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THEORY AND COMPUTATION FOR MULTIPLE POSITIVE SOLUTIONS OF NON-LOCAL PROBLEMS AT RESONANCE

Adela Novac[1] and Radu Precup[2,†]

Abstract Resonance non-positone and non-isotone problems for first order differential systems subjected to non-local boundary conditions are reduced to the non-resonance positone and isotone case by changes of variables. This allows us to prove the existence of multiple positive solutions. The theory is illustrated by two examples for which three positive numerical solutions are obtained using the Mathematica shooting program.

Keywords Nonlinear ordinary differential equation, general boundary condition, positive solution, multiple solutions, numerical computation.

MSC(2010) 34B15, 34B10, 34C25, 65L10.

1. Introduction and preliminaries

In this paper we deal with the existence, localization and multiplicity of positive solutions to non-local problems of the form

$$\begin{cases} u' = f(t,u), \quad t \in [0,1] \\ u(0) = \alpha\left(u\right), \end{cases} \tag{1.1}$$

where $f : [0,1] \times \mathbb{R}^n_+ \to \mathbb{R}^n$ is a continuous function and $\alpha : C([0,1], \mathbb{R}^n) \to \mathbb{R}^n$ is a continuous linear mapping. We are interested in *positive solutions*, i.e., solutions $u \in C^1\left([0,1], \mathbb{R}^n\right) \cap C\left([0,1], \mathbb{R}^n_+\right)$. Here the *non-local boundary condition* $u(0) = \alpha\left(u\right)$ is a general one which covers both discrete *multi-point boundary conditions*,

$$u(0) = \sum_{i=1}^{m} c_i u(t_i), \tag{1.2}$$

where $0 < t_1 < t_2 < \cdots < t_m \leq 1$, and *continuous boundary conditions* expressed by Stieltjes integrals,

$$u(0) = \int_0^1 u(s) d\xi(s).$$

[†]the corresponding author. Email address: r.precup@math.ubbcluj.ro (R. Precup)
[1]Department of Mathematics, Technical University, 28 Memorandumului Street, 400114 Cluj-Napoca, Romania
[2]Department of Mathematics, Babeş–Bolyai University, 1 M. Kogălniceanu Street, 400084 Cluj-Napoca, Romania

Differential equations with general boundary conditions of multi-point or integral type have a long history (see, e.g., Cioranescu [4], Whyburn [15], Conti [5]), and have gained a special attention in the last decades motivated by concrete applications in different domains (see, e.g., [1–3] and [6–14]). One of the most common approaches to study the problem (1.1) is to rewrite it as an integral equation, or equivalently as a fixed point problem in $C\left([0,1],\mathbb{R}_+^n\right)$, namely

$$u(t) = (I-\alpha[1])^{-1}\alpha\left(\int_0^{\cdot} f(s,u(s))ds\right) + \int_0^t f(s,u(s))ds, \ \ t\in[0,1], \tag{1.3}$$

where I is the identity matrix of size n and $\alpha[1]$ is the matrix whose columns are

$$\alpha(1,0,...,0), \ \alpha(0,1,...,0), \ ..., \ \alpha(0,0,...,1).$$

This representation is possible provided that the matrix $I-\alpha[1]$ is non-singular, i.e.,

$$\det(I-\alpha[1]) \neq 0. \tag{1.4}$$

We call (1.4), the *non-resonance* condition. Otherwise, that is if $I-\alpha[1]$ is singular, we say that the the non-local condition, or the non-local problem (1.1) is in the *resonance* case. For example, the *periodic boundary condition* $u(0)=u(1)$, for which $\alpha(u)=u(1)$ and $\alpha[1]=I$, is a resonance condition. More generally, condition (1.2) is in the non-resonance case if $\sum_{i=1}^m c_i \neq 1$, and in the resonance case if $\sum_{i=1}^m c_i = 1$. Looking for positive solutions of the fixed point problem (1.3) we are led to assume that the nonlinear operator N associated to the right hand-side of (1.3) maps the positive cone $C\left([0,1],\mathbb{R}_+^n\right)$ into itself. This happens in the *positone* case which is characterized by the following positivity conditions:

$$f \ \text{ is non-negative, i.e., } \ f:[0,1]\times\mathbb{R}_+^n \to \mathbb{R}_+^n;$$

$$\alpha(u)\in\mathbb{R}_+^n \ \text{ for all } \ u\in C^1([0,1],\mathbb{R}^n) \ \text{ with } \ u'\in C\left([0,1],\mathbb{R}_+^n\right) \ \text{ and } \ u(0)=0; \tag{1.5}$$

$$\text{matrix } \ I-\alpha[1] \ \text{ is inverse-positive,} \tag{1.6}$$

i.e., the elements of its inverse are non-negative.

