Journal of Applied Analysis and Computation
Volume 7, Number 4, November 2017, 1613–1623
Website:http://jaac-online.com/
DOI:10.11948/2017098

ANALYTICAL SOLITARY WAVE SOLUTIONS FOR THE NONLINEAR ANALOGUES OF THE BOUSSINESQ AND SIXTH-ORDER MODIFIED BOUSSINESQ EQUATIONS

Sherif Amirov[1] and Mustafa Anutgan[2,†]

Abstract Using tanh function and polynomial function methods, analytical solitary wave solutions have been found for the nonlinear analogues of Boussinesq and sixth-order modified Boussinesq equations where the nonlinearity is in the time-derivative term.

Keywords Nonlinear sixth-order modified Boussinesq equation, Tanh function method, Polynomial function method.

MSC(2010) 35Q51, 35Q53.

1. Introduction

The Boussinesq-type partial differential equations (PDEs) are used to describe the longitudinal waves on nonlinear elastic rods [17], plasma waves expressed with hydrodynamical equations [4, 11, 13, 14], heat transfer in a porous medium [5], ultrashort pulse propagation [3] and long solitary waves on shallow waters [2, 7, 12, 16, 18, 19] together with their interactions [10]. In addition, nonlinear analogues of the Boussinesq equation arise in the signal propagation along the transmission lines including diode-like components whose capacitance exhibits nonlinear dependance on the displacement voltage [6].

The nonlinear term(s) in the nonlinear analogues of the Boussinesq equation studied so far in the literature was (were) mostly included in the sought function and/or in the derivative of that sought function with respect to the space variables [8, 9, 15]. In our previous work, we proved the existence and uniqueness of the solution of boundary value problem for the nonlinear analogues of the Boussinesq equation where the nonlinearity was included in the time-derivative of the sought function [1]. In the present work, analytical solitary wave solutions for such equations are demonstrated.

The nonlinear analogue of the Boussinesq equation studied in this work has the following general form:

$$\frac{\partial^2 q(u)}{\partial t^2} - u_{xx} - u_{xxtt} - \mu u_{xxxxxx} = 0 \tag{1.1}$$

[†]the corresponding author. Email address: mustafaanutgan@karabuk.edu.tr (M. Anutgan)
[1]Mathematics Department, Science Faculty, Karabuk University, 78050, Karabuk, Turkey
[2]Mechatronics Engineering Department, Technology Faculty, Karabuk University, Karabuk, 78050, Turkey

Here, $q(\xi)$ is a sufficiently differentiable monotonous function. In the present paper, $\mu = 0$ and $\mu = 1$ cases are investigated corresponding to nonlinear analogues of the Boussinesq and sixth-order modified Boussinesq equations, respectively.

2. Formulation of the problem and method of the solution

2.1. Equations to be investigated

In (1.1), the analytical solitary wave solutions are to be shown for $q(\xi)=\xi^3$ and $\mu = 0, 1$ where the equation takes the following forms:

$$(u^3)_{tt} - u_{xx} - u_{xxtt} = 0, \tag{2.1}$$

$$(u^3)_{tt} - u_{xx} - u_{xxtt} - u_{xxxxxx} = 0. \tag{2.2}$$

2.2. Common steps to reach the solutions

Step 1. Consider the following nonlinear PDE with two independent variables x and t:

$$P(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0, \tag{2.3}$$

where P is a polynomial in $u(x,t)$ and its several partial derivatives including any kind of linear and nonlinear terms. In order to obtain the solitary wave solutions, we perform a change of variables as

$$\xi = x - ct \tag{2.4}$$

with c being the speed of the wave. Hence, $u(x,t)$ becomes $u(\xi)$.
Step 2. After the change of variables, the PDE given in (2.3) is transformed to the following ordinary differential equation (ODE):

$$P(u, -cu', u', c^2u'', u'', \ldots) = 0. \tag{2.5}$$

Step 3. (2.5) is integrated as many times as possible and the integration constants are set to zero for simplicity.
Step 4. The homogeneous balance between the highest order derivatives and the highest order nonlinear terms in (2.5) must be taken into account in order to obtain m, which will be explained in Step 5. The degrees of the relevant terms to be equated to each other can be calculated using the general expression:

$$\deg\left[u^q\left(\frac{d^r u}{d\xi^r}\right)^s\right] = mq + s\,(m+r)\,. \tag{2.6}$$

After this step, tanh function and polynomial function methods are separated, which are explained in the following subsections.

