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DYNAMICS OF STOCHASTIC HEROIN EPIDEMIC MODEL WITH LÉVY JUMPS*

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Abstract People have paid the surge of attention to the prevention and the control of the heroin epidemic for the number of drug addicts is increasing dramatically. In the study of the heroin epidemic, modeling is an important tool. So far many heroin epidemic models are often characterized by ordinary differential equations (ODEs) and many results about them have been obtained. But unfortunately, there is little literature of stochastic heroin epidemic model with jumps. Based on this point, this paper establishes a class of heroin epidemic models—stochastic heroin epidemic model with Lévy jumps. Under some given conditions, the existence of the global positive solution of such model is first obtained. We then study the asymptotic behavior of this model by applying the Lyapunov technique.

Keywords Stochastic heroin model, global positive solution, asymptotic behavior, Lévy noise.

MSC(2010) 92D25, 93D20, 93E03.

1. Introduction

In recent years, the number of drug-poisoning deaths involving heroin has increased dramatically and the use of heroin has become a serious social problem. Burns [5] indicated that 243 million individuals around the word suffer from drug addiction, which leads to an increase in health care costs, lost productivity and crime. In [5], the author also gave an estimation that 5% of the world's population remains plagued with addiction, and this number is increasing at an alarming rate. Hedegaard et al. [13] presented the age-adjusted rate for drug-poisoning deaths involving heroin increased from 0.7 deaths per 100,000 to 2.7 deaths per 100,000 during 2000–2013. Wingo et al. [27] showed that global treatment of drug addiction costs society billions of dollars annually, but current psychopharmacological therapies have not been successful at desired rates. Though the death rate involving heroin is alarming and heroin users are at high risk for addiction, it is difficult for us to do experiments on human body to obtain the statistic data to prevent and control the heroin prevalence. Based on this fact, mathematical modelling provides a useful tool to investigate the heroin treatment. White and Comiskey [26] first presented an ODE

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model to investigate the heroin epidemic. The authors in [26] identified parameters of interest in the model with a view to informing and assisting policy-makers in targeting prevention and treatment resources for maximum effectiveness, and they also examined the stability of the model by analyzing the basic reproduction number. Later, Mulone and Straughan [19] revised the ODE model and proved that the positive equilibrium of the model in [26] is stable. In 2011, Wang et al. [28] studied the model of the following form:

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \beta_1 S(t)U_1(t) - \mu S(t), \\ \dfrac{dU_1(t)}{dt} = \beta_1 S(t)U_1(t) + \beta_3 U_1(t)U_2(t) - (p + \mu + \delta_1)U_1(t), \\ \dfrac{dU_2(t)}{dt} = pU_1(t) - \beta_3 U_1(t)U_2(t) - (\mu + \delta_2)U_2(t). \end{cases} \tag{1.1}$$

Here $S(t)$ denotes the number of susceptible individuals at time t in the population. $U_1(t)$ denotes the number of drug users not in treatment: initial and relapsed drug users. $U_2(t)$ denotes the number of drug users in treatment. Λ denotes the number of individuals in the general population entering the susceptible population. μ denotes the natural death rate of the general population. p denotes the proportion of drug users who enter treatment. β_1 and β_3 represent the probability of becoming a drug user and the probability of a drug user in treatment relapsing to untreated use, respectively. δ_1 denotes a removal rate that includes drug-related deaths of users not in treatment and a spontaneous recovery rate: individuals not in treatment who stop using drugs but are no longer susceptible. δ_2 denotes a removal rate that includes the drug-related deaths of users in treatment and a rate of successful "cure" that corresponds to recovery to a drug free life and immunity to drug addiction for the duration of the modelling time period. All parameter values in model (1.1) are assumed to be nonnegative and the total population $N(t) = S(t) + U_1(t) + U_2(t)$ is very according to the time. The authors in [28] obtained that the drug-free equilibrium is global asymptotically stable under some conditions and the positive equilibrium is globally asymptotically stable by using the second matrix. The time delay factor is also considered in the heroin models by some researchers and many results on such models can be found in [2, 8, 12, 16, 24] and the references therein. Also, the number of drug users may depend on the age structure (see [9, 29] and the references therein). Recently, Ma et al. [20] investigated the bifurcation of the heroin model. Yang et al. [30] proposed a heroin epidemic model on complex networks. The authors in [30] showed that the drug transmission always spreads if the degree of the network is large enough, and verified that sensitivity analysis of the basic reproduction number with the various parameters in the model plays an important role in controlling the drug transmission.

