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THE REVISED GENERALIZED TIKHONOV METHOD FOR THE BACKWARD TIME-FRACTIONAL DIFFUSION EQUATION*

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Abstract In this paper, we solve the backward problem for a time-fractional diffusion equation with variable coefficients in a bounded domain by using the revised generalized Tikhonov regularization method. Convergence estimates under an *a-priori* and *a-posteriori* regularization parameter choice rules are given. Numerical example shows that the proposed method is effective and stable.

Keywords Inverse problem, fractional diffusion equation, Tikhonov regularization, convergence analysis.

MSC(2010) 35R30, 35R11, 65M32.

1. Introduction

The fractional diffusion equation is a generalization of the classical diffusion equation which models anomalous diffusive phenomena. The forward problems of fractional diffusion equations have been studied extensively, one can consult books [3,6]. In recent years, inverse problems for time-fractional diffusion equation have become very active, interdisciplinary research area. The term inverse problem refers to the problem of determining unknown quantities based on observations of their effects. This is in contrast to the corresponding direct problem, the solution of which involves finding effects based on a complete description of their physical parameters. Inverse problems have found wide application in science and engineering, industry, medicine, finance as well as in life and earth sciences.

In this paper, we consider the backward problem for the fractional diffusion

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equation of the form

$$\left.\begin{aligned} {}_0\partial_t^\alpha u(x,t) - (Lu)(x,t) = 0, &\qquad x \in \Omega, \quad t \in (0,T),\\ u(x,t) = 0, &\qquad x \in \partial\Omega, \quad t \in (0,T),\\ u(x,T) = g(x), &\qquad x \in \bar{\Omega}, \end{aligned}\right\} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with sufficient smooth boundary $\partial\Omega$, ${}_0\partial_t^\alpha u$ is the left-sided Caputo fractional derivative of order $\alpha (0 < \alpha < 1)$ defined by

$${}_0\partial_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^\alpha} \frac{\partial u(x,s)}{\partial s} ds, 0 < \alpha < 1, \tag{1.2}$$

and $-L$ is a symmetric uniformly elliptic operator defined on $D(-L) = H^2(\Omega) \cap H_0^1(\Omega)$ by

$$Lu(x) = \sum_{i=1}^d \frac{\partial}{\partial x_i} \left(\sum_{j=1}^d a_{ij}(x) \frac{\partial}{\partial x_j} u(x) \right) + c(x)u(x), x \in \Omega, \tag{1.3}$$

the coefficients satisfy $a_{ij} = a_{ji} \in C^1(\bar{\Omega})$, $\sum_{i,j=1}^d a_{ij}\xi_i\xi_j \geq \theta \sum_{i=1}^d |\xi_i|^2 (\theta > 0)$, $c(x) \leq 0$, $c(x) \in C(\bar{\Omega})$.

The backward problem is to find $u(x,0) := f(x) \in L^2(\Omega)$ from a final data $g(x)$. Since the data $g(x)$ is measured, there must be measurement errors and we assume the measured data function $g^\delta(x) \in L^2(\Omega)$ which satisfies

$$\left\| g - g^\delta \right\| \leq \delta \tag{1.4}$$

where $\|\cdot\|$ refers to the L^2 norm and the constant $\delta > 0$ represents noise level.

According to the Hadamard requirements (existence, uniqueness and stability of the solution), the inverse problem is ill-posed mathematically [4]. The generalized Tikhonov regularization [7], the revised generalized Tikhonov regularization [15] have been proposed for solving the inverse problems for usual partial differential equations. Sakamoto and Yamamoto [9] derived the regularity and qualitative properties of solution to fractional diffusion-wave equation and discussed some inverse problems. Cheng et al. [1] obtained the uniqueness in determining diffusion coefficient on the basis of Gel'fand-Levitan theory. Wang et al. [13] used Tikhonov regularization and a simplified Tikhonov regularization method to solve the inverse source problem for fractional diffusion equation. A number of techniques including Tikhonov regularization method [11], regularization by projection method [8],quasi-reversibility method [5], quasi-boundary value method [12], modified quasi-boundary value method [14] and generalized Tikhonov method [16] have been applied for backward time-fractional diffusion problem.