2.2.1. Tanh function method

Step 5. Defining a new variable y=tanh(ξ), the solution of (2.5) is sought in the following form:

$$u(\xi) = S(y) = a_0 + \sum_{i=1}^{m} a_i y^i + b_i y^{i-1} \sqrt{\sigma \left(1 + \frac{y^2}{\mu}\right)}, \tag{2.7}$$

where $a_0, a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m$, are coefficients to be determined, while σ and μ are constants.

Introducing this new variable requires the derivatives in (2.5) to be modified as:

$$\begin{aligned}
\frac{d}{d\xi} &= \left(1 - y^2\right) \frac{d}{dy};\\
\frac{d^2}{d\xi^2} &= \left(1 - y^2\right) \left(\left(1 - y^2\right) \frac{d^2}{dy^2} - 2y \frac{d}{dy}\right);\\
\frac{d^3}{d\xi^3} &= \left(1 - y^2\right)^3 \frac{d^3}{dy^3} + 4y^2 \left(1 - y^2\right) \frac{d}{dy} + \left(1 - y^2\right)^2 \left(-6y \frac{d^3}{dy^3} - 2\frac{d}{dy}\right);\\
\frac{d^4}{d\xi^4} &= \left(1 - y^2\right)^4 \frac{d^4}{dy^4} - 4y \left(3y^2 - 1\right) \left(1 - y^2\right) \frac{d}{dy}\\
&\quad + \left(1 - y^2\right)^3 \left(-12y \frac{d^3}{dy^3} - 6\frac{d^2}{dy^2}\right)\\
&\quad + \left(1 - y^2\right)^2 \left(24y^2 \frac{d^2}{dy^2} + 2\left(3y^2 - 1\right) \frac{d^2}{dy^2} + 12y \frac{d}{dy}\right).
\end{aligned} \tag{2.8}$$

Step 6. Using the m value found in Step 4, the function (2.7) is substituted into (2.5) to get a polynomial function in y^i and $y^i \sqrt{\sigma \left(1 + \frac{y^2}{\mu}\right)}$. Since the right-hand-side of this equation is zero, the left-hand-side has to be a zero polynomial. Thus, all the coefficients of this zero polynomial is set equal to zero, which results in an overdetermined system of homogeneous algebraic equations with respect to the constants a_0, a_i, b_i, c, σ and μ.

Step 7. This equation system can be solved in a mathematics software like Mathematica, Matlab, or Maple. We prefer to use Mathematica as it is more practical in symbolic calculations. Then, possible values for the coefficients in the function (2.7) are determined.

Step 8. After the coefficients determined are substituted into the function (2.7) and taking the change of variables in (2.4) into account, the solitary wave solution $u(x, t)$ of (2.3) is found.

2.2.2. Polynomial function method

Step 5. The general solution of (2.5) is sought in the following form:

$$u(x, t) = \sum_{i=0}^{m} a_i \phi^i, \tag{2.9}$$

where the coefficients a_i are unknown constants for now.

Step 6. Here, the function ϕ is the solution of the following ODE:

$$(\phi'(\xi))^2 = \alpha\phi^2(\xi) + \beta\phi^3(\xi) + \gamma\phi^4(\xi), \tag{2.10}$$

where α, β and γ are unknown constants for now.
Step 7. Substituting the function (2.9) into (2.5), a polynomial in powers of ϕ is obtained. Equating the coefficients of this polynomial to zero, an overdetermined system of algebraic equations is found depending on the unknowns a_i, α, β and γ.
Step 8. This system of algebraic equations is solved using Mathematica and the unknowns a_i, α, β and γ are determined.
Step 9. These coefficients are substituted into the function (2.9) and the change of variables in (2.4) is taken into account to find the solitary wave solution $u(x,t)$ of (2.3).