As a matter of fact, epidemic models are inevitably affected by the environmental noise. Therefore, it is important to study the effect of random disturbance on epidemic models. Britton [6] presented a survey on stochastic epidemic models in a closed community. He pointed out that deterministic models may not be suitable in some cases and the related stochastic epidemic models need to be considered. So far, stochastic epidemic models with white noise are investigated by more and more researchers and many results on such stochastic epidemic models have been established (see [7, 10, 14, 15, 17, 18, 25, 31] and the references therein). However, ODE models and stochastic epidemic models only with white noise can not describe

the massive diseases like avian influenza and SARS for these diseases may break the continuous of solutions. In order to explain these phenomena, introducing a jump process into the epidemic models provides a feasible and more realistic model. Therefore, it is interesting and beneficial to study the stochastic epidemic models with Lévy noise. Zhang and Wang [33] established a stochastic SIR model with jumps to describe such massive diseases and took the lead in using the stochastic differential equation with jumps to study the asymptotic behavior of such model. Later, they also established a stochastic SEIR model with jumps and proved that the positive solution of the model is stochastically asymptotically stable by applying Lyapunov method in [34]. Since then, many results on the epidemic models with jumps have been reported (e.g. [3, 11, 32, 35]).

Due to the decline in the immunity of the drug addicts, the number of drug users will change suddenly when encountered with toxic pollutants, SARS, avian influenza, etc. Moreover, when the drug is suddenly reduced as a result of earthquakes, hurricanes, flood, drought and some human factors, the number of drug users will also suddenly fluctuate greatly. How can we describe such phenomena? Motivated by the epidemic models with jumps mentioned above, we hence consider the heroin epidemic model (1.1) with jumps to describe these phenomena. Moreover, as far as we know, many scholars have adopted an approach to introduce stochasticity, that is they assumed that environmental white noise and jumps are directly proportional to the solution of epidemic models, and many literature related to this approach have been obtained (see [14, 15, 23, 33–35]). Based on these points, model (1.1) changes into the following model, which is the model that we consider in this paper:

$$\begin{cases} dS(t) =(\Lambda - \beta_1 S(t)U_1(t) - \mu S(t))dt \\ \qquad\quad + b_1 S(t)dB_1(t) + \displaystyle\int_Z C_1(z)S(t-)\tilde{N}(dt,dz), \\ dU_1(t) =(\beta_1 S(t)U_1(t) + \beta_3 U_1(t)U_2(t) - (p+\mu+\delta_1)U_1(t))dt \\ \qquad\quad + b_2 U_1(t)dB_2(t) + \displaystyle\int_Z C_2(z)U_1(t-)\tilde{N}(dt,dz), \\ dU_2(t) =(pU_1(t) - \beta_3 U_1(t)U_2(t) - (\mu+\delta_2)U_2(t))dt \\ \qquad\quad + b_3 U_2(t)dB_3(t) + \displaystyle\int_Z C_3(z)U_3(t-)\tilde{N}(dt,dz), \end{cases} \tag{1.2}$$

where $X(t-)$ means the left limit of $X(t)$, $C_i(z) > -1 (i = 1, 2, 3)$. Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. its right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $B(t) = (B_1(t), B_2(t), B_3(t))^T$ be a three-dimensional standard Brownian motion defined on the complete probability space with its intensity $b_i > 0 (i = 1, 2, 3)$. N is a Poisson random measure defined on $R_+ \times Z$ with the compensator $\tilde{N}$ and intensity measure ν. We assume that N is independent of B and ν is a Lévy measure such that $\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$, where $\nu(dz) < \infty$ and $\int_Z (|z|^2 \wedge 1)\nu(dz) < \infty$. Usually, the pair (B, N) is called a Lévy noise. While, if $C_i = 0$ and $b_i S(t)dB_i(t) = b_i dB_i(t) (i = 1, 2, 3)$, model (1.2) will change into the system (4.2) in [28], but the authors didn't investigate the asymptotical behavior of the solutions to such model around the drug-free and endemic equilibrium of the corresponding determined model. It is also easy to see that model (1.2) is the determined model (1.1) when $b_i = 0$ and $C_i = 0 (i = 1, 2, 3)$. This is the first paper to

establish the stochastic heroin epidemic model with Lévy jumps, and to investigate the asymptotical behavior of the solutions to model (1.2) around the drug-free and endemic equilibrium of model (1.1).

The organization of this paper is as follows. Section 2 proves that model (1.2) has a unique global positive solution. Section 3 studies the asymptotical behavior of the solutions to model (1.2) around the drug-free equilibrium E_0 of model (1.1). Section 4 analyzes the asymptotical behavior of the solutions to model (1.2) near the endemic equilibrium of model (1.1) by the obtained Lyapunov function. Finally, a brief conclusion is drawn in the last section.

2. Existence and uniqueness of positive solution

In order to investigate the dynamical behavior of model (1.2), we first need to know whether the solution is global. In this section, we show model (1.2) has a unique global (i.e. no explosion in finite time) positive solution for any given positive initial value by using the Lyapunov technique [4, 21, 33].