However it should be emphasized that the revised generalized Tikhonov regularization method is mainly concerned with inverse source problems for the heat equation and there have been no attempts made for studying the backward time-fractional diffusion problem. In this article, we propose a revised generalized Tikhonov regularization method for backward time-fractional diffusion problem and establish a convergence estimate.

The main novelty of this paper lies in the following aspects: In Section 2, we simply recall some preliminaries. In Section 3, we propose the revised generalized Tikhonov regularization method. In Section 4, we present convergence estimates under an *a-priori* and *a-posteriori* regularization parameter choice rules. Finally numerical example and their simulation are exploited to demonstrate the usefulness and effectiveness of the method.

2. Preliminaries

In this section, we recall basic definitions and lemmas.

Definition 2.1 ([6]). The Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, z \in \mathbb{C},$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are arbitrary constants.

Lemma 2.1 ([11]). *Let* $\lambda > 0$; *then we have*

$${}_0\partial_t^{\alpha} E_{\alpha,1}(-\lambda t^{\alpha}) = -\lambda E_{\alpha,1}(-\lambda t^{\alpha}), t > 0, 0 < \alpha < 1.$$

Lemma 2.2 ([11]). *For any* λ_n *satisfying* $\lambda_n \geq \lambda_1 > 0$, *there exist positive constants* $\underline{C}, \overline{C}$ *depending on* α, T, λ_1 *such that*

$$\frac{\underline{C}}{\lambda_n} \leq E_{\alpha,1}(-\lambda_n T^{\alpha}) \leq \frac{\overline{C}}{\lambda_n}.$$

Lemma 2.3. *For constants* $p > 0, \mu > 0, s \geq \lambda_1 > 0$, *we have*

$$F(s) = \frac{s}{\underline{C}(1 + \mu s^{p+1})} \leq C_1(\underline{C}, p)\mu^{-\frac{1}{p+1}}, \tag{2.1}$$

$$G(s) = \frac{\mu s^{\frac{p}{2}+1}}{1 + \mu s^{p+1}} \leq C_2(p)\mu^{\frac{p}{2p+2}}, \tag{2.2}$$

$$H(s) = \frac{\overline{C}\mu s^{\frac{p}{2}}}{1 + \mu s^{p+1}} \leq C_3(\overline{C}, p)\mu^{\frac{p+2}{2p+2}}. \tag{2.3}$$

Proof. We know that, $\lim_{s\to 0} F(s) = \lim_{s\to\infty} F(s) = 0$, thus $F(s) \leq \sup_{s>0} F(s) \leq F(s_0)$, where $s_0 > 0$ such that $F'(s_0) = 0$. It is easy to prove that $s_0 = \left(\frac{1}{p\mu}\right)^{1/p+1} > 0$, then we have

$$F(s) \leq F(s_0) = \frac{p}{\underline{C}(p+1)} p^{-\frac{1}{p+1}} \mu^{-\frac{1}{p+1}} = C_1(\underline{C}, p)\mu^{-\frac{1}{p+1}}.$$

Similarly, we can prove (2.2) and (2.3). □

3. The revised generalized Tikhonov regularization method

Denote the eigenvalues of the operator $-L$ as λ_n which satisfy

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \cdots, \lim_{n\to\infty} \lambda_n = +\infty,$$

and the corresponding eigenfunctions as $X_n(x) \in H^2(\Omega) \cap H_0^1(\Omega)$ form an orthonormal basis in $L^2(\Omega)$.

Define

$$D((-L)^\gamma) = \{\psi \in L^2(\Omega); \sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(\psi, X_n)|^2 < \infty\},$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, then $D((-L)^\gamma)$ is a Hilbert space with the norm

$$\|\psi\|_{D((-L)^\gamma)} = \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(\psi, X_n)|^2 \right)^{\frac{1}{2}}.$$

From Theorem 4.1 in [9], we know there exists a unique weak solution $u \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ to problem (1.1) if $g \in H^2(\Omega) \cap H_0^1(\Omega)$. By the separation of variables and Lemma 2.1, we know that the formal solution for (1.1) can be written as

$$u(x,t) = \sum_{n=1}^{\infty} (u(x,0), X_n) E_{\alpha,1}(-\lambda_n t^\alpha) X_n(x).$$