3. Results

3.1. Application to nonlinear Boussinesq equation

The nonlinear Boussinesq equation given in (2.1) takes the following general ODE form after Step 3 of Section 2.2:

$$3c^2\left(u^2 u_\xi\right) - u_\xi - c^2 u_{\xi\xi\xi} = 0. \tag{3.1}$$

Let us determine the value of m as explained in Step 4:

$$2m + m + 1 = m + 3. \tag{3.2}$$

Here, we find m=1.

3.1.1. Application of the tanh function method

Taking the equalities in (2.9) into account and substituting the function (2.7) into (3.1), the following equation is acquired:

$$\begin{aligned} &3c^2\left(1-y^2\right)S(y)^2 S'(y) - c^2\left(\left(1-y^2\right)^3 S^{(3)}(y)\right) + c^2\left(6y\left(1-y^2\right)^2 S''(y)\right) \\ &-c^2\left(2\left(3y^2-1\right)\left(1-y^2\right)S'(y)\right) - \left(1-y^2\right)S'(y) = 0. \end{aligned} \tag{3.3}$$

Since m=1, the function $S(y)$ becomes:

$$S(y) = a_0 + a_1 y + b_1\sqrt{\sigma\left(1+\frac{y^2}{\mu}\right)}. \tag{3.4}$$

Substituting the function (3.4) in (3.3), we get a polynomial in y^i and $y^i\sqrt{\sigma\left(1+\frac{y^2}{\mu}\right)}$ whose coefficients depend on a_0, a_1, b_1, c, σ and μ. Setting the coefficients of this polynomial equal to zero, the overdetermined system of algebraic equations can be solved for the three different cases of $\{a_0{=}0\}$, $\{a_0{=}0 \ \& \ b_1{=}0\}$ and $\{a_0{=}0 \ \& \ a_1{=}0\}$.

Case i. The system of equations for a_0=0 is as follows:

$$\begin{aligned}
&c^2(3b_1^2\sigma+2)-1=0,\\
&-\mu+c^2(3+6a_1^2\mu^2+\mu(8+3b_1^2\sigma))=0,\\
&c^2(\mu(3a_1^2-3b_1^2\sigma-8)+15b_1^2\sigma+4)+\mu-2=0,\\
&-2+\mu+c^2(1-6a_1^2\mu^2+9b_1^2\sigma+\mu(-20+21a1^2-3b1^2\sigma))=0,\\
&-1+2\mu+c^2(2-3(-2+a_1^2)\mu^2+21b_1^2\sigma+\mu(-16+6a_1^2-15b_1^2\sigma))=0,\\
&-1+2\mu+c^2(2-3(-4+7a_1^2)\mu^2+9b_1^2\sigma+\mu(-19+24a_1^2-9b_1^2\sigma))=0,\\
&-6(-2+a_1^2)c^2\mu^2+9b_1^2c^2\sigma+\mu(1+c^2(-8+3a_1^2-21b_1^2\sigma))=0,\\
&-3(-5+8a_1^2)c^2\mu^2+3b_1^2c^2\sigma+\mu(1+c^2(-8+9a_1^2-9b_1^2\sigma))=0,\\
&(-2+a_1^2)\mu+3b_1^2\sigma=0,\\
&(-2+3a_1^2)\mu+b_1^2\sigma=0.
\end{aligned}\tag{3.5}$$

Solving this equation system with Mathematica results in a_1=$\pm\frac{1}{\sqrt{2}}$, $b_1\neq$0, c=$\pm\sqrt{2}$, σ=$\frac{-c^2-4}{12b_1^2}$, μ=$\frac{1}{6}\left(-4-c^2\right)$. Using these values and considering the change of variables in (2.4), the solitary wave solution is acquired as follows:

$$u(x,t)=\frac{\sqrt{\tanh^2\left(\sqrt{2}t-x\right)-1}}{\sqrt{2}}-\frac{\tanh\left(\sqrt{2}t-x\right)}{\sqrt{2}}\tag{3.6}$$

whose behavior is shown in Fig. 1-a.

