left| a^{-1}x_0(T) - x_0(0) \right| + |x_0(0)| \right),$$

which indicates that $\{x_0(mT)\}$ is bounded for any $m \in \mathbb{N}$.

Analogously,

$$\begin{aligned} \left| a^{-(i+1)}x_0(s+(i+1)T) - a^{-i}x_0(s+mT) \right| \le & |a|^{-i}\mathcal{O}(s+iT) \left| a^{-1}x_0(T) - x_0(0) \right| \\ \le & |a|^{\varepsilon(i)} \left| x_0(T) - x_0(0) \right| \end{aligned}$$

for all $s \in [0,T)$ and then

$$\begin{aligned} \left| a^{-(m+1)}x_0(s+(m+1)T) \right| &\le \sum_{i=0}^{m} \left| a^{-(i+1)}x_0(s+(i+1)T) - a^{-i}x_0(s+iT) \right| + |x_0(s)| \\ &\le (m|a|^{\varepsilon}+1) \left| a^{-1}x_0(T) - x_0(0) \right| + |x_0(s)|. \end{aligned}$$

Hence

$$|x_0(s+(m+1)T)| \le |a|^{m+1} \left((m|a|^{\varepsilon}+1) \left| a^{-1}x_0(T) - x_0(0) \right| + |x_0(s)| \right),$$

which indicates that $\{x_0(s+mT)\}$ is bounded for all $s \in [0,T)$ and any $m \in \mathbb{N}$.

Furthermore, for any possible solution $x(t)$ to (1.1)-(1.2), we have

$$|x(t) - x_0(t)| \le \mathcal{O}(t)|x(0) - x_0(0)|,$$

thus

$$|x(t)| \le |x_0(t)| + \mathcal{O}(t)|x(0) - x_0(0)|,$$

which indicates that $x(t)$ is bounded in $[0,\infty)$.

Set

$$\mathrm{B} = \{x \in \mathbb{R} : \|x\| \le r\},$$

where

$$\|x\| = \max_{t\in[0,\infty)} |x(t)|, \quad \forall x \in \mathrm{B}.$$

Define the Poincaré mapping

$$\begin{aligned} P : \mathrm{B} &\longrightarrow \mathrm{B}, \\ p &\longrightarrow a^{-1}x(T;p), \end{aligned}$$

where $x(t;p)$ is the semiflow for (1.1)-(1.2) with respect to initial value p, i.e., $x(0;p) = p$. Then we claim that P is a contractive mapping. Indeed,

$$|P(p_1) - P(p_2)| = |a|^{-1}|x(T;p_1) - x(T;p_2)| \le |a|^{-1}\mathcal{O}(T)|p_1 - p_2| \le |a|^{\varepsilon}|p_1 - p_2|$$

for all $p_i \in \mathrm{B}(i=1,2)$. Using Banach contraction mapping principle, there exists a unique fixed point $p^* \in \mathrm{B}$ such that

$$a^{-1}x(T;p^*) = p^*.$$

Furthermore

$$\begin{aligned} x(t+T;p^*) &= p^* + \int_0^{t+T} f(s,x(s;p^*))\mathrm{d}s \\ &= p^* + \int_0^{T} f(s,x(s;p^*))\mathrm{d}s + \int_T^{t+T} f(s,x(s;p^*))\mathrm{d}s \\ &= p^* + \int_0^{T} f(s,x(s;p^*))\mathrm{d}s + \int_0^{t} f(s+T,x(s+T;p^*))\mathrm{d}s \\ &= x(T;p^*) + a\int_0^{t} f(s,a^{-1}x(s+T;p^*))\mathrm{d}s, \end{aligned}$$

hence

$$a^{-1}x(t+T;p^*) = a^{-1}x(T;p^*) + \int_0^t f(s,a^{-1}x(s+T;p^*))\mathrm{d}s,$$

that is

$$a^{-1}x(t+T;p^*) = p^* + \int_0^t f(s,a^{-1}x(s+T;p^*))\mathrm{d}s.$$

By the uniqueness of the solution with respect to initial value,

$$x(t;p^*) = a^{-1}x(t+T;p^*),$$

that is

$$x(t+T;p^*) = ax(t;p^*).$$

This completes the proof of Theorem 2.1. □

From the proof of Theorem 2.1, we have the following corollary:

