Journal of Applied Analysis and Computation
Volume 8, Number 6, December 2018, 1694–1706

Website:http://jaac.ijournal.cn
DOI:10.11948/2018.1694

NEW EXISTING RESULTS FOR A SYSTEM OF NONLINEAR THIRD-ORDER DIFFERENTIAL EQUATION VIA FIXED POINT INDEX*

Chengbo Zhai[1,†], Li Zhao[1], Shunyong Li[1] and H. R. Marasi[2]

Abstract By using fixed point index theory, we investigate a system of nonlinear third-order differential equation. We give some sufficient conditions for the existence of at least one or two positive solutions to the system of nonlinear third-order differential equation. As applications, we also present two examples to demonstrate the main results.

Keywords System of third-order differential equation, positive solution, fixed point index theory, existence.

MSC(2010) 34B10, 34B18.

1. Introduction

There are some areas of applied mathematics and physics involving third-order differential equations with different boundary conditions, such as the deflection of a curved beam having a constant or varying cross section, electromagnetic waves, three-layer beams, gravity driven flows see [9]. So third-order differential equations with different boundary conditions have been paid much attention during the past several decades. Especially, the existence of positive solutions for third-order boundary value problems has been studied widely, see [1–8, 10, 12–28] and the references therein. Recently, In [15] the authors established an existence result for positive solutions to the following system of third-order differential equation

$$\begin{cases} -u'''(t) = a(t)f(t,v), t \in (0,1), \\ -v'''(t) = b(t)g(t,u), t \in (0,1), \\ u(0) = u'(0) = 0, \ u'(1) = \alpha u'(\eta), \\ v(0) = v'(0) = 0, \ v'(1) = \alpha v'(\eta). \end{cases}$$

The method is Krasnoselskii's fixed point theorem. However, we found that the authors replaced $u(t), v(t)$ by some integral expressions in the proof of the main

[†]the corresponding author. Email address:cbzhai@sxu.edu.cn(C. Zhai)
[1]School of Mathematical Sciences, Shanxi University, Taiyuan 030006, Shanxi, China
[2]University of Tabriz, Tabriz, Iran
[*]The research was supported by the Youth Science Foundation of China (11201272) and Shanxi Province Science Foundation (2015011005), 131 Talents Project of Shanxi Province (2015). The third author was supported by Shanxi Scholarship Council of China (2016-009).

result. Evidently, this is false. So we need continue to study this kind of system for nonlinear third-order differential equation. In addition, there are very few works on a system of nonlinear third-order differential equation in literature. In this paper, we will use fixed point index theory to study the following system:

$$\begin{cases} -u^{'''}(t) = a(t)f(t,u,v), t \in (0,1), \\ -v^{'''}(t) = b(t)g(t,u,v), t \in (0,1), \\ u(0) = u'(0) = 0, u'(1) = \alpha u'(\eta), \\ v(0