Case ii. Similarly, a system of equations for the particular case of a_0=0 and b_1=0 can be obtained as follows:

$$\begin{aligned}
&-a_1+2a_1c^2=0,\\
&a_1-8a_1c^2+3a_1^3c^2=0,\\
&6a_1c^2-3a_1^3c^2=0.
\end{aligned}\tag{3.7}$$

This equation can be simply solved to have a_1=$\pm\sqrt{2}$, and c=$\pm\frac{1}{\sqrt{2}}$. Then, the solitary wave solution is written as:

$$u(x,t)=\sqrt{2}\tanh\left(\frac{t}{\sqrt{2}}-x\right).\tag{3.8}$$

The behavior of (3.8) is plotted in Fig. 1-b.

Case iii. For a_0=0 and a_1=0, the overdetermined system of algebraic equations is as follows:

$$\begin{aligned}
&3b_1c^2\mu^2\sigma-b_1\mu^3\sigma+8b_1c^2\mu^3\sigma+3b_1^3c^2\mu^3\sigma^2=0,\\
&-2b_1\mu^2\sigma+b_1c^2\mu^2\sigma+b_1\mu^3\sigma-20b_1c^2\mu^3\sigma+9b_1^3c^2\mu^2\sigma^2-3b_1^3c^2\mu^3\sigma^2=0,\\
&-b_1\mu\sigma+2b_1c^2\mu\sigma+2b_1\mu^2\sigma-19b_1c^2\mu^2\sigma+12b_1c^2\mu^3\sigma+9b_1^3c^2\mu\sigma^2-9b_1^3c^2\mu^2\sigma^2=0.\\
&b_1\mu\sigma-8b_1c^2\mu\sigma+15b_1c^2\mu^2\sigma+3b_1^3c^2\sigma^2-9b_1^3c^2\mu\sigma^2=0,\\
&6b_1c^2\mu\sigma-3b_1^3c^2\sigma^2=0.
\end{aligned}\tag{3.9}$$

Solution of this equation system leads to the values of the unknowns b_1≠0, c=$\pm i$, σ=$-\frac{2\left(c^2+4\right)}{3b_1^2}$ and μ=$\frac{1}{3}\left(-c^2-4\right)$. Using these values, the solitary wave solution is

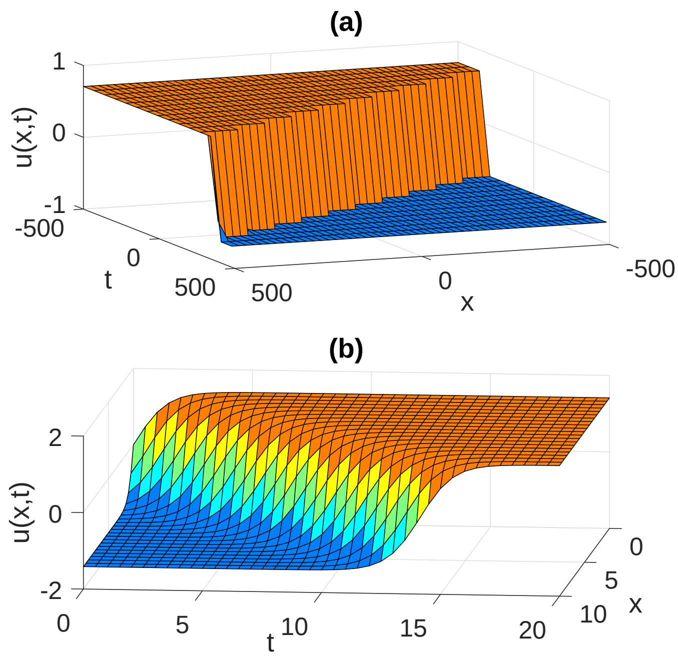

Figure 1. Solitary wave solutions of the nonlinear Boussinesq equation using the tanh function method where (a) a_0=0, (b) a_0=0 and b_1=0.

found as the following:

$$u(x,t) = \sqrt{2}\sqrt{-1-\tan^2(t+ix)}. \tag{3.10}$$

3.1.2. Application of the polynomial function method

Since m=1, the solitary wave solutions of (2.1) is sought using the polynomial function method in the following form:

$$u(x,t) = u(\xi) = a_0 + a_1\phi(\xi). \tag{3.11}$$

Depending on the choice of the function $\phi(\xi)$, two possible solitary wave solutions can be found as given below.