It follows from [22] that if the coefficients of the stochastic differential equation satisfy the local Lipschitz condition and the linear growth condition, then the stochastic differential equation admits a unique global solution for any given initial value. However, the coefficients of model (1.2) only satisfy the local Lipschitz condition but do not meet the linear growth condition, hence the solution of model (1.2) may explode at a finite time. Because $S(t)$, $U_1(t)$ and $U_2(t)$ represent the size of susceptible individuals in the population, the size of drug users not in treatment and the size of drug users in treatment at time t in model (1.2), we are only interested in the case that they are positive. Next, we will prove that model (1.2) admits a unique global positive solution for any given positive initial value.

Through out this paper, we assume that for each $N > 0$, there exists $L_N > 0$ such that jump diffusion coefficient satisfies

$(\mathbf{A_1})$ $\int_Z |H_i(x,z) - H_i(y,z)| \leq L_N|x-y|^2$, $i = 1,2,3$, where $H_1(x,z) = C_1(z)S(t-)$, $H_2(x,z) = C_2(z)U_1(t-)$, $H_3(x,z) = C_3(z)U_2(t-)$ with $|x| \vee |y| \leq N$.

$(\mathbf{A_2})$ $|\ln(1 + C_i(z))| \leq M_1$, for $C_i(z) > -1$, $i = 1,2,3$, where M_1 is a positive constant.

Theorem 2.1. *Let Assumptions* $(\mathbf{A_1})$ *and* $(\mathbf{A_2})$ *hold, then for any given value* $(S(0), U_1(0), U_2(0)) \in R^3_+$ *and* $t \geq 0$, *model* (1.2) *has a unique global solution* $(S(t), U_1(t), U_2(t)) \in R^3_+$ *almost surely.*

Proof. Since $(\mathbf{A_1})$ and the coefficients of model (1.2) satisfy the local Lipschitz condition, for any given $(S(0), U_1(0), U_2(0)) \in R^3_+$, there is a unique local solution $(S(t), U_1(t), U_2(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time [1, 22]. In order to show this solution is global, we need to prove $\tau_e = \infty$ a.s. Assume that $k_0 (k_0 > 0)$ is sufficiently large and $S(0)$,$U_1(0)$ and $U_2(0)$ all lie within the interval $[1/k_0, k_0]$. Define the stopping time

$$\tau_k = \inf\{t \in [0, \tau_e) : \min\{S(t), U_1(t), U_2(t)\} \leq 1/k \text{ or } \max\{S(t), U_1(t), U_2(t)\} \geq k\},$$

for each integer $k \geq k_0$. We set $\inf \emptyset = \infty$(as usual $\emptyset$ denotes the empty set). It is clear that τ_k is increasing as $k \uparrow \infty$. Denote $\tau_\infty = \lim\limits_{k\to\infty} \tau_k$. Obviously, $\tau_\infty \leq \tau_e$ a.s. If we can prove that $\tau_\infty = \infty$ is true, then $\tau_e = \infty$ and $(S(t), U_1(t), U_2(t)) \in R^3_+$

a.s. for all $t \geq 0$. In other words, we need to show $\tau_\infty = \infty$ a.s. If $\tau_\infty = \infty$ is false, then there exists a pair of constants $T > 0$ and $\varepsilon \in (0,1)$ satisfying

$$P\{\tau_\infty \leq T\} \geq \varepsilon.$$

Therefore, there is an integer $k_1 \geq k_0$ such that

$$P\{\tau_k \leq T\} \geq \varepsilon, \quad \text{for all } k \geq k_1. \tag{2.1}$$

Choose a C^2-function

$$V(S,U_1,U_2) = \left(S - a - a\log\frac{S}{a}\right) + (U_1 - 1 - \log U_1) + \left(U_2 - b - b\log\frac{U_2}{b}\right), \tag{2.2}$$

where a and b are two positive constants to be determined later. It is easy to see that this function is nonnegative for $u - 1 - \log u > 0 (u \geq 0)$. By employing the Itô formula, one obtains

$$\begin{aligned} dV(S(t),U_1(t),U_2(t)) =& LV(S(t),U_1(t),U_2(t)) + b_1(S(t)-a)dB_1(t) \\ &+ b_2(U_1(t)-1)dB_2(t) + b_3(U_2(t)-b)dB_3(t) \\ &+ \int_Z [C_1(z)S(t-) - a\log(1+C_1(z)) + C_2(z)U_1(t-) \\ &- \log(1+C_2(z)) + C_3(z)U_2(t-) - b\log(1+C_3(z))]\tilde{N}(dt,dz) \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} LV(S,U_1,U_2) =& \left(\Lambda + \mu(a+1) + p + \delta_1 + b(\mu+\delta_2) + \frac{1}{2}ab_1^2 + \frac{1}{2}b_2^2 + \frac{1}{2}bb_3^2\right) \\ &+ [a\beta_1 - (\mu+\delta_1) + b\beta_3]U_1 - (\mu+\beta_1)S - (\mu+\delta_2+\beta_3)U_2 - \frac{a}{S}\Lambda - bp\frac{U_1}{U_2} \\ &+ \int_Z [aC_1(z) - a\log(1+C_1(z)) + C_2(z) - \log(1+C_2(z)) + bC_3(z) \\ &\quad - b\log(1+C_3(z))]\nu(dz). \end{aligned} \tag{2.4}$$

