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MICROPOLAR FLUID FLOWS WITH DELAY ON 2D UNBOUNDED DOMAINS*

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Abstract In this paper, we investigate the incompressible micropolar fluid flows on 2D unbounded domains with external force containing some hereditary characteristics. Since Sobolev embeddings are not compact on unbounded domains, first, we investigate the existence and uniqueness of the stationary solution, and further verify its exponential stability under appropriate conditions – essentially the viscosity $\delta_1 := \min\{\nu, c_a + c_d\}$ is asked to be large enough. Then, we establish the global well-posedness of the weak solutions via the Galerkin method combined with the technique of truncation functions and decomposition of spatial domain.

Keywords Micropolar fluid flow, truncation function, well-posedness, exponential stability.

MSC(2010) 35B40, 35Q35, 35A01, 35A02.

1. Introduction

The micropolar fluid model is firstly derived by Eringen [12] in 1966, which is used to describe the fluids consisting of randomly oriented particles suspended in a viscous medium. The model can be described by the following equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} - (\nu + \nu_r)\Delta u - 2\nu_r \mathrm{rot}\omega + (u \cdot \nabla)u + \nabla p = f, \\ \nabla \cdot u = 0, \\ \dfrac{\partial \omega}{\partial t} - (c_a + c_d)\Delta\omega + 4\nu_r\omega + (u \cdot \nabla)\omega - (c_0 + c_d - c_a)\nabla \mathrm{div}\omega - 2\nu_r \mathrm{rot} u = \tilde{f}, \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3)$ is the velocity, $\omega = (\omega_1, \omega_2, \omega_3)$ is the angular velocity field of rotation of particles, p represents the pressure, $f = (f_1, f_2, f_3)$ and $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$ stand for the external force and moments, respectively. The positive parameters ν, ν_r, c_0, c_a and c_d are the viscosity coefficients. Actually, ν represents the usual Newtonian viscosity and ν_r is called the microrotation viscosity. Note that when the gyration is neglected, the micropolar fluid equations are reduced to the classical Navier-Stokes equations.

Micropolar fluid model takes an important role in the fields of applied and computational mathematics, there is a wide literature on the mathematical theory of

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micropolar fluid model (1.1). Here we only illustrate some known results. First, we must mention that Lukaszewicz [15] obtained fruitful results, including the existence and uniqueness of solutions for the stationary problems and the existence of weak and strong solutions for the evolutionary problems, as well as the global existence of solution for the heat-conducting flows and the applications of the micropolar fluids in lubrication theory and in porous media, etc. Also, there are some other papers concentrating on the existence and uniqueness of solutions for the micropolar fluid flows. Galdi and Rionero [13] showed the existence and uniqueness theorems, known in the theorem of the Navier-Stokes equations, are valid for the incompressible micropolar equations too. Yamaguchi [25] established the existence of global strong solution in 3D bounded domain. Boldrini, Durán and Rojas-Medar [2] proved the existence and uniqueness of strong solution in a bounded or unbounded domain $\Omega \subseteq \mathbb{R}^3$ having a compact $\mathcal{C}^2$-boundary. Zhang [28] investigated the global existence and uniqueness of classical solutions to the 2D micropolar fluid flows with fix partial dissipation and angular viscosity. Dong and Zhang [11] proved the global existence and uniqueness of smooth solutions to the 2D micropolar fluid flows with zero angular viscosity on unbounded domains. At the same time, the long time behavior of solutions for the micropolar fluid model has been investigated from various aspects, see, e.g. [7–10, 16, 17, 21, 26, 29]. However, to our knowledge, there are very few articles about the micropolar fluid model with time delay. To date, we have not found in the literature any work that considers the combination of delay terms and unbounded domains.

In the real world, delay terms appear naturally, for instance as effects in wind tunnel experiments (see [18]), in the equations describing the motions of the fluids. The delay situations may also occur, for example, when we want to control the system via applying a force which considers not only the present state but also the history state of the system. There are some articles concerning the pullback asymptotic behavior of solutions to the nonlinear evolution equations with delays on bounded or unbounded domains, see, e.g. [3–6, 14, 20, 23, 24].

In this paper, we consider the situation that the velocity component in the x_3-direction is zero and the axes of rotation of particles are parallel to the x_3 axis. That is, $u = (u_1, u_2, 0)$, $\omega = (0, 0, \omega_3)$, $f = (f_1, f_2, 0)$, $\tilde{f} = (0, 0, \tilde{f}_3)$, $g = (g_1, g_2, 0)$ and $\tilde{g} = (0, 0, \tilde{g}_3)$. Let $\Omega \subseteq \mathbb{R}^2$ be an open set with boundary Γ that is not necessarily bounded but satisfies the following Poincaré inequality:

$$\text{There exists } \lambda_1 > 0 \text{ such that } \lambda_1 \|\varphi\|^2_{L^2(\Omega)} \leqslant \|\nabla\varphi\|^2_{L^2(\Omega)}, \quad \forall \varphi \in H^1_0(\Omega). \tag{1.2}$$

Then, we discuss the following equations of 2D non-autonomous incompressible micropolar fluid flows:

$$\begin{cases} \dfrac{\partial u}{\partial t} - (\nu + \nu_r)\Delta u - 2\nu_r \nabla \times \omega + (u \cdot \nabla)u + \nabla p = f + g(t, u_t), \text{ in } (0,T) \times \Omega, \\ \dfrac{\partial \omega}{\partial t} - \bar{\alpha}\Delta\omega + 4\nu_r\omega - 2\nu_r \nabla \times u + (u \cdot \nabla)\omega = \tilde{f} + \tilde{g}(t, \omega_t), \text{ in } (0,T) \times \Omega, \\ \nabla \cdot u = 0, \quad \text{in } (0,T) \times \Omega, \\ u = 0, \ \omega = 0, \quad \text{on } (0,T) \times \Gamma, \\ (u(0,\cdot), \omega(0,\cdot)) = (u^0(\cdot), \omega^0(\cdot)), \\ (u(t,\cdot), \omega(t,\cdot)) = (\phi_1(t,\cdot), \phi_2(t,\cdot)), \quad t \in (-h, 0), \ x \in \Omega, \end{cases} \tag{1.3}$$

where $T > 0$ is given, $\bar{\alpha} := c_a + c_d$, $x := (x_1, x_2) \in \Omega$, (u^0, ω^0) is the initial velocity filed, g and $\tilde{g}$ stand for the external force containing some hereditary characteristics u_t and ω_t, which are defined on $[-h, 0]$ as follows:

$$u_t = u_t(\cdot) := u(t + \cdot),\ \omega_t = \omega_t(\cdot) := \omega(t + \cdot),\ \forall t \geqslant 0. \tag{1.4}$$

In addition, (ϕ_1, ϕ_2) represents the initial datum in the interval of time $(-h, 0)$, where h is a positive fixed number, and

$$\nabla \times u := \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \quad \nabla \times \omega := (\frac{\partial \omega_3}{\partial x_2}, -\frac{\partial \omega_3}{\partial x_1}).$$

For the sake of convenience, we introduce the following useful operators:

$$\begin{cases} \langle Aw, \phi\rangle := \tilde{\nu}(\nabla u, \nabla \Phi) + \bar{\alpha}(\nabla \omega, \nabla \phi_3),\ \forall w = (u, \omega), \varphi = (\Phi, \phi_3) \in \widehat{V}, \\ \langle B(u, w), \phi\rangle := ((u \cdot \nabla)w, \phi),\ \forall u \in V,\ w = (u, \omega) \in \widehat{V},\ \forall \phi \in \widehat{V}, \\ N(w) := (-2\nu_r \nabla \times \omega, -2\nu_r \nabla \times u + 4\nu_r \omega),\ \forall w = (u, \omega) \in \widehat{V}, \end{cases} \tag{1.5}$$

where $\tilde{\nu} = (\nu + \nu_r)$ and the notation $\widehat{V}$ will be defined later. Then, we can formulate the weak version of equations (1.3) as follows:

$$\begin{cases} \dfrac{\partial w}{\partial t} + Aw + B(u, w) + N(w) = F(t, x) + G(t, w_t), \quad \text{in } (0, T) \times \Omega, \\ \nabla \cdot u = 0, \quad \text{in } (0, T) \times \Omega, \\ w = (u, \omega) = 0, \quad \text{on } (0, T) \times \Gamma, \\ w(0, x) = w^0(x) = (u^0(x), \omega^0(x)),\ w(t, x) = \phi(t, x), \quad t \in (-h, 0),\ x \in \Omega, \end{cases} \tag{1.6}$$

where $w := (u, \omega), F(t) = F(t, x) := (f, \tilde{f})$ and $G(t, w_t) := (g(t, u_t), \tilde{g}(t, \omega_t))$.

There are two goals in writing this thesis.

The first goal is to prove the existence and uniqueness of the stationary solution and to verify its exponential stability, exactly, we reveal that when the viscosity $\delta_1 := \min\{\nu, c_a + c_d\}$ is large enough, the weak solution of the evolutionary system (1.6) exponentially approaches the stationary solution as time increasing infinity. In this part, we need pay enough attention and give careful analysis for each term. In addition, the delay term will also increase the difficulty of the estimates.

The second goal is to establish the global well-posedness of the weak solution of system (1.6). Due to the lack of compact embedding in an unbounded domain, which will result in some obstacles in the process of using the classical Galerkin method to prove the existence of solutions. To overcome this difficulty, we utilize the Galerkin method combined with the technique of truncation function and the decomposition of spatial domain, and classical method to complete our purpose.

It is worth to mentioning that the existence and uniqueness of the weak solutions for the Navier-Stokes with delay on smooth bounded domains has been established by Caraballo and Real in [3]. Later, they investigated the asymptotic behaviour of the weak solutions in [4]. Afterwards, Garrido-Atienza and Marín-Rubio in [14] extended the results of [3] to unbounded domains. Moreover, they studied the existence and uniqueness of the stationary solution and its stability. We want to point out that the main idea of this paper originates from paper [14, 20, 27].

Compared with the Navier-Stokes equations studied in [14], the angular velocity field ω of the micropolar particles in the micropolar fluid flows leads to a different nonlinear term $B(u,w)$ and an additional term $N(w)$ in the abstract equation (see (1.6)). For that reason, more delicate estimates and analysis are required in our studies.