Note that condition (1.5) is less restrictive than the positivity of α in the sense that $\alpha(u)\in\mathbb{R}_+^n$ for all $u\in C\left([0,1],\mathbb{R}_+^n\right)$. For instance, for $n=1$, the functional $\alpha(u)=-au(t_1)+bu(t_2)$, where $0<a\leq b$ and $0<t_1<t_2\leq 1$ is not positive but satisfies (1.5) as shown in [13, Remark 2.2].

Also note that the positivity of f implies that for any $u\in C\left([0,1],\mathbb{R}_+^n\right)$, $N(u)$ is increasing componentwise and consequently

$$N(u)(0)\leq N(u)(t)\leq N(u)(1)$$

for every $t\in[0,1]$, where the ordering $\leq$ in $\mathbb{R}^n$ is on components. These properties, via the fixed point equation $u=N(u)$, are passed on the solutions making possible to localize them only in terms of their initial and final values, which is extremely convenient for their numerical computation. The following existence, localization and multiplicity result for the positone non-resonance case was obtained in [13] using fixed point index theory.

To state the result, let us consider the cone

$$K := \{u \in C\left([0,1], \mathbb{R}_+^n\right): \ u(t) \geq u(0) \quad \text{for all} \quad t \in [0,1]\}$$

and for any vectors $r \in \mathbb{R}_+^n$ and $R \in (0,\infty)^n$, the set

$$K_{rR} := \{u \in K: \ r \leq u(0) \ \text{ and } \ |u|_\infty \leq R\}.$$

Here, for a function $u = (u_1, u_2, ..., u_n) \in C([0,1], \mathbb{R}^n)$, by $|u|_\infty$ we mean the vector-valued norm

$$|u|_\infty = (|u_1|_\infty, \ |u_2|_\infty, \ldots, \ |u_n|_\infty),$$

where

$$|u_i|_\infty = \max_{t\in[0,1]} |u_i(t)| \quad (1 \leq i \leq n).$$

Also, the notation $r < R$ will signify that $r_i < R_i$ for all $i \in \{1,2,...,n\}$.

Theorem 1.1. *Assume that there are vectors $r \in \mathbb{R}_+^n$ and $R \in (0,\infty)^n$ such that $r < R$ and the following conditions are satisfied:*

$$N(u)(0) \geq r \quad \text{for all} \quad u \in K_{rR}, \tag{1.7}$$

$$N(u)(1) \leq R \quad \text{for all} \quad u \in K \quad \text{with} \quad |u|_\infty \leq R. \tag{1.8}$$

Then problem (1.1) has a solution $u \in C\left([0,1], \mathbb{R}_+^n\right)$ such that

$$r \leq u(0) \quad \text{and} \quad u(1) \leq R. \tag{1.9}$$

If in addition inequality (1.7) is strict and there is a vector $\tau \in \mathbb{R}_+^n$, $\tau \neq R$ such that for every i with $\tau_i \neq R_i$, one has $\tau_i < r_i$ and

$$N_i(u)(1) < \tau_i \ \text{ for all } u \in K \text{ with } |u|_\infty \leq \tau,$$

then there exist two other solutions $v, w \in C\left([0,1], \mathbb{R}_+^n\right)$ with

$$r \not\leq v(0), \quad v(1) \not\leq \tau, \quad v(1) \leq R \quad \text{and} \quad w(1) \leq \tau. \tag{1.10}$$

Remark 1.1. It is worth noting that Theorem 1.1 immediately yields the existence of many positive solutions provided that its assumptions are fulfilled for several pairs r^k, R^k of vectors such that $r^k < R^k < r^{k+1} < R^{k+1}$.

Remark 1.2. In the isotone case, that is when in addition $f(t,u)$ is non-decreasing in its second vector variable u on $\mathbb{R}_+^n$, more exactly

$$u, v \in \mathbb{R}_+^n, \quad u \leq v \ \text{ implies } \ f(t,u) \leq f(t,v) \text{ for every } t \in [0,1],$$

the conditions (1.7), (1.8) hold provided that the following inequalities in terms of problem's data are respectively satisfied:

$$(I - \alpha[1])^{-1} \alpha \left(\int_0^{\cdot} f(s,r)\, ds \right) \geq r,$$

$$(I - \alpha[1])^{-1} \alpha \left(\int_0^{\cdot} f(s,R)\, ds \right) + \int_0^1 f(s,R)\, ds \leq R.$$

As shown in [13] on some examples, the radii τ, r and R from the localization formulas (1.9) and (1.10) are essential to start the numerical algorithms for the computation of the solutions u, v and w, and to distinguish one solution from the others. Thus it is a challenge to obtain localizations of solutions as precise as possible.