Let $a = \frac{\mu+\delta_1}{2\beta_1}$ and $b = \frac{\mu+\delta_1}{2\beta_3}$. By $(\mathbf{A_2})$ and $x - \log(x+1) \geq 0 (x > -1)$, then

$$\begin{aligned} LV(S,U_1,U_2) =& \left(\Lambda + \mu(a+1) + p + \delta_1 + b(\mu+\delta_2) + \frac{1}{2}ab_1^2 + \frac{1}{2}b_2^2 + \frac{1}{2}bb_3^2\right) \\ &- (\mu+\beta_1)S - (\mu+\delta_2+\beta_3)U_2 - \frac{a}{S}\Lambda - bp\frac{U_1}{U_2} \\ &+ \int_Z [aC_1(z) - a\log(1+C_1(z)) + C_2(z) - \log(1+C_2(z)) + bC_3(z) \\ &\quad - b\log(1+C_3(z))]\,\nu(dz). \\ \leq& \Lambda + \mu(a+1) + p + \delta_1 + b(\mu+\delta_2) + \frac{1}{2}ab_1^2 + \frac{1}{2}b_2^2 + \frac{1}{2}bb_3^2 + 3M_2 =: M \end{aligned} \tag{2.5}$$

where $M_2 = \max\{\int_Z [aC_1(z) - a\log(1+C_1(z))]\nu(dz), \int_Z [C_2(z) - \log(1+C_2(z))]\nu(dz), \int_Z [bC_3(z) - b\log(1+C_3(z))]\nu(dz\}$. It follows that

$$\begin{aligned}
&\int_0^{\tau_k\wedge T} dV(S(t), U_1(t), U_2(t)) \\
\leq &\int_0^{\tau_k\wedge T} M dt + \int_0^{\tau_k\wedge T} b_1(S(t)-a)dB_1(t) \\
&+ \int_0^{\tau_k\wedge T} b_2(U_1(t)-1)dB_2(t) + \int_0^{\tau_k\wedge T} b_3(U_2(t)-b)dB_3(t) \\
&+ \int_0^{\tau_k\wedge T}\int_Z [C_1(z)S(t-) - a\log(1+C_1(z)) \\
&+ C_2(z)U_1(t-) - \log(1+C_2(z)) + C_3(z)U_2(t-) - b\log(1+C_3(z))]\tilde{N}(dt,dz).
\end{aligned} \tag{2.6}$$

Hence, one yields

$$EV(S(\tau_k\wedge T), U_1(\tau_k\wedge T), U_2(\tau_k\wedge T)) \leq V(S(0), U_1(0), U_2(0)) + MT. \tag{2.7}$$

Set $\Omega_k = \{\tau_k \leq T\}$ for $k \geq k_0$ and it is easy to see from (2.1) that $P(\Omega_k) \geq \varepsilon$. Note that for every $\omega \in \Omega_k$, $S(\tau_k, \omega)$ or $U_1(\tau_k, \omega)$ or $U_2(\tau_k, \omega)$ equals either k or $\frac{1}{k}$. Then by (2.2) and (2.7), one obtains

$$\begin{aligned}
V(S(0), U_1(0), U_2(0)) + MT \geq &E[I_{\Omega_k}(\omega)V(S(\tau_k), U_1(\tau_k), U_2(\tau_k))] \\
\geq &\varepsilon\left\{\left(k - a - a\log\frac{k}{a}\right) \wedge \left(\frac{1}{k} - a - a\log\frac{1}{ak}\right)\right. \\
&\wedge (k - 1 - \log k) \wedge \left(\frac{1}{k} - 1 - \log\frac{1}{k}\right) \\
&\left.\wedge \left(k - b - b\log\frac{k}{b}\right) \wedge \left(\frac{1}{k} - b - b\log\frac{1}{bk}\right)\right\}
\end{aligned} \tag{2.8}$$

where I_{Ω_k} is the indicator function of Ω_k. It is easy to see that (2.8) implies the contradiction

$$\infty > V(S(0), U_1(0), U_2(0)) + MT = \infty, \text{ as } k \to \infty.$$

Therefore, $\tau_\infty = \infty$ must hold, which implies $(S(t), U_1(t), U_2(t)) \in R^3_+$ almost surely. The proof is complete.