Throughout this paper, we denote the usual Lebesgue space and Sobolev space (see [1]) by $L^p(\Omega)$ and $W^{m,p}(\Omega)$ endowed with norms $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$, respectively.

$$\|\varphi\|_p := (\int_\Omega |\varphi|^p \mathrm{d}x)^{1/p} \quad \text{and} \quad \|\varphi\|_{m,p} := (\sum_{|\beta|\leqslant m} \int_\Omega |D^\beta \varphi|^p \mathrm{d}x)^{1/p}.$$

Especially, we denote $H^m(\Omega) := W^{m,2}(\Omega)$ and $H_0^1(\Omega)$ the closure of $\{\varphi \in \mathcal{C}_0^\infty(\Omega)\}$ with respect to $H^1(\Omega)$ norm.

$$\begin{aligned}
&\mathcal{V} := \mathcal{V}(\Omega) := \{\varphi \in \mathcal{C}_0^\infty(\Omega) \times \mathcal{C}_0^\infty(\Omega) |\, \varphi = (\varphi_1, \varphi_2), \nabla \cdot \varphi = 0\},\\
&\widehat{\mathcal{V}} := \widehat{\mathcal{V}}(\Omega) := \mathcal{V} \times \mathcal{C}_0^\infty(\Omega),\\
&H := H(\Omega) := \text{closure of } \mathcal{V} \text{ in } L^2(\Omega) \times L^2(\Omega), \text{ with norm } \|\cdot\|_H \text{ and dual space } H^*,\\
&V := V(\Omega) := \text{closure of } \mathcal{V} \text{ in } H^1(\Omega) \times H^1(\Omega), \text{ with norm } \|\cdot\|_V \text{ and dual space } V^*,\\
&\widehat{H} := \widehat{H}(\Omega) := \text{closure of } \widehat{\mathcal{V}} \text{ in } (L^2(\Omega))^3, \text{ with norm } \|\cdot\|_{\widehat{H}} \text{ and dual space } \widehat{H}^*,\\
&\widehat{V} := \widehat{V}(\Omega) := \text{closure of } \widehat{\mathcal{V}} \text{ in } (H^1(\Omega))^3, \text{ with norm } \|\cdot\|_{\widehat{V}} \text{ and dual space } \widehat{V}^*,
\end{aligned}$$

where $\|\cdot\|_H$, $\|\cdot\|_V$, $\|\cdot\|_{\widehat{H}}$ and $\|\cdot\|_{\widehat{V}}$ are defined by

$$\begin{aligned}
&\|(u,v)\|_H := (\|u\|_2^2 + \|v\|_2^2)^{1/2}, && \|(u,v)\|_V := (\|u\|_{H^1}^2 + \|v\|_{H^1}^2)^{1/2},\\
&\|(u,v,w)\|_{\widehat{H}} := (\|(u,v)\|_H^2 + \|w\|_2^2)^{1/2}, && \|(u,v,w)\|_{\widehat{V}} := (\|(u,v)\|_V^2 + \|w\|_{H^1}^2)^{1/2}.
\end{aligned}$$

$(\cdot,\cdot)-$ the inner product in $L^2(\Omega), H$ or $\widehat{H}$, $\langle\cdot,\cdot\rangle-$ the dual pairing between V and V^* or between $\widehat{V}$ and $\widehat{V}^*$. Throughout this article, we simplify the notations $\|\cdot\|_2$, $\|\cdot\|_H$ and $\|\cdot\|_{\widehat{H}}$ by the same notation $\|\cdot\|$ if there is no confusion. Furthermore, we denote

$L^p(I;X) :=$ space of strongly measurable functions on the closed interval I, with values in the Banach space X, endowed with norm

$$\|\varphi\|_{L^p(I;X)} := (\int_I \|\varphi\|_X^p \mathrm{d}t)^{1/p}, \quad \text{for } 1 \leqslant p < \infty,$$

$\mathcal{C}(I;X) :=$ space of continuous functions on the interval I, with values in the Banach space X, endowed with the usual norm,

$\hookrightarrow\hookrightarrow$ $-$ the compact embedding between spaces.

The rest of this paper is organized as follows. In section 2, we first make some preliminaries. Then, we concentrate on establishing the existence and uniqueness of the stationary, and further verifying its exponential stability, that is, under suitable conditions, the weak solution, the existence of which could be ensured by section 3, exponentially approaches the stationary solutions as time goes to $+\infty$. In section 3, we show the global well-posedness of the weak solutions.

2. Stability of stationary solutions

We divide this section into two subsections. In the first subsection, we make some necessary preliminaries. In the other subsection, we prove the stability of the stationary solutions.

2.1. Preliminaries

To begin with, let us give some useful properties and estimates about the operators defined in (1.5). That is,

Lemma 2.1.

(1) *The operator A is linear continuous both from $\widehat{V}$ to $\widehat{V}^*$ and from $D(A)$ to $\widehat{H}$, and so is for the operator $N(\cdot)$ from $\widehat{V}$ to $\widehat{H}$, where $D(A) := \widehat{V} \cap (H^2(\Omega))^3$.*

(2) *The operator $B(\cdot,\cdot)$ is continuous from $V \times \widehat{V}$ to $\widehat{V}^*$. Moreover, for any $u \in V$ and $w \in \widehat{V}$, there holds*

$$\langle B(u,\psi),\varphi\rangle = -\langle B(u,\varphi),\psi\rangle, \ \ \forall\, u \in V, \ \ \forall\, \psi \in \widehat{V}, \ \ \forall\, \varphi \in \widehat{V}. \tag{2.1}$$

Proof. (1) The continuity of the operators A and $N(\cdot)$ can be deduced directly from their definition. The linearity of the operator A is evident. So we only need check the linearity of the operator $N(\cdot)$. Indeed, for any $\phi = (\Phi, \phi_3) \in \widehat{V}$ with $\Phi = (\phi_1, \phi_2)$ and $\psi = (\Psi, \psi_3) \in \widehat{V}$ with $\Psi = (\psi_1, \psi_2)$, we have

$$\begin{aligned}
&N(\phi) - N(\psi)\\
=&\big(-2\nu_r(\nabla\times\phi_3 - \nabla\times\psi_3), -2\nu_r(\nabla\times\Phi - \nabla\times\Psi) + 4\nu_r(\phi_3-\psi_3)\big)\\
=&\big(-2\nu_r(\frac{\partial\phi_3}{\partial x_2} - \frac{\partial\psi_3}{\partial x_2}, -\frac{\partial\phi_3}{\partial x_1} + \frac{\partial\psi_3}{\partial x_1}),\\
&\qquad\qquad -2\nu_r(\frac{\partial\phi_2}{\partial x_1} - \frac{\partial\phi_1}{\partial x_2} - \frac{\partial\psi_2}{\partial x_1} + \frac{\partial\psi_1}{\partial x_2}) + 4\nu_r(\phi_3-\psi_3)\big)\\
=&\big(-2\nu_r(\frac{\partial(\phi_3-\psi_3)}{\partial x_2}, -\frac{\partial(\phi_3-\psi_3)}{\partial x_1}),\\
&\qquad\qquad -2\nu_r(\frac{\partial(\phi_2-\psi_2)}{\partial x_1} - \frac{\partial(\phi_1-\psi_1)}{\partial x_2}) + 4\nu_r(\phi_3-\psi_3)\big)\\
=&\big(-2\nu_r\nabla^{\perp}(\phi_3-\psi_3), -2\nu_r\nabla\times(\Phi-\Psi) + 4\nu_r(\phi_3-\psi_3)\big) = N(\phi-\psi).
\end{aligned}$$

(2) The continuity of the operator $B(\cdot,\cdot)$ can be also obtained from its definition. Next, we verify (2.1). In fact, for any $u \in V, w \in \widehat{V}$, we have

$$\begin{aligned}
&\langle B(u,w),w\rangle = ((u\cdot\nabla)w,w)\\
=&\int_\Omega (u_1\frac{\partial}{\partial x_1} + u_2\frac{\partial}{\partial x_2} + u_3\frac{\partial}{\partial x_3})(w_1,w_2,w_3)(w_1,w_2,w_3)\mathrm{d}x = \sum_{j=1}^3\sum_{i=1}^3\int_\Omega u_i\frac{\partial w_j}{\partial x_i}w_j\mathrm{d}x\\
=&\sum_{j=1}^3\sum_{i=1}^3\frac{1}{2}\int_\Omega u_i\frac{\partial w_j^2}{\partial x_i}\mathrm{d}x = \frac{1}{2}\sum_{j=1}^3\sum_{i=1}^3(u_iw_j^2|_{\partial\Omega} - \int_\Omega w_j^2D_iu_i\mathrm{d}x)\\
=&-\frac{1}{2}\sum_{j=1}^3\sum_{i=1}^3\int_\Omega w_j^2D_iu_i\mathrm{d}x = -\frac{1}{2}\sum_{j=1}^3\int_\Omega w_j^2(\nabla\cdot u)\mathrm{d}x = 0.
\end{aligned} \tag{2.2}$$

Hence, (2.1) is valid as a consequence of (2.2). The proof is complete. □

We further have

Lemma 2.2 (see [16, 19, 26]).

(1) *There are two positive constants* c_1 *and* c_2 *such that*

$$c_1\langle Aw, w\rangle \leqslant \|w\|_{\widehat{V}}^2 \leqslant c_2\langle Aw, w\rangle, \ \ \forall w \in \widehat{V}. \tag{2.3}$$

(2) *There exists some positive constant* λ_0 *which depends only on* Ω*, such that for any* $(u, \psi, \varphi) \in V \times \widehat{V} \times \widehat{V}$ *there holds*

$$|\langle B(u,\psi),\varphi\rangle| \leqslant \begin{cases} \lambda_0\|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{1}{2}}\|\varphi\|^{\frac{1}{2}}\|\nabla\varphi\|^{\frac{1}{2}}\|\nabla\psi\|, \\ \lambda_0\|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{1}{2}}\|\psi\|^{\frac{1}{2}}\|\nabla\psi\|^{\frac{1}{2}}\|\nabla\varphi\|. \end{cases} \tag{2.4}$$

(3) *There exists a positive constant* $c(\nu_r)$ *such that*

$$\|N(\psi)\| \leqslant c(\nu_r)\|\psi\|_{\widehat{V}}, \ \ \forall \psi \in \widehat{V}. \tag{2.5}$$

In addition,

$$\delta_1\|\psi\|_{\widehat{V}}^2 \leqslant \langle A\psi,\psi\rangle + \langle N(\psi),\psi\rangle, \ \ \forall\, \psi \in \widehat{V}, \tag{2.6}$$

where $\delta_1 := \min\{\nu, \bar{\alpha}\}$*.*

Next, we recall a key lemma from [14] as follows.

Lemma 2.3. *Let* I *be a bounded open set of* $\mathbb{R}^d$*, and* X*,* E *are two Banach spaces with* $X \hookrightarrow\hookrightarrow E$*. Consider* $1 \leqslant r < q \leqslant \infty$*. Suppose* $F \subset L^r(I; E)$ *satisfies*

(i) $\forall \omega \subset\subset I, \ \sup_{f\in F} \|\Pi_h f - f\|_{L^r(\omega;E)} \to 0$ *as* $h \to 0$*,*
where $\Pi_h f$ *is the translation function:* $(\Pi_h f)(t) = f(t+h)$*,*

(ii) F *is bounded in* $L^q(I; E) \cap L^1(I; X)$*.*

Then F *is precompact in* $L^r(I; E)$*.*

Finally, we end this subsection with the definition of weak solution of (1.6).

Definition 2.1. For each $T > 0$, function w is called a weak solution of (1.6) if, $w = (u, \omega) \in \mathcal{C}^0([0,T]; \widehat{H}) \cap L^2(-h, T; \widehat{V})$ is such that for any $t \in (0,T)$ and any $\varphi \in \widehat{V}$,

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(w(t),\varphi) + \langle Aw,\varphi\rangle + \langle B(u,w),\varphi\rangle + \langle N(w),\varphi\rangle = \langle F(t,x),\varphi\rangle + (G(t,w_t),\varphi), \\ w(0) = w^0, \ w(t) = \phi(t), \quad t \in (-h, 0) \end{cases}$$

holds in the distribution sense of $\mathcal{D}'(0,T)$.

2.2. Stability of stationary solutions

In this subsection, we prove the existence and uniqueness of stationary solutions to the micropolar fluid flows provided the viscosity is large enough, when the delay term has a special form. Furthermore, in a little stronger conditions, we verify its exponential stability.