Denote $f(x) = u(x,0)$, $f_n = (f, X_n)$, $g_n = (g, X_n)$; then letting $t = T$, we have

$$u(x,T) = g(x) = \sum_{n=1}^{\infty} f_n E_{\alpha,1}(-\lambda_n T^\alpha) X_n(x), \tag{3.1}$$

and

$$g_n(x) = f_n E_{\alpha,1}(-\lambda_n T^\alpha).$$

From this equation we get

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{E_{\alpha,1}(-\lambda_n T^\alpha)} g_n X_n(x). \tag{3.2}$$

From (3.1), the problem of finding $f(x)$ can be formulated as an integral equation:

$$(Kf)(x) = \int_\Omega k(x,\xi) f(\xi) d\xi = g(x), \tag{3.3}$$

where K is a Fredholm integral operator with the first kind and k is a kernel function defined by

$$k(x,\xi) = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n T^\alpha) X_n(x) X_n(\xi).$$

From $k(x,\xi) = k(\xi,x)$, we know that K is a linear self-adjoint. From Theorem 2.1 in [9], if $f \in L^2(\Omega)$, we have $g \in H^2(\Omega)$. Because $H^2(\Omega)$ is compactly imbedded into $L^2(\Omega)$, so $K : L^2(\Omega) \to L^2(\Omega)$ is compact operator and the problem (3.3) is ill-posed. For stable solution of problem (3.3) with noisy data g^δ, [11] used $\|f\|^2_{L^2(\Omega)}$ as the regularization term. Instead of the regularization term, [16] used

$\|f\|^2_{D((-L)^{\frac{p}{2}})}$, $(p > 0)$ to construct a generalized Tikhonov regularization method which minimizes the quantity

$$\left\|Kf - g^{\delta}\right\|^2_{L^2(\Omega)} + \mu \|f\|^2_{D((-L)^{\frac{p}{2}})}, p > 0. \tag{3.4}$$

Let f_{μ}^{δ} be a solution of the problem (3.4) which satisfies the following normal equation

$$K^*Kf_{\mu}^{\delta} + \mu(-L)^p f_{\mu}^{\delta} = K^* g^{\delta}.$$

By singular value decomposition for compact self-adjoint operator, we have

$$f_{\mu}^{\delta}(x) = \sum_{n=1}^{\infty} \frac{E_{\alpha,1}(-\lambda_n T^{\alpha})}{E^2_{\alpha,1}(-\lambda_n T^{\alpha}) + \mu\lambda_n^p} g_n^{\delta} X_n(x).$$

Therefore

$$f_{\mu}^{\delta}(x) = \sum_{n=1}^{\infty} \frac{\frac{1}{E_{\alpha,1}(-\lambda_n T^{\alpha})}}{1 + \mu\lambda_n^p \frac{1}{E^2_{\alpha,1}(-\lambda_n T^{\alpha})}} g_n^{\delta} X_n(x). \tag{3.5}$$

Comparing (3.2) and (3.5), we note that the generalized Tikhonov regularization procedure consists in replacing the unkown $g(x)$ with an appropriately filtered noised data $g^{\delta}(x)$. The filter in (3.5) attenuates the coefficient g_n^{δ} of $g^{\delta}(x)$ in a manner consistent with the goal of minimizing quantity (3.4). By this idea, we can use a much better filter $\frac{1}{1+\mu\lambda_n^{p+1}}$ to replace the filter $\frac{1}{1+\mu\lambda_n^p \frac{1}{E^2_{\alpha,1}(-\lambda_n T^{\alpha})}}$ and propose a revised generalized Tikhonov regularized solution $f^{\delta}_{\mu,REV}(x)$ for noisy data g_{δ} as

$$f^{\delta}_{\mu,REV}(x) = \sum_{n=1}^{\infty} \frac{1}{E_{\alpha,1}(-\lambda_n T^{\alpha})(1 + \mu\lambda_n^{p+1})} g_n^{\delta} X_n(x). \tag{3.6}$$

4. Convergence estimates under *a-priori* and *a-posteriori* parameter choice rules

To obtain the estimates, we usually need some *a-priori* bounded condition:

$$\|f\|_{D((-L)^{\frac{p}{2}})} \le E, p > 0, \tag{4.1}$$

where $E > 0$ is a constant. From (3.2), we obtain the following conditional stability.