□

3. Asymptotic behavior around the drug-free equilibrium of the deterministic model

In this section, we will study the asymptotical behavior of the solutions to model (1.2) around the drug-free equilibrium $E_0(S_0, 0, 0) = (\frac{\Lambda}{\mu}, 0, 0)$ of model (1.1). If $R_0 = \beta_1 S_0/(\mu + p + \delta_1) < 1$, E_0 is globally asymptotically stable. While for model (1.2), E_0 is no longer the drug-free equilibrium and the solution of model (1.2) does not converge to E_0. In the following, we present the result.

Theorem 3.1. *Let assumptions (A_1) and (A_2) hold. If $R_0 < 1$ and the following conditions are satisfied*

$$\begin{aligned}
&\mu > b_1^2 + 3\int_Z C_1^2(z)\nu(dz),\\
&\mu + \delta_1 > \frac{1}{2}b_2^2 + \frac{3}{2}\int_Z C_2^2(z)\nu(dz),\\
&\mu + \delta_2 > \frac{1}{2}b_3^2 + \frac{3}{2}\int_Z C_3^2(z)\nu(dz).
\end{aligned} \tag{3.1}$$

Then the solution $(S(t), U_1(t), U_2(t))$ of model (1.2) with any initial value $(S(0), U_1(0), U_2(0)) \in R_+^3$ satisfies

$$\limsup_{t\to\infty} \frac{1}{t} E\int_0^t [(S(r) - S_0)^2 + U_1(t)^2 + U_2^2(t)]dr \le \frac{S_0^2}{K}\left(b_1^2 + 3\int_Z C_1^2(z)\nu(dz)\right), \tag{3.2}$$

where

$$\begin{aligned}
K = \min\Big\{&\mu - b_1^2 - 3\int_Z C_1^2(z)\nu(dz), \mu + \delta_1 - \frac{1}{2}b_2^2 - \frac{3}{2}\int_Z C_2^2(z)\nu(dz),\\
&\mu + \delta_2 - \frac{1}{2}b_3^2 - \frac{3}{2}\int_Z C_3^2(z)\nu(dz)\Big\}.
\end{aligned}$$

Proof. Set $u = S - S_0,\ v = U_1,\ w = U_2$, then model (1.2) can be rewritten as

$$\begin{cases}
du(t) = &[-\beta_1(u(t) + S_0)v(t) - \mu u(t)]dt\\
&+ b_1(u(t) + S_0)dB_1(t) + \displaystyle\int_Z C_1(z)(u(t-) + S_0)\tilde{N}(dt, dz),\\
dv(t) = &[\beta_1(u(t) + S_0)v(t) + \beta_3 v(t)w(t) - (p + \mu + \delta_1)v(t))dt\\
&+ b_2 v(t)dB_2(t) + \displaystyle\int_Z C_2(z)v(t-)\tilde{N}(dt, dz),\\
dw(t) = &[pv(t) - \beta_3 v(t)w(t) - (\mu + \delta_2)w(t)]dt\\
&+ b_3 w(t)dB_3(t) + \displaystyle\int_Z C_3(z)w(t-)\tilde{N}(dt, dz).
\end{cases} \tag{3.3}$$

Choose the C^2-function $G(u, v, w) = \frac{1}{2}(u+v+w)^2$. By the Itô formula, one derives

$$\begin{aligned}
&dG(u, v, w)\\
=&LG(u, v, w)dt + (u + v + w)(b_1(u + S_0)dB_1(t) + b_2 v dB_2(t) + b_3(t)w(t)dB_3(t))\\
&+ \int_Z \{\frac{1}{2}[C_1(z)(u(t-) + S_0) + C_2(z)v(t-) + C_3(z)w(t-)]^2\\
&+ (u(t-)+v(t-)+w(t-))[C_1(z)(u(t-)+S_0)+C_2(z)v(t-)+C_3(z)w(t-)]\}\tilde{N}(dt, dz)
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
LG(u, v, w) =&(u + v + w)[-\mu u - (\mu + \delta_1)v - (\mu + \delta_2)w]\\
&+ \frac{1}{2}b_1^2(u + S_0)^2 + \frac{1}{2}b_2^2 v^2 + \frac{1}{2}b_3^2 w^2
\end{aligned}$$

$$+\frac{1}{2}\int_Z [C_1(z)(u(t-)+S_0)+C_2(z)v(t-)+C_3(z)w(t-)]^2\nu(dz). \tag{3.5}$$