From now on, we suppose that $\rho(\cdot) \in \mathcal{C}^1([0,T])$, $\rho(t) \geqslant 0$ for all $t \in [0,T]$, $h = \max_{t\in[0,T]} \rho(t) > 0$ and $\rho_* = \max_{t\in[0,T]} \rho'(t) < 1$, and the external force F is independent of time, the delay term $G(t, w_t) = \widehat{G}(w(t-\rho(t)))$ with $\widehat{G}: \ \widehat{H} \mapsto \widehat{H}$ satisfies

$$\begin{cases} \text{(i) } \widehat{G}(0) = 0, \\ \text{(ii) there exists } L_0 > 0 \text{ such that } \|\widehat{G}(w) - \widehat{G}(v)\| \leqslant L_0\|w - v\|, \ \forall\, w,\, v \in \widehat{H}. \end{cases} \tag{2.7}$$

In the following, we concentrate on establishing the existence and uniqueness of the stationary of (1.6). That is, to find a function $w^* = (u^*, \omega^*) \in \widehat{V}$ such that

$$\langle Aw^*, v\rangle + \langle B(u^*, w^*), v\rangle + \langle N(w^*), v\rangle = \langle F, v\rangle + (\widehat{G}(w^*), v), \text{ for all } v \in \widehat{V}. \tag{2.8}$$

Theorem 2.1. *Suppose that $\widehat{G}$ satisfies* (2.7) *and $\delta_1 > \lambda_1^{-1} L_0$. Then, for any $F \in \widehat{V}^*$,*

1. *there exists at least one solution to* (2.8)*,*
2. *under the extra condition: $\lambda_1^{\frac{1}{2}}(\delta_1 - \lambda_1^{-1}L_0)^2 > \lambda_0\|F\|_{\widehat{V}^*}$, there corresponds at most one solution to* (2.8)*.*

Where the constant λ_0 comes from (2.4)*.*

Proof. (1) *Existence of stationary solutions.* Firstly, we take an orthonormal basis $\{v_j\}_{j=1}^{\infty} \subset \widehat{\mathcal{V}}$ of $\widehat{H}$ such that the span$\{v_1, v_2, \cdots, v_n, \cdots\}$ is dense in $\widehat{V}$. Denote $\widehat{V}_m := \text{span}\{v_1, v_2, \cdots, v_m\}$ and consider the following problem:

$$\begin{cases} \text{To find } w^m \in \widehat{V}_m \text{ such that, for a fixed } q^m = (q^m_{(1)}, q^m_{(2)}, q^m_{(3)}) = (p^m, q^m_{(3)}) \in \widehat{V}_m, \\ \sigma(w^m, v^m) = \langle F, v^m\rangle + (\widehat{G}(q^m), v^m), \quad \forall\, v^m \in \widehat{V}_m, \end{cases} \tag{2.9}$$

where $\sigma(w^m, v^m) := \langle Aw^m, v^m\rangle + \langle B(p^m, w^m), v^m\rangle + \langle N(w^m), v^m\rangle$. On one hand, it is not difficult to check that, for a fixed $q^m = (p^m, q^m_{(3)}) \in \widehat{V}_m$, the function $\sigma(\cdot,\cdot)$ is bilinear, continuous and coercive in $\widehat{V}_m \times \widehat{V}_m$. On the other hand, the function $v^m \mapsto \langle F, v^m\rangle + (G(q^m), v^m)$ is obviously linear and continuous. Therefore, by Lax-Milgram theorem, for each fixed $q^m = (p^m, q^m_{(3)}) \in \widehat{V}_m$, there exists a unique solution to problem (2.9), which we denote w^m. Then, consider the map $E_m(\cdot): \widehat{V}_m \mapsto \widehat{V}_m$, which is defined by

$$E_m(q^m) = E_m((p^m, q^m_{(3)})) = (u^m, \omega^m) = w^m.$$

Next, we are devoted to proving that, for each m, there exists at least one fixed point of the mapping $E_m(\cdot)$. This implies that there exists a $w^m \in \widehat{V}_m$ satisfying

$$\begin{aligned} &\langle Aw^m, v^m\rangle + \langle B(u^m, w^m), v^m\rangle + \langle N(w^m), v^m\rangle \\ &\qquad = \langle F, v^m\rangle + (\widehat{G}(w^m), v^m), \quad \forall\, v^m \in \widehat{V}_m. \end{aligned} \tag{2.10}$$

In order to proceed, taking $v^m = w^m$ in (2.9), and using (1.2), (2.2), (2.6) and (2.7), we deduce

$$\delta_1\|w^m\|^2_{\widehat{V}} \leqslant \langle F, w^m\rangle + (\widehat{G}(q^m), w^m) \leqslant \|F\|_{\widehat{V}^*}\|w^m\|_{\widehat{V}} + \|\widehat{G}(q^m)\|\|w^m\|$$

$$\leqslant \|F\|_{\widehat{V}^*}\|w^m\|_{\widehat{V}} + \lambda_1^{-1}L_0\|q^m\|_{\widehat{V}}\|w^m\|_{\widehat{V}},$$

which implies

$$\delta_1\|w^m\|_{\widehat{V}} \leqslant \|F\|_{\widehat{V}^*} + \lambda_1^{-1}L_0\|q^m\|_{\widehat{V}}. \tag{2.11}$$

Since $\delta_1 > \lambda_1^{-1}L_0$, we can take $k > 0$ such that $k(\delta_1 - \lambda_1^{-1}L_0) \geqslant \|F\|_{\widehat{V}^*}$, then,

$$\delta_1\|w^m\|_{\widehat{V}} \leqslant k(\delta_1 - \lambda_1^{-1}L_0) + \lambda_1^{-1}L_0\|q^m\|_{\widehat{V}}. \tag{2.12}$$

Now, define $K_m = \{q \in \widehat{V}_m \mid \|q\|_{\widehat{V}} \leqslant k\}$, which is a convex compact set of $\widehat{V}$. Then, it follows from (2.12) that $E_m(\cdot)$ maps K_m to K_m. In the following, we are going to apply the Brouwer fixed point theorem to $E_m(\cdot)\big|_{K_m}$. For this end, it only remains to show that E_m is continuous. Indeed, take $q_i^m \in K_m$, $i = 1, 2$, and denote $w_i^m = (u_i^m, \omega_i^m) = E_m(q_i^m) = E_m((p_i^m, q_{i(3)}^m))$ the respective solutions of (2.9). Then, it holds that

$$\begin{aligned}&\langle A(w_1^m - w_2^m), v^m\rangle + \langle B(p_1^m, w_1^m) - B(p_2^m, w_2^m), v^m\rangle + \langle N(w_1^m - w_2^m), v^m\rangle\\ =&(\widehat{G}(q_1^m) - \widehat{G}(q_2^m), v^m), \quad \forall\, v^m \in \widehat{V}_m.\end{aligned}$$

Particularly, choose $v^m = w_1^m - w_2^m$, with the aid of (1.2), (2.2), (2.4), (2.6) and (2.7), the above inequality gives

$$\begin{aligned}&\delta_1\|w_1^m - w_2^m\|_{\widehat{V}}^2\\ \leqslant&\langle B(p_2^m, w_2^m) - B(p_1^m, w_1^m), w_1^m - w_2^m\rangle + (\widehat{G}(q_1^m) - \widehat{G}(q_2^m), w_1^m - w_2^m)\\ \leqslant&\langle B(p_2^m - p_1^m, w_2^m), w_1^m - w_2^m\rangle + \|\widehat{G}(q_1^m) - \widehat{G}(q_2^m)\|\|w_1^m - w_2^m\|\\ \leqslant&\lambda_0\|p_2^m - p_1^m\|^{\frac{1}{2}}\|p_2^m - p_1^m\|_{\widehat{V}}^{\frac{1}{2}}\|w_2^m\|_{\widehat{V}}\|w_2^m - w_1^m\|^{\frac{1}{2}}\|w_2^m - w_1^m\|_{\widehat{V}}^{\frac{1}{2}}\\ &+ L_0\|q_1^m - q_2^m\|\|w_1^m - w_2^m\|\\ \leqslant&\lambda_1^{-\frac{1}{2}}\lambda_0\|p_1^m - p_2^m\|_{\widehat{V}}\|w_2^m\|_{\widehat{V}}\|w_1^m - w_2^m\|_{\widehat{V}} + \lambda_1^{-1}L_0\|q_1^m - q_2^m\|_{\widehat{V}}\|w_1^m - w_2^m\|_{\widehat{V}}\\ \leqslant&(\lambda_1^{-\frac{1}{2}}\lambda_0 k + \lambda_1^{-1}L_0)\|q_1^m - q_2^m\|_{\widehat{V}}\|w_1^m - w_2^m\|_{\widehat{V}},\end{aligned} \tag{2.13}$$

where we also used Hölder inequality and the facts $q_i^m = (p_i^m, q_{i(3)}^m)$, $w_i^m = E_m(q_i^m) \in K_m$. The inequality (2.13) implies E_m is continuous. At this stage, we can conclude that, for each $m \in \mathbb{N}$, there exists a $w^m = (u^m, \omega^m) \in \widehat{V}_m$ satisfying (2.10).

Finally, we will pass to the limit in (2.10) to obtain the existence of solutions of (2.8). Similar to (2.11), taking $v^m = w^m$ in (2.10), we obtain

$$\delta_1\|w^m\|_{\widehat{V}} \leqslant \|F\|_{\widehat{V}^*} + \lambda_1^{-1}L_0\|w^m\|_{\widehat{V}},$$

that is,

$$\|w^m\|_{\widehat{V}} \leqslant \frac{1}{\delta_1 - \lambda_1^{-1}L_0}\|F\|_{\widehat{V}^*}. \tag{2.14}$$

So, we may extract a subsequence (denoting by the same symbol) $\{w^m\}$ such that

$$w^m \rightharpoonup w \quad \text{weakly in } \widehat{V}. \tag{2.15}$$

Moreover, for any regular bounded set $\mathcal{Q} \subset \Omega$, we have the same uniform bounds of $w^m\big|_{\mathcal{Q}}$, which means, using the compact injection, that

$$w^m\big|_{\mathcal{Q}} \to w\big|_{\mathcal{Q}} \quad \text{strongly in } (L^2(\mathcal{Q}))^3. \tag{2.16}$$

Based on the above argument, for a fixed $v_j \in \{v_j\}_{j=1}^{\infty}$, denote by $\mathcal{Q}_j$ the support of v_j (which is compact) and take $\mathcal{Q} \subset \Omega$ a bounded open set with smooth boundary containing $\mathcal{Q}_j$, then we not only have the weak convergence $w^m \rightharpoonup w$ in $\widehat{V}(\mathcal{Q}_j)$ but the strong convergence $w^m \to w$ in $(L^2(\mathcal{Q}_j))^3$. Furthermore, it holds that

$$\begin{aligned} |(\widehat{G}(w^m), v_j) - (\widehat{G}(w), v_j)| \leqslant& \|\widehat{G}(w^m) - \widehat{G}(w)\|_{(L^2(\mathcal{Q}_j))^3} \|v_j\| \\ \leqslant& L_0 \|w^m - w\|_{(L^2(\mathcal{Q}_j))^3} \|v_j\| \to 0, \quad \text{as } m \to \infty. \end{aligned}$$

At last, combining with (2.15), we may pass to the limit with respect to m for every term in (2.10) to obtain

$$\langle Aw, v_j \rangle + \langle B(u, w), v_j \rangle + \langle N(w), v_j \rangle = \langle F, v_j \rangle + (\widehat{G}(w), v_j). \tag{2.17}$$

Since span$\{v_1, v_2, \cdots, v_n, \cdots\}$ is dense in $\widehat{V}$, we conclude that there exists at least one function $w^* := w$ satisfies (2.8).

(2) *Uniqueness of the stationary solution.* Now, we prove uniqueness of solution to (2.8) under the extra condition $\lambda_1^{\frac{1}{2}}(\delta_1 - \lambda_1^{-1} L_0)^2 > \lambda_0 \|F\|_{\widehat{V}^*}$.