However, for most cases the non-resonance, positone and isotone conditions do not hold. In such situations a change of variable could be a solution as shown in Section 4 of the same paper [13]. The suggested change of variable was $z = e^{\sigma t}u$, where $\sigma > 0$, and has led to the new problem

$$\begin{cases} z' = g(t, z), & t \in [0, 1] \\ z(0) = \beta(z), \end{cases} \tag{1.11}$$

where

$$g(t, z) = e^{\sigma t} f\left(t, e^{-\sigma t} z\right) + \sigma z \quad \text{and} \quad \beta(z) = \alpha\left(e^{-\sigma t} z\right).$$

This new problem has a chance to be in the non-resonance positone and isotone case. This happens provided that conditions (1.5), (1.6) hold for β,

$$f(t, u) + \sigma u \in \mathbb{R}^n_+ \ \text{ for every } \ u \in \mathbb{R}^n_+,$$

and

$$f(t, u) + \sigma u \quad \text{is non-decreasing in the variable } \ u \ \text{ on } \mathbb{R}^n_+.$$

Obviously, these radii τ, r, R are connected with the change of variable, specifically with σ. The present paper, considers a general change of variables to the aim of a better localization of solutions, in support of computer implementation of numerical methods. The theoretical results combined with a Mathematica shooting procedure allow us to obtain multiple positive solutions for a class of non-local boundary value problems.

2. Main results

2.1. Change of variables by a t-dependent general matrix

In this section we assume that the problem (1.1) can be reduced to the non-resonance, positone and isotone case by a general linear change of variables depending on t,

$$z(t) = a(t)\, u(t),$$

where $a \in C^1([0, 1], \mathcal{M}_n(\mathbb{R}))$ is a matrix-valued function such that for each $t \in [0, 1]$, the matrix $a(t)$ is non-singular and inverse-positive, i.e., its inverse $a(t)^{-1}$ has non-negative entries. Denote by a^{-1} the matrix-valued function defined as $a^{-1}(t) = a(t)^{-1}$ $(t \in [0, 1])$. Then, in view of

$$u(t) = a^{-1}(t)\, z(t),$$

u is positive if z is positive. Making in (1.1) the change of variables yields the problem (1.11), where

$$g(t, z) = a(t)\, f\left(t, a^{-1}(t)\, z\right) + a'(t)\, a^{-1}(t)\, z$$

and

$$\beta(z) = a(0)\,\alpha\left(a^{-1}z\right).$$

The new problem is in the non-resonance, positone and isotone case if the following conditions are satisfied:

(i) $I - \beta[1]$ is inverse-positive and $\beta(z) \in \mathbb{R}^n_+$ for every $z \in C^1([0,1], \mathbb{R}^n)$ with $z' \in C\left([0,1], \mathbb{R}^n_+\right)$ and $z(0) = 0$;

(ii) $g(t,z) \in \mathbb{R}^n_+$ for every $z \in \mathbb{R}^n_+$ and $t \in [0,1]$;

(iii) $g(t,z)$ is non-decreasing in its second vector variable z on $\mathbb{R}^n_+$.

Then Theorem 1.1 and Remark 1.2 yield the following existence, localization and multiplicity result. To state the result, for a vector $v \in \mathbb{R}^n$, we shall denote its i^{th} component by $v{\downarrow_i}$.

Theorem 2.1. *Assume that the conditions* (i)-(iii) *hold. If there are vectors* $r \in \mathbb{R}^n_+$ *and* $R \in (0,\infty)^n$ *such that* $r < R$,

$$(I - \beta[1])^{-1}\,\beta\left(\int_0^{\cdot} g(s,r)\,ds\right) \geq r \tag{2.1}$$

and

$$(I - \beta[1])^{-1}\,\beta\left(\int_0^{\cdot} g(s,R)\,ds\right) + \int_0^1 g(s,R)\,ds \leq R,$$

then problem (1.1) *has a solution* $u \in C\left([0,1], \mathbb{R}^n_+\right)$ *such that*

$$a^{-1}(0)\,r \leq u(0) \quad \textit{and} \quad u(1) \leq a^{-1}(1)\,R.$$

If in addition inequality (2.1) *is strict and there is a vector* $\tau \in \mathbb{R}^n_+$, $\tau \neq R$ *such that for every* i *with* $\tau_i \neq R_i$, *one has* $\tau_i < r_i$ *and*