Using the basic inequality $(a+b+c)^2 \le 3a^2+3b^2+3c^2$, one yields

$$\begin{aligned} &LG(u,v,w)\\ &\le -\left(\mu - b_1^2 - 3\int_Z C_1^2(z)\nu(dz)\right)u^2 - \left(\mu+\delta_1-\frac{1}{2}b_2^2-\frac{3}{2}\int_Z C_2^2(z)\nu(dz)\right)v^2\\ &\quad -\left(\mu+\delta_2-\frac{1}{2}b_3^2-\frac{3}{2}\int_Z C_3^2(z)\nu(dz)\right)w^2 + b_1^2S_0^2 + 3S_0^2\int_Z C_1^2(z)\nu(dz). \end{aligned} \tag{3.6}$$

Integrating both sides of (3.4) and then taking expectation, one gains

$$EG(u(t),v(t),\omega(t)) = G(u(0),v(0),\omega(0)) + E\int_0^t LGdr. \tag{3.7}$$

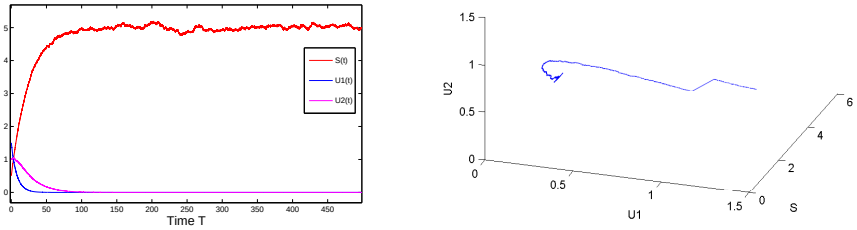
According to the condition (3.1), it follows (3.7) that

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t [(S(r)-S_0)^2+U_1^2(r)+U_2^2(r)]dr \le \frac{S_0^2}{K}\left(b_1^2+3\int_Z C_1^2(z)\nu(dz)\right),$$

where K is defined as before. The proof is complete. □

Remark 3.1. We see from Theorem 3.1 that the solutions of model (1.2) are stochastic vibration around the drug-free equilibrium E_0 of model (1.1), and the vibration intensity is relevant to the strength of b_i and C_i (i=1,2,3). That is, if the vibration intensity of Lévy noise is smaller, the solutions of model (1.2) and the drug-free equilibrium E_0 of model (1.1) are nearer. In this case, the heroin epidemic is nearly extinct and will not spread in the society.

In the following, set $(S(0),U_1(0),U_2(0)) = (0.5,1.5,1)$, $\Lambda = 0.2, \beta_1 = 0.004, \beta_3 = 0.003, \mu = 0.04, \delta_1 = 0.05, \delta_2 = 0.01, p = 0.05,\ b_1 = 0.005, b_2 = 0.007, b_3 = 0.008$, $C_i(z) = -k_i z^2/(1+z^2)(i=1,2,3),\ z\in[-1,1],\ k_1 = 0.1, k_2 = 0.2, k_3 = 0.3$. The trajectory of the solution to model (1.2) and the corresponding phase portrait are shown in the left and right of Figure 1, respectively.



(a) The sample path of solution to model (1.2). (b) Phase portrait.

Figure 1. $R_0 < 1$

4. Asymptotic behavior around the endemic equilibrium of the deterministic model

In this section, we assume that $R_0 = \beta_1 S_0/(\mu + p + \delta_1) > 1$. From [28], let $E^* = (S^*, U_1^*, U_2^*)$ be the positive solution of model (1.1) and satisfy $S^* \le \frac{p+\mu+\delta_1}{\beta_1}$, then the solution $E^* = (S^*, U_1^*, U_2^*)$ is unique. However, E^* is not the endemic equilibrium of model (1.2), because model (1.2) does not have the endemic equilibrium. We also study the asymptotic behavior of the solutions to model (1.2) near E^*. The result is as follows.

Theorem 4.1. *Let assumptions (A_1) and (A_2) hold. If $R_0 > 1$ and the following conditions are satisfied*

$$\begin{aligned}
&\mu > \frac{b_1^2}{2} + \frac{\epsilon}{2}(4\mu + \delta_1 + \delta_2) + \frac{1}{2}(1+\epsilon)^2 \int_Z C_1^2(z)\nu(dz),\\
&\mu + \delta_1 > \frac{1}{2}b_2^2 + \frac{\epsilon}{2}(4\mu + 2\delta_1 + \delta_2) + \frac{(1+\epsilon)^2}{2\epsilon} \int_Z C_2^2(z)\nu(dz),\\
&\mu + \delta_2 > \frac{1}{2}b_3^2 + \frac{\epsilon}{2}(4\mu + \delta_1 + 2\delta_2) + \frac{\epsilon+1}{2\epsilon} \int_Z C_3^2(z)\nu(dz),
\end{aligned} \tag{4.1}$$

where $\epsilon \in (0,1)$, then the solution $(S(t), U_1(t), U_2(t))$ of model (1.2) with any initial value $(S(0), U_1(0), U_2(0)) \in R_+^3$ has the property