Theorem 4.1 ([11]). *If the a-priori bounded condition* (4.1) *holds, then*

$$\|f\| \le C_4 E^{\frac{2}{p+2}} \|g\|^{\frac{p}{p+2}}, p > 0, \tag{4.2}$$

where C_4 is a constant depending on α, T, p, λ_1.

4.1. An *a-priori* parameter choice rule

Theorem 4.2. *Suppose that the a-priori condition* (4.1) *and the noise assumption* (1.4) *hold. If we choose the regularization parameter $\mu = \left(\frac{\delta}{E}\right)^{\frac{2p+2}{p+2}}$, we have the convergence estimate*

$$\left\|f^{\delta}_{\mu,REV}(x) - f(x)\right\| \le \bar{D} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, \tag{4.3}$$

where $\bar{D} = C_1(\underline{C}, p) + C_2(p)$.

Proof. We know

$$\left\|f^{\delta}_{\mu,REV}(x)-f(x)\right\| \le \left\|f^{\delta}_{\mu,REV}(x)-f_{\mu,REV}(x)\right\| + \|f_{\mu,REV}(x)-f(x)\| . \tag{4.4}$$

From (1.4) and Lemma 2.2, we have

$$\begin{aligned}\left\|f^{\delta}_{\mu,REV}(x)-f_{\mu,REV}(x)\right\|^2 &= \sum_{n=1}^{\infty}\left(\frac{1}{E_{\alpha,1}(-\lambda_n T^{\alpha})(1+\mu\lambda_n^{p+1})}\right)^2 (g_n^{\delta}-g_n)^2 \\ &\le \sum_{n=1}^{\infty}\left(\frac{1}{\frac{\underline{C}}{\lambda_n}(1+\mu\lambda_n^{p+1})}\right)^2 (g_n^{\delta}-g_n)^2 \\ &\le \delta^2(\sup_n A(n))^2.\end{aligned}$$

Using Lemma 2.3, we get

$$A(n)=\frac{\lambda_n}{\underline{C}(1+\mu\lambda_n^{p+1})} \le C_1(\underline{C},p)\mu^{-\frac{1}{p+1}};$$

then

$$\left\|f^{\delta}_{\mu,REV}(x)-f_{\mu,REV}(x)\right\| \le \delta C_1(\underline{C},p)\mu^{-\frac{1}{p+1}}. \tag{4.5}$$

On the other hand

$$\begin{aligned}\|f_{\mu,REV}(x)-f(x)\|^2 &= \sum_{n=1}^{\infty}\left(\frac{g_n}{E_{\alpha,1}(-\lambda_n T^{\alpha})}\right)^2 \lambda_n^p \left(\frac{-\mu\lambda_n^{p+1}}{(1+\mu\lambda_n^{p+1})}\right)^2 \frac{1}{\lambda_n^p} \\ &= \sum_{n=1}^{\infty}\left(\frac{-\mu\lambda_n^{\frac{p}{2}+1}}{1+\mu\lambda_n^{p+1}}\right)^2 \lambda_n^p f_n^2 \\ &\le E^2(\sup_n B(n))^2.\end{aligned}$$

From Lemma 2.3, we get

$$B(n)=\frac{\mu\lambda_n^{\frac{p}{2}+1}}{1+\mu\lambda_n^{p+1}} \le C_2(p)\mu^{\frac{p}{2p+2}},$$

hence

$$\|f_{\mu,REV}(x)-f(x)\| \le EC_2(p)\mu^{\frac{p}{2p+2}}. \tag{4.6}$$

From (4.4)–(4.6), with $\mu=\left(\frac{\delta}{E}\right)^{\frac{2p+2}{p+2}}$, the estimate (4.3) can be obtained. $\square$

4.2. An *a-posteriori* parameter choice rule

According to the Morozov's discrepancy principle, choose the regularization parameter μ as the solution of the equation

$$\left\|Kf^{\delta}_{\mu,REV}(x)-g^{\delta}\right\| = \tau\delta, \tag{4.7}$$

where $\tau>1$ is a constant.

