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THE DYNAMIC BEHAVIOR OF DETERMINISTIC AND STOCHASTIC DELAYED SIQS MODEL*

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Abstract In this paper, we present the deterministic and stochastic delayed SIQS epidemic models. For the deterministic model, the basic reproductive number R_0 is given. Moreover, when $R_0 < 1$, the disease-free equilibrium is globally asymptotical stable. When $R_0 > 1$ and additional conditions hold, the endemic equilibrium is globally asymptotical stable. For the stochastic model, a sharp threshold $\hat{R}_0$ which determines the extinction or persistence in the mean of the disease is presented. Sufficient conditions for extinction and persistence in the mean of the epidemic are established. Numerical simulations are also conducted in the analytic results.

Keywords Random perturbations, Itô′s formula, the threshold, time delay.

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1. Introduction

Quarantine (Isolation) is an important intervention means to control the spread of infectious diseases. It can reduce transmissions of the infections to susceptibles. More recently, Quarantine is popularly used to fight against the spread of some emerging and re-emerging human and animal infectious diseases, such as the swine influenza pandemic, foot-and-mouth disease, the severe acute respiratory syndrome (SARS), ebola and so on (see [6, 17, 32, 37] and the references therein).

Numerous mathematical models have been designed to study effects of quarantine on controlling the spread of infectious disease in human and animal populations (see [5, 8, 28–31, 34, 36] and the references therein). Hsieh etc [13] studied impact of quarantine on the 2003 SARS outbreak. Dobay etc [7] investigated a SIR model with the quarantine by analyzing an epidemic of syphilis. Liu etc [24] discussed the stability of an SIQS model with the effects of transposrt-related infection and exit-entry screenings. Chen etc [4] addressed the stability analysis and the estima-

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tion of domain of attraction for the endemic equilibrium of a class of susceptible-exposed-infected-quarantine (SEIQ) epidemic models. Herbert etc [12] introduced the following SIQS model

$$\begin{aligned} S'(t) &= A - \beta IS - \mu S + \gamma I + \varepsilon Q, \\ I'(t) &= \beta IS - (\mu + \alpha_2 + \delta + \gamma) I, \\ Q'(t) &= \delta I - (\mu + \alpha_3 + \varepsilon) Q. \end{aligned} \tag{1.1}$$

The meanings of all variables and parameters in the above model are as follows:

$S(t)$: the number of susceptible individuals in the population at time t;

$I(t)$: the number of infectious individuals in the population at time t;

$Q(t)$: the number of quarantined individuals in the population at time t

A: the recruitment rate of susceptibles corresponding to births and immigration;

μ: the natural death rate;

δ: the rate of individuals leaving the infective compartment I for the quarantined compartment Q;

α_2: the disease-related death rate in I ;

α_3: the disease-related death rate in Q;

γ and ε: the rate of individuals recovering and returning to susceptible compartment S from compartments I and Q, respectively.

In [12], the authors assumed that the individuals in the quarantined compartment $Q(t)$ return to the susceptible compartment S at constant rate. However, in reality, the infectious are usually quarantined for a fixed time period in term of the types of infectious diseases. For this reason, the fixed quarantined period is incorporated in our model by introducing the term $\delta I(t-\tau) e^{-(\mu+\alpha_3)\tau}$, where τ is the length of quarantined period. That is, we suppose that a infectious individual is quarantined in the quarantined compartment Q for a fixed finite time period τ, then return to susceptible compartment S. Consequently, the model (1.2) can be rewritten as follows:

$$\begin{aligned} \frac{dS(t)}{dt} &= A - \beta IS - \mu S + \gamma I + \delta I(t-\tau) e^{-(\mu+\alpha_3)\tau}, \\ \frac{dI(t)}{dt} &= \beta IS - (\mu + \alpha_2 + \delta + \gamma) I, \\ \frac{dQ(t)}{dt} &= \delta I - (\mu + \alpha_3) Q - \delta I(t-\tau) e^{-(\mu+\alpha_3)\tau}, \end{aligned} \tag{1.2}$$

Moreover, we assume that all parameters are positive except τ which is non-negative.

As usual, the initial condition of (1.2) is given as

$$\begin{aligned} &S(\varpi) = \psi_1(\varpi), I(\varpi) = \psi_2(\varpi), Q(\varpi) = \psi_3(\varpi), \ -\tau \le \varpi \le 0, \\ &\psi_1(\varpi) \ge 0, \psi_2(\varpi) \ge 0, \psi_3(\varpi) \ge 0, \ -\tau \le \varpi \le 0, \\ &\psi_1(0) > 0, \psi_2(0) > 0, \psi_3(0) > 0, \end{aligned} \tag{1.3}$$

where $\psi = (\psi_1, \psi_2, \psi_3)^T \in C\left([-\tau, 0], \mathbb{R}^3_{+0}\right)$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^3_{+0}$, where $\mathbb{R}^3_{+0} = \{(u_1, u_2, u_3) : u_1, u_2, u_3 \ge 0\}$.

On the other hand, any system is always subject to environmental noise. Recently, many scholars introduce environmental noise to epidemic models and study their influence on the dynamics of infectious disease (see [1,10,11,15,16,18–21,25,27,

38–40]). Gray etc [9] studied a stochastic differential equation SIS epidemic model. Teng etc [33] discussed persistence and extinction for a class of stochastic SIS epidemic model with nonlinear incidence rate. Liu etc [22] analysed the deterministic and stochastic SIRS epidemic models with nonlinear incidence. Wei etc [35] considered an SIQS epidemic model with saturated incidence and independent random perturbations. Zhang etc [41] introduced a deterministic and stochastic SIQS model with nonlinear incidence and gave sufficient conditions for the extinction and the existence of a unique stationary distribution of a disease.

Inspired by the above literature, we introduce environmental noise into (1.2), consequently, obtain the following stochastic SIQS model:

$$\begin{aligned} dS(t) &= \left(A - \beta IS - \mu S + \gamma I + \delta I(t-\tau)\, e^{-(\mu+\alpha_3)\tau}\right) dt + \sigma_1 S dB_1(t), \\ dI(t) &= \left(\beta IS - (\mu + \alpha_2 + \delta + \gamma) I\right) dt + \sigma_2 I dB_2(t), \\ dQ(t) &= \left(\delta I - (\mu + \alpha_3) Q - \delta I(t-\tau)\, e^{-(\mu+\alpha_3)\tau}\right) dt + \sigma_3 Q dB_3(t), \end{aligned} \tag{1.4}$$

where all variables and parameters have the same meaning in (1.2) . $B_i(t)$ $(i = 1, 2, 3)$ are mutually independent standard Brownian motions and σ_i $(i = 1, 2, 3)$ represent corresponding the intensities, respectively.

Because the first two equations are independent of the third equation of system (1.4), so we only need consider following system:

$$\begin{aligned} dS(t) &= \left(A - \beta IS - \mu S + \gamma I + \delta I(t-\tau)\, e^{-(\mu+\alpha_3)\tau}\right) dt + \sigma_1 S dB_1(t), \\ dI(t) &= \left(\beta IS - (\mu + \alpha_2 + \delta + \gamma) I\right) dt + \sigma_2 I dB_2(t). \end{aligned} \tag{1.5}$$

For system (1.5), we are concerned about a sharp threshold which determines extinction or persistence of the disease. As far as we know, the results of this study are very few. Cai etc [3] studied a stochastic SIRS epidemic model with infectious force under intervention strategies and gave a sharp threshold of extinction of disease and endemic stationary distribution. Ji etc [42] investigated threshold of a stochastic SIR model with bilinear incidence. Zhao etc [43] studied a sharp threshold of a stochastic SIRS epidemic model with saturated incidence and then considered a sharp threshold of a stochastic SIS epidemic model with vaccination (see [44]). Recently, they investigated a sharp threshold of a stochastic SIRS epidemic model in a population with varying size (see [14]) and a sharp threshold of a stochastic SIVS epidemic model with nonlinear saturated incidence. Liu [23] analysed the a sharp threshold of a stochastic delayed SIR epidemic model.

The purpose of the current study is to analyse effect of isolation time τ and environmental noise on dynamic behavior of the model (1.2) and (1.5). To proceed, the rest of the paper is arranged as follows. In Section 2, the basic reproduction number R_0 of the model (1.2) is presented. Moreover, the global stability of the disease-free equilibrium is established, if $R_0 < 1$. Hopf bifurcations at the endemic equilibrium is discussed. Sufficient conditions are derived for the global stability of the endemic equilibrium. In Section 3, the existence and uniqueness of a global positive solution of system (1.5) is proved. A sharp threshold $\hat{R}_0$ of system (1.5) is proposed. Moreover, it is showed that when $\hat{R}_0 < 1$, the disease will died out and when $\hat{R}_0 > 1$, the disease will persist. In Section 4, we give some discussions.

2. Asymptotical behavior of the deterministic model (1.2)

Obviously, for the model (1.2), the disease free equilibrium $P_0\left(\frac{A}{\mu},0,0\right)$ always exists.

Now, we analyze the behavior of the system (1.2) near P_0. The characteristic equation of the linearization of (1.2) near P_0 is

$$\det\begin{pmatrix} \lambda+\mu & -\gamma+\frac{\beta A}{\mu}-\delta e^{-(\mu+\alpha_3+\lambda)\tau} & 0 \\ 0 & \lambda-\frac{\beta A}{\mu}+(\mu+\alpha_2+\delta+\gamma) & 0 \\ 0 & \left(1-e^{-(\mu+\alpha_3+\lambda)\tau}\right) & (\lambda+\mu+\alpha_3) \end{pmatrix}=0,$$

that is,

$$(\lambda+\mu)(\lambda+\mu+\alpha_3)\left(\lambda-\frac{\beta A}{\mu}+(\mu+\alpha_2+\delta+\gamma)\right)=0. \tag{2.1}$$

The roots of (2.1) are $\lambda=-\mu$, $\lambda=-\mu-\alpha_3$ and $\lambda=\frac{\beta A}{\mu}-(\mu+\alpha_2+\delta+\gamma)$.