Case i. Let $\phi(\xi)$ be the solution of the ODE in (2.10) when β=0. Now, we substitute function (3.11) into (3.1) and we get a polynomial in $\phi(\xi)$ whose coefficients include the constants a_0, a_1, α, γ and c. Setting these coefficients equal to zero, we obtain the below overdetermined system of algebraic equations. When this equation system is processed in Mathematica, we see that a_0=0, $c\neq 0$ and $a_1\neq 0$ in order to have a nontrivial solution.

$$\begin{aligned} &-1-\alpha c^2 = 0,\\ &a_1^2 - 2\gamma = 0. \end{aligned} \tag{3.12}$$

The solution of this equation system results in $\alpha = -\frac{1}{c^2}$ and $\gamma = \frac{a_1^2}{2}$ which will be inserted into (2.10). Considering the following solution of (2.10)

$$\phi(\xi) = \pm\sqrt{2}\cot\xi\sqrt{\tan^2\xi + 1}, \tag{3.13}$$

whose constant is equated to zero for simplicity, the solitary wave solution (3.11) is acquired as:

$$u(x,t) = -\sqrt[4]{3}\cot\left(\frac{x-ct}{2\sqrt{2}}\right)\sqrt{\frac{1}{\cos\left(\frac{x-ct}{\sqrt{2}}\right)+1}}. \tag{3.14}$$

The behavior of (3.14) is given in Fig. 2-a.
Case ii. Now, let the function (3.11) be sought when

$$\phi(\xi) = \frac{4\beta}{2\beta^2 c_1\xi + \beta^2 c_1^2 + \beta^2\xi^2 - 4\gamma} \tag{3.15}$$

is the solution of the ODE (2.10) for α=0. In this solution, c_1 is taken as equal to 1 for simplicity. Following the same procedure given in the previous case, the below

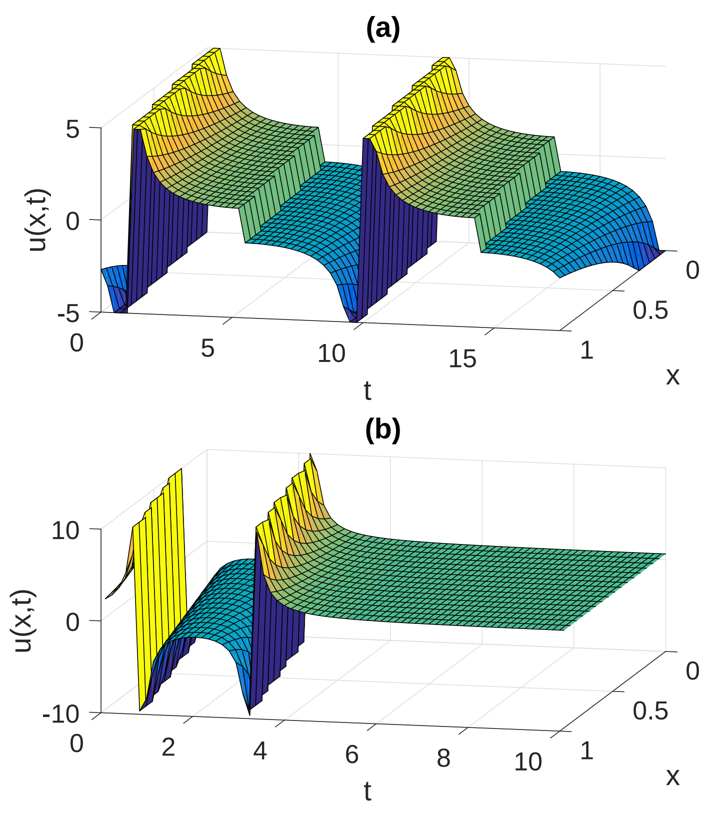

Figure 2. Solitary wave solution for the nonlinear Boussinesq equation with the polynomial function method where (a) β=0, (b) α=0.