Lemma 4.1. *Let* $\rho(\mu)=\left\|Kf^{\delta}_{\mu,REV}(x)-g^{\delta}(x)\right\|$. *If* $0<\delta<\left\|g^{\delta}\right\|$, *then the following results hold:*

(a) $\rho(\mu)$ *is a continuous function;*

(b) $\lim_{\mu\to 0}\rho(\mu)=0$*;*

(c) $\lim_{\mu\to+\infty}\rho(\mu)=\left\|g^\delta\right\|$*;*

(d) For $\mu\in(0,+\infty),\rho(\mu)$ *is a strictly increasing function.*

Proof. The above results are straightforward by setting

$$\rho(\mu)=\left(\sum_{n=1}^{\infty}\left(\frac{\mu\lambda_n^{p+1}}{1+\mu\lambda_n^{p+1}}\right)^2(g_n^\delta)^2\right)^{\frac{1}{2}}. \tag{4.8}$$

□

Lemma 4.1 indicates that there exists a unique solution for (4.7) if $0<\tau\delta<\left\|g^\delta\right\|$.

Theorem 4.3. *Suppose that the a-priori condition* (4.1) *and the noise assumption* (1.4) *hold, and there exists* $\tau>1$ *such that* $0<\tau\delta<\left\|g^\delta\right\|$. *The regularization parameter* $\mu>0$ *is chosen by Morozov's discrepancy principle* (4.7). *Then we have the following convergence estimate*

$$\left\|f_{\mu,REV}^\delta(x)-f(x)\right\|\le\tilde{D}E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}, \tag{4.9}$$

where $\tilde{D}=D(\underline{C},\overline{C},p,\tau)+C_4(\tau+1)^{\frac{p}{p+2}}$.

Proof. Similar to (4.4), we have

$$\left\|f_{\mu,REV}^\delta(x)-f(x)\right\|\le\left\|f_{\mu,REV}^\delta(x)-f_{\mu,REV}(x)\right\|+\left\|f_{\mu,REV}(x)-f(x)\right\|. \tag{4.10}$$

From (4.7), there holds

$$\begin{aligned}\tau\delta&=\left\|\sum_{n=1}^{\infty}\frac{\mu\lambda_n^{p+1}}{1+\mu\lambda_n^{p+1}}g_n^\delta X_n(x)\right\|\\&\le\left\|\sum_{n=1}^{\infty}\frac{\mu\lambda_n^{p+1}}{1+\mu\lambda_n^{p+1}}(g_n^\delta-g_n)X_n(x)\right\|+\left\|\sum_{n=1}^{\infty}\frac{\mu\lambda_n^{p+1}}{1+\mu\lambda_n^{p+1}}g_nX_n(x)\right\|\\&\le\delta+J.\end{aligned} \tag{4.11}$$

By using (4.1), we obtain

$$\begin{aligned}J&=\left\|\sum_{n=1}^{\infty}\frac{\mu\lambda_n^{\frac{p}{2}+1}E_{\alpha,1}(-\lambda_nT^\alpha)}{1+\mu\lambda_n^{p+1}}\frac{g_n}{E_{\alpha,1}(-\lambda_nT^\alpha)}\lambda_n^{\frac{p}{2}}X_n(x)\right\|\\&\le E\sup_n C(n).\end{aligned} \tag{4.12}$$

From Lemma 2.3, we have

$$\begin{aligned}C(n)&=\frac{\mu\lambda_n^{\frac{p}{2}+1}E_{\alpha,1}(-\lambda_nT^\alpha)}{1+\mu\lambda_n^{p+1}}\\&\le\frac{\mu\lambda_n^{\frac{p}{2}+1}\frac{\overline{C}}{\lambda_n}}{1+\mu\lambda_n^{p+1}}\\&\le C_3(\overline{C},p)\mu^{\frac{p+2}{2p+2}}.\end{aligned} \tag{4.13}$$