$$(I - \beta[1])^{-1}\,\beta\left(\int_0^{\cdot} g(s,\tau)\,ds\right)\Big\downarrow_i + \int_0^1 g_i(s,\tau)\,ds < \tau_i$$

then there exist two other solutions $v, w \in C\left([0,1], \mathbb{R}^n_+\right)$ *with*

$$a^{-1}(0)\,r \not\leq v(0), \quad v(1) \not\leq a^{-1}(1)\,\tau, \quad v(1) \leq a^{-1}(1)\,R \quad \textit{and} \quad w(1) \leq a^{-1}(1)\,\tau.$$

2.2. Change of variables by a diagonal matrix

It is useful to see what Theorem 2.1 becomes in case that the change of variables is given by a diagonal matrix

$$a(t) = \varphi(t)\,I,$$

where $\varphi \in C^1([0,1], \mathbb{R})$ and $\varphi(t) > 0$ for every $t \in [0,1]$. Then $a^{-1}(t) = (1/\varphi(t))\,I$,

$$g(t,z) = \varphi(t)\,f\left(t, \frac{z}{\varphi(t)}\right) + \frac{\varphi'(t)}{\varphi(t)} z \quad \text{and} \quad \beta(z) = \varphi(0)\,\alpha\left(\frac{z}{\varphi}\right).$$

Consequently the conditions (i)-(iii) can be written in terms of α and f as follows:

(i') $\alpha(u) \in \mathbb{R}_+^n$ for every $u \in C^1([0,1], \mathbb{R}^n)$ with $(\varphi u)' \in C([0,1], \mathbb{R}_+^n)$ and $u(0) = 0$; and the matrix $I - \varphi(0)\alpha[1/\varphi]$ is inverse-positive, where $\alpha[1/\varphi]$ is the matrix whose columns are

$$\alpha(1/\varphi, 0, ..., 0), \ \alpha(0, 1/\varphi, ..., 0), \ ..., \ \alpha(0, 0, ..., 1/\varphi).$$

(ii') $f(t,u) + (\varphi'(t)/\varphi(t))u \in \mathbb{R}_+^n$ for every $u \in \mathbb{R}_+^n$ and $t \in [0,1]$;

(iii') $f(t,u) + (\varphi'(t)/\varphi(t))u$ is non-decreasing in u on $\mathbb{R}_+^n$.

Denote

$$M = \varphi(0)(I - \varphi(0)\alpha[1/\varphi])^{-1}$$

and

$$h(t,u) = \int_0^t \left(\varphi(s) f\left(s, \frac{u}{\varphi(s)}\right) + \frac{\varphi'(s)}{\varphi(s)}u\right) ds \quad \left(u \in \mathbb{R}_+^n\right).$$

Corollary 2.1. *Assume that the conditions* (i')-(iii') *hold. If there are vectors* $r \in \mathbb{R}_+^n$ *and* $R \in (0,\infty)^n$ *such that* $r < R$,

$$M\alpha(h(\cdot, r)) \geq r \tag{2.2}$$

and

$$M\alpha(h(\cdot, R)) + h(1, R) \leq R,$$

then problem (1.1) *has a solution* $u \in C([0,1], \mathbb{R}_+^n)$ *such that*

$$\frac{1}{\varphi(0)}r \leq u(0) \quad \text{and} \quad u(1) \leq \frac{1}{\varphi(1)}R.$$

If in addition inequality (2.2) *is strict and there is a vector* $\tau \in \mathbb{R}_+^n$, $\tau \neq R$ *such that for every* i *with* $\tau_i \neq R_i$, *one has* $\tau_i < r_i$ *and*

$$M\alpha(h(\cdot, \tau))\downarrow_i + h_i(1, \tau) < \tau_i$$

then there exist two other solutions $v, w \in C([0,1], \mathbb{R}_+^n)$ *with*

$$\frac{1}{\varphi(0)}r \not\leq v(0), \quad v(1) \not\leq \frac{1}{\varphi(1)}\tau, \quad v(1) \leq \frac{1}{\varphi(1)}R \quad \text{and} \quad w(1) \leq \frac{1}{\varphi(1)}\tau.$$

Clearly, the change of variables proposed in [13] corresponds to the choice $\varphi(t) = \exp(\sigma t)$.