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t} E \int_0^t \left[\left(S(r) - \frac{\mu}{a_1} S^* \right)^2 + \left(U_1(r) - \frac{\mu+\delta_1}{a_2} U_1^* \right)^2 + \left(U_2(r) - \frac{\mu+\delta_2}{a_3} U_2^* \right)^2 \right] dr\\
&\le \frac{L_1 + L_2}{a}.
\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}
a_1 &= \mu - \frac{b_1^2}{2} - \frac{\epsilon}{2}(4\mu + \delta_1 + \delta_2) - \frac{1}{2}(1+\epsilon)^2 \int_Z C_1^2(z)\nu(dz),\\
a_2 &= \mu + \delta_1 - \frac{1}{2}b_2^2 - \frac{\epsilon}{2}(4\mu + 2\delta_1 + \delta_2) - \frac{(1+\epsilon)^2}{2\epsilon} \int_Z C_2^2(z)\nu(dz),\\
a_3 &= \mu + \delta_2 - \frac{1}{2}b_3^2 - \frac{\epsilon}{2}(4\mu + \delta_1 + 2\delta_2) - \frac{\epsilon+1}{2\epsilon} \int_Z C_3^2(z)\nu(dz),\\
L_1 &= \frac{1}{2\epsilon}(4\mu + \delta_1 + \delta_2)(S^*)^2 + \frac{1}{2\epsilon}(4\mu + 2\delta_1 + \delta_2)(U_1^*)^2 + \frac{1}{2\epsilon}(4\mu + \delta_1 + 2\delta_2)(U_2^*)^2,\\
L_2 &= \frac{\mu^2}{a_1}(S^*)^2 + \frac{(\mu+\delta_1)^2}{a_2}(U_1^*)^2 + \frac{(\mu+\delta_2)^2}{a_3}(U_2^*)^2,
\end{aligned}$$

and $a = \min\{a_1, a_2, a_3\}$.

Proof. For E^* is the solution of model (1.1), thus one can get

$$\Lambda = \mu S^* + (\mu + \delta_1)U_1^* + (\mu + \delta_2)U_2^*. \tag{4.3}$$

Define the positive function $H(S, U_1, U_2) = \frac{1}{2}(S - S^* + U_1 - U_1^* + U_2 - U_2^*)^2$.

Employing the Itô formula, one obtains

$$\begin{aligned}
&dH(S(t),U_1(t),U_2(t))\\
=&LH(S,U_1,U_2)dt\\
&+(S-S^*+U_1-U_1^*+U_2-U_2^*)\left(b_1S(t)dB_1(t)+b_2U_1(t)dB_2(t)+b_3U_2(t)dB_3(t)\right)\\
&+\int_Z\{\frac{1}{2}[C_1(z)S(t-)+C_2(z)U_1(t-)+C_3(z)U_2(t-)]^2\\
&+(C_1(z)S(t-)+C_2(z)U_1(t-)+C_3(z)U_2(t-))(S-S^*+U_1-U_1^*+U_2-U_2^*))\}
\end{aligned}\tag{4.4}$$

where

$$\begin{aligned}
LH(S,U_1,U_2)=&(S-S^*+U_1-U_1^*+U_2-U_2^*)[\Lambda-\mu S-(\mu+\delta_1)U_1-(\mu+\delta_2)U_2]\\
&+\frac{1}{2}b_1^2S^2+\frac{1}{2}b_2^2U_1^2+\frac{1}{2}b_3^2U_2^2\\
&+\frac{1}{2}\int_Z[C_1(z)S(t-)+C_2(z)U_1(t-)+C_3(z)U_2(t-)]^2\nu(dz).
\end{aligned}\tag{4.5}$$