Suppose there are two solutions w_1, w_2 to (2.8). Taking the difference, it holds that

$$\begin{aligned} &\langle Aw_1 - Aw_2, v \rangle + \langle B(u_1, w_1) - B(u_2, w_2), v \rangle + \langle N(w_1) - N(w_2), v \rangle \\ &\qquad = (\widehat{G}(w_1) - \widehat{G}(w_2), v), \ \forall \, v \in \widehat{V}. \end{aligned}$$

In particular, taking $v = w_1 - w_2$, similar to (2.13), we have

$$\begin{aligned} \delta_1 \|w_1 - w_2\|_{\widehat{V}}^2 \leqslant& \lambda_1^{-\frac{1}{2}} \lambda_0 \|u_1 - u_2\|_{\widehat{V}} \|w_2\|_{\widehat{V}} \|w_1 - w_2\|_{\widehat{V}} + \lambda_1^{-1} L_0 \|w_1 - w_2\|_{\widehat{V}}^2 \\ \leqslant& \frac{\lambda_1^{-\frac{1}{2}} \lambda_0}{\delta_1 - \lambda_1^{-1} L_0} \|F\|_{\widehat{V}^*} \|w_1 - w_2\|_{\widehat{V}}^2 + \frac{L_0}{\lambda_1} \|w_1 - w_2\|_{\widehat{V}}^2, \end{aligned}$$

where, in the second inequality, we have used (2.14) and the fact $\|u_1 - u_2\|_{\widehat{V}} \leqslant \|w_1 - w_2\|_{\widehat{V}}$. It follows from the above inequality that

$$\big[(\delta_1 - \lambda_1^{-1} L_0)^2 - \lambda_1^{-\frac{1}{2}} \lambda_0 \|F\|_{\widehat{V}^*}\big] \|w_1 - w_2\|_{\widehat{V}}^2 \leqslant 0.$$

Hence, the uniqueness follows as long as $\lambda_1^{\frac{1}{2}}(\delta_1 - \lambda_1^{-1} L_0)^2 > \lambda_0 \|F\|_{\widehat{V}^*}$. This completes the proof. □

Next, under a little stronger condition than that in Theorem 2.1, which ensures the existence and uniqueness of the stationary solution w^* of (2.8), we prove that the weak solution of the evolutionary problem (1.6) exponentially approaches w^* as time increases to infinity. That is, the following theorem.

Theorem 2.2. *Assume that $F \in \widehat{V}^*$, $\widehat{G}$ satisfies (2.7) and the delay term $G(t, w_t)$ in (1.6) is given by $G(t, w_t) = \widehat{G}(w(t - \rho(t)))$. Suppose also that*

$$\delta_1 \lambda_1 > L_0 \quad \textit{and} \quad \delta_1 \lambda_1 > \frac{L_0}{\sqrt{1 - \rho_*}} + \frac{\lambda_1^{\frac{1}{2}} \lambda_0 \|F\|_{\widehat{V}^*}}{\delta_1 - \lambda_1^{-1} L_0}. \tag{2.18}$$

Then, for all $w^0 \in \widehat{H}$ and $\phi \in L^2(-h, 0; \widehat{V})$, the solution $w(t)$ of (1.6) with $F(t) \equiv F$ exponentially approaches the solution w^* of (2.8) as t goes to $+\infty$. To be exact, there exists two positive constants r_1 and r_2 such that

$$\|w(t) - w^*\|^2 \leqslant r_1 e^{-r_2 t} \big(\|w^0 - w^*\|^2 + \|\phi - w^*\|_{L^2(-h,0;\widehat{V})}\big), \quad \forall \, t \geqslant 0. \tag{2.19}$$

Proof. Set $\bar{w}(t) = w(t) - w^*$, and observe that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\bar{w}(t), v) + \langle A\bar{w}(t), v\rangle + \langle B(u(t), w(t)), v\rangle - \langle B(u^*, w^*), v\rangle + \langle N(\bar{w}(t)), v\rangle \\ =&(\widehat{G}(w(t-\rho(t))), v) - (\widehat{G}(w^*), v) \quad \text{for all } v \in \widehat{V}.\end{aligned}$$

In particular, it holds that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{w}(t)\|^2 + \langle A\bar{w}(t), \bar{w}(t)\rangle + \langle N(\bar{w}(t)), \bar{w}(t)\rangle \\ =&(\widehat{G}(w(t-\rho(t))), \bar{w}(t)) - (\widehat{G}(w^*), \bar{w}(t)) - \langle B(u(t), w(t)), \bar{w}(t)\rangle + \langle B(u^*, w^*), \bar{w}(t)\rangle.\end{aligned} \tag{2.20}$$

From (1.2), (2.2) and (2.4), we see that

$$\begin{aligned}&|-\langle B(u(t), w(t)), \bar{w}(t)\rangle + \langle B(u^*, w^*), \bar{w}(t)\rangle| = |\langle B(u(t)-u^*, \bar{w}(t)), w^*\rangle| \\ \leqslant&\lambda_0\|u(t)-u^*\|^{\frac{1}{2}}\|u(t)-u^*\|_{\widehat{V}}^{\frac{1}{2}}\|\bar{w}(t)\|_{\widehat{V}}\|w^*\|^{\frac{1}{2}}\|w^*\|_{\widehat{V}}^{\frac{1}{2}} \\ \leqslant&\lambda_1^{-\frac{1}{2}}\lambda_0\|\bar{w}(t)\|_{\widehat{V}}^2\|w^*\|_{\widehat{V}}.\end{aligned} \tag{2.21}$$

In addition, it follows from (1.2) and (2.7) that

$$\begin{aligned}&(\widehat{G}(w(t-\rho(t))), \bar{w}(t)) - (\widehat{G}(w^*), \bar{w}(t)) \leqslant \|\widehat{G}(w(t-\rho(t))) - \widehat{G}(w^*)\|\|\bar{w}(t)\| \\ \leqslant&L_0\|w(t-\rho(t)) - w^*\|\|\bar{w}(t)\| \leqslant \lambda_1^{-1}L_0\|\bar{w}(t-\rho(t))\|_{\widehat{V}}\|\bar{w}(t)\|_{\widehat{V}}.\end{aligned} \tag{2.22}$$

Taking (2.6), (2.20)-(2.22) and the estimate $\|w^*\|_{\widehat{V}} \leqslant \frac{\|F\|_{\widehat{V}^*}}{\delta_1 - \lambda_1^{-1}L_0}$ (which can be deduced from (2.14)) into account, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{w}(t)\|^2 + 2\delta_1\|\bar{w}(t)\|_{\widehat{V}}^2 &\leqslant 2\lambda_1^{-1}L_0\|\bar{w}(t-\rho(t))\|_{\widehat{V}}\|\bar{w}(t)\|_{\widehat{V}} + \frac{2\lambda_1^{-\frac{1}{2}}\lambda_0\|F\|_{\widehat{V}^*}}{\delta_1 - \lambda_1^{-1}L_0}\|\bar{w}(t)\|_{\widehat{V}}^2 \\ &\leqslant \Big(\frac{\delta L_0}{\lambda_1} + \frac{2\lambda_1^{-\frac{1}{2}}\lambda_0\|F\|_{\widehat{V}^*}}{\delta_1 - \lambda_1^{-1}L_0}\Big)\|\bar{w}(t)\|_{\widehat{V}}^2 + \frac{L_0}{\delta\lambda_1}\|\bar{w}(t-\rho(t))\|_{\widehat{V}}^2,\end{aligned}$$

where δ is a positive constant specified later. Obviously, the above inequality yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(e^{r_2t}\|\bar{w}(t)\|^2) \\ \leqslant&r_2e^{r_2t}\|\bar{w}(t)\|^2 - 2\delta_1e^{r_2t}\|\bar{w}(t)\|_{\widehat{V}}^2 + e^{r_2t}\Big(\frac{\delta L_0}{\lambda_1} + \frac{2\lambda_1^{-\frac{1}{2}}\lambda_0\|F\|_{\widehat{V}^*}}{\delta_1 - \lambda_1^{-1}L_0}\Big)\|\bar{w}(t)\|_{\widehat{V}}^2 \\ &+ \frac{L_0}{\delta\lambda_1}e^{r_2t}\|\bar{w}(t-\rho(t))\|_{\widehat{V}}^2 \\ \leqslant&\lambda_1^{-1}e^{r_2t}\Big(r_2 - 2\delta_1\lambda_1 + \delta L_0 + \frac{2\lambda_1^{\frac{1}{2}}\lambda_0\|F\|_{\widehat{V}^*}}{\delta_1 - \lambda_1^{-1}L_0}\Big)\|\bar{w}(t)\|_{\widehat{V}}^2 + \frac{L_0}{\delta\lambda_1}e^{r_2t}\|\bar{w}(t-\rho(t))\|_{\widehat{V}}^2,\end{aligned}$$

where r_2 is a positive constant which will be specified later. Consequently, for any $t \in [0, T]$,

$$e^{r_2t}\|\bar{w}(t)\|^2 \leqslant\|\bar{w}(0)\| + \lambda_1^{-1}\Big(r_2 - 2\delta_1\lambda_1 + \delta L_0 + \frac{2\lambda_1^{\frac{1}{2}}\lambda_0\|F\|_{\widehat{V}^*}}{\delta_1 - \lambda_1^{-1}L_0}\Big)\int_0^t e^{r_2\theta}\|\bar{w}(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta$$

$$+\frac{L_0}{\delta\lambda_1}\int_0^t e^{r_2\theta}\|\bar{w}(\theta-\rho(\theta))\|_{\widehat{V}}^2\mathrm{d}\theta. \tag{2.23}$$

Now, we give a further estimate for (2.23). Set $\eta(t) := t-\rho(t)$, then $\eta(t)$ is a strictly increasing function. Since $\rho(t)\in[0,h]$, we may conclude that $\eta^{-1}(s)\leqslant s+h$. Observe that

$$\begin{aligned}&\int_0^t e^{r_2\theta}\|\bar{w}(\theta-\rho(\theta))\|_{\widehat{V}}^2\mathrm{d}\theta\\ =&\int_{-\rho(0)}^{t-\rho(t)} e^{r_2\eta^{-1}(s)}\|\bar{w}(s)\|_{\widehat{V}}^2\cdot\frac{1}{1-\rho'(\eta^{-1}(s))}\mathrm{d}s\leqslant\frac{e^{r_2h}}{1-\rho_*}\int_{-h}^t e^{r_2s}\|\bar{w}(s)\|_{\widehat{V}}^2\mathrm{d}s.\end{aligned}$$

Substituting the above inequality into (2.23), we obtain

$$\begin{aligned}&e^{r_2t}\|\bar{w}(t)\|^2\\ \leqslant&\|\bar{w}(0)\|^2+\frac{L_0e^{r_2h}}{\delta\lambda_1(1-\rho_*)}\int_{-h}^t e^{r_2s}\|\bar{w}(s)\|_{\widehat{V}}^2\mathrm{d}s\\ &+\lambda_1^{-1}\big(r_2-2\delta_1\lambda_1+\delta L_0+\frac{2\lambda_1^{\frac12}\lambda_0\|F\|_{\widehat{V}^*}}{\delta_1-\lambda_1^{-1}L_0}\big)\int_0^t e^{r_2\theta}\|\bar{w}(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta\\ \leqslant&\|\bar{w}(0)\|^2+\frac{L_0e^{r_2h}}{\delta\lambda_1(1-\rho_*)}\int_{-h}^0 \|\bar{w}(s)\|_{\widehat{V}}^2\mathrm{d}s\\ &+\lambda_1^{-1}\big(\frac{L_0e^{r_2h}}{\delta(1-\rho_*)}+r_2-2\delta_1\lambda_1+\delta L_0+\frac{2\lambda_1^{\frac12}\lambda_0\|F\|_{\widehat{V}^*}}{\delta_1-\lambda_1^{-1}L_0}\big)\int_0^t e^{r_2\theta}\|\bar{w}(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta.\end{aligned}$$

Thanks to (2.18), choosing an appropriate δ such that $\frac{L_0}{\delta(1-\rho_*)}+\delta L_0=\frac{2L_0}{\sqrt{1-\rho_*}}$, then there exists $r_2>0$ small enough such that

$$\frac{L_0}{\delta(1-\rho_*)}e^{r_2h}+r_2-2\delta_1\lambda_1+\delta L_0+\frac{2\lambda_1^{\frac12}\lambda_0\|F\|_{\widehat{V}^*}}{\delta_1-\lambda_1^{-1}L_0}\leqslant 0.$$

Consequently, we deduce that

$$\|\bar{w}(t)\|^2\leqslant e^{-r_2t}\|\bar{w}(0)\|^2+\frac{L_0e^{r_2h}}{\delta\lambda_1(1-\rho_*)}e^{-r_2t}\int_{-h}^0\|\bar{w}(s)\|_{\widehat{V}}^2\mathrm{d}s.$$

Therefore, (2.19) is satisfied with $r_1=\max\{1,\frac{L_0e^{r_2h}}{\delta\lambda_1(1-\rho_*)}\}$. This completes the proof. □

3. Global well-posedness of the weak solutions

In this section, we concentrate on proving the global existence, uniqueness and stability of the weak solution to system (1.6).