2.3. The case of a single equation

It is also useful to state the above results for a single equation, that is, for $n = 1$. Thus in this case $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$ is a continuous function, $\alpha : C([0,1], \mathbb{R}) \to \mathbb{R}$ is a continuous linear functional, and for the change of variable $z(t) = \varphi(t)u(t)$ with $\varphi \in C^1([0,1], \mathbb{R})$, $\varphi(t) > 0$ for all $t \in [0,1]$, one has

$$\alpha[1/\varphi] = \alpha(1/\varphi),$$

and the conditions (i')-(iii') become

(i$_1$) $\alpha(u) \geq 0$ for every $u \in C^1([0,1], \mathbb{R})$ with $(\varphi u)' \in C([0,1], \mathbb{R}_+)$ and $u(0) = 0$; and $\alpha(1/\varphi) < 1/\varphi(0)$;

(ii$_1$) $f(t,u) + (\varphi'(t)/\varphi(t))\, u \in \mathbb{R}_+$ for every $u \in \mathbb{R}_+$ and $t \in [0,1]$;

(iii$_1$) $f(t,u) + (\varphi'(t)/\varphi(t))\, u$ is non-decreasing in u on $\mathbb{R}_+$.

Corollary 2.2. *Assume that the conditions* (i$_1$)-(iii$_1$) *hold. If there are numbers r and R such that $0 \leq r < R$,*

$$M\alpha(h(\cdot, r)) \geq r \tag{2.3}$$

and

$$M\alpha(h(\cdot, R)) + h(1, R) \leq R, \tag{2.4}$$

then problem (1.1) *has a solution $u \in C([0,1], \mathbb{R}_+)$ such that*

$$\frac{1}{\varphi(0)} r \leq u(0) \quad \textit{and} \quad u(1) \leq \frac{1}{\varphi(1)} R.$$

If in addition inequality (2.2) *is strict, $r > 0$ and there is a number τ such that $0 < \tau < r$ and*

$$M\alpha(h(\cdot, \tau)) + h(1, \tau) < \tau, \tag{2.5}$$

then there exist two other solutions $v, w \in C([0,1], \mathbb{R}_+)$ with

$$\frac{1}{\varphi(0)} r > v(0), \quad v(1) > \frac{1}{\varphi(1)} \tau, \quad v(1) \leq \frac{1}{\varphi(1)} R \quad \textit{and} \quad w(1) \leq \frac{1}{\varphi(1)} \tau.$$

Example 2.1. Consider the non-local problem with a multi-point boundary condition

$$\begin{cases} u' = \frac{5}{1+e^{-8(u-1)}} - \frac{1}{2} e^{\frac{t}{2}} u, & t \in [0,1] \\ u(0) = \frac{1}{2} u(\frac{1}{2}) + \frac{1}{2} u(1). \end{cases} \tag{2.6}$$

Here $\alpha(u) = (u(1/2) + u(1))/2$. The problem is in the resonance non-positone and non-isotone case.

In [13] the reduction to the non-resonance, positone and isotone case was achieved via the substitution $z(t) = \varphi(t)\, u(t)$, with $\varphi(t) = \exp(t\sqrt{e}/2)$. Let us now take

$$\varphi(t) = \exp(\sigma(t)).$$

Since $\varphi'(t)/\varphi(t) = \sigma'(t)$, one has

$$f(t,u) + (\varphi'(t)/\varphi(t))\, u = \frac{5}{1 + e^{-8(u-1)}} + \left(\sigma'(t) - \frac{1}{2} e^{\frac{t}{2}}\right) u.$$

It follows that the conditions (ii$_1$) and (iii$_1$) hold if

$$\sigma'(t) \geq \frac{1}{2} e^{\frac{t}{2}}. \tag{2.7}$$

Condition (i$_1$), $\alpha(u) \geq 0$, holds even for every non-negative function u, and the condition $\alpha(1/\varphi) < 1/\varphi(0)$ reads as $\left(e^{-\sigma(1/2)} + e^{-\sigma(1)}\right)/2 < e^{-\sigma(0)}$, or equivalently

$$\left(e^{-(\sigma(1/2) - \sigma(0))} + e^{-(\sigma(1) - \sigma(0))}\right)/2 < 1,$$

which is true since in virtue of (2.7) $\sigma(t)$ is increasing. Also

$$M = 2e^{\sigma(0)} / \left(2 - e^{\sigma(0)-\sigma(1/2)} - e^{\sigma(0)-\sigma(1)}\right)$$

and

$$\begin{aligned} h(t,u) &= \int_0^t \left(e^{\sigma(s)} f\left(s, e^{-\sigma(s)}u\right) + \sigma'(s)\, u\right) ds \\ &= (\sigma(t) - \sigma(0))\, u + \int_0^t e^{\sigma(s)} f\left(s, e^{-\sigma(s)}u\right) ds. \end{aligned}$$