overdetermined system of algebraic equations is reached:

$$\begin{aligned}&\beta - 3a_0^2\beta c^2 = 0,\\&-6a_0a_1\beta c^2 + 3\beta^2c^2 + \gamma - 3a_0^2c^2\gamma = 0,\\&-3\beta c^2(a_1^2 - 2\gamma) - 6a_0a_1c^2\gamma + 3\beta c^2\gamma = 0,\\&a_1^2 - 2\gamma = 0.\end{aligned} \tag{3.16}$$

Solving this equation system in Mathematica gives out $c{\neq}0$, $a_1{\neq}0$, $a_0{=}\pm\frac{1}{\sqrt{3}c}$, $\beta{=}2a_0a_1$, $\gamma{=}\frac{a_1^2}{2}$. Substituting the values of these constants in the function (3.11), the below solitary wave solution is found:

$$u(x,t) = \frac{1}{\sqrt{3}} + \frac{8}{\sqrt{3}\left(\frac{4}{3}(x-ct)^2 + \frac{8}{3}(x-ct) - \frac{2}{3}\right)}. \tag{3.17}$$

The behavior of (3.17) is given in Fig. 2-b.

3.2. Application to nonlinear sixth-order modified Boussinesq equation

After Step 3 of Section 2.2, the nonlinear sixth-order modified Boussinesq equation (2.2) takes the following ODE form:

$$c^2u^3 - u - c^2u_{\xi\xi} - u_{\xi\xi\xi\xi} = 0. \tag{3.18}$$

As stated in (2.6), the m value of the polynomial function (2.7) can be calculated from

$$3m = m + 4, \tag{3.19}$$

where we find $m{=}2$.

3.2.1. Application of the polynomial function method

Since the tanh function method given in Section 2.2.1 did not produce a solitary wave solution for (3.18), we directly show the application of the polynomial function method given in Section 2.2.2 The polynomial function (2.9) is sought in the following form:

$$u(x,t) = u(\xi) = a_0 + a_1\phi(\xi) + a_2\phi^2(\xi). \tag{3.20}$$

Two possible solitary wave solutions can be found for two different $\phi(\xi)$ functions. **Case i.** Here, $\phi(\xi)$ is the solution of the ODE given in (2.10) for $\gamma{=}0$. When the polynomial function (3.20) is inserted into (3.18), a polynomial in $\phi(\xi)$ is obtained. Next, the coefficients of this polynomial, which depend on the unknown constants a_0, a_1, a_2, α, β and c, are equated to zero. When the resultant overdetermined system of algebraic equations is processed in Mathematica, the real valued solutions of this equation system can be reached for $a_0{=}0$ and $a_2{=}0$. The simplified equation system after this process is given as:

$$\begin{aligned}&1 + \alpha^2 + \alpha c^2 = 0,\\&-\frac{3}{2}a_1\beta(5\alpha + c^2) = 0,\\&\frac{1}{2}a_1(-15\beta^2 + 2a_1^2c^2) = 0.\end{aligned} \tag{3.21}$$

The nontrivial solution of this equation system results in $a_1 \neq 0$, $c = \pm\sqrt{\frac{5}{2}}$, $\alpha = \frac{-c^2}{5}$ and $\beta = \pm\sqrt{\frac{2}{15}} a_1 c$. Using these constants and taking the solution

$$\phi(\xi) = \frac{1}{2}\left(\sqrt{3} + \sqrt{3}\tan^2\left(\frac{1}{4}\left(2\sqrt{3}c_1 + \sqrt{2}\xi\right)\right)\right) \tag{3.22}$$

of (2.10) into account together with $c_1=0$ for simplicity, the solitary wave solution (3.20) can be written as follows:

$$u(x,t) = \frac{1}{2}\sqrt{3}\sec^2\left(\frac{x - \sqrt{\frac{5}{2}}\,t}{2\sqrt{2}}\right), \tag{3.23}$$

whose behavior is plotted in Fig. 3-a.