Combining (4.11)-(4.13), we get

$$(\tau-1)\delta \le C_3(\overline{C},p)E\mu^{\frac{p+2}{2p+2}}.$$

This yields
$$\frac{1}{\mu} \le \left(\frac{C_3(\overline{C},p)}{(\tau-1)}\right)^{\frac{2p+2}{p+2}} \left(\frac{E}{\delta}\right)^{\frac{2p+2}{p+2}}. \tag{4.14}$$

Using (4.5) and (4.14), we obtain

$$\left\|f^{\delta}_{\mu,REV}(x) - f_{\mu,REV}(x)\right\| \le D(\underline{C},\overline{C},p,\tau)E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}. \tag{4.15}$$

On the other hand

$$\begin{aligned} K(f_{\mu,REV}(x)-f(x)) &= \sum_{n=1}^{\infty}\frac{-\mu\lambda_n^{p+1}}{(1+\mu\lambda_n^{p+1})}g_nX_n(x) \\ &= \sum_{n=1}^{\infty}\frac{-\mu\lambda_n^{p+1}}{(1+\mu\lambda_n^{p+1})}(g_n-g_n^{\delta})X_n(x) \\ &\quad + \sum_{n=1}^{\infty}\frac{-\mu\lambda_n^{p+1}}{(1+\mu\lambda_n^{p+1})}g_n^{\delta}X_n(x). \end{aligned}$$

Using (4.7), we get
$$\|K(f_{\mu,REV}(x)-f(x))\| \le \delta + \tau\delta. \tag{4.16}$$

Applying the *a-priori* bound condition for $f(x)$, we obtain

$$\begin{aligned} \|f_{\mu,REV}(x)-f(x)\|_{D((-L)^{\frac{p}{2}})} &= \left(\sum_{n=1}^{\infty}\left(\frac{g_n}{E_{\alpha,1}(-\lambda_nT^{\alpha})}\frac{-\mu\lambda_n^{p+1}}{(1+\mu\lambda_n^{p+1})}\lambda_n^{\frac{p}{2}}\right)^2\right)^{\frac{1}{2}} \\ &\le \left(\sum_{n=1}^{\infty}\lambda_n^{p}\left(\frac{g_n}{E_{\alpha,1}(-\lambda_nT^{\alpha})}\right)^2\right)^{\frac{1}{2}} \le E. \end{aligned} \tag{4.17}$$

From (4.16), (4.17) and the conditional stability (4.2), we deduce that

$$\|f_{\mu,REV}(x)-f(x)\| \le C_4(\tau+1)^{\frac{p}{p+2}}E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}. \tag{4.18}$$

Combining (4.15) with (4.18), the convergence estimate can be established. □

5. Numerical Example

After obtaining the theoretical results, we propose the numerical schemes for the inverse problem. The regularization parameter plays a major role in the numerical simulation. In fact, the effectiveness of a regularization method depends strongly on the choice of the regularization parameter. As the analytic solution of problem (1.1) is difficult to derive, we construct the final data $g(x)$ by solving the following forward problem

$$\left.\begin{aligned} {}_0\partial_t^{\alpha}u(x,t)-(Lu)(x,t) &= 0, \quad x\in\Omega, \quad t\in(0,T), \\ u(x,t) &= 0, \quad x\in\partial\Omega, \quad t\in(0,T), \\ u(x,0) &= f(x), \quad x\in\bar{\Omega}, \end{aligned}\right\} \tag{5.1}$$

with the given data $f(x)$ by a finite difference scheme [2,10,16]. For simplification, we consider $d=1$. The noisy data is generated by adding a random perturbation, that is, $g^\delta = g+\varepsilon g(2\,\mathrm{rand}(\mathrm{size}(g))-1)$. The corresponding noise level is calculated by $\delta=\varepsilon\,\|g\|$.

In our computations, we choose $\Omega=(0,1)$ and the grid sizes for time and space variables are $\Delta t=\frac{T}{N}$ and $\Delta x=\frac{1}{M}$ respectively. The grid points in the time interval $[0,T]$ are labeled $t_n=n\Delta t, n=0,1,\cdots,N$; the grid points in the space interval $[0,1]$ are $x_i=i\Delta x, i=0,1,2,\cdots,M$, and set $u_i^n=u(x_i,t_n)$.