Note that the conditions (2.3) (with strict inequality), (2.4) and (2.5) can be written as

$$F(r) > 0, \quad G(R) \le 0 \quad \text{and} \quad G(\tau) < 0, \tag{2.8}$$

where $F, G : \mathbb{R}_+ \to \mathbb{R}$,

$$\begin{aligned} F(u) &= \alpha(h(\cdot, u)) - M^{-1}u, \\ G(u) &= \alpha(h(\cdot, u)) + M^{-1}h(1,u) - M^{-1}u. \end{aligned} \tag{2.9}$$

or more explicitly

$$\begin{aligned} F(u) &= \frac{1}{2}\left(h\left(\frac{1}{2}, u\right) + h(1,u)\right) - M^{-1}u, \\ G(u) &= \frac{1}{2}\left(h\left(\frac{1}{2}, u\right) + h(1,u)\right) + M^{-1}h(1,u) - M^{-1}u. \end{aligned}$$

The intervals of positivity of F and negativity of G show us how to find the numbers τ, r and R such that the condition (2.8) be satisfied.

Therefore, according to Corollary 2.2, if $\sigma \in C^1[0,1]$ satisfies (2.7) and there exist numbers τ, r and R, $0 < \tau < r < R$ such that inequalities (2.8) hold, then the problem (2.6) has three positive solutions u, v and w such that

$$\begin{aligned} &e^{-\sigma(0)}r \le u(0), \quad u(1) \le e^{-\sigma(1)}R; \\ &v(0) < e^{-\sigma(0)}r, \quad e^{-\sigma(1)}\tau < v(1) \le e^{-\sigma(1)}R; \\ &w(1) \le \tau e^{-\sigma(1)}. \end{aligned} \tag{2.10}$$

For computer simulation we take

$$\sigma(t) = \sigma t, \quad \text{where} \;\; \sigma = 0.9, \;\; \tau = 0.2, \;\; r = 5 \;\; \text{and} \;\; R = 26.$$

To find these three solutions we use the shooting method. This method needs a starting value for $u(0)$. This is obtained by taking the linear approximation of the solution $u(t)$, $\widetilde{u}(t) := u(0) + tu'(0)$. Then the boundary condition for $\widetilde{u}$,

$$\widetilde{u}(0) = \frac{1}{2}\widetilde{u}\left(\frac{1}{2}\right) + \frac{1}{2}\widetilde{u}(1)$$

yields $u'(0) = 0$, whence, from the differential equation,

$$f(0, u(0)) = 0.$$

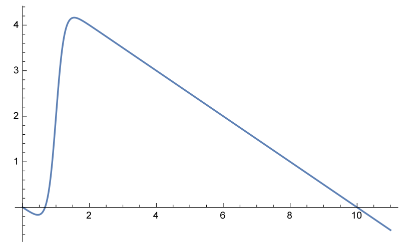

Figure 1. Graph of function $f(0, \cdot)$.

Thus the starting value u_0 for $u(0)$ is a root of the equation $f(0, u_0) = 0$, or

$$\frac{5}{1+e^{-8(u_0-1)}} - \frac{u_0}{2} = 0.$$

Using the command $\texttt{Plot}[f[0, u_0], 0, 11]$ the graph of the function $f(0, \cdot)$ is first obtained (Fig. 1) showing the existence of three positive roots of $f(0, u_0)$. We find these roots by the commands

$\texttt{FindRoot}[f[0, u_0] == 0, \{u_0, 0.1\}],\ u_0 = 0.00344724,$
$\texttt{FindRoot}[f[0, u_0] == 0, \{u_0, 1\}]\ u_0 = 0.670982,$
$\texttt{FindRoot}[f[0, u_0] == 0, \{u_0, 2\}]\ u_0 = 10.$

Next for each root u_0, we use the Shooting command of Mathematica:

$\texttt{u=NDSolveValue}[\{u'[t] = f[t, u[t]], u[0] = \frac{1}{2}u[\frac{1}{2}] + \frac{1}{2}u[1]\}, u, \{t, 0, 1\}, \texttt{Method}$ $\rightarrow \text{"}\texttt{Shooting}\text{"}, \text{"}\texttt{StartingInitialConditions}\text{"}\{u[0] = u_0\}, \texttt{WorkingPrecision} \rightarrow$ $54]\ \texttt{graficU = Plot}[U[t], \{t, 0, 1\}, \texttt{AxesLabel} \rightarrow \{\text{"}t\text{"}, \text{"}u(t)\text{"}\}, \texttt{PlotLabel} \rightarrow \text{"}u\text{"}]$