In order to establish the global well-posedness of the weak solutions, the following assumption is required.

(A) Assume that $G:[0,T]\times L^2(-h,0;\widehat{H})\mapsto(L^2(\Omega))^3$ satisfies:

(i) For any $\xi\in L^2(-h,0;\widehat{H})$, the mapping $[0,T]\ni t\mapsto G(t,\xi)\in(L^2(\Omega))^3$ is measurable.

(ii) $G(\cdot, 0) = (0, 0, 0)$.

(iii) There exists a constant $L_G > 0$ such that for any $t \in [0, T]$ and any $\xi, \eta \in L^2(-h, 0; \widehat{H})$,

$$\|G(t, \xi) - G(t, \eta)\| \leqslant L_G \|\xi - \eta\|_{L^2(-h,0;\widehat{H})}.$$

(iv) There exists $C_G \in (0, \delta_1)$ such that, for any $t \in [0, T]$ and any $w, v \in L^2(-h, T; \widehat{H})$,

$$\int_0^t \|G(\theta, w_\theta) - G(\theta, v_\theta)\|^2 \mathrm{d}\theta \leqslant C_G^2 \int_{-h}^t \|w(\theta) - v(\theta)\|^2 \mathrm{d}\theta.$$

Moreover, for any $t \in [0, T]$, there exists a $\gamma \in (0, 2\delta_1 - 2C_G)$ such that

$$\int_0^t e^{\gamma\theta} \|G(\theta, w_\theta)\|^2 \mathrm{d}\theta \leqslant C_G^2 \int_{-h}^t e^{\gamma\theta} \|w(\theta)\|^2 \mathrm{d}\theta, \ \forall\, w \in L^2(-h, T; \widehat{H}).$$

(v) If w^m converges to w weakly in $L^2(-h, T; \widehat{V})$, weakly star in $L^\infty(0, T; \widehat{H})$ and strongly in $L^2(-h, T; (L^2(\mathcal{Q}))^3)$ for a bounded open set $\mathcal{Q} \subset \Omega$ with smooth boundary, then $G(\cdot, w^m_\cdot)$ converges weakly to $G(\cdot, w_\cdot)$ in $L^2(0, T; \widehat{H}(\mathcal{Q}))$.

Now, we show the existence of the weak solutions in the following theorem.

Theorem 3.1 (Existence). *Assume that $F(t, x) \in L^2(0, T; \widehat{V}^*)$ and $G(t, w_t)$ satisfies (A), then, for any given initial data $w^0 \in \widehat{H}, \phi \in L^2(-h, 0; \widehat{V})$, there corresponds at least one weak solution to system* (1.6).

Proof. We will divide the proof into three steps.

Step One: *Local existence and uniqueness of the Galerkin approximate solutions.* Consider an orthonormal basis $\{v_j\}_{j=1}^\infty \subset \widehat{\mathcal{V}}$ of $\widehat{H}$ such that

$$\text{span } \{v_1, v_2, \cdots, v_n, \cdots\} \text{ is dense in } \widehat{V}.$$

Denote $\widehat{V}_m$:=span$\{v_1, v_2, \cdots, v_m\}$ and consider the projector

$$P_m w := \sum_{j=1}^m (w, v_j) v_j, \ w \in \widehat{H} \text{ or } \widehat{V}.$$

For each $T > 0$, define $w^m(t) := \sum\limits_{j=1}^m \beta_{m,j}(t) v_j$, where the coefficients $\beta_{m,j}(t)$ are desired to satisfy the following Cauchy problem of ordinary differential equations:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(w^m(t), v_j) + \langle Aw^m(t), v_j \rangle + \langle B(u^m(t), w^m(t)), v_j \rangle + \langle N(w^m(t)), v_j \rangle \\ \qquad\qquad = \langle F(t, x), v_j \rangle + (G(t, w_t^m), v_j), \ 1 \leqslant j \leqslant m, \ t \in [0, T], \\ w^m(0) = P_m w^0, \ w^m(t) = P_m \phi(t), \quad t \in (-h, 0). \end{cases} \tag{3.1}$$

Based on Theorem 2.1 in [14], the existence and uniqueness of the Galerkin approximate solution follows.

Step Two: *A priori estimates of the Galerkin approximate solutions.*

We now deduce a priori estimates to obtain the global existence of the Galerkin approximate solutions. Multiplying $(3.1)_1$ by $\beta_{m,j}(t)$, summing up for j from 1 to m and using (2.2) and (2.6), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w^m(t)\|^2+\delta_1\|w^m(t)\|_{\widehat{V}}^2\\ \leqslant&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w^m(t)\|^2+\langle Aw^m,w^m\rangle+\langle N(w^m(t)),w^m(t)\rangle+\langle B(u^m(t),w^m(t)),w^m(t)\rangle\\ =&\langle F(t),w^m(t)\rangle+(G(t,w_t^m),w^m(t)).\end{aligned}\tag{3.2}$$

Multiplying (3.2) by $e^{\gamma t}$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(e^{\gamma t}\|w^m(t)\|^2)-\gamma e^{\gamma t}\|w(t)\|^2+2\delta_1 e^{\gamma t}\|w^m(t)\|_{\widehat{V}}^2&\\ \leqslant 2e^{\gamma t}\langle F(t),w^m(t)\rangle+2e^{\gamma t}(G(t,w_t^m),w^m(t)).&\end{aligned}$$

Let $0\leqslant\theta\leqslant t\leqslant T$. Replacing the time variable t with θ in the above inequality, then integrating it for θ over $[0,t]$ gives

$$\begin{aligned}&e^{\gamma t}\|w^m(t)\|^2+(2\delta_1-\gamma)\int_0^t e^{\gamma\theta}\|w(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta\\ \leqslant&\|w^m(0)\|^2+2\int_0^t e^{\gamma\theta}\langle F(\theta),w^m(\theta)\rangle\mathrm{d}\theta+2\int_0^t e^{\gamma\theta}(G(\theta,w_\theta^m),w^m(\theta))\mathrm{d}\theta.\end{aligned}\tag{3.3}$$

By Young's inequality and assumption (A), we see that

$$\begin{aligned}&2\int_0^t e^{\gamma\theta}(G(\theta,w_\theta^m),w^m(\theta))\mathrm{d}\theta\leqslant 2\int_0^t e^{\gamma\theta}\|G(\theta,w_\theta^m)\|\|w^m(\theta)\|\mathrm{d}\theta\\ \leqslant&2\big(\int_0^t e^{\gamma\theta}\|G(\theta,w_\theta^m)\|^2\mathrm{d}\theta\big)^{\frac{1}{2}}\big(\int_0^t e^{\gamma\theta}\|w^m(\theta)\|^2\mathrm{d}\theta\big)^{\frac{1}{2}}\\ \leqslant&C_G\int_{-h}^0 e^{\gamma\theta}\|w^m(\theta)\|^2\mathrm{d}\theta+2C_G\int_0^t e^{\gamma\theta}\|w^m(\theta)\|^2\mathrm{d}\theta,\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}&2\int_0^t e^{\gamma\theta}\langle F(\theta),w^m(\theta)\rangle\mathrm{d}\theta\leqslant 2\int_0^t e^{\gamma\theta}\|F(\theta)\|_{\widehat{V}^*}\|w^m(\theta)\|_{\widehat{V}}\mathrm{d}\theta\\ \leqslant&\alpha^{-1}\int_0^t e^{\gamma\theta}\|F(\theta)\|_{\widehat{V}^*}^2\mathrm{d}\theta+\alpha\int_0^t e^{\gamma\theta}\|w^m(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta,\end{aligned}\tag{3.5}$$

where $\alpha\in(0,2\delta_1-\gamma-2C_G)$. Substituting (3.4) and (3.5) into (3.3), we have

$$\begin{aligned}&e^{\gamma t}\|w^m(t)\|^2+\beta\int_0^t e^{\gamma\theta}\|w(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta\\ \leqslant&\|w^m(0)\|^2+C_G\|P_m\phi\|_{L^2(-h,0;\widehat{H})}^2+\alpha^{-1}\int_0^t e^{\gamma\theta}\|F(\theta)\|_{\widehat{V}^*}^2\mathrm{d}\theta,\end{aligned}$$

where $\beta:=2\delta_1-\gamma-2C_G-\alpha>0$. It is obtained easily from the above inequality that there exist two constants k_1 and k_2 (depending on $w^0,\phi,\delta_1,F,G,h,T$, but not on m nor $t_*\leqslant T$) such that

$$\sup_{t\in[0,t_*]}\|w^m(t)\|^2\leqslant k_1,\quad\int_0^{t_*}\|w^m(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta\leqslant k_2.\tag{3.6}$$

Moreover, observe that $w^m = P_m\phi$ in $(-h,0)$ converges to ϕ in $L^2(-h,0;\widehat{V})$. Thus, we can take $t_* = T$ and obtain that

$$\{w^m\} \text{ is bounded in } L^2(-h,T;\widehat{V}) \cap L^\infty(0,T;\widehat{H}), \tag{3.7}$$

which together with the local existence obtained in step one gives to the global existence of the Galerkin approximate solution for all time $t \in [0,T]$.

Step Three: *Existence of the global weak solutions.*

We will prove that the limit function of the Galerkin approximate solutions is a weak solution of (1.6). Using the diagonal procedure, we deduce from (3.7) that there exists a subsequence (which is still denoted by) $\{w^m\}$, an element $w \in L^\infty(0,T;\widehat{H}) \cap L^2(0,T;\widehat{V})$ such that

$$\begin{cases} w^m \rightharpoonup^* w \text{ weakly star in } L^\infty(0,T;\widehat{H}) & \text{as } m \to \infty, \\ w^m \rightharpoonup w \text{ weakly in } L^2(0,T;\widehat{V}) & \text{as } m \to \infty. \end{cases} \tag{3.8}$$

Based on the above argument, we claim that, for any bounded open set $\mathcal{Q} \subset \Omega$, there exists a subsequence (depending on $\mathcal{Q}$ which we relabel) satisfying

$$w^m(t) \to w(t) \quad \text{strongly in} \quad L^2(0,T;\widehat{H}(\mathcal{Q})) \quad \text{as} \quad m \to \infty. \tag{3.9}$$

For the sake of clarity, we give the proof of (3.9) in the back of the present theorem.