Corollary 3.1. *The existence of* (T,a)*-affine-periodic solution to* (1.1)-(1.2) *is equivalent to the existence of solution to* (1.1)-(1.2) *with* (T,a)*-affine-periodic boundary condition*

$$ax(0) = x(T).$$

3.2. Proof of Theorem 2.2

Proof. Consider the auxiliary system

$$x' = -\lambda kx + (1-\lambda)f(t,x), \quad 0 \le \lambda \le 1 \tag{3.1}$$

with (T,a)-affine-periodic boundary condition

$$ax(0) = x(T), \tag{3.2}$$

where

$$k > \max\left\{-(\lambda T)^{-1}\ln a, \max_{t\in[0,T]}\left\{-\frac{\alpha'(t)}{\alpha(t)}, -\frac{\beta'(t)}{\beta(t)}\right\}\right\} \quad \text{for} \quad 0<\lambda<1.$$

Then the solution to (3.1)-(3.2) is

$$x(t) = e^{-\lambda kt}\left(a - e^{-\lambda kT}\right)^{-1}(1-\lambda)\int_0^T e^{-\lambda k(T-s)} f(s,x(s))\mathrm{d}s$$
$$+ (1-\lambda)\int_0^t e^{-\lambda k(t-s)} f(s,x(s))\mathrm{d}s. \tag{3.3}$$

Set

$$C_{a,T} = \{x \in C(\mathbb{R};\mathbb{R}) : \quad x(t+T) = ax(t), \quad \forall t \in [0,\infty)\}$$

with the usual maximum norm $\|\cdot\| = \max_{[0,T]}|\cdot|$, then $(C_{(T,a)}, \|\cdot\|)$ is a Banach space.

Define a mapping

$$R_\lambda : [\alpha,\beta] \longrightarrow C_{a,T},$$
$$\phi \longrightarrow R_\lambda\phi,$$

where

$$[\alpha,\beta] = \{x \in C_{a,T} : \quad \alpha(t) \le x(t) \le \beta(t), \quad \forall t \in [0,T]\}$$

and

$$(R_\lambda\phi)(t) = e^{-\lambda kt}\left(a - e^{-\lambda kT}\right)^{-1}(1-\lambda)\int_0^T e^{-\lambda k(T-s)} f(s,\phi(s))\mathrm{d}s$$
$$+ (1-\lambda)\int_0^t e^{-\lambda k(t-s)} f(s,\phi(s))\mathrm{d}s.$$

Next, we claim that R_λ is well-defined, i.e., $R_\lambda\phi$ is a (T,a)-affine-periodic function. In fact

$$(R_\lambda\phi)(0) = (a - e^{-\lambda kT})^{-1}(1-\lambda)\int_0^T e^{-\lambda k(T-s)} f(s,\phi(s))\mathrm{d}s$$

and

$$\begin{aligned}(R_\lambda\phi)(T) &= (1-\lambda)\left(e^{-\lambda kT}(a - e^{-\lambda kT})^{-1} + 1\right)\int_0^T e^{-\lambda k(T-s)} f(s,\phi(s))\mathrm{d}s \\ &= a(a - e^{-\lambda kT})^{-1}(1-\lambda)\int_0^T e^{-\lambda k(T-s)} f(s,\phi(s))\mathrm{d}s \\ &= a(R_\lambda\phi)(0).\end{aligned}$$

Furthermore

$$\begin{aligned}(R_\lambda\phi)(t+T) &= e^{-\lambda k(t+T)}(a - e^{-\lambda kT})^{-1}(1-\lambda)\int_0^T e^{-\lambda k(T-s)} f(s,\phi(s))\mathrm{d}s \\ &\quad + (1-\lambda)\int_0^{t+T} e^{-\lambda k(t+T-s)} f(s,\phi(s))\mathrm{d}s \\ &= e^{-\lambda kt}e^{-\lambda kT}(a - e^{-\lambda kT})^{-1}(1-\lambda)\int_0^T e^{-\lambda k(T-s)} f(s,\phi(s))\mathrm{d}s \\ &\quad + (1-\lambda)e^{-\lambda kt}\int_0^T e^{-\lambda k(T-s)} f(s,\phi(s))\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+(1-\lambda)\int_T^{t+T} e^{-\lambda k(t+T-s)} f(s,\phi(s))\mathrm{d}s \\
&= e^{-\lambda kt}(e^{-\lambda kT}(a-e^{-\lambda kT})^{-1}+1)(1-\lambda)\int_0^T e^{-\lambda k(T-s)} f(s,\phi(s))\mathrm{d}s \\
&+(1-\lambda)\int_0^t e^{-\lambda k(t-s)} f(s+T,\phi(s+T))\mathrm{d}s \\
&= e^{-\lambda kt}a(a-e^{-\lambda kT})^{-1}(1-\lambda)\int_0^T e^{-\lambda k(T-s)} f(s,\phi(s))\mathrm{d}s \\
&+(1-\lambda)a\int_0^t e^{-\lambda k(t-s)} f(s,\phi(s))\mathrm{d}s \\
&= a(R_\lambda\phi)(t).
\end{aligned}
$$