Case ii. Now, let $\beta=0$ and $\phi(\xi)$ be the solution of the corresponding ODE (2.10). Similar to the procedure followed in the previous case (Case i), a polynomial in $\phi(\xi)$ is obtained after the polynomial function (3.20) is substituted into (3.18). Again, the coefficients of this polynomial are equated to zero to have an overdetermined system of algebraic equations. Real valued solutions to this equation system are found by Mathematica only for $a_0=0$ and $a_1=0$. Consequently, the equation system

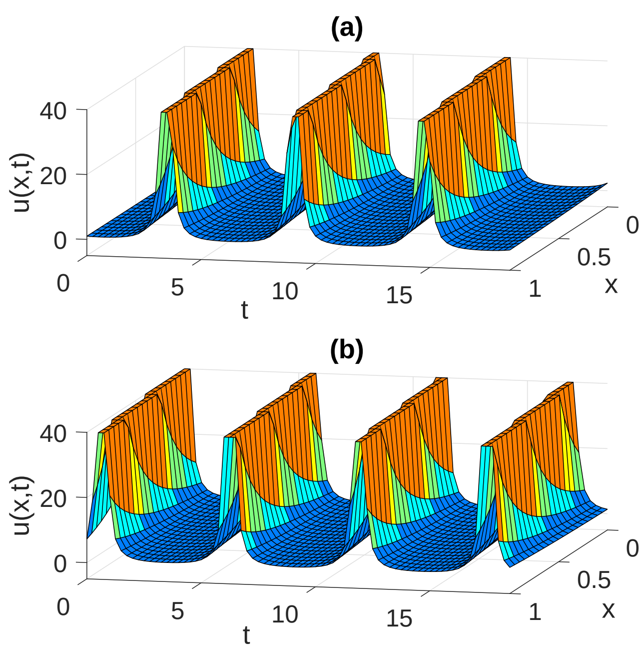

Figure 3. Solitary wave solution for the nonlinear sixth-order modified Boussinesq equation with the polynomial function method where (a) $\gamma=0$, (b) $\beta=0$.

to be solved becomes:

$$\begin{aligned} &-16\alpha^2 - 4\alpha c^2 - 1 = 0, \\ &20\alpha + c^2 = 0, \\ &a_2^2 c^2 - 120\gamma^2 = 0, \end{aligned} \tag{3.24}$$

which leads to $a_2 \neq 0$, $c = \pm\sqrt{\frac{5}{2}}$, $\alpha = -\frac{c^2}{20}$, $\gamma = \pm\frac{a_2 c}{2\sqrt{30}}$. With these constants and the solution

$$\phi(\xi) = \pm\frac{\cot\left(\frac{1}{4}\left(\left(-\sqrt{2}\right)\xi - 4\sqrt{3}c_1\right)\right)}{\sqrt{2}} \times \frac{\sqrt{\sqrt{3}\tan^2\left(\frac{1}{4}\left(\left(-\sqrt{2}\right)\xi - 4\sqrt{3}c_1\right)\right) + \sqrt{3}}}{\sqrt{2}} \tag{3.25}$$

of (2.10), the solitary wave solution (3.20) is acquired as follows after c_1 is set equal to zero for simplicity:

$$u(x,t) = \frac{1}{2}\sqrt{3}\csc^2\left(\frac{x - \sqrt{\frac{5}{2}}\,t}{2\sqrt{2}}\right). \tag{3.26}$$

The graph of (3.26) is given in Fig. 3-b.

4. Conclusion

The analytical solitary wave solutions are found for the nonlinear analogue of the Boussinesq and sixth-order modified Boussinesq equation where the nonlinearity is within the term which includes the derivative with respect to time. Both tanh function and polynomial function methods are used to reach the solitary wave solutions for the nonlinear Boussinesq equation, while those for the nonlinear sixth-order modified Boussinesq equation are obtained using the polynomial function method.