Now, fix an element v_j and let $\varphi \in \mathcal{C}^1([0,T])$ with $\varphi(T) = 0$. Then, it follows from (3.1) that

$$\begin{aligned} &-\int_0^T (w^m(t), v_j\varphi'(t))\mathrm{d}t + \int_0^T \langle Aw^m(t), v_j\varphi(t)\rangle \mathrm{d}t \\ &\qquad + \int_0^T \langle B(u^m(t), w^m(t)), v_j\varphi(t)\rangle \mathrm{d}t + \int_0^T \langle N(w^m(t)), v_j\varphi(t)\rangle \mathrm{d}t \\ =&(w^m(0), v_j)\varphi(0) + \int_0^T \langle F(t), v_j\varphi(t)\rangle \mathrm{d}t + \int_0^T (G(t,w_t^m), v_j\varphi(t))\mathrm{d}t. \end{aligned} \tag{3.10}$$

In the following, we are committed to passing to the limit in (3.10) to obtain a weak solution. Choose a subsequence (denoted again by w^m) by using diagonal procedure that satisfies (3.9) for a sequence of regular bounded open sets $\mathcal{Q}_j \subset \Omega$ containing all supports of functions v_j of the basis. Observe that for every $v \in \widehat{V}$, by the density, there exists a sequence $\{v_j\} \subset \widehat{\mathcal{V}}$ such that $v_j \to v$ in $\widehat{V}$ as $j \to \infty$. Namely, for any $\epsilon > 0$, there exists a $n_\epsilon > 0$ such that

$$\|v_j - v\|_{\widehat{V}} < \epsilon \quad \text{for all} \quad j \geqslant n_\epsilon. \tag{3.11}$$

Thus, from (3.8), (3.9) and assumption (A), it is not difficult to see that for the above ϵ, there exists a $m_\epsilon \geqslant n_\epsilon$ such that, for all $m \geqslant m_\epsilon$,

$$\begin{aligned} &\Big|\int_0^T (G(t,w_t^m), v_{n_\epsilon}\varphi(t))\mathrm{d}t - \int_0^T (G(t,w_t), v\varphi(t))\mathrm{d}t\Big| \\ =&\Big|\int_0^T \int_\Omega (G(t,w_t^m) - G(t,w_t)) \cdot v_{n_\epsilon}\varphi(t)\mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega G(t,w_t)\cdot(v_{n_\epsilon} - v)\varphi(t)\mathrm{d}x\mathrm{d}t\Big| \\ =&\Big|\int_0^T \int_{\mathcal{Q}_{n_\epsilon}} (G(t,w_t^m) - G(t,w_t)) \cdot v_{n_\epsilon}\varphi(t)\mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega G(t,w_t)\cdot(v_{n_\epsilon} - v)\varphi\mathrm{d}x\mathrm{d}t\Big| \end{aligned}$$

$$\begin{aligned}&\leqslant \|G(t,w_t^m)-G(t,w_t)\|_{L^2(0,T;\widehat{H}(\mathcal{Q}_{n_\epsilon}))}\|\varphi(t)\|_{L^2(0,T)}\|v_{n_\epsilon}\|\\&\quad+\|v_{n_\epsilon}-v\|\int_0^T\|(G(t,w_t)\|\cdot|\varphi(t)|\mathrm{d}t\\&\leqslant C_1\epsilon.\end{aligned}$$

Similarly, it holds that

$$\Big|\int_0^T(w^m(t),v_{n_\epsilon}\varphi'(t))\mathrm{d}t-\int_0^T(w(t),v\varphi'(t))\mathrm{d}t\Big|\leqslant C_2\epsilon,$$
$$\Big|\int_0^T\langle F(t),v_{n_\epsilon}\varphi(t)\rangle\mathrm{d}t-\int_0^T\langle F(t),v\varphi(t)\rangle\mathrm{d}t\Big|\leqslant C_3\epsilon$$

and

$$\Big|(w^m(0),v_{n_\epsilon})\varphi(0)-(w(0),v)\varphi(0)\Big|\leqslant C_4\epsilon.$$

In addition, by (1.5), (2.4), (2.5), (3.7)-(3.9), (3.11) and Lemma 2.1, we have

$$\begin{aligned}&\Big|\int_0^T\langle Aw^m(t),v_{n_\epsilon}\varphi(t)\rangle\mathrm{d}t-\int_0^T\langle Aw(t),v\varphi(t)\rangle\mathrm{d}t\Big|\\&\leqslant\max\{\nu+\nu_r,\bar{\alpha}\}\Big(\Big|\int_0^T\int_\Omega\nabla(w^m-w)\cdot\nabla(v_{n_\epsilon}\cdot\varphi(t))\mathrm{d}x\mathrm{d}t\Big|\\&\qquad+\int_0^T\|\nabla w(t)\|\|\nabla(v_{n_\epsilon}-v)\||\varphi(t)|\mathrm{d}t\Big)\\&\leqslant\max\{\nu+\nu_r,\bar{\alpha}\}\Big(\Big|\int_0^T\int_\Omega\nabla(w^m-w)\cdot\nabla(v_{n_\epsilon}\cdot\varphi(t))\mathrm{d}x\mathrm{d}t\Big|\\&\qquad+\|v_{n_\epsilon}-v\|_{\widehat{V}}\int_0^T\|\nabla w(t)\||\varphi(t)|\mathrm{d}t\Big)\leqslant C_5\epsilon,\end{aligned}$$

$$\begin{aligned}&\Big|\int_0^T\langle B(u^m,w^m),v_{n_\epsilon}\varphi(t)\rangle\mathrm{d}t-\int_0^T\langle B(u,w),v\varphi(t)\rangle\mathrm{d}t\Big|\\&=\Big|\int_0^T\Big(\langle B(u^m-u,w^m),v_{n_\epsilon}\varphi(t)\rangle+\langle B(u,w^m-w),v_{n_\epsilon}\varphi(t)\rangle\\&\qquad+\langle B(u,w),(v_{n_\epsilon}-v)\varphi(t)\rangle\Big)\mathrm{d}t\Big|\\&\leqslant\lambda_0\sup_{t\in[0,T]}|\varphi(t)|\Big\{\int_0^T\|u^m-u\|^{\frac12}_{H(\mathcal{Q}_{n_\epsilon})}\|\nabla(u^m-u)\|^{\frac12}\|w^m\|^{\frac12}\|\nabla w^m\|^{\frac12}\mathrm{d}t\ \|\nabla v_{n_\epsilon}\|\\&\qquad+\int_0^T\|u\|^{\frac12}\|\nabla u\|^{\frac12}\|w^m-w\|^{\frac12}_{\widehat{H}(\mathcal{Q}_{n_\epsilon})}\|\nabla(w^m-w)\|^{\frac12}\mathrm{d}t\ \|\nabla v_{n_\epsilon}\|\\&\qquad+\int_0^T\|u\|^{\frac12}\|\nabla u\|^{\frac12}\|w\|^{\frac12}\|\nabla w\|^{\frac12}\mathrm{d}t\ \|v_{n_\epsilon}-v\|_{\widehat{V}}\Big\}\\&\leqslant\lambda_0\sup_{t\in[0,T]}|\varphi(t)|\\&\quad\cdot\Big\{\|u^m-u\|^{\frac14}_{L^2(0,T;H(\mathcal{Q}_{n_\epsilon}))}\|u^m-u\|^{\frac14}_{L^2(0,T;V)}\|w^m\|^{\frac14}_{L^2(0,T;\widehat{H})}\|w^m\|^{\frac14}_{L^2(0,T;\widehat{V})}\|v_{n_\epsilon}\|_{\widehat{V}}\\&\quad+\|u\|^{\frac14}_{L^2(0,T;H)}\|u\|^{\frac14}_{L^2(0,T;V)}\|w^m-w\|^{\frac14}_{L^2(0,T;\widehat{H}(\mathcal{Q}_{n_\epsilon}))}\|w^m-w\|^{\frac14}_{L^2(0,T;\widehat{V}(\mathcal{Q}_{n_\epsilon}))}\|v_{n_\epsilon}\|_{\widehat{V}}\end{aligned}$$

$$+\|u\|^{\frac{1}{4}}_{L^2(0,T;H)}\|u\|^{\frac{1}{4}}_{L^2(0,T;V)}\|w\|^{\frac{1}{4}}_{L^2(0,T;\widehat{H})}\|w\|^{\frac{1}{4}}_{L^2(0,T;\widehat{V})}\|v_{n_\epsilon}-v\|_{\widehat{V}}\Big\}\leqslant C_6\epsilon$$

and

$$\begin{aligned}
&\Big|\int_0^T\langle N(w^m(t)),v_{n_\epsilon}\varphi(t)\rangle\mathrm{d}t-\int_0^T\langle N(w(t)),v\varphi(t)\rangle\mathrm{d}t\Big|\\
=&\Big|\int_0^T\int_\Omega N(w^m(t)-w(t))\cdot v_{n_\epsilon}\varphi(t)\mathrm{d}x\mathrm{d}t+\int_0^T\int_\Omega N(w(t))\cdot(v_{n_\epsilon}-v)\varphi(t)\mathrm{d}x\mathrm{d}t\Big|\\
\leqslant&\Big|\int_0^T\int_{\mathcal{Q}_{n_\epsilon}}(-2\nu_r\nabla\times(\omega^m-\omega),-2\nu_r\nabla\times(u^m-u)+4\nu_r(\omega^m-\omega))\cdot v_{n_\epsilon}\mathrm{d}x\varphi(t)\mathrm{d}t\Big|\\
&\qquad\qquad+\|v_{n_\epsilon}-v\|\int_0^T\|N(w(t))\|\cdot|\varphi(t)|\mathrm{d}t\\
\leqslant&4\nu_r\int_0^T\|w^m(t)-w(t)\|_{\widehat{H}(\mathcal{Q}_{n_\epsilon})}\|v_{n_\epsilon}\|_{\widehat{V}(\mathcal{Q}_{n_\epsilon})}|\varphi(t)|\mathrm{d}t+c(\nu_r)\|v_{n_\epsilon}-v\|\int_0^T\|w(t)\|_{\widehat{V}}|\varphi(t)|\mathrm{d}t\\
\leqslant&4\nu_r\|w^m(t)-w(t)\|_{L^2(0,T;\widehat{H}(\mathcal{Q}_\epsilon))}\|v_{n_\epsilon}\|_{\widehat{V}(\mathcal{Q}_{n_\epsilon})}\|\varphi(t)\|_{L^2(0,T)}\\
&+c(\nu_r)\|v_{n_\epsilon}-v\|\|w(t)\|_{L^2(0,T;\widehat{V})}\|\varphi(t)\|_{L^2(0,T)}\\
\leqslant&C_7\epsilon,
\end{aligned}$$

where $C_i, i=1,2,3,4,5,6,7$ are positive constants. According to the above seven inequalities and the arbitrariness of ϵ, we can pass to the limit in (3.10) and obtain

$$\begin{aligned}
&-\int_0^T(w(t),v\varphi'(t))\mathrm{d}t+\int_0^T\langle Aw(t),v\varphi(t)\rangle\mathrm{d}t\\
&\qquad+\int_0^T\langle B(u,w),v\varphi(t)\rangle\mathrm{d}t+\int_0^T\langle N(w(t)),v\varphi(t)\rangle\mathrm{d}t\\
=&(w^0,v)\varphi(0)+\int_0^T\langle F(t),v\varphi(t)\mathrm{d}t+\int_0^T(G(t,w_t),v\varphi(t))\mathrm{d}t,\ \forall\, v\in\widehat{V}.
\end{aligned}\tag{3.12}$$

Since $(w(t),v_j)\varphi(t)\in H^1(0,T)$, we also can obtain an analogous expression to (3.12) with $(w(0),v)$ instead of (w^0,v). This implies $(w(0)-w^0,v)=0$ for all $v\in\widehat{V}$, hence $w(0)=w^0$. It makes sense at time $t=0$. Writing (3.12) for $\varphi\in\mathcal{D}(0,T)$, w satisfies (1.6) in the distribution sense. This completes the proof.