We find the error of approximation by the command:

$\texttt{Print}[\text{"}\texttt{maxerr} = \text{"},$

$\texttt{Max}[\texttt{Table}[\texttt{Derivative}[1][\texttt{U}][\texttt{t}] - (\frac{5}{1+e^{-8(u[t]-1)}} - \frac{1}{2}e^{\frac{t}{2}}u[t]), \{t, 0, 1, .0001\}]]]$

Thus, the errors of the numerical solutions u, v and w (see Fig. 2) are of 6.09965×10^{-9}, 2.38987×10^{-9} and 9.78437×10^{-12}, respectively.

Finally, by direct computation, we can check that the functions $\exp(0.9t)\,u(t)$, $\exp(0.9t)\,v(t)$ and $\exp(0.9t)\,w(t)$ are nondecreasing on $[0, 1]$ and that the following localization conditions

$$\begin{aligned} &r \leq u(0), \quad u(1) \leq e^{-0.9}R; \\ &v(0) < r, \qquad e^{-0.9}\tau < v(1) \leq e^{-0.9}R; \\ &w(1) \leq \tau e^{-0.9} \end{aligned}$$

are satisfied. Thus the theory is confirmed.

Example 2.2. Consider the periodic problem

$$\begin{cases} u' = \frac{5}{1+e^{-8(u-1)}} - \frac{1}{2}e^{\frac{t}{2}}u, \quad t \in [0, 1] \\ u(0) = u(1). \end{cases} \tag{2.11}$$

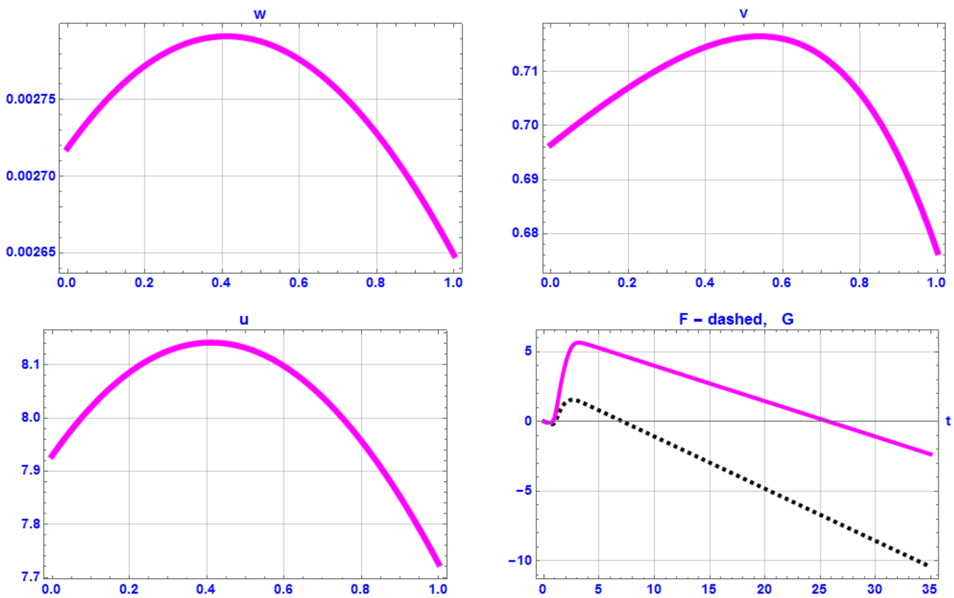

Figure 2. Three numerical solutions of a multi-point boundary value problem. (a) Solution w satisfying $w(1) \leq \tau e^{-0.9}$; (b) Solution v satisfying $v(0) < r$, $e^{-0.9}\tau < v(1) \leq e^{-0.9}R$; (c) Solution u with $r \leq u(0)$, $u(1) \leq e^{-0.9}R$; (d) Graphs of F and G which show the intervals of their positivity and negativity, respectively, helping to choose radii τ, r and R.

Here $\alpha(u) = u(1)$ and it is easy to see that the problem is in the resonance non-positone case. However, as in Example 2.1, by the change of variables, $z(t) = \varphi(t)\, u(t)$, where $\varphi(t) = \exp(\sigma(t))$, and $\sigma \in C^1[0,1]$ satisfies (2.7), it becomes a non-resonance positone and isotone problem. Condition (i_1) holds if $\sigma(1) > \sigma(0)$, which is true since in virtue of (2.7) $\sigma(t)$ is increasing.