References

[1] S. Amirov and A. I. Kozhanov, *Global solvability of initial boundary-value problems for nonlinear analogs of the Boussinesq equation*, Mathematical Notes, 2016, 99(1–2), 183–191.

[2] H. A. Basha and S. F. Maalouf, *Theoretical and conceptual models of subsurface hillslope flows*, Water Resources Research, 2005, 41(7).

[3] M. A. Helal and A. R. Seadawy, *Variational method for the derivative nonlinear Schrdinger equation with computational applications*, Physica Scripta, 2009, 80(3), 035004.

[4] V. I. Karpman, *Non-Linear Waves in Dispersive Media: International Series of Monographs in Natural Philosophy (Vol. 71)*, Pergamon Press, Hungary, 1975, 15–18.

[5] X. L. Li and Y. Zheng, *The Effects of the Boussinesq Model to the Rising of the Explosion Clouds*, Nuclear Electronics & Detection Technology, 2010, 30(11), 1454–1458.

[6] K. E. Lonngren, *Observation of solitons on nonlinear dispersive transmission lines*, Solitons in Action, 1978, 25, 127–152.

[7] P. A. Madsen, H. B. Bingham and H. A. Schäffer, *Boussinesq-type formulations for fully nonlinear and extremely dispersive water waves: derivation and analysis*, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, 459(2033), 2003, 1075–1104.

[8] K. N. Moutsopoulos, *The analytical solution of the Boussinesq equation for flow induced by a step change of the water table elevation revisited*, Transport in Porous Media, 2010, 85(3), 919-940.

[9] D. E. Rupp and J. S. Selker, *On the use of the Boussinesq equation for interpreting recession hydrographs from sloping aquifers*, Water Resources Research, 2006, 42(12), W12421 1–15.

[10] A. R. Seadawy and K. El-Rashidy, *Traveling wave solutions for some coupled nonlinear evolution equations*, Mathematical and Computer Modelling, 2013, 57(5), 1371–1379.

[11] A. R. Seadawy, *Stability analysis for Zakharov-Kuznetsov equation of weakly nonlinear ion-acoustic waves in a plasma*, Computers & Mathematics with Applications, 2014, 67(1), 172–180.

[12] A. R. Seadawy, *Fractional solitary wave solutions of the nonlinear higher-order extended KdV equation in a stratified shear flow: Part I*, Computers & Mathematics with Applications, 2015, 70(4), 345–352.

[13] A. R. Seadawy, *Three-dimensional nonlinear modified Zakharov-Kuznetsov equation of ion-acoustic waves in a magnetized plasma*, Computers & Mathematics with Applications, 2016, 71(1), 201–212.

[14] A. R. Seadawy, *Stability analysis solutions for nonlinear three-dimensional modified Korteweg-de Vries-Zakharov-Kuznetsov equation in a magnetized electron-positron plasma*, Physica A: Statistical Mechanics and its Applications, 2016, 455, 44–51.

[15] A. R. Seadawy, O. H. El-Kalaawy and R. B. Aldenari, *Water wave solutions of Zufiria's higher-order Boussinesq type equations and its stability*, Applied Mathematics and Computation, 2016, 280, 57–71.

[16] A. R. Seadawy, *Travelling-wave solutions of a weakly nonlinear two-dimensional higher-order Kadomtsev-Petviashvili dynamical equation for dispersive shallow-water waves*, The European Physical Journal Plus, 2017, 132(1), 29.

[17] M. P. Soerensen, P. L. Christiansen and P. S. Lomdahl, *Solitary waves on nonlinear elastic rods. I*, The Journal of the Acoustical Society of America, 1984, 76(3), 871–879.

[18] M. Wang, *Solitary wave solutions for variant Boussinesq equations*, Physics Letters A, 1995, 199(3–4), 169–172.

[19] J. A. Zufiria, *Weakly nonlinear non-symmetric gravity waves on water of finite depth*, Journal of Fluid Mechanics, 1987, 180, 371–385.