Furthermore, we claim that there admits a positive constant $M>0$ such that

$$\|R_\lambda[\alpha,\beta]]\| \leq M.$$

If not, there exists a sequence $\{\phi_n\}_{n=1}^\infty \subseteq [\alpha,\beta]$ such that

$$\|R_\lambda\phi_n\| \to \infty, \quad n\to\infty.$$

On the one hand, set

$$\overline{\phi}_n = \frac{R_\lambda\phi_n}{\|R_\lambda\phi_n\|} = \frac{\varphi_n}{\|\varphi_n\|},$$

where φ_n is the (T,a)-affine-periodic solution to

$$x^{'} = -\lambda kx + (1-\lambda)f\left(t,\phi_n(t)\right).$$

Then $\overline{\phi}_n$ is the (T,a)-affine-periodic solution of

$$x^{'} = -\lambda kx + \frac{1}{\|\varphi_n\|}(1-\lambda)f(t,\phi_n(t)).$$

On the other hand, there exist a subsequence $\{\overline{\phi}_{n_j}\}_{j=1}^\infty \subseteq \{\overline{\phi}_n\}_{n=1}^\infty$ and $\overline{\phi}_0 \in C_{(T,a)}$ such that

$$\overline{\phi}_{n_j} \to \overline{\phi}_0, \quad j\to\infty.$$

Thus $\overline{\phi}_0$ is the (T,a)-affine-periodic solution to

$$x^{'} = -\lambda kx. \tag{3.4}$$

However, equation (3.4) has one and only one (T,a)-affine-periodic solution 0, i.e., $\overline{\phi}_0 \equiv 0$, which is a contradiction to $\|\overline{\phi}_0\| = 1$.

Finally, we prove that

$$0 \notin H_\lambda(\partial(\alpha,\beta)), \quad \forall\lambda\in[0,1],$$

where

$$H_\lambda = I - R_\lambda$$

is completely continuous by Arzela-Ascoli lemma, and

$$(\alpha,\beta) = \{x\in C_{a,T}: \quad \alpha(t)<x(t)<\beta(t), \quad \forall t\in[0,T]\}$$

is a nonempty open set in $C_{a,T}$.

Since $x \equiv 0 \in (\alpha,\beta)$ is the desired dissipative-(T,a)-affine-periodic solution when $\lambda = 1$, then it follows the complete continuity of H_λ that there exists $\Lambda \to 1^-$ such that

$$0 \notin H_\lambda(\partial(\alpha,\beta)), \quad \forall \lambda \in [\Lambda, 1].$$

Thus we only need to illustrate

$$0 \notin H_\lambda(\partial(\alpha,\beta)), \quad \forall \lambda \in [0, \Lambda].$$

If not, then there exists $\phi \in \partial(\alpha,\beta)$ such that

$$(H_\lambda \phi)(t) = 0,$$

that is, ϕ satisfies equation

$$\dot{\phi} = -\lambda k \phi + (1-\lambda) f(t,\phi).$$

Set

$$\delta = \alpha - \phi \quad \text{or} \quad \beta - \phi$$

then there exists $t_0 \in [0,T]$ such that δ reaches its maximal value or minimal value 0, i.e.,

$$\alpha(t_0) = \phi(t_0) \quad \text{or} \quad \beta(t_0) = \phi(t_0).$$

Without loss of generality, we only discuss the left case, and the right case is analogous. Furthermore

(i) If $t_0 \in (0,T)$, then $\delta^{'}(t_0) = 0$. However,

$$\begin{aligned} \alpha^{'}(t_0) &= (\alpha - \delta)^{'}(t_0) \\ &= \phi^{'}(t_0) \\ &= -\lambda k \phi(t_0) + (1-\lambda) f(t_0, \phi(t_0)) \\ &= -\lambda k \alpha(t_0) + (1-\lambda) f(t_0, \alpha(t_0)) \\ &> -\lambda k \alpha(t_0) + (1-\lambda)\alpha^{'}(t_0) \\ &\geq \alpha^{'}(t_0), \end{aligned}$$

a contradiction.