The time-fractional derivative and the value of Lu are approximated by

$$ {}_0\partial_t^\alpha u(x_i,t_n)\approx\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\left[u_i^n-\sum_{j=1}^{n-1}(w_{n-j-1}-w_{n-j})u_i^j-w_{n-1}u_i^0\right], \tag{5.2}$$

$$ Lu(x_i,t_n)\approx\frac{1}{(\Delta x)^2}\left(a_{i+\frac12}u_{i+1}^n-(a_{i+\frac12}+a_{i-\frac12})u_i^n+a_{i-\frac12}u_{i-1}^n\right)+c(x_i)u_i^n, \tag{5.3}$$

where $i=1,2,\cdots,M-1,\quad n=1,2,\cdots,N$, $w_j=(j+1)^{1-\alpha}-(j)^{1-\alpha}$ and $a_{i+\frac12}=a(x_{i+\frac12})$ with $x_{i+\frac12}=\frac{x_i+x_{i+1}}{2}$.

Applying (5.2) and (5.3) in (5.1), we can get numerical solution to the direct problem. From this, we take $g=u_i^N$ as the exact final data.

In our numerical experiments, we only use the *a-posteriori* regularization parameter choice rule (4.7) for chose the regularization parameter with $\tau=1.1$, we always fix $T=1, M=50, N=100$.

To test the accuracy of our methods, we compute the L^2 error denoted by

$$ e(f,\varepsilon)=||f(x)-f^\delta_{\mu,REV}(x)||, \tag{5.4}$$

and the relative L^2 error denoted by

$$ e_r(f,\varepsilon)=||f(x)-f^\delta_{\mu,REV}(x)||/||f(x)||. \tag{5.5}$$

Example 5.1. Let $a(x)=x^2+1, c(x)=-(x+1)$. Take the initial function $f(x)=x^\alpha(1-x)^\alpha e^{x^\alpha}\sin(7\pi x)$.

Table 1. The numerical results for different α with $\varepsilon=0.01, p=1$.

α	0.05	0.1	0.3	0.7	0.9	0.95
$e(f,\varepsilon)$	0.0490	0.0437	0.0313	0.0148	0.0104	0.00967
$e_r(f,\varepsilon)$	0.0296	0.0301	0.0347	0.0381	0.0395	0.0403

Table 2. The numerical results for different α with $\varepsilon=0.01, p=2$.

α	0.05	0.1	0.3	0.7	0.9	0.95
$e(f,\varepsilon)$	0.0406	0.0385	0.0318	0.0214	0.0150	0.0142
$e_r(f,\varepsilon)$	0.0244	0.0265	0.0353	0.0550	0.0568	0.0594

In Tables 1-2, we show the numerical errors for different α with $p=1,2$ and $\varepsilon=0.01$; it can be seen that the numerical results depend on α and p.

For fixed $\alpha=0.6$, the absolute and relative errors with $p=1,2$, for various ε are shown in Tables 3-4. From Tables 3-4, it can be noted that the computational effect

Table 3. The numerical results for different ε with $\alpha = 0.6, p = 1$.

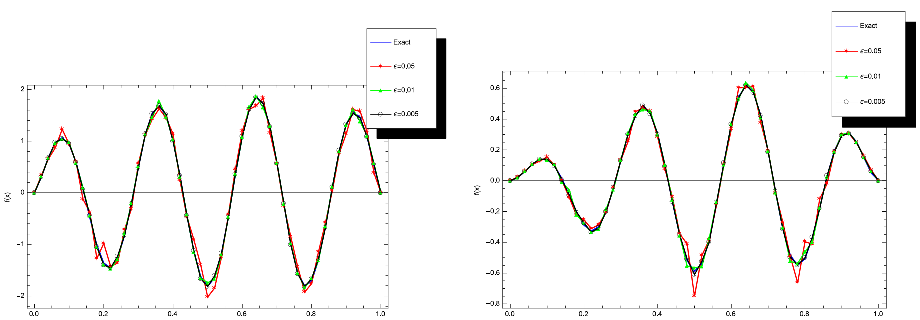
ε	0.0005	0.001	0.002	0.004	0.008	0.016	0.032	0.064
$e(f,\varepsilon)$	0.0024	0.0036	0.0054	0.0077	0.0165	0.0240	0.0370	0.0589
$e_r(f,\varepsilon)$	0.0051	0.0077	0.0115	0.0162	0.0346	0.0505	0.0778	0.1236

Table 4. The numerical results for different ε with $\alpha = 0.6, p = 2$.