Together with (4.3), one can obtain

$$\begin{aligned}
&LH(S,U_1,U_2)\\
\le&-\left[\mu-\frac{b_1^2}{2}-\frac{\epsilon}{2}(4\mu+\delta_1+\delta_2)-\frac{1}{2}(1+\epsilon)^2\int_Z C_1^2(z)\nu(dz)\right]S^2\\
&-\left[\mu+\delta_1-\frac{1}{2}b_2^2-\frac{\epsilon}{2}(4\mu+2\delta_1+\delta_2)-\frac{(1+\epsilon)^2}{2\epsilon}\int_Z C_2^2(z)\nu(dz)\right]U_1^2\\
&-\left[\mu+\delta_2-\frac{1}{2}b_3^2-\frac{\epsilon}{2}(4\mu+\delta_1+2\delta_2)-\frac{\epsilon+1}{2\epsilon}\int_Z C_3^2(z)\nu(dz)\right]U_2^2\\
&+2\mu SS^*+2(\mu+\delta_1)U_1U_1^*+2(\mu+\delta_2)U_2U_2^*\\
&+\frac{1}{2\epsilon}(4\mu+\delta_1+\delta_2)(S^*)^2+\frac{1}{2\epsilon}(4\mu+2\delta_1+\delta_2)(U_1^*)^2+\frac{1}{2\epsilon}(4\mu+\delta_1+2\delta_2)(U_2^*)^2\\
=&-a_1\left(S-\frac{\mu}{a_1}S^*\right)^2-a_2\left(U_1-\frac{\mu+\delta_1}{a_2}U_1^*\right)^2-a_3\left(U_2-\frac{\mu+\delta_2}{a_3}U_2^*\right)^2\\
&+L_1+L_2,
\end{aligned}\tag{4.6}$$

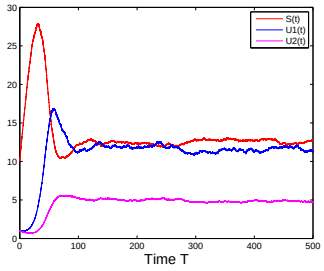
where $a_1, a_2, a_3, a, L_1, L_2$ are defined as before, and condition (4.1) implies $a_1, a_2, a_3>0$. Integrating both sides of (4.4) from 0 to t and then taking expectation, one gains

$$0\le EH(S(t),U_1(t),U_2(t))=H(S(0),U_1(0),U_2(0))+E\int_0^t LH(S(r),U_1(r),U_2(r))dr$$

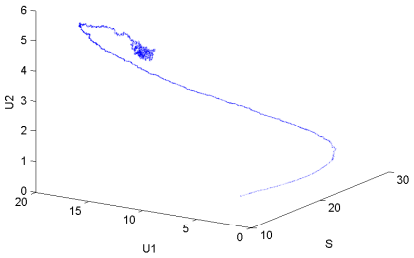
which together with (4.6) that one can get (4.2). The proof is therefore complete. □

Remark 4.1. It follows from Theorem 4.1 that the solutions of model (1.2) fluctuate around a certain level which is relevant to $(\frac{\mu}{a_1}S^*, \frac{\mu+\delta_1}{a_2}U_1^*, \frac{\mu+\delta_2}{a_3}U_2^*)$, b_i and C_i (i=1,2,3). This theorem reveals that the heroin epidemic is lasting and will spread in the society.

In model (1.2), choose the following parameters $(S(0), U_1(0), U_2(0)) = (10, 1, 1)$, $\Lambda = 1, \beta_1 = 0.006, \beta_3 = 0.001, \mu = 0.01, \delta_1 = 0.04, \delta_2 = 0.05, p = 0.03$, $b_1 = 0.005, b_2 = 0.007, b_3 = 0.008$, $C_i(z) = -k_i z^2/(1+z^2)(i = 1, 2, 3)$, $z \in [-1, 1]$, $k_1 = 0.1, k_2 = 0.2, k_3 = 0.3$. Then the trajectory of the solution to model (1.2) and the corresponding phase portrait are given in (a) and (b) of Figure 2, respectively.



(a) The sample path of the solution to model (1.2).



(b) Phase portrait.

Figure 2. $R_0 > 1$

5. Conclusions

In recent years, people often use ordinary differential equations (ODEs) to characterize the heroin epidemic. However, in some cases which we have stated above, ODE models are insufficient to characterize these phenomena. Based on this point and motivated by the results on the stochastic disease epidemic models with jumps, we establish a stochastic heroin epidemic model with Lévy jumps. Meanwhile, we investigate the dynamics of the established model. Precisely, first of all it is shown that the stochastic heroin epidemic model with Lévy jumps admits a unique global positive solution for any given positive initial value. When $R_0 < 1$, we then investigate the asymptotical behavior of the solution to the established model around the drug-free equilibrium E_0 of the corresponding deterministic model, and we find that the solution oscillates around the drug-free equilibrium, which reveals that the heroin epidemic is nearly extinct and will not spread in the society. When $R_0 > 1$, we also study the asymptotic behavior of the solution to the established model around the endemic equilibrium $E^* = (S^*, U_1^*, U_2^*)$ of the corresponding deterministic model, and we obtain that the solution goes around $(\frac{\mu}{a_1}S^*, \frac{\mu+\delta_1}{a_2}U_1^*, \frac{\mu+\delta_2}{a_3}U_2^*)$, which reveals that the heroin epidemic is lasting in the society. Our results in this paper can identify parameters in the study of the heroin epidemic, which will be a help for informing and assisting policy-makers in targeting prevention and treatment resources for maximum effectiveness.

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