Define the basic reproduction number $R_0=\frac{\beta A}{\mu(\gamma+\delta+\mu+\alpha_2)}$. If $R_0<1$, then the roots of (2.1) are negative and the disease free equilibrium P_0 is locally asymptotically stable. If $R_0>1$, then (2.1) has a positive root and hence the disease free equilibrium P_0 is unstable.

Next, we will discuss globally asymptotical behavior of the disease-free equilibrium P_0.

Theorem 2.1. *If $R_0<1$, then the disease-free equilibrium $P_0\left(\frac{A}{\mu},0,0\right)$ of system (1.2) is globally asymptotical stable.*

Proof. Since $R_0<1$, we may choose $\varepsilon>0$ sufficiently small such that

$$\beta\left(\frac{A}{\mu}+\varepsilon\right)<(\gamma+\delta+\mu+\alpha_2). \tag{2.2}$$

Denote the total population $N(t)=S(t)+I(t)+Q(t)$. From (1.2), we have

$$\begin{aligned}\frac{dN(t)}{dt}=&A-\mu N-\alpha_2 I-\alpha_3 Q\\ \le&A-\mu N,\end{aligned}$$

which yields

$$\lim_{t\to\infty}\sup N(t)\le\frac{A}{\mu}. \tag{2.3}$$

Hence, $\lim\limits_{t\to\infty}\sup S(t)\le\frac{A}{\mu}$. This implies that for any ε satisfying (2.2), there exists a time $t_1>0$ such that when $t>t_1$, $S(t)<\frac{A}{\mu}+\varepsilon$.

For any ε satisfying (2.2) and $t > t_1$, it follows from the second equation of system (1.2) that

$$\frac{dI(t)}{dt} \leq \left[\beta\left(\frac{A}{\mu}+\varepsilon\right)-(\mu+\alpha_2+\delta+\gamma)\right]I.$$

Noting that (2.2) holds, it is easy to see that

$$\lim_{t\to\infty}\sup I(t)=0. \tag{2.4}$$

Similarly, for any ε satisfying (2.2), there exists a time $t_2 > t_1$ such that when $t > t_2$, $I(t) < \varepsilon$.

Moreover, for $t > t_2 + \tau$, from the first equation of system (1.2) we get that

$$\frac{dS(t)}{dt} \geq A-\beta\varepsilon S-\mu S,$$

which deduce that

$$\lim_{t\to\infty}\inf S(t) \geq \frac{A}{\mu+\beta\varepsilon}.$$

Letting $\varepsilon \to 0$, we have that

$$\lim_{t\to\infty}\inf S(t) \geq \frac{A}{\mu}.$$

This together with (2.3), yields that

$$\lim_{t\to\infty} S(t)=\frac{A}{\mu}.$$

It is easy to confirm that the third equation of system equivalent to

$$Q(t)=\delta\int_{t-\tau}^{t} I(r)\,e^{-(\mu+\alpha_3)(t-\tau)}dr.$$

Thus, for $t > \tau$, using L'Hospital's rule and (2.4) yields that

$$\lim_{t\to\infty} Q(t)=0.$$

Furthermore, noting that when $R_0 < 1$, the disease-free equilibrium $P_0\left(\frac{A}{\mu},0,0\right)$ of system is locally asymptotically stable. Hence, P_0 is globally asymptotically stable. This finishes the proof. □

When $R_0 > 1$, it is easy to confirm that there exists the endemic equilibrium $P_1(S^*,I^*,R^*) = \left(\frac{\mu+\alpha_2+\delta+\gamma}{\beta}, \frac{\beta(R_0-1)}{\mu(\gamma+\delta+\mu+\alpha_2)\left(\mu+\alpha_2+\delta-\delta e^{-(\mu+\alpha_3)\tau}\right)}, \frac{\delta\left(1-e^{-(\mu+\alpha_3)\tau}\right)}{\mu+\alpha_3}I^*\right)$.
Next, let us discuss the asymptotical behavior of the system (1.2) near P_1.

The characteristic equation of the linearization of (1.2) near P_1 is

$$g(\lambda)=(\lambda+\mu+\alpha_3)\left(\lambda^2+a\lambda+b+ce^{-\lambda\tau}\right)=0,$$

where

$$\begin{aligned} a =& \beta I^* + \mu, \\ b =& \beta I^* (\mu + \alpha_2 + \delta), \\ c =& -\beta I^* \delta e^{-(\mu+\alpha_3)\tau}. \end{aligned}$$

If $\tau = 0$, it is easy to see

$$g(\lambda) = (\lambda + \mu + \alpha_3)\left(\lambda^2 + (\beta I^* + \mu)\lambda + \beta I^*(\mu + \alpha_2)\right)$$

and all roots of $g(\lambda) = 0$ have negative real parts. Hence, when $\tau = 0$, the endemic equilibrium P_1 is locally asymptotically stable.

In addition, as $\tau \to \infty$,

$$g(\lambda) = (\lambda + \mu + \alpha_3)\left(\lambda^2 + (\beta I^* + \mu)\lambda + \beta I^*(\mu + \alpha_2 + \delta)\right),$$

of which all roots have negative real parts. So, the endemic equilibrium P_1 is also locally asymptotically stable, as $\tau \to \infty$.

Denote

$$h(\lambda) = \lambda^2 + a\lambda + b + ce^{-\lambda\tau}. \tag{2.5}$$

Let $\lambda = iy$ with $y > 0$ and $\tau > 0$ finite and $h(iy) = 0$, we obtain that

$$\begin{cases} \sin y\tau = -\frac{(\beta I^* + \mu)y}{\beta I^* \delta e^{-(\mu+\alpha_3)\tau}} = \frac{ay}{c}, \\ \cos y\tau = \frac{\beta I^*(\mu+\alpha_2+\delta) - y^2}{\beta I^* \delta e^{-(\mu+\alpha_3)\tau}} = \frac{y^2 - b}{c}. \end{cases} \tag{2.6}$$

Squaring and adding the equations in (2.6) gives

$$y^4 + \left(a^2 - 2b\right)y^2 + b^2 - c^2 = 0. \tag{2.7}$$

Obviously,

$$\begin{aligned} b^2 - c^2 =& \left[\beta I^*(\mu + \alpha_2 + \delta)\right]^2 - \left(\beta I^* \delta e^{-(\mu+\alpha_3)\tau}\right)^2 > 0, \\ a^2 - 2b =& \beta^2 I^{*2} + 2\beta I^* \mu + \mu^2 - 2\beta I^*(\mu + \alpha_2 + \delta) \\ =& \beta^2 I^{*2} - 2\beta(\alpha_2 + \delta) I^* + \mu^2. \end{aligned}$$

If (2.7) has no positive root, then the endemic equilibrium P_1 is locally asymptotically stable. Consequently, we have the following sufficient condition for stability of the endemic equilibrium P_1.

Theorem 2.2. *Suppose that*

$$2b - a^2 < 2\sqrt{b^2 - c^2}, \tag{2.8}$$

then the endemic equilibrium P_1 of system (1.2) is locally asymptotically stable.

Proof. If $a^2 - 2b \geq 0$, then (2.8) hold and (2.7) does not admit any positive root. If $a^2 - 2b < 0$, then (2.8) implies that the discriminant of (2.7) $\triangle_1 = \left(2b - a^2\right)^2 - 4\left(b^2 - c^2\right) < 0$. Therefore, (2.7) has no positive root and the endemic equilibrium P_1 of (1.2) system is locally asymptotically stable. □

Define

$$f(I) = \beta^2 I^2 - 2\beta(\alpha_2 + \delta) I + \mu^2,$$

then $a^2 - 2b = f(I^*)$.

Let $\triangle_2$ be the discriminant of $f(I)$, then

$$\begin{aligned}\triangle_2 =& 4\beta^2(\alpha_2 + \delta)^2 - 4\beta^2\mu^2 \\ =& 4\beta^2(\alpha_2 + \delta + \mu)(\alpha_2 + \delta - \mu).\end{aligned}$$

Clearly, if $\alpha_2 + \delta - \mu > 0$, then $f(I) > 0$ which implies $a > 0$. Namely, when $\alpha_2 + \delta - \mu > 0$, the positive equilibrium P_1 of system is locally asymptotically stable for all $\tau > 0$.

If $2b - a^2 = 2\sqrt{b^2 - c^2}$, then (2.7) has a unique positive root $y_1 = \sqrt[4]{b^2 - c^2}$.

If $2b - a^2 > 2\sqrt{b^2 - c^2}$, then (2.7) has two positive roots y_- and y_+ satisfying

$$y_\pm^2(\tau) = \frac{1}{2}\left(2b - a^2 \pm \sqrt{(2b - a^2)^2 - 4(b^2 - c^2)}\right).$$

In order to find that τ values of stability switches, we use the procedure described in Beretta etc [2]. For each positive root $y(\tau)$ of (2.7), we define the angle $\theta(\tau) \in (0, 2\pi)$ as a solution of

$$\begin{cases} \sin\theta(\tau) = \frac{a(\tau)y(\tau)}{c(\tau)}, \\ \cos\theta(\tau) = \frac{y^2(\tau) - b(\tau)}{c(\tau)}. \end{cases}$$

For each $y(\tau)$ satisfying (2.7), define

$$S_n(\tau) = \tau - \frac{\theta(\tau) + 2n\pi}{y(\tau)}, \ n = 0, \pm 1, \cdots.$$

Following Beretta etc [2], we have the following result.