ε	0.0005	0.001	0.002	0.004	0.008	0.016	0.032	0.064
$e(f,\varepsilon)$	0.0094	0.0128	0.0143	0.0164	0.0199	0.0236	0.0270	0.0317
$e_r(f,\varepsilon)$	0.0198	0.0269	0.0300	0.0346	0.0419	0.0497	0.0567	0.0666

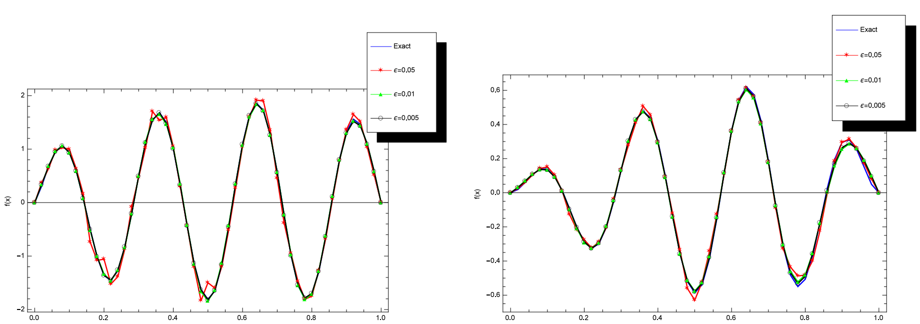
is satisfying and the error is decreasing as ε becomes smaller moreover compared to Tikhonov method [11] for large value of noise level ε we get better computational result when p is increasing.

Figure 1-3 illustrates the exact initial data and the regularized approximations given by the *a-posteriori* parameter choice rule with $p = 1, 2, 3$ and $\varepsilon = 0.005, 0.01, 0.05$ in case of $\alpha = 0.2, 0.8$. We can see that the numerical results are in good agreement with the exact shape.



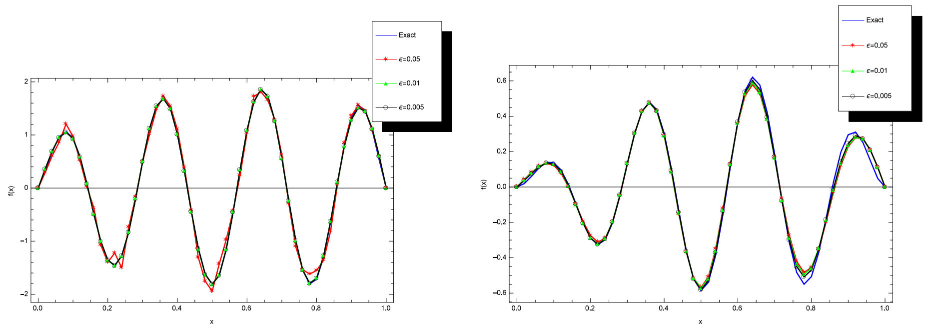
(a) $\alpha = 0.2$ (b) $\alpha = 0.8$

Figure 1. Exact and regularized solutions for p=1.



(a) $\alpha = 0.2$ (b) $\alpha = 0.8$

Figure 2. Exact and regularized solutions for p=2.



(a) $\alpha = 0.2$ (b) $\alpha = 0.8$

Figure 3. Exact and regularized solutions for p=3.

6. Conclusion

In this paper, we solve the backward problem for a time-fractional diffusion equation with variable coefficients by using the revised generalized Tikhonov regularization method. Based on the properties of Mittag-Leffler function and conditional stability we derive convergence estimates under an *a-priori* and *a-posteriori* regularization parameter choice rules. Numerical example shows that the proposed method is effective and stable.

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