Also

$$M = e^{\sigma(0)} / \left(1 - e^{\sigma(0)-\sigma(1)}\right)$$

and

$$\begin{aligned} h(t,u) &= \int_0^t \left(e^{\sigma(s)} f\left(s, e^{-\sigma(s)}u\right) + \sigma'(s)\, u\right) ds \\ &= (\sigma(t) - \sigma(0))\, u + \int_0^t e^{\sigma(s)} f\left(s, e^{-\sigma(s)}u\right) ds. \end{aligned}$$

Here the functions (2.9) are

$$F(u) = \int_0^1 e^{\sigma(s)} f(s, e^{-\sigma(s)}u) ds - u(e^{\sigma(1)-\sigma(0)} - 1 - \sigma(1) + \sigma(0)), \qquad (2.12)$$

$$G(u) = \int_0^1 e^{\sigma(s)} f(s, e^{-\sigma(s)}u) ds - u(1 - e^{\sigma(0)-\sigma(1)} - \sigma(1) + \sigma(0)).$$

As in the previous example, the intervals of positivity of F and negativity of G show us how to find the numbers τ, r and R such that the condition (2.8) be satisfied. For machine computation using Mathematica packages we take as an example $\sigma(t) = e^{t/2}$. The corresponding graphs of the functions F and G allow us to find numbers τ, r and R such that (2.8) holds. Thus we may choose $\tau = 0.5,\ r = 6$ and $R = 40$. Using the Shooting program of Mathematica we obtain three numerical solutions u, v and w (Fig. 3) with residuals 5.94249×10^{-9}, 2.11344×10^{-9} and 9.83916×10^{-12}, respectively.

Again by direct computation, the localization conditions (2.10), that is,

$$\begin{aligned}
&e^{-1}r \leq u(0), \quad u\,(1) \leq e^{-\sqrt{e})}R;\\
&v\,(0) < e^{-1}r, \quad e^{-\sqrt{e}}\tau < v\,(1) \leq e^{-\sqrt{e}}R;\\
&w(1) \leq \tau e^{-\sqrt{e}}
\end{aligned}$$

are satisfied as given by the theory.

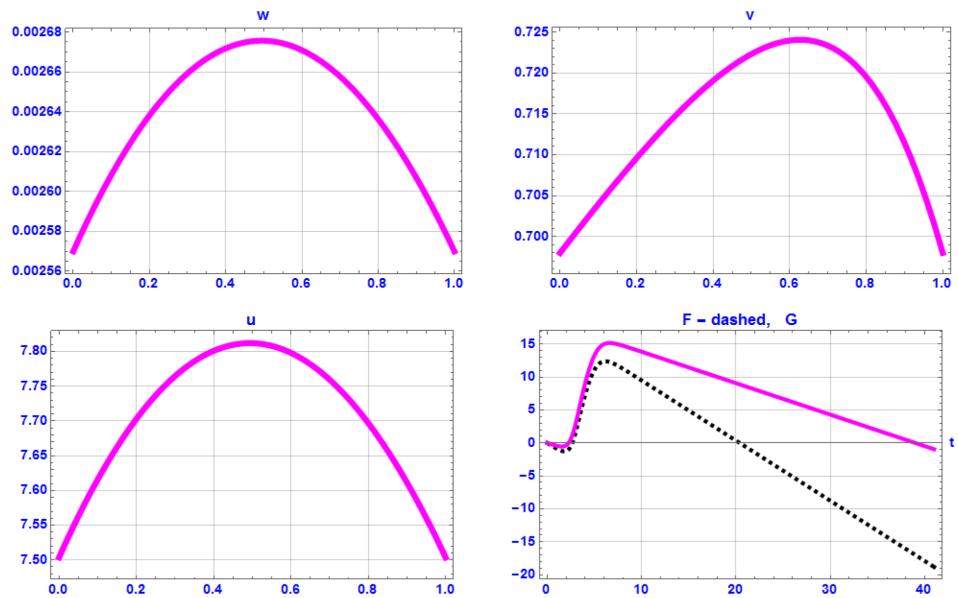

Figure 3. Three numerical solutions of a periodic boundary value problem. (a) Solution w satisfying $w(1) \leq \tau e^{-\sqrt{e}}$; (b) Solution v satisfying $v\,(0) < e^{-1}r,\ e^{-\sqrt{e}}\tau < v\,(1) \leq e^{-\sqrt{e}}R$; (c) Solution u with $e^{-1}r \leq u(0),\ u\,(1) \leq e^{-\sqrt{e})}R$; (d) Graphs of F and G.

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