Theorem 2.3. *If $R_0 > 1$. For system , we have*

(1) *If $\beta^2 I^{*2} - 2\beta(\alpha_2 + \delta) I^* + \mu^2 > 0$, then the endemic equilibrium P_1 of system (1.2) is locally asymptotically stable for all $\tau \geq 0$.*

(2) *Suppose that there is $\tau^* > 0$ satisfying $S_n(\tau^*) = 0$ for some $n \in N$ and that (2.7) has a pair of simple and conjugate pure imaginary roots $\lambda = \pm y(\tau^*) i$ with $y(\tau^*) > 0$.*

(a) when $y(\tau^*) = y_+(\tau^*)$, this pair of simple pure imaginary roots crosses the imaginary axis from left to right (as τ increases) if $\xi_+(\tau^*) > 0$ and from right to left if $\xi_+(\tau^*) < 0$, where

$$\xi_+(\tau^*) = sign\left\{\frac{d(Re\lambda)}{d\tau}\bigg|_{\lambda = iy_+(\tau^*)}\right\} = sign\left\{\frac{dS_n(\tau)}{d\tau}\bigg|_{\tau=\tau^*}\right\}.$$

(b) when $y(\tau^*) = y_-(\tau^*)$, this pair of simple pure imaginary roots crosses the imaginary axis from left to right (as τ increases) if $\xi_-(\tau^*) > 0$ and from right to left if $\xi_-(\tau^*) < 0$, where

$$\xi_-(\tau^*) = sign\left\{\frac{d(Re\lambda)}{d\tau}\bigg|_{\lambda = iy_-(\tau^*)}\right\} = -sign\left\{\frac{dS_n(\tau)}{d\tau}\bigg|_{\tau=\tau^*}\right\}.$$

To illustrate possible behaviors of system (1.2), we give some examples and perform numerical simulation.

Example 2.1. In system (1.2), choose $A = 0.25$, $\beta = 4$, $\mu = 0.1$, $\alpha_2 = 0.1$, $\alpha_3 = 0.001$, $\gamma = 0.001$, $\delta = 5$. These give $R_0 = 1.5382$. System (1.2) has a unique endemic equilibrium P_1. Let $\tau = 1$ in system (1.2) with the other coefficients above, numerical simulation shows that the endemic equilibrium P_1 of system (1.2) is stable (see Fig. 1). Let $\tau = 3$ in system (1.2) with the other coefficients above, the system (1.2) admits a periodic solution (see Fig. 2). As the quarantine period increase from $\tau = 3$ to $\tau = 4$, the amplitude of these oscillations increases correspondingly (see Fig. 3). Further increase $\tau = 30$, the periodic solution disappears and the endemic equilibrium P_1 becomes stable (Fig. 4). This implies that there exist a interval in which system admits a periodic solution.

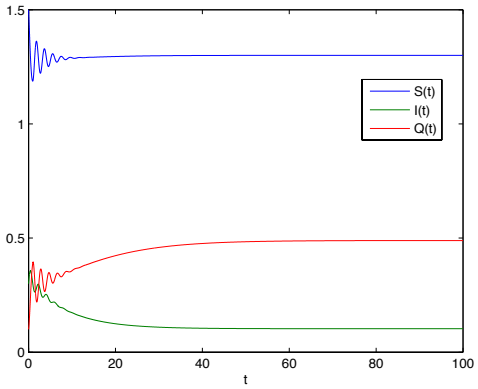

Figure 1. $A = 0.25$, $\beta = 4$, $\mu = 0.1$, $\alpha_2 = 0.1$, $\alpha_3 = 0.001$, $\gamma = 0.001$, $\delta = 5$, $\tau = 1$.

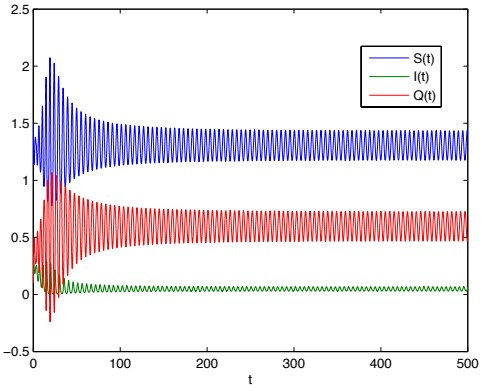

Figure 2. $A = 0.25$, $\beta = 4$, $\mu = 0.1$, $\alpha_2 = 0.1$, $\alpha_3 = 0.001$, $\gamma = 0.001$, $\delta = 5$, $\tau = 3$.

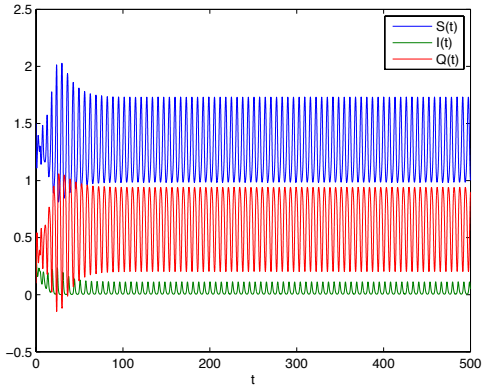

Figure 3. $A = 0.25$, $\beta = 4$, $\mu = 0.1$, $\alpha_2 = 0.1$, $\alpha_3 = 0.001$, $\gamma = 0.001$, $\delta = 5$, $\tau = 4$.

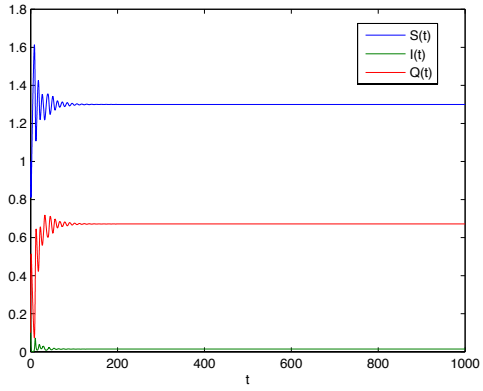

Figure 4. $A = 0.25$, $\beta = 4$, $\mu = 0.1$, $\alpha_2 = 0.1$, $\alpha_3 = 0.001$, $\gamma = 0.001$, $\delta = 5$, $\tau = 30$.

Next, we present a lemma to be used to prove globally asymptotical stability of the endemic equilibrium P_1.

Lemma 2.1. *Let the initial value for system* (1.2) *satisfy* (1.3). *Then* $S(t) \leq \max\left\{\frac{A}{\mu}, S(0)+I(0)+R(0)\right\} \triangleq N_\infty$.

Proof. From system (1.2), we know that the total population $N(t)$ satisfy

$$\frac{dN(t)}{dt} = A - \mu N - \alpha_2 I - \alpha_3 Q \leq A - \mu N.$$

Now, Consider the following auxiliary equation

$$\begin{cases} \frac{du(t)}{dt} = A - \mu u(t), \\ u(0) = S(0) + I(0) + R(0). \end{cases}$$

It is easy to know that

$$u(t) = \frac{A}{\mu} + \left(u(0) - \frac{A}{\mu}\right) e^{-ut} \to \frac{A}{\mu} \text{ as } t \to \infty.$$

This implies that $u(t) \leq \max\left\{\frac{A}{\mu}, u(0)\right\}$. Hence, it follows from the comparison principle that

$$N(t) \leq u(t) \leq \max\left\{\frac{A}{\mu}, S(0)+I(0)+R(0)\right\}.$$

Consequently,

$$S(t) \leq N(t) \leq N_\infty.$$

□

Theorem 2.4. *Suppose that* $R_0 > 1$. *If* $\mu - \frac{1}{2}\delta e^{-(\mu+\alpha_3)\tau} > 0$, $\mu + \alpha_3 - \frac{1}{2}\delta - \frac{1}{2}\delta e^{-(\mu+\alpha_3)\tau} > 0$ *and* $c(\mu+\alpha_2+\delta)+(\mu+\alpha_2+\delta+\gamma)-\beta N_\infty - \frac{1}{2}\delta - \frac{1+3c}{2}\delta e^{-(\mu+\alpha_3)\tau} > 0$, *then the endemic equilibrium* $P_1(S^*, I^*, Q^*)$ *of system* (1.2) *is globally asymptotical stable, where* $N_\infty = \max\left\{\frac{A}{\mu}, S(0)+I(0)+R(0)\right\}$ *and* $c = \frac{\beta I^*}{2\mu+\alpha_2+\delta}$.

Proof. We center system (1.2) at the endemic equilibrium $P_1(S^*, I^*, Q^*)$ by introducing new variables as

$$x = S - S^*, y = I - I^* \text{ and } z = Q - Q^*.$$

After substituting these variables, system (1.2) can be rewritten in the following form:

$$\begin{cases} \frac{dx}{dt} = -\beta I^* x - \beta S y - \mu x + \gamma y + \delta y(t-\tau) e^{-(\mu+\alpha_3)\tau}, \\ \frac{dy(t)}{dt} = \beta I^* x + \beta S y - (\mu + \alpha_2 + \delta + \gamma) y, \\ \frac{dz(t)}{dt} = \delta y - (\mu + \alpha_3) z - \delta y(t-\tau) e^{-(\mu+\alpha_3)\tau}. \end{cases} \tag{2.9}$$