(ii) If $t_0 \in \{0\} \Longleftrightarrow \delta(0) = 0 \Longleftrightarrow \delta(T) = 0 \Longleftrightarrow t_0 \in \{T\}$, then

$$\alpha^{'}(0) \leq \phi^{'}(0), \quad \alpha^{'}(T) \geq \phi^{'}(T).$$

Notice that

$$a\alpha^{'}(0) = \alpha^{'}(T) \geq \phi^{'}(T) = a\phi^{'}(0).$$

Thus

$$\alpha^{'}(0) = \phi^{'}(0).$$

However

$$\begin{aligned}\alpha'(0) &= (\alpha-\delta)'(0)\\ &= \phi'(0)\\ &= -\lambda k\phi(0)+(1-\lambda)f(0,\phi(0))\\ &= -\lambda k\alpha(0)+(1-\lambda)f(0,\alpha(0))\\ &> -\lambda k\alpha(0)+(1-\lambda)\alpha'(0)\\ &\geq \alpha'(0),\end{aligned}$$

this leads to a contradiction.

Since $x \equiv 0$ is the unique a-affine T-periodic solution and R_1 is linear, then

$$\deg_{LS}(H_1,(\alpha,\beta),0)\neq 0.$$

Using the homotopy invariance of Leray-Schauder degree, we have

$$\deg_{LS}(H_0,(\alpha,\beta),0)=\deg_{LS}(H_1,(\alpha,\beta),0)\neq 0,$$

which indicates that there exists a fixed point $\phi^* \in (\alpha,\beta)$ to R, i.e., ϕ^* is the desired a-affine T-periodic solution. This completes the proof of Theorem 2.2. □

Remark 3.1. The proof of Theorem 2.3 is similar to one of Theorem 2.2, and we derive a contradiction at points $t = 0$ and $t = 2T$ in Theorem 2.3 rather than at points $t = 0$ and $t = T$ in Theorem 2.2.

4. Applications

In this section, we give some examples to illustrate Theorem 2.2.

Example 4.1. Consider the ODE

$$x' = -2x + e^{-t}. \tag{4.1}$$

Set

$$f(t,x) = -2x+e^{-t}, \quad -e^{-t}=\alpha(t)<0<\beta(t)=2e^{-t}, \quad a=e^{-T}.$$

Obviously,

$$\begin{aligned}&f(t+T,x) = -2x+e^{-(t+T)} = e^{-T}\left(-2(e^{T}x)+e^{-t}\right) = af(t,a^{-1}x),\\ &\alpha(t+T) = -e^{-(t+T)} = e^{-T}\left(-e^{-t}\right) = a\alpha(t),\\ &\beta(t+T) = 2e^{-(t+T)} = e^{-T}(2e^{-t}) = a\beta(t),\\ &\alpha'(t) = e^{-t} < 3e^{-t} = f(t,\alpha(t)), \quad \beta'(t) = -2e^{-t} > -3e^{-t} = f(t,\beta(t)).\end{aligned}$$

By Theorem 2.2, equation (4.1) has a dissipative-(T, e^{-T})-affine-periodic solution in the order interval $(-e^{-t}, 2e^{-t})$.

Example 4.2. Consider the ODE

$$x' = -2x + e^{-t}\sin t. \tag{4.2}$$

Set

$$f(t,x) = -2x + e^{-t}\sin t, \quad a = e^{-2\pi},$$

and

$$-(1+\varepsilon)e^{-t} = \alpha(t) < 0 < \beta(t) = (1+\varepsilon)e^{-t}, \quad \varepsilon > 0.$$

Obviously,

$$\begin{aligned}
&f(t+2\pi) = -2x + e^{-(t+2\pi)}\sin(t+2\pi) = e^{-2\pi}(-2e^{2\pi}x + e^{-t}\sin t) = af(t, a^{-1}x),\\
&\alpha(t+2\pi) = -(1+\varepsilon)e^{-(t+2\pi)} = e^{-2\pi}\left(-(1+\varepsilon)e^{-t}\right) = a\alpha(t),\\
&\beta(t+2\pi) = (1+\varepsilon)e^{-(t+2\pi)} = e^{-2\pi}((1+\varepsilon)e^{-t}) = a\beta(t),\\
&\alpha^{'}(t) = (1+\varepsilon)e^{-t} < (2+2\varepsilon+\sin t)e^{-t} = f(t,\alpha(t)),\\
&\beta^{'}(t) = -(1+\varepsilon)e^{-t} > (-2-2\varepsilon+\sin t)e^{-t} = f(t,\beta(t)).
\end{aligned}$$

By Theorem 2.2, equation (4.2) has a dissipative-$(2\pi, e^{-2\pi})$-affine-periodic solution in the order interval $(-(1+\varepsilon)e^{-t}, (1+\varepsilon)e^{-t})$.

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