Proof of (3.9). For this purpose, we will check the situation fits to Lemma 2.3 with $r=2, q=+\infty, I=(0,T)$.

(1) if $\mathcal{Q}\subset\subset\Omega$, then there exists a finite covering of balls, denoted by $\tilde{\mathcal{Q}}\subset\Omega$, which is bounded and open, such that $X=(H^1(\tilde{\mathcal{Q}}))^3\hookrightarrow\hookrightarrow E=(L^2(\tilde{\mathcal{Q}}))^3$.

(2) For a general $\mathcal{Q}\subset\Omega$ the above comment may be not true since $\mathcal{Q}$ and Ω can share part of their boundaries. The compact injection from H^1 may not hold for lack of regularity on the boundary Γ, but it does in H^1_0. Therefore, we consider a truncation argument.

Taking a function $\chi(\cdot)\in\mathcal{C}^1(\mathbb{R}^2), \chi(x)\in[0,1], \forall\, x\in\mathbb{R}^2$ such that

$$\chi(x)=\begin{cases}1, & |x|\leqslant 1,\\ 0, & |x|\geqslant 4.\end{cases}$$

Let $l > 0$ be large enough such that $\mathcal{Q} \subset B(0,l)$ and denote

$$\tilde{\mathcal{Q}} := \Omega \cap B(0,2l),\ \ w^{m,l}(x) := w^m(x)\chi(\frac{|x|^2}{l^2}).$$

Then $X = (H_0^1(\tilde{\mathcal{Q}}))^3 \hookrightarrow\hookrightarrow E = (L^2(\tilde{\mathcal{Q}}))^3$. We conserve the original functions $w^m(\cdot)$ on $\Omega \cap B(0,l)$ and, for clarity, continue the proof directly with $w^m(\cdot)$ instead of $w^{m,l}(\cdot)$.

Obviously, the condition (ii) in Lemma 2.3 follows from (3.7). In the following, we concentrate on verifying the condition (i). In fact, we will prove that, for the whole domain Ω,

$$\sup_{m\in\mathbb{N}} \|\Pi_h w^m - w^m\|_{L^2(0,T-h;(L^2(\Omega))^3)} \to 0 \text{ as } h \to 0. \tag{3.13}$$

Consider $h > 0$ arbitrarily small. From (3.1), we see that, for $(t, t+h) \subset (0,T)$,

$$\begin{aligned}(w^m(t+h) - w^m(t), v_j) + \int_t^{t+h} \langle Aw^m(\theta), v_j\rangle \mathrm{d}\theta + \int_t^{t+h} \langle B(u^m(\theta), w^m(\theta)), v_j\rangle \mathrm{d}\theta \\ + \int_t^{t+h} \langle N(w^m(\theta)), v_j\rangle \mathrm{d}\theta \\ = \int_t^{t+h} \langle F(\theta), v_j\rangle \mathrm{d}\theta + \int_t^{t+h} (G(\theta, w^m_\theta), v_j)\mathrm{d}\theta.\end{aligned}$$

Multiplying the above inequality by $\beta_{m,j}(t+h) - \beta_{m,j}(t)$ and summing in j, we obtain

$$\begin{aligned}&\|w^m(t+h) - w^m(t)\|^2 \\ =& - \int_t^{t+h} \langle Aw^m(\theta), w^m(t+h) - w^m(t)\rangle \mathrm{d}\theta \\ &- \int_t^{t+h} \langle B(u^m(\theta), w^m(\theta)), w^m(t+h) - w^m(t)\rangle \mathrm{d}\theta \\ &- \int_t^{t+h} \langle N(w^m(\theta)), w^m(t+h) - w^m(t)\rangle \mathrm{d}\theta + \int_t^{t+h} \langle F(\theta), w^m(t+h) - w^m(t)\rangle \mathrm{d}\theta \\ &+ \int_t^{t+h} (G(\theta, w^m_\theta), w^m(t+h) - w^m(t))\mathrm{d}\theta.\end{aligned} \tag{3.14}$$

In the following, we estimate the terms on the right-half side of (3.14) one by one. First, according to the definitions of the operators in (1.5), Lemma 2.1, Lemma 2.2, and using Hölder inequality, we have

$$\begin{aligned}&- \int_t^{t+h} \langle Aw^m(\theta), w^m(t+h) - w^m(t)\rangle \mathrm{d}\theta \\ \leqslant& c_1^{-1} \|w^m(t+h) - w^m(t)\|_{\widehat{V}} \int_t^{t+h} \|w^m(\theta)\|_{\widehat{V}} \mathrm{d}\theta,\end{aligned} \tag{3.15}$$

$$\begin{aligned}&- \int_t^{t+h} \langle B(u^m(\theta), w^m(\theta)), w^m(t+h) - w^m(t)\rangle \mathrm{d}\theta \\ \leqslant& \lambda_0 \|w^m(t+h) - w^m(t)\|_{\widehat{V}} \int_t^{t+h} \|u^m(\theta)\|^{\frac{1}{2}} \|\nabla u^m(\theta)\|^{\frac{1}{2}} \|w^m(\theta)\|^{\frac{1}{2}} \|\nabla w^m(\theta)\|^{\frac{1}{2}} \mathrm{d}\theta\end{aligned}$$

$$\leqslant \lambda_0 \|w^m(t+h) - w^m(t)\|_{\widehat{V}} \int_t^{t+h} \|w^m(\theta)\| \|w^m(\theta)\|_{\widehat{V}} \mathrm{d}\theta, \tag{3.16}$$

and

$$\begin{aligned} & - \int_t^{t+h} \langle N(w^m(\theta)), w^m(t+h) - w^m(t) \rangle \mathrm{d}\theta \\ \leqslant & \|w^m(t+h) - w^m(t)\| \int_t^{t+h} \|N(w^m(\theta))\| \mathrm{d}\theta \\ \leqslant & c(\nu_r) \|w^m(t+h) - w^m(t)\|_{\widehat{V}} \int_t^{t+h} \|w^m(\theta)\|_{\widehat{V}} \mathrm{d}\theta. \end{aligned} \tag{3.17}$$

Next, it is easy to know from Schwartz's inequality that

$$\int_t^{t+h} \langle F(\theta), w^m(t+h) - w^m(t) \rangle \mathrm{d}\theta \leqslant \|w^m(t+h) - w^m(t)\|_{\widehat{V}} \int_t^{t+h} \|F(\theta)\|_{\widehat{V}^*} \mathrm{d}\theta. \tag{3.18}$$

Finally, using (1.2) and Schwartz's inequality, we obtain

$$\begin{aligned} & \int_t^{t+h} (G(\theta, w_\theta^m), w^m(t+h) - w^m(t)) \mathrm{d}\theta \\ \leqslant & \|w^m(t+h) - w^m(t)\| \int_t^{t+h} \|G(\theta, w_\theta^m)\| \mathrm{d}\theta \\ \leqslant & \lambda_1^{-\frac{1}{2}} \|w^m(t+h) - w^m(t)\|_{\widehat{V}} \int_t^{t+h} \|G(\theta, w_\theta^m)\| \mathrm{d}\theta. \end{aligned} \tag{3.19}$$

Now, substituting (3.15)-(3.19) into (3.14) and integrating the resultant inequality from 0 to $T-h$, we have

$$\begin{aligned} \|\Pi_h w^m(t) - w^m(t)\|^2_{L^2(0,T-h;(L^2(\Omega))^3)} &= \int_0^{T-h} \int_\Omega |\Pi_h w^m(t) - w^m(t)|^2 \mathrm{d}x \mathrm{d}t \\ \leqslant \int_0^{T-h} \|w^m(t+h) - w^m(t)\|_{\widehat{V}} \int_t^{t+h} H_m(\theta) \mathrm{d}\theta \mathrm{d}t, \end{aligned}$$

where

$$H_m := (c_1^{-1} + c(\nu_r)) \|w^m(\theta)\|_{\widehat{V}} + \lambda_0 \|w^m(\theta)\| \|w^m(\theta)\|_{\widehat{V}} + \|F(\theta)\|_{\widehat{V}^*} + \lambda_1^{-\frac{1}{2}} \|G(\theta, w_\theta^m)\|.$$

Set

$$\bar{\theta} = \begin{cases} 0, & \text{if } \theta \leqslant 0, \\ \theta, & \text{if } 0 < \theta \leqslant T-h, \\ T-h, & \text{if } \theta > T-h. \end{cases}$$

Then, since $0 \leqslant \bar{\theta} - \overline{\theta - h} \leqslant h$, with the aid of (3.6) and the Fubini theorem, we obtain

$$\begin{aligned} & \int_0^{T-h} \|w^m(t+h) - w^m(t)\|_{\widehat{V}} \int_t^{t+h} H_m(\theta) \mathrm{d}\theta \mathrm{d}t \\ \leqslant & \int_0^T H_m(\theta) \int_{\overline{\theta-h}}^{\bar{\theta}} \|w^m(t+h) - w^m(t)\|_{\widehat{V}} \mathrm{d}t \mathrm{d}\theta \end{aligned}$$

$$\leqslant \int_0^T H_m(\theta)\Big(\int_{\overline{\theta-h}}^{\bar\theta}\|w^m(t+h)-w^m(t)\|_{\widehat V}^2\mathrm{d}t\Big)^{\frac12}\Big(\int_{\overline{\theta-h}}^{\bar\theta}\mathrm{d}t\Big)^{\frac12}\mathrm{d}\theta$$
$$\leqslant \int_0^T H_m(\theta)\cdot 2h^{\frac12}\Big(\int_0^{T-h}\|w^m\|_{\widehat V}^2\mathrm{d}t\Big)^{\frac12}\mathrm{d}\theta \leqslant 2(hk_2)^{\frac12}\int_0^T H_m(\theta)\mathrm{d}\theta. \tag{3.20}$$

From (3.7) and the assumption (A), it is not difficult to conclude that $\int_0^T H_m(\theta)\mathrm{d}\theta$ is bounded. Therefore, (3.13) is valid, that is, the condition (i) in Lemma 2.3 holds. Consequently, with the help of Lemma 2.3, (3.9) follows. □

Remark 3.1. On the assumption (A)(v). When condition (A)(v) is satisfied in all "good" $\mathcal{Q}\subset\Omega$ (actually, it is enough for all $\mathcal{Q}_j\supset supp(v_j)$), then the convergence holds in $L^2(0,T;\widehat H)$. Indeed, consider $\varphi\in\widehat H$. We check that $\lim_{m\to\infty}(G(\cdot,w_\cdot^{(m)}),\varphi)=(G(\cdot,w_\cdot),\varphi)$. Take a sequence $\varphi_n\in\widehat{\mathcal{V}}$ such that $\varphi_n\to\varphi$ in $\widehat H$, and fix $\epsilon>0$. Consider n_ϵ such that $\|G(\cdot,w_\cdot^m)\|\|\varphi_n-\varphi\|\leqslant\frac{\epsilon}{2}$ and $\|G(\cdot,w_\cdot)\|\|\varphi_n-\varphi\|\leqslant\frac{\epsilon}{2}$ for all $n\geqslant n_\epsilon$. Observe that $(G(\cdot,w_\cdot^m)-G(\cdot,w_\cdot),\varphi_{n_\epsilon})\to 0$, so it is possible to choose m_ϵ to conclude that the claim is true.