Now, let us introduce the following functional:

$$V_1(x, y, z) = \frac{1}{2} c (x+y)^2 + \frac{1}{2}\left(y^2 + z^2\right),$$

where $c = \frac{\beta I^*}{2\mu+\alpha_2+\delta}$. Differentiating $V(x,y,z)$ and using (2.9) gives

$$\begin{aligned}\frac{dV_1}{dt} =& c(x+y)\begin{bmatrix} -\beta I^* x - \beta S y - \mu x + \gamma y + \delta y(t-\tau)e^{-(\mu+\alpha_3)\tau} \\ +\beta I^* x + \beta S y - (\mu+\alpha_2+\delta+\gamma)y \end{bmatrix} \\ &+ y[\beta I^* x + \beta S y - (\mu+\alpha_2+\delta+\gamma)y] \\ &+ z\left[\delta y - (\mu+\alpha_3)z - \delta y(t-\tau)e^{-(\mu+\alpha_3)\tau}\right] \\ =& -c\mu x^2 + [\beta S - c(\mu+\alpha_2+\delta) - (\mu+\alpha_2+\delta+\gamma)]y^2 - (\mu+\alpha_3)z^2 \\ &+ [-c(\mu+\alpha_2+\delta) - c\mu + \beta I^*]xy + \delta yz + c\delta xy(t-\tau)e^{-(\mu+\alpha_3)\tau} \\ &+ c\delta y y(t-\tau)e^{-(\mu+\alpha_3)\tau} - \delta z y(t-\tau)e^{-(\mu+\alpha_3)\tau}.\end{aligned}$$

Using $c = \frac{\beta I^*}{2\mu+\alpha_2+\delta}$ and Cauchy-Schwartz inequality, we obtian that

$$\begin{aligned}\frac{dV_1}{dt} \le& -c\mu x^2 + [\beta S - c(\mu+\alpha_2+\delta) - (\mu+\alpha_2+\delta+\gamma)]y^2 \\ &- (\mu+\alpha_3)z^2 + \frac{1}{2}\delta\left(y^2+z^2\right) + \frac{1}{2}c\delta e^{-(\mu+\alpha_3)\tau}x^2 \\ &+ \frac{1}{2}c\delta e^{-(\mu+\alpha_3)\tau}y^2(t-\tau) + \frac{1}{2}c\delta e^{-(\mu+\alpha_3)\tau}y^2 \\ &+ \frac{1}{2}c\delta e^{-(\mu+\alpha_3)\tau}y^2(t-\tau) + \frac{1}{2}\delta e^{-(\mu+\alpha_3)\tau}z^2 + \frac{1}{2}\delta e^{-(\mu+\alpha_3)\tau}y^2(t-\tau) \\ =& -c\left(\mu - \frac{1}{2}\delta e^{-(\mu+\alpha_3)\tau}\right)x^2 \\ &+ \left[c(\mu+\alpha_2+\delta) + (\mu+\alpha_2+\delta+\gamma) - \beta S - \frac{1}{2}\delta - \frac{1}{2}c\delta e^{-(\mu+\alpha_3)\tau}\right]y^2 \\ &- \left(\mu+\alpha_3 - \frac{1}{2}\delta - \frac{1}{2}\delta e^{-(\mu+\alpha_3)\tau}\right)z^2 + \left(c+\frac{1}{2}\right)\delta e^{-(\mu+\alpha_3)\tau}y^2(t-\tau).\end{aligned}$$

Choose the Lyapunov functional

$$V(x,y,z) = V_1(x,y,z) + \left(c+\frac{1}{2}\right)\delta e^{-(\mu+\alpha_3)\tau}\int_{t-\tau}^{t} y^2(r)\,dr.$$

Using Lemma 2.1, we get

$$\begin{aligned}\frac{dV}{dt} \le& -c\left(\mu - \frac{1}{2}\delta e^{-(\mu+\alpha_3)\tau}\right)x^2 - \begin{bmatrix} c(\mu+\alpha_2+\delta) + (\mu+\alpha_2+\delta+\gamma) \\ -\beta N_\infty - \frac{1}{2}\delta - \frac{1}{2}c\delta e^{-(\mu+\alpha_3)\tau} \end{bmatrix} y^2 \\ &- \left(\mu+\alpha_3 - \frac{1}{2}\delta - \frac{1}{2}\delta e^{-(\mu+\alpha_3)\tau}\right)z^2 + \left(c+\frac{1}{2}\right)\delta e^{-(\mu+\alpha_3)\tau}y^2 \\ =& -c\left(\mu - \frac{1}{2}\delta e^{-(\mu+\alpha_3)\tau}\right)x^2 - \left(\mu+\alpha_3 - \frac{1}{2}\delta - \frac{1}{2}\delta e^{-(\mu+\alpha_3)\tau}\right)z^2 \\ &- \left[c(\mu+\alpha_2+\delta) + (\mu+\alpha_2+\delta+\gamma) - \beta N_\infty - \frac{1}{2}\delta - \frac{1+3c}{2}\delta e^{-(\mu+\alpha_3)\tau}\right]y^2.\end{aligned}$$

By the conditions of the Theorem 2.4 and the Lyapunov-LaSalle type theorem, it is easy to see $\lim_{t\to\infty} x(t) = 0$, $\lim_{t\to\infty} y(t) = 0$ and $\lim_{t\to\infty} z(t) = 0$. □

3. Asymptotical behavior of the stochastic model (1.5)

In this section, we will discuss the dynamic behavior of the model (1.5). First, we give some symbols and instructions.

In what follows, let $\left(\Omega, \mathscr{F}, \{\mathscr{F}\}_{t\geq 0}, P\right)$ be a complete probability space with a filtration $\{\mathscr{F}\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathscr{F}_0$ contains all $P-$null sets).

In general, the d-dimensional stochastic system:

$$dX(t) = f(t, X(t))\, dt + g(t, X(t))\, dB_t, \tag{3.1}$$

where $f(t,x)$ is an function in $\mathbb{R}^d$ defined in $[t_0, \infty] \times \mathbb{R}^d$, and $g(t,x)$ is an $d \times m$ matrix, f, g are locally Lipschitz functions in x. B_t is an m-dimensional standard Wiener process defined on the above probability space.

Denote by $C^{2,1}\left(\mathbb{R}^d \times [t_0, \infty]; \mathbb{R}_+\right)$ the family of all nonnegative functions $V(x,t)$ defined on $\mathbb{R}^d \times [t_0, \infty]$ such that they are continuously twice differentiable in x and once in t. Set $\mathbb{R}^d_+ = \left\{x \in \mathbb{R}^d,\ x_i > 0,\ 1 \leq 1 \leq d\right\}$. The differential operator L of Eq. (3.1) is defined [26] by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(t) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \left[g^T(x,t)\, g(x,t)\right]_{ij} \frac{\partial^2}{\partial x_i \partial x_j}. \tag{3.2}$$

If L acts on a function $V \in C^{2,1}\left(\mathbb{R}^d \times [t_0, \infty]; \mathbb{R}_+\right)$, then

$$LV(x,t) = V_t(x,t) + V_x(x,t)\, f(x,t) + \frac{1}{2} trace\left[g^T(x,t)\, V_{xx} g(x,t)\right],$$

where $V_t(x,t) = \frac{\partial V}{\partial t}$, $V_x(x,t) = \left(\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_d}\right)$, $V_{xx} = \left(\frac{\partial^2 V}{\partial x_i x_j}\right)_{d\times d}$. By Itô′s formula, if $x(t) \in \mathbb{R}^d$, then $dV(x,t) = LV(x,t)\, dt + V_x(x,t)\, g(x,t)\, dB_t$.

3.1. Existence and uniqueness of the positive solution

In this section, we show there is a unique global positive solution of model (1.5).

Theorem 3.1. *For any given initial value $S(0) > 0$ and $I(\varpi) \geq 0$ for all $\varpi \in [-\tau, 0)$ with $I(0) > 0$, there is a unique positive solution $(S(t), I(t))$ of model (1.5) on $t \geq 0$ and the solution will remain in $\in R^2_+$ with probability 1, namely $(S(t), I(t)) \in R^2_+$ for $t \geq 0$ almost surely.*

Proof. Since the coefficients of the model (1.5) are locally Lipschitz continuous, for any given initial value $S(0) > 0$ and $I(\varpi) \geq 0$ for all $\varpi \in [-\tau, 0)$ with $I(0) > 0$, there is a unique local solution $(S(t), I(t))$ on $t \in [-\tau, \tau_e)$, where τ_e is the explosion time (see [27]). To show this solution is global, we need to show that $\tau_e = \infty$ a.s. Let $k_0 \geq 0$ be sufficiently large so that $S(0), I(0) \in [1/k_0, k_0]$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\left\{t \in [0, \tau_e) : \min(S(t), I(t)) \leq 1/k \text{ or } \max(S(t), I(t)) \geq k\right\},$$

where throughout this paper we set $\inf \phi = \infty$ (as usual ϕ denotes the empty set). Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$ whence $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s. (almost surely), then $\tau_e = \infty$ and $(S(t), I(t)) \in \mathbb{R}^2_+$ a.s. for all $t \geq 0$. In other words, to complete the proof, we need to show is that $\tau_\infty = \infty$ a.s. If this statement is false, then there is a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leq T\} > \epsilon.$$

Hence, there is an integer $k_1 \geq k_0$ such that,

$$P\{\tau_k \leq T\} \geq \epsilon \text{ for all } k \geq k_1. \tag{3.3}$$

Define a C^2-function V: $\mathbb{R}^2_+ \to \mathbb{R}^1_+$

$$V = \left(S - \lambda - \lambda \log \frac{S}{\lambda}\right) + (I - 1 - \log I) + \delta e^{-(\mu+\alpha_3)\tau} \int_{t-\tau}^{t} I(r)\, dr, \tag{3.4}$$

where λ is a positive constant to be determined later. The non-negativity of this function can be seen from $u - 1 - \log u \ \forall\ u > 0$. Using Itô′s formula, we get

$$\begin{aligned} dV =& \left(1 - \frac{\lambda}{S}\right)\left[\left(A - \beta IS - \mu S + \gamma I + \delta I(t-\tau) e^{-(\mu+\alpha_3)\tau}\right) dt + \sigma_1 S dB_1(t)\right] \\ &+ \left(1 - \frac{1}{I}\right)\left[(\beta IS - (\mu + \alpha_2 + \delta + \gamma) I)\, dt + \sigma_2 I dB_2(t)\right] \\ &+ \frac{\lambda\sigma_1^2}{2} dt + \frac{\sigma_2^2}{2} dt + \delta e^{-(\mu+\alpha_3)\tau} I dt - \delta e^{-(\mu+\alpha_3)\tau} I(t-\tau)\, dt \\ =& LV dt + \left(1 - \frac{\lambda}{S}\right)\sigma_1 S dB_1(t) + \left(1 - \frac{1}{I}\right)\sigma_2 I dB_2(t), \end{aligned}$$

where

$$\begin{aligned} LV =& \left(1 - \frac{\lambda}{S}\right)\left(A - \beta IS - \mu S + \gamma I + \delta I(t-\tau) e^{-(\mu+\alpha_3)\tau}\right) + \frac{\lambda\sigma_1^2}{2} \\ &+ \left(1 - \frac{1}{I}\right)(\beta IS - (\mu + \alpha_2 + \delta + \gamma) I) + \frac{\sigma_2^2}{2} \\ &+ \delta e^{-(\mu+\alpha_3)\tau} I - \delta e^{-(\mu+\alpha_3)\tau} I(t-\tau) \\ =& A - \mu S - (\mu + \alpha_2 + \delta) I - \frac{\lambda A}{S} + \lambda\beta I + \lambda\mu - \lambda\gamma \frac{I}{S} - \lambda \frac{\delta I(t-\tau) e^{-(\mu+\alpha_3)\tau}}{S} \\ &- \beta S + \mu + \alpha_2 + \delta + \gamma + \frac{\lambda\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \delta e^{-(\mu+\alpha_3)\tau} I \\ \leq& A - (\mu + \alpha_2 + \delta) I + \lambda\beta I + \lambda\mu + \mu + \alpha_2 + \delta + \gamma + \frac{\lambda\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \delta e^{-(\mu+\alpha_3)\tau} I \\ =& A + \left\{\lambda\beta - \left[\mu + \alpha_2 + \delta\left(1 - e^{-(\mu+\alpha_3)\tau}\right)\right]\right\} I \\ &+ \lambda\mu + \mu + \alpha_2 + \delta + \gamma + \frac{\lambda\sigma_1^2}{2} + \frac{\sigma_2^2}{2}. \end{aligned}$$

Choose $\lambda = \frac{\left[\mu+\alpha_2+\delta\left(1-e^{-(\mu+\alpha_3)\tau}\right)\right]}{\beta}$, we have

$$LV \leq A + \lambda\mu + \mu + \alpha_2 + \delta + \gamma + \frac{\lambda\sigma_1^2}{2} + \frac{\sigma_2^2}{2} \triangleq M.$$

The remainder of the proof follows that in Mao [27]. □

3.2. Extinction

For a infective disease, it is importance to obtain the conditions for the spread and extinction of the disease. In this section, we will analyse the condition for the extinction of the disease.

For convenience, we define $\langle x(t)\rangle = \frac{1}{t}\int_0^t x(u)\,du$.

First, we present two lemmas which will be used to discuss extinction of the disease.

Denote $w(t) = S(t) + I(t) + \delta e^{-(\mu+\alpha_3)t}\int_{t-\tau}^t e^{(\mu+\alpha_3)r} I(r)\,dr$.

Lemma 3.1. *Assume that $\mu > \frac{\sigma^2}{2}$. Let $(S(t), I(t))$ be any solution of model (1.5) with initial value $S(0) > 0$ and $I(\varpi) \geq 0$ for all $\varpi \in [-\tau, 0)$ with $I(0) > 0$. Then $\lim\limits_{t\to\infty}\frac{w(t)}{t} = 0$, $\lim\limits_{t\to\infty}\frac{S(t)}{t} = 0$, $\lim\limits_{t\to\infty}\frac{I(t)}{t} = 0$ and $\lim\limits_{t\to\infty}\frac{\delta e^{-(\mu+\alpha_3)t}\int_{t-\tau}^t e^{(\mu+\alpha_3)r} I(r)dr}{t} = 0$ a.s., moreover, $\lim\limits_{t\to\infty}\frac{\ln w(t)}{t} = 0$, $\lim\limits_{t\to\infty}\frac{\ln S(t)}{t} = 0$, $\lim\limits_{t\to\infty}\frac{\ln I(t)}{t} = 0$ and*

$$\lim_{t\to\infty}\frac{\ln\left(\delta e^{-(\mu+\alpha_3)t}\int_{t-\tau}^t e^{(\mu+\alpha_3)r} I(r)\,dr\right)}{t} = 0 \text{ a.s., where } \sigma^2 = \max\left(\sigma_1^2, \sigma_2^2\right).$$

Proof. Define

$$\Phi(w) = (1+w)^\rho,$$

where $\rho \in \left(2, 1+\frac{2\mu}{\sigma^2}\right)$ is a positive constant. Using Itô′s formula, we have

$$d\Phi(w) = L\Phi(w)\,dt + \rho(1+w)^{\rho-1}\left(\sigma_1 S dB_1(t) + \sigma_2 I dB_2(t)\right),$$

where

$$\begin{aligned}
L\Phi(w) =& \rho(1+w)^{\rho-1}\begin{pmatrix} A-\mu S \\ -(\mu+\alpha_3)\delta e^{-(\mu+\alpha_3)t}\int_{t-\tau}^t e^{(\mu+\alpha_3)r} I(r)\,dr \\ -(\mu+\alpha_2) I \end{pmatrix} \\
&+ \frac{\rho(\rho-1)}{2}(1+w)^{\rho-2}\left(\sigma_1^2 S^2 + \sigma_2^2 I^2\right) \\
\leq& \rho(1+w)^{\rho-1}\left(A - \mu S - \mu\delta e^{-(\mu+\alpha_3)t}\int_{t-\tau}^t e^{(\mu+\alpha_3)r} I(r)\,dr - \mu I\right) \\
&+ \frac{\rho(\rho-1)}{2}(1+w)^{\rho-2}\left(\sigma_1^2 S^2 + \sigma_2^2 I^2\right) \\
=& \rho(1+w)^{\rho-2}\left((A-\mu w)(1+w) + \frac{\rho-1}{2}\left(\sigma_1^2 S^2 + \sigma_2^2 I^2\right)\right) \\
\leq& \rho(1+w)^{\rho-2}\left((A-\mu w)(1+w) + \frac{\rho-1}{2}\sigma^2 w^2\right) \\
=& \rho(1+w)^{\rho-2}\left(-\left(\mu - \frac{\rho-1}{2}\sigma^2\right)w^2 + (A-\mu)w + A\right).
\end{aligned}$$

Since $\rho \in \left(2, 1+\frac{2\mu}{\sigma^2}\right)$, $\mu - \frac{\rho-1}{2}\sigma^2 > 0$. Denote $\mu - \frac{\rho-1}{2}\sigma^2 = \eta$. Then

$$L\Phi(w) \leq \rho(1+w)^{\rho-2}\left(-\eta w^2 + (A-\mu)w + A\right), \tag{3.5}$$

yields that

$$\begin{aligned} d\Phi(w) \leq & \rho(1+w)^{\rho-2}\left(-\eta w^2+(A-\mu)w+A\right)dt \\ & +\rho(1+w)^{\rho-1}\left(\sigma_1 S dB_1(t)+\sigma_2 I dB_2(t)\right). \end{aligned} \tag{3.6}$$

Let λ be a positive constant and $\lambda < \rho\eta$. Then

$$\begin{aligned} d\left(\exp(\lambda t)\Phi(w)\right) = & L\left(\exp(\lambda t)\Phi(w)\right)dt+\exp(\lambda t)\rho(1+w)^{\rho-1} \\ & \times\left(\sigma_1 S dB_1(t)+\sigma_2 I dB_2(t)\right). \end{aligned}$$

where

$$\begin{aligned} & L\left(\exp(\lambda t)\Phi(w)\right) \\ = & \lambda\exp(\lambda t)\Phi(w)+\exp(\lambda t)L\Phi(w) \\ \leq & \lambda\exp(\lambda t)(1+w)^{\rho}+\exp(\lambda t)\rho(1+w)^{\rho-2}\left(-\eta w^2+(A-\mu)w+A\right) \\ = & \exp(\lambda t)(1+w)^{\rho-2}\left\{-(\rho\eta-\lambda)w^2+(A\rho-\mu\rho+2\lambda)w+\rho A+\lambda\right\}. \end{aligned}$$

Denoting $M=\sup\limits_{w\in\mathbb{R}_+}(1+w)^{\rho-2}\left\{-(\rho\eta-\lambda)w^2+(A\rho-\mu\rho+2\lambda)w+\rho A+\lambda\right\}$, we have

$$E\left(\exp(\lambda t)\Phi(w)\right)\leq\Phi(w(0))+M\exp(\lambda t),$$

which leads to

$$\lim_{t\to\infty}\sup E(1+w)^{\rho}\leq M,\ \text{a.s.}$$

This together with the continuity of $w(t)$ means that there exists a constant $H>0$ such that for $t>0$

$$E(1+w)^{\rho}\leq H,\ \text{a.s.} \tag{3.7}$$

Let ϑ be a positive constant and $n=1,2,\ldots$, from (3.6) we obtain for $n\vartheta\leq t\leq(n+1)\vartheta$

$$\begin{aligned} \Phi(w(t)) \leq & \Phi(w(n\vartheta))+\int_{n\vartheta}^{t}\rho(1+w(s))^{\rho-2}\left(-\eta w^2(s)+(A-\mu)w(s)+A\right)ds \\ & +\int_{n\vartheta}^{t}\rho(1+w(s))^{\rho-1}\left(\sigma_1 S dB_1(s)+\sigma_2 I dB_2(s)\right). \end{aligned}$$

Then

$$\begin{aligned} & E\left[\sup_{n\vartheta\leq t\leq(n+1)\vartheta}(1+w(t))^{\rho}\right] \\ \leq & E\left[(1+w(n\vartheta))^{\rho}\right]+f_1(t)+f_2(t) \\ \leq & H+G_1(t)+G_2(t), \end{aligned}$$

where

$$\begin{aligned}
G_1(t) =& E\left[\sup_{n\vartheta\le t\le(n+1)\vartheta}\left|\int_{n\vartheta}^{t}\rho\left(1+w(s)\right)^{\rho-2}\left(-\eta w(s)^2+(A-\mu)w(s)+A\right)ds\right|\right]\\
\le& l_1 E\left[\sup_{n\vartheta\le t\le(n+1)\vartheta}\left|\int_{n\vartheta}^{t}\rho\left(1+w(s)\right)^{\rho}ds\right|\right]\\
\le& l_1 E\left[\int_{n\vartheta}^{(n+1)\vartheta}\rho\left(1+w(s)\right)^{\rho}ds\right]\le l_1\vartheta E\left[\sup_{n\vartheta\le t\le(n+1)\vartheta}\left(1+w(t)\right)^{\rho}\right],
\end{aligned}$$

l_1 is a positive constant, and

$$\begin{aligned}
G_2(t) =& E\left[\sup_{n\vartheta\le t\le(n+1)\vartheta}\left|\int_{n\vartheta}^{t}\rho\left(1+w(s)\right)^{\rho-1}\left(\sigma_1 S dB_1(s)+\sigma_2 I dB_2(s)\right)ds\right|\right]\\
=&\sqrt{32}E\left[\left|\int_{n\vartheta}^{t}\rho^2\left(1+w(s)\right)^{2\rho-2}\left(\sigma_1^2S^2+\sigma_2^2I^2\right)ds\right|\right]^{\frac{1}{2}}\\
\le&\sigma\rho\sqrt{32}E\left[\int_{n\vartheta}^{t}\left(1+w(s)\right)^{2\rho-2}\left(S^2+I^2\right)ds\right]^{\frac{1}{2}}\\
\le&\sigma\rho\sqrt{32}E\left[\int_{n\vartheta}^{t}\left(1+w(s)\right)^{2\rho-2}\left(1+w\right)^2ds\right]^{\frac{1}{2}}\\
\le&\sigma\rho\sqrt{32}E\left[\int_{n\vartheta}^{(n+1)\vartheta}\sup_{n\vartheta\le s\le(n+1)\vartheta}\left(1+w(s)\right)^{2\rho}ds\right]^{\frac{1}{2}}\\
=&\sigma\rho\sqrt{32\vartheta}E\left[\sup_{n\vartheta\le s\le(n+1)\vartheta}\left(1+w(s)\right)^{\rho}\right],
\end{aligned}$$

where Burkholder-Davis-inequality is used. Hence

$$E\left[\sup_{n\vartheta\le t\le(n+1)\vartheta}\left(1+w(t)\right)^{\rho}\right]\le H+\left(l_1\vartheta+\sigma\rho\sqrt{32\vartheta}\right)E\left[\sup_{n\vartheta\le s\le(n+1)\vartheta}\left(1+w(s)\right)^{\rho}\right].$$

Choose $\vartheta>0$ such that $l_1\vartheta+\sigma\rho\sqrt{32\vartheta}\le\frac{2}{3}$, then

$$E\left[\sup_{n\vartheta\le t\le(n+1)\vartheta}\left(1+w(t)\right)^{\rho}\right]\le 3H.$$

Let $\varsigma>0$ be arbitrary. It follows from Chebyshev's inequality that

$$\begin{aligned}
&P\left\{\sup_{n\vartheta\le t\le(n+1)\vartheta}\left(1+w(t)\right)^{\rho}>(n\vartheta)^{1+\varsigma}\right\}\\
\le&\frac{E\left[\sup_{n\vartheta\le t\le(n+1)\vartheta}\left(1+w(t)\right)^{\rho}\right]}{(n\vartheta)^{1+\varsigma}}\le\frac{3H}{(n\vartheta)^{1+\varsigma}},\ n=1,2,\cdots.
\end{aligned}$$

The Borel-Cantelli lemma yields that for almost all $\omega\in\Omega$ there exists a positive integer $n_0=n_0(\omega)$ such that

$$\sup_{n\vartheta\le t\le(n+1)\vartheta}\left(1+w(t)\right)^{\rho}\le(n\vartheta)^{1+\varsigma},\ \text{whenever } n>n_0(\omega).$$

So, for $n\vartheta \le t \le (n+1)\,\vartheta$ $n > n_0(\omega)$, we have

$$\frac{\ln(1+w(t))^{\rho}}{\ln t} \le \frac{(1+\varsigma)\ln(n\vartheta)}{\ln(n\vartheta)} = (1+\varsigma).$$

Hence,

$$\lim_{t\to\infty}\sup\frac{\ln(1+w(t))^{\rho}}{\ln t} \le (1+\varsigma) \text{ a.s.}$$

Letting $\varsigma \to 0$, we get

$$\lim_{t\to\infty}\sup\frac{\ln(1+w(t))^{\rho}}{\ln t} \le 1 \text{ a.s.}$$

Then

$$\lim_{t\to\infty}\sup\frac{\ln w(t)}{\ln t} \le \lim_{t\to\infty}\sup\frac{\ln(1+w(t))}{\ln t} \le \frac{1}{\rho} \text{ a.s.} \tag{3.8}$$

It follows from (3.8) and $\rho \in \left(2, 1+\frac{2\mu}{\sigma^2}\right)$ that for arbitrary $\varrho \in \left(0, 1-\frac{1}{\rho}\right)$ there exist a time $T_1 = T_1(\omega)$ and a set Ω_1 such that $P(\Omega_1) \ge 1-\varrho$ and for $t > T_1(\omega)$

$$\frac{\ln w(t)}{\ln t} \le \frac{1}{\rho} + \varrho < 1.$$

Hence

$$\lim_{t\to\infty}\sup\frac{w(t)}{t} \le \lim_{t\to\infty}\sup\frac{t^{\frac{1}{\rho}+\varrho}}{t} = 0,$$

which together with positivity of the solution yields that

$$\lim_{t\to\infty}\frac{w(t)}{t} = \lim_{t\to\infty}\frac{S(t)+I(t)+\delta e^{-(\mu+\alpha_3)t}\int_{t-\tau}^{t}e^{(\mu+\alpha_3)r}I(r)\,dr}{t} = 0, \text{ a.s.} \tag{3.9}$$

Therefore

$$\lim_{t\to\infty}\frac{S(t)}{t} = 0, \ \lim_{t\to\infty}\frac{I(t)}{t} = 0, \ \lim_{t\to\infty}\frac{\delta e^{-(\mu+\alpha_3)t}\int_{t-\tau}^{t}e^{(\mu+\alpha_3)r}I(r)\,dr}{t} = 0, \text{ a.s.}$$

It follows from (3.9) that for arbitrary $\epsilon > 0$ there exist a time $T_3 = T_3(\omega)$ and a set Ω_3 such that $P(\Omega_3) \ge 1-\epsilon$ and for $t > T_3(\omega)$

$$w(t) \le \epsilon t.$$

Then, for $t > T_3(\omega)$ and $\omega \in \Omega_3$, we have

$$\ln w(t) \le \ln\epsilon + \ln t,$$

leads to

$$\lim_{t\to\infty}\frac{\ln w(t)}{t} = 0, \text{ a.s.}$$

Hence, $\lim_{t\to\infty}\frac{\ln S(t)}{t} = 0$, $\lim_{t\to\infty}\frac{\ln I(t)}{t} = 0$, $\lim_{t\to\infty}\frac{\ln\left(\delta e^{-(\mu+\alpha_3)t}\int_{t-\tau}^{t}e^{(\mu+\alpha_3)r}I(r)dr\right)}{t} = 0$ a.s. This finishes the proof of Lemma 3.1. □

Lemma 3.2. *Assume that* $\mu > \frac{\sigma^2}{2}$. *Let* $(S(t), I(t))$ *be any solution of model* (1.4) *with initial value* $S(0) > 0$ *and* $I(\varpi) \geq 0$ *for all* $\varpi \in [-\tau, 0)$ *with* $I(0) > 0$. *Then* $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t S(u)\, dB_1(u) = 0$, $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t I(u)\, dB_2(u) = 0$ *a.s., where* $\sigma^2 = \max\left(\sigma_1^2, \sigma_2^2\right)$.

Proof. Assign $\rho \in \left(2, 1 + \frac{2\mu}{\sigma^2}\right)$. By virtue of Burkholder-Davis-Gundy inequality and (3.7)

$$\begin{aligned} E\sup_{0\leq s\leq t}\left|\int_0^s S(u)\, dB_1(u)\right|^\rho \leq & C_\rho E\sup_{0\leq s\leq t}\left|\int_0^t S^2(u)\, du\right|^{\frac{\rho}{2}} \leq C_\rho t^{\frac{\rho}{2}} E\left[\sup_{0\leq u\leq t} S^2(u)\right]^{\frac{\rho}{2}} \\ = & C_\rho t^{\frac{\rho}{2}} E\left[\sup_{0\leq u\leq t} S^\rho(u)\right] \leq C_\rho t^{\frac{\rho}{2}} H. \end{aligned}$$

Let $\varkappa$ be an arbitrary positive constant. Using Doob's martingle inequality [26], we obtain

$$\begin{aligned} & P\left\{\sup_{n\vartheta\leq t\leq(n+1)\vartheta}\left|\int_0^s S(u)\, dB_1(u)\right|^\rho > (n\vartheta)^{1+\frac{\rho}{2}+\varkappa}\right\} \\ \leq & \frac{E\left(\left|\int_0^{(n+1)\vartheta} S(u)\, dB_1(u)\right|^\rho\right)}{(n\vartheta)^{1+\frac{\rho}{2}+\varkappa}} \leq \frac{E\sup_{0\leq s\leq(n+1)\vartheta}\left|\int_0^s S(u)\, dB_1(u)\right|^\rho}{(n\vartheta)^{1+\frac{\rho}{2}+\varkappa}} \\ \leq & \frac{C_\rho\left[(n+1)\vartheta\right]^{\frac{\rho}{2}} H}{(n\vartheta)^{1+\frac{\rho}{2}+\varkappa}} \leq \frac{C_\rho\left[2n\vartheta\right]^{\frac{\rho}{2}} H}{(n\vartheta)^{1+\frac{\rho}{2}+\varkappa}} = \frac{C_\rho 2^{\frac{\rho}{2}} H}{(n\vartheta)^{1+\varkappa}}. \end{aligned}$$

Then it follows from the Borel-Cantelli lemma that for almost all $\omega \in \Omega$ there exists a positive integer $n_1 = n_1(\omega)$ such that

$$\sup_{n\vartheta\leq t\leq(n+1)\vartheta}\left|\int_0^s S(u)\, dB_1(u)\right|^\rho < (n\vartheta)^{1+\frac{\rho}{2}+\varkappa}, \text{ whenever } n > n_1(\omega).$$

Therefore

$$\frac{\ln\left|\int_0^s S(u)\, dB_1(u)\right|^\rho}{\ln t} \leq \frac{\left(1+\frac{\rho}{2}+\varkappa\right)\ln(n\vartheta)}{\ln(n\vartheta)} = 1 + \frac{\rho}{2} + \varkappa,$$

which leads to

$$\lim_{t\to\infty}\sup\frac{\ln\left|\int_0^s S(u)\, dB_1(u)\right|^\rho}{\ln t} \leq 1 + \frac{\rho}{2} + \varkappa.$$

Letting $\varkappa \to 0$, we have

$$\lim_{t\to\infty}\sup\frac{\ln\left|\int_0^s S(u)\, dB_1(u)\right|}{\ln t} \leq \frac{1+\frac{\rho}{2}}{\rho} = \frac{1}{2} + \frac{1}{\rho},$$

which means that for arbitrary $\upsilon \in \left(0, \frac{1}{2} - \frac{1}{\rho}\right)$, there exist a random time $T_1 = T_1(\omega) > 0$ and a set Ω_2 such that $P(\Omega_2) \geq 1 - \upsilon$ and for $t > T_1(\omega)$, $\omega \in \Omega_2$,

$$\ln\left|\int_0^s S(u)\, dB_1(u)\right| \leq \left(\frac{1}{2} + \frac{1}{\rho} + \upsilon\right)\ln t.$$

Hence,

$$\lim_{t\to\infty}\sup\frac{\left|\int_0^s S(u)\,dB_1(u)\right|}{t}\le\lim_{t\to\infty}\sup\frac{t^{\frac{1}{2}+\frac{1}{\rho}+\upsilon}}{t}=0.$$

which leads to

$$\lim_{t\to\infty}\frac{\left|\int_0^s S(u)\,dB_1(u)\right|}{t}=0,\ \text{a.s.},$$

that is

$$\lim_{t\to\infty}\frac{\int_0^s S(u)\,dB_1(u)}{t}=0,\ \text{a.s.}$$

Similarly, we have $\lim_{t\to\infty}\frac{\int_0^s I(u)dB_1(u)}{t}=0$ a.s. This completes the proof of Lemma 3.2. □

Denote $\hat{R}_0=\frac{\beta A}{\mu(\mu+\alpha_2+\delta+\gamma)}-\frac{\sigma_2^2}{2(\mu+\alpha_2+\delta+\gamma)}$.

Theorem 3.2. *Assume that $\mu>\frac{\sigma^2}{2}$. Let $(S(t),I(t))$ be any solution of model (1.5) with initial $S(0)>0$ and $I(\varpi)\ge 0$ for all $\varpi\in[-\tau,0)$ with $I(0)>0$. If $\hat{R}_0<1$, then $\lim_{t\to\infty}\frac{\ln I(t)}{t}\le(\mu+\alpha_2+\delta+\gamma)\left(\hat{R}_0-1\right)$, $\lim_{t\to\infty}\langle S(t)\rangle=\frac{A}{\mu}$ a.s., where $\sigma^2=\max\left(\sigma_1^2,\sigma_2^2\right)$. That is, the disease dies out with probability one.*

Proof. From (1.5), we get

$$\begin{aligned}
&\frac{S(t)-S(0)}{t}+\frac{I(t)-I(0)}{t}+\frac{1}{t}\delta e^{-(\mu+\alpha_3)\tau}\left(\int_{t-\tau}^t I(r)\,dr-\int_{-\tau}^0 I(r)\,dr\right)\\
=&A-\mu\langle S(t)\rangle-(\mu+\alpha_2+\delta)\langle I(t)\rangle+\frac{1}{t}\delta e^{-(\mu+\alpha_3)\tau}\int_0^t I(r-\tau)\,dr\\
&+\frac{1}{t}\left(\int_{t-\tau}^t I(r)\,dr-\int_{-\tau}^0 I(r)\,dr\right)+\frac{\sigma_1^2}{t}\int_0^t S(u)\,dB_1(u)+\frac{\sigma_2^2}{t}\int_0^t I(u)\,dB_2(u)\\
=&A-\mu\langle S(t)\rangle-\left[\mu+\alpha_2+\delta\left(1-e^{-(\mu+\alpha_3)\tau}\right)\right]\langle I(t)\rangle\\
&+\frac{\sigma_1^2}{t}\int_0^t S(u)\,dB_1(u)+\frac{\sigma_2^2}{t}\int_0^t I(u)\,dB_2(u).
\end{aligned}$$

That is,

$$\langle S(t)\rangle=\frac{A}{\mu}-\frac{1}{\mu}\left[\mu+\alpha_2+\delta\left(1-e^{-(\mu+\alpha_3)\tau}\right)\right]\langle I(t)\rangle+h(t),\tag{3.10}$$

where $h(t)=\frac{1}{\mu}\begin{pmatrix}\frac{\sigma_1^2}{t}\int_0^t S(u)\,dB_1(u)+\frac{\sigma_2^2}{t}\int_0^t I(u)\,dB_2(u)-\frac{S(t)-S(0)}{t}\\-\frac{I(t)-I(0)}{t}-\frac{1}{t}\delta e^{-(\mu+\alpha_3)\tau}\left(\int_{t-\tau}^t I(r)\,dr-\int_{-\tau}^0 I(r)\,dr\right)\end{pmatrix}$.

According to Lemma 3.1, 3.2 and the law of large number,

$$\lim_{t\to\infty}h(t)=0\ \text{a.s.}\tag{3.11}$$

Applying Itô′s formula, we have

$$d\ln I(t)=\left(\beta S-(\mu+\alpha_2+\delta+\gamma)-\frac{1}{2}\sigma_2^2\right)dt+\sigma_2 dB_2(t),$$

which yields

$$\frac{\ln I(t)-\ln I(0)}{t}=\beta\langle S(t)\rangle-(\mu+\alpha_2+\delta+\gamma)-\frac{1}{2}\sigma_2^2+\sigma_2\frac{B_2(t)}{t}.$$

Together with (3.10), we have

$$\begin{aligned}\frac{\ln I(t)}{t}=&\frac{\beta A}{\mu}-(\mu+\alpha_2+\delta+\gamma)-\frac{1}{2}\sigma_2^2\\&-\beta\frac{1}{\mu}\left[\mu+\alpha_2+\delta\left(1-e^{-(\mu+\alpha_3)\tau}\right)\right]\langle I(t)\rangle\\&+\beta h(t)+\sigma_2\frac{B_2(t)}{t}+\frac{\ln I(0)}{t}\\=&(\mu+\alpha_2+\delta+\gamma)\left(\hat{R}_0-1\right)\\&-\beta\frac{1}{\mu}\left[\mu+\alpha_2+\delta\left(1-e^{-(\mu+\alpha_3)\tau}\right)\right]\langle I(t)\rangle\\&+\beta h(t)+\sigma_2\frac{B_2(t)}{t}+\frac{\ln I(0)}{t}\\\leq&(\mu+\alpha_2+\delta+\gamma)\left(\hat{R}_0-1\right)\\&+\beta h(t)+\sigma_2\frac{B_2(t)}{t}+\frac{\ln I(0)}{t}.\end{aligned}\tag{3.12}$$

By the law of large number, we have $\lim\limits_{t\to\infty}\frac{B_2(t)}{t}=0$ a.s. which together with (3.11) leads to

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t}\leq(\mu+\alpha_2+\delta+\gamma)\left(\hat{R}_0-1\right)<0 \text{ a.s.}$$

This implies

$$\lim_{t\to\infty}I(t)=0 \text{ a.s.}\tag{3.13}$$

By (3.10) and (3.13), we have

$$\lim_{t\to\infty}\langle S(t)\rangle=0 \text{ a.s.}$$

This finishes the proof the Theorem 3.2.

□

Next, we perform numerical simulation to support our results.

Example 3.1. In system (1.5), choose $A=0.25$, $\beta=0.13$, $\mu=0.1$, $\alpha_2=0.1$, $\alpha_3=0.001$, $\gamma=0.001$, $\delta=0.1$, $\tau=1$, $\sigma_1=0.01$, $\sigma_2=0.2$. These give $\hat{R}_0=0.9468$ and $R_0=1.0797$. Then, from Theorem 3.2 we have $\lim\limits_{t\to\infty}\langle S(t)\rangle=2.5$ and $\lim\limits_{t\to\infty}I(t)=0$ a.s., namely, the disease dies out with probability one. Fig. 5 confirms these.

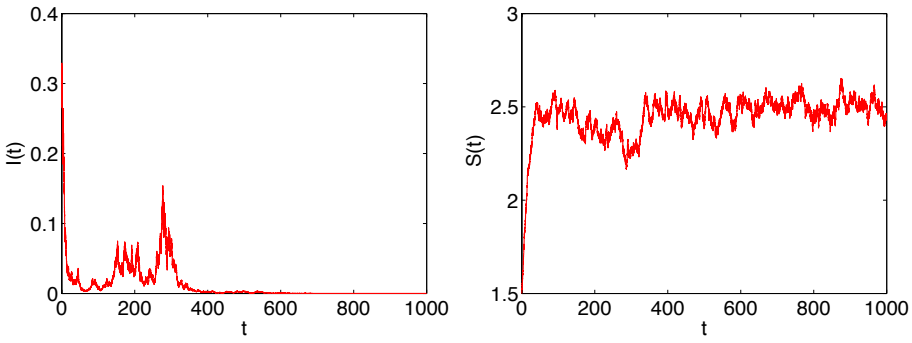

Figure 5. $A = 0.25$, $\beta = 0.13$, $\mu = 0.1$, $\alpha_2 = 0.1$, $\alpha_3 = 0.001$, $\gamma = 0.001$, $\delta = 0.1$, $\tau = 1$, $\sigma_1 = 0.01$, $\sigma_2 = 0.2$, $\hat{R}_0 = 0.9468$ and $R_0 = 1.0797$.

3.3. Persistence

In this section, we study the persistence of the model (1.5).

Theorem 3.3. *Suppose that* $\mu > \frac{\sigma^2}{2}$. *Let* $(S(t), I(t))$ *be any solution of model* (1.5) *with initial value* $S(0) > 0$ *and* $I(\varpi) \geq 0$ *for all* $\varpi \in [-\tau, 0)$ *with* $I(0) > 0$. *If* $\hat{R}_0 > 1$, *then* $\lim_{t\to\infty} \langle S(t)\rangle = \frac{A}{\mu} - \frac{(\mu+\alpha_2+\delta+\gamma)\left(\hat{R}_0-1\right)}{\beta}$, $\lim_{t\to\infty} \langle I(t)\rangle = \frac{(\mu+\alpha_2+\delta+\gamma)\left(\hat{R}_0-1\right)}{\beta\left(\frac{(\mu+\alpha_3)\delta}{\mu(\mu+\alpha_3+\varepsilon)}+\frac{\mu+\alpha_2}{\mu}\right)}$ *a.s.*

Proof. From (3.12), we have

$$\frac{\ln I(t)}{t} = (\mu+\alpha_2+\delta+\gamma)\left(\hat{R}_0-1\right) - \frac{\beta}{\mu}\left[\mu+\alpha_2+\delta\left(1-e^{-(\mu+\alpha_3)\tau}\right)\right]\langle I(t)\rangle + \beta h(t) + \sigma_2\frac{B_2(t)}{t} + \frac{\ln I(0)}{t}.$$

By the law of large number, Lemma 3.1 and (3.11), we have

$$\lim_{t\to\infty}\langle I(t)\rangle = \frac{\mu(\mu+\alpha_2+\delta+\gamma)\left(\hat{R}_0-1\right)}{\beta\left(\mu+\alpha_2+\delta\left(1-e^{-(\mu+\alpha_3)\tau}\right)\right)}. \tag{3.14}$$

Using (3.10) and (3.11), we have

$$\langle S(t)\rangle = \frac{A}{\mu} - \frac{1}{\mu}\left[\mu+\alpha_2+\delta\left(1-e^{-(\mu+\alpha_3)\tau}\right)\right]\langle I(t)\rangle + h(t),$$

$$\lim_{t\to\infty}\langle S(t)\rangle = \frac{A}{\mu} - \frac{1}{\mu}\left[\mu+\alpha_2+\delta\left(1-e^{-(\mu+\alpha_3)\tau}\right)\right]\lim_{t\to\infty}\langle I(t)\rangle.$$

Consequently,

$$\lim_{t\to\infty}\langle S(t)\rangle = \frac{A}{\mu} - \frac{(\mu+\alpha_2+\delta+\gamma)\left(\hat{R}_0-1\right)}{\beta}.$$

This completes the proof of Theorem 3.3. □

Now, we perform numerical simulation to support our results.

Example 3.2. In system (1.5), we choose the same parameters in Example 3.1 except $\sigma_2 = 0.1$ such that $\hat{R}_0 = 1.0465 > 1$. Obviously, $R_0 = 1.0797$ is the same as in Example 3.1. From Theorem 3.3 we have $\lim\limits_{t\to\infty} \langle S(t)\rangle = 2.3923$ and $\lim\limits_{t\to\infty} \langle I(t)\rangle = 0.0538$ a.s., that is, the disease is persistent. Fig. 6 confirms these.

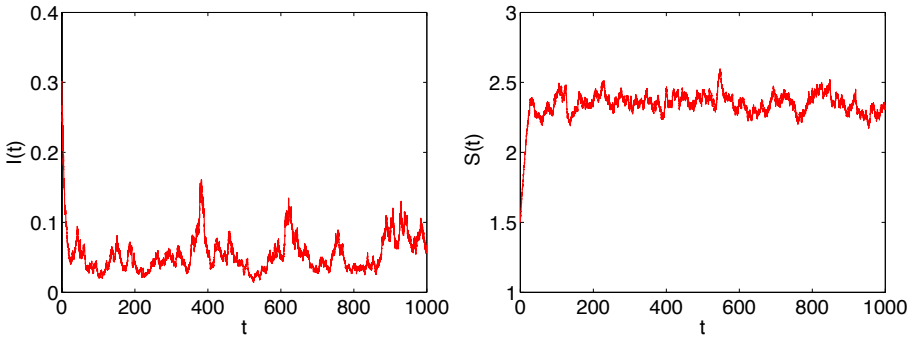

Figure 6. $A = 0.25$, $\beta = 0.13$, $\mu = 0.1$, $\alpha_2 = 0.1$, $\alpha_3 = 0.001$, $\gamma = 0.001$, $\delta = 0.1$, $\tau = 1$, $\sigma_1 = 0.01$, $\sigma_2 = 0.1$, $\hat{R}_0 = 1.0465$ and $R_0 = 1.0797$.

4. Discussion

In this paper, we formulated a deterministic and stochastic delayed SIQS model. For the deterministic system (1.2), we presented the basic reproduction number $R_0 = \frac{\beta A}{\mu(\gamma+\delta+\mu+\alpha_2)}$. In the case $R_0 < 1$ we have showed that the disease-free equilibrium $P_0\left(\frac{A}{\mu}, 0, 0\right)$ is globally asymptotically stable for any time delay, while in case $R_0 > 1$ it has been proved that the endemic equilibrium is existent, and the disease-free equilibrium becomes unstable. Using Lyapunov functional technique, we have proved that under certain restrictions on the parameter values and the delay time, the endemic equilibrium is globally asymptotically (see Theorem 2.4). To further investigate system (1.2), we resorted to numerical simulations. From Fig. 1, we know that when $R_0 > 1$ and the quarantine period τ is small enough, the endemic equilibrium of system (1.2) is stable. Then, let the quarantine period τ increase, the solutions of system (1.2) have a periodic solution with small amplitude oscillations near the endemic equilibrium P_1 (see Fig. 2). Further, let the quarantine period τ increase, the amplitude of these oscillations increase correspondingly (see Fig. 3). As the quarantine period τ increases to large enough, system (1.2) returns the stable steady-state (see Fig. 4). This shows the dependence of a long term dynamics of solutions of system (1.2) on the time τ. This shows the dependence of a long term dynamics of solutions of system (1.2) on the quarantine period τ. It is interesting to theoretically study the existence of a Hopf bifurcation at the endemic equilibrium of system (1.2). We leave this for further work.

For the stochastic system (1.5), We show that the system has a unique global positive solution and calculate its corresponding sharp threshold $\hat{R}_0 = R_0 - \frac{\sigma_2^2}{2(\mu+\alpha_2+\delta+\gamma)}$ which can be used to govern the stochastic dynamics of the model (1.5) as follows:

(1) If $\hat{R}_0 < 1, \mu > \frac{1}{2}\max\left(\sigma_1^2, \sigma_2^2\right)$, then $\lim\limits_{t\to\infty} \frac{\ln I(t)}{t} \leq (\mu + \alpha_2 + \delta + \gamma)\left(\hat{R}_0 - 1\right)$, $\lim\limits_{t\to\infty} \langle S(t)\rangle = \frac{A}{\mu}$, a.s. Namely, the disease dies out (see Fig. 5).

(2) If $\hat{R}_0 > 1$, $\mu > \frac{1}{2}\max\left(\sigma_1^2, \sigma_2^2\right)$, then $\lim\limits_{t\to\infty} \langle S(t)\rangle = \frac{A}{\mu} - \frac{(\mu+\alpha_2+\delta+\gamma)\left(\hat{R}_0-1\right)}{\beta}$, $\lim\limits_{t\to\infty} \langle I(t)\rangle = \frac{(\mu+\alpha_2+\delta+\gamma)\left(\hat{R}_0-1\right)}{\beta\left(\frac{(\mu+\alpha_3)\delta}{\mu(\mu+\alpha_3+\varepsilon)}+\frac{\mu+\alpha_2}{\mu}\right)}$, a.s. That is, the disease will persists (see Fig. 6).

From the above results, we can find that noise can suppress the disease outbreak. $\hat{R}_0 = R_0 - \frac{\sigma_2^2}{2(\mu+\alpha_2+\delta+\gamma)} < R_0$ implies that noise decrease the reproduction number. When $\hat{R}_0 < 1 < R_0$, the stochastic model (1.5) has disease extinction with probability one while the corresponding deterministic model (1.2) has endemic.

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