In the following, we investigate the uniqueness of the weak solution.

Theorem 3.2 (Uniqueness)**.** *Under the conditions of Theorem 3.1, for any $T>0$ and initial data $w^0\in\widehat H,\phi\in L^2(-h,0;\widehat V)$, there corresponds at most one weak solution to system* (1.6)*.*

Proof. Let $w^{(1)}=(u^{(1)},\omega^{(1)})$ and $w^{(2)}=(u^{(2)},\omega^{(2)})$ be two solutions of (1.6) in the interval $[-h,T]$ with the same initial data. Denote $w=(u,\omega)=w^{(1)}-w^{(2)}$, then w satisfies

$$\begin{aligned}&\frac{\partial w}{\partial t}+Aw+B(u^{(1)},w^{(1)})-B(u^{(2)},w^{(2)})+N(w)\\ =&G(t,w_t^{(1)})-G(t,w_t^{(2)}),\quad t\in[0,T].\end{aligned}\tag{3.21}$$

Testing (3.21) by $w(t)$, we have

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2+\langle Aw(t),w(t)\rangle+\langle B(u^{(1)},w^{(1)})-B(u^{(2)},w^{(2)}),w(t)\rangle+\langle N(w(t)),w(t)\rangle\\ =&(G(t,w_t^{(1)})-G(t,w_t^{(2)}),w(t)),\ t\in[0,T].\end{aligned}\tag{3.22}$$

By (2.2), (2.4), Young's inequality and the facts

$$\|u(t)\|\leqslant\|w(t)\| \text{ and } \|\nabla u(t)\|\leqslant\|\nabla w(t)\|,$$

we deduce that

$$\begin{aligned}&|\langle B(u^{(1)},w^{(1)})-B(u^{(2)},w^{(2)}),w(t)\rangle|\\ \leqslant&\lambda_0\|u(t)\|^{\frac12}\|\nabla u\|^{\frac12}\|w(t)\|^{\frac12}\|\nabla w(t)\|^{\frac12}\|\nabla w^{(1)}(t)\|\\ \leqslant&\lambda_0\|w(t)\|\|w(t)\|_{\widehat V}\|w^{(1)}(t)\|_{\widehat V}\leqslant\frac{\lambda_0^2}{4\delta_1}\|w^{(1)}(t)\|_{\widehat V}^2\|w(t)\|^2+\delta_1\|w(t)\|_{\widehat V}^2.\end{aligned}\tag{3.23}$$

In addition, it is easy to see that

$$(G(t,w_t^{(1)})-G(t,w_t^{(2)}),w(t))\leqslant\|G(t,w_t^{(1)})-G(t,w_t^{(2)})\|\|w(t)\|$$

$$\leqslant a_1\|w(t)\|^2 + a_2\|G(t,w_t^{(1)}) - G(t,w_t^{(2)})\|^2, \tag{3.24}$$

where the positive constants a_1 and a_2 satisfy $a_1a_2 \geqslant \frac{1}{4}$. Taking (2.6) and (3.22)-(3.24) into account, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2 + 2\delta_1\|w(t)\|_{\widehat{V}}^2 \leqslant & \frac{\lambda_0^2}{2\delta_1}\|w^{(1)}(t)\|_{\widehat{V}}^2\|w(t)\|^2 + 2\delta_1\|w(t)\|_{\widehat{V}}^2 + 2a_1\|w(t)\|^2 \\ & + 2a_2\|G(t,w_t^{(1)}) - G(t,w_t^{(2)})\|^2. \end{aligned}\tag{3.25}$$

Integrating the above inequality over $[0,t]$, we obtain from assumption (A) that

$$\begin{aligned}\|w(t)\|^2 \leqslant & \frac{\lambda_0^2}{2\delta_1}\int_0^t \|w^{(1)}(\theta)\|_{\widehat{V}}^2\|w(\theta)\|^2\mathrm{d}\theta + 2a_1\int_0^t \|w(\theta)\|^2\mathrm{d}\theta \\ & + 2a_2\int_0^t \|G(\theta,w_\theta^{(1)}) - G(\theta,w_\theta^{(2)})\|^2\mathrm{d}\theta \\ \leqslant & \int_0^t \big(\frac{\lambda_0^2}{2\delta_1}\|w^{(1)}(\theta)\|_{\widehat{V}}^2 + 2a_1 + 2a_2C_G^2\big)\|w(\theta)\|^2\mathrm{d}\theta. \end{aligned}\tag{3.26}$$

Using the Gronwall inequality to (3.26) yields

$$\|w(t)\| = 0, \quad \forall\, t \in [0,T].$$

This completes the proof. □

Finally, we verify the stability of the weak solutions with respect to the initial data.

Theorem 3.3 (Stability)**.** *Assume that the conditions of Theorem 3.1 hold, and let $w^{(i)}(\cdot)$ with $i = 1,2$ be two solutions of* (1.6) *in the interval [-h,T] with initial data $w^{(i)}(0) = w^{0^{(i)}}$ and $w^{(i)}(s,x) = \phi^{(i)}, s \in (-h,0)$, respectively. Then*

$$\begin{aligned}\|w^{(1)}(t) - w^{(2)}(t)\|^2 \leqslant & \big(\|w^{0^{(1)}} - w^{0^{(2)}}\|^2 + 2a_2C_G^2\|\phi^{(1)} - \phi^{(2)}\|_{L^2(-h,0;\widehat{H})}^2\big) \\ & \times \exp\big\{\int_0^t (\frac{\lambda_0^2}{\delta_1}\|w^{(1)}(\theta)\|_{\widehat{V}}^2 + 2a_1 + 2a_2C_G^2)\mathrm{d}\theta\big\}, \end{aligned}\tag{3.27}$$

$$\begin{aligned}\int_0^t \|w^{(1)}(\theta) - w^{(2)}(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta \leqslant & \delta_1^{-1}\|w^{0^{(1)}} - w^{0^{(2)}}\|^2 + \frac{\delta_1}{2a_2}C_G^2\|\phi^{(1)} - \phi^{(2)}\|_{L^2(-h,0;\widehat{H})}^2 \\ & \times \big[1 + \int_0^t (\frac{\lambda_0^2}{\delta_1}\|w^{(1)}(\theta)\|_{\widehat{V}}^2 + 2a_1 + 2a_2C_G^2) \\ & \times \exp\big\{\int_0^\theta (\frac{\lambda_0^2}{\delta_1}\|w^{(1)}(s)\|_{\widehat{V}}^2 + 2a_1 + 2a_2C_G^2)\mathrm{d}s\big\}\mathrm{d}\theta\big], \end{aligned}\tag{3.28}$$

where the positive constants a_1 and a_2 satisfy $a_1a_2 \geqslant \frac{1}{4}$.

Proof. Set $u(\cdot) = u^{(1)}(\cdot) - u^{(2)}(\cdot), \omega(\cdot) = \omega^{(1)}(\cdot) - \omega^{(2)}(\cdot), w(\cdot) = (u(\cdot),\omega(\cdot)) = w^{(1)}(\cdot) - w^{(2)}(\cdot)$, then $w(t)$ is a solution of the following system:

$$\begin{cases}\dfrac{\partial w(t)}{\partial t} + Aw + B(u^{(1)},w^{(1)}) - B(u^{(2)},w^{(2)}) + N(w) = G(t,w_t^{(1)}) - G(t,w_t^{(2)}), \\ w(0) = w^{(1)}(0) - w^{(2)}(0),\ \ w(t) = \phi^{(1)} - \phi^{(2)}, \quad t \in (-h,0). \end{cases}\tag{3.29}$$

Testing $(3.29)_1$ by $w(t)$, it holds that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2+\langle Aw(t),w(t)\rangle+\langle N(w(t)),w(t)\rangle\\=&(G(t,w_t^{(1)})-G(t,w_t^{(2)}),w(t))-\langle B(u^{(1)}(t),w^{(1)}(t))+B(u^{(2)}(t),w^{(2)}(t)),w(t)\rangle.\end{aligned}\tag{3.30}$$

Similar to (3.25), we can deduce from (3.30) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2+\delta_1\|w(t)\|_{\widehat{V}}^2\leqslant&\frac{\lambda_0^2}{\delta_1}\|w^{(1)}(t)\|_{\widehat{V}}^2\|w(t)\|^2+2a_1\|w(t)\|^2\\&+2a_2\|G(t,w_t^{(1)})-G(t,w_t^{(2)})\|^2,\end{aligned}$$

where a_1, a_2 comes from (3.25). Integrating the above inequality over $[0,t]$ and using assumption (A), we obtain

$$\begin{aligned}\|w(t)\|^2+\delta_1\int_0^t\|w(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta\leqslant&\|w(0)\|^2+\frac{\lambda_0^2}{\delta_1}\int_0^t\|w^{(1)}(\theta)\|_{\widehat{V}}^2\|w(\theta)\|^2\mathrm{d}\theta+2a_1\int_0^t\|w(\theta)\|^2\mathrm{d}\theta\\&+2a_2\int_0^t\|G(\theta,w_\theta^{(1)})-G(\theta,w_\theta^{(2)})\|^2\mathrm{d}\theta\\\leqslant&\|w^0\|^2+\int_0^t\big(\frac{\lambda_0^2}{\delta_1}\|w^{(1)}(\theta)\|_{\widehat{V}}^2+2a_1+2a_2C_G^2\big)\|w(\theta)\|^2\mathrm{d}\theta\\&+2a_2C_G^2\int_{-h}^0\|\phi^{(1)}-\phi^{(2)}\|^2\mathrm{d}\theta.\end{aligned}\tag{3.31}$$

Hence,

$$\begin{aligned}\|w(t)\|^2\leqslant&\|w^0\|^2+2a_2C_G^2\|\phi^{(1)}-\phi^{(2)}\|_{L^2(-h,0;\widehat{H})}^2\\&+\int_0^t\big(\frac{\lambda_0^2}{\delta_1}\|w^{(1)}(\theta)\|_{\widehat{V}}^2+2a_1+2a_2C_G^2\big)\|w(\theta)\|^2\mathrm{d}\theta.\end{aligned}$$

Using the Gronwall's inequality to the above inequality yields (3.27). Furthermore, (3.28) follows from (3.27) and (3.31). This completes the proof. □

Remark 3.2. The existence and uniqueness of the stationary solution of (2.8) and the existence of the weak solutions to system (1.6) can be obtained in the same way in 3D unbounded domains. The key is the nonlinear term $B(u,w)$, the estimates (which can be deduced by Hölder and Gagliardo-Nirenberg inequality) for this term is different in the cases fo 2D and 3D domains. For more detail, one can refer to [22, §2, §3 and §10].

Remark 3.3. Existence, uniqueness and stability of the solution have been established under different conditions – essentially the viscosity $\delta_1 := \min\{\nu, \bar{\alpha}\}$ is asked to be large enough. We also want to point out that there still much work to be done concerning the micropolar fluid flows with delay on unbounded domains. For example, we could study the attractors, further, investigate the regularity, boundedness and tempered behavior of the pullback attractors. These issues will be the topics of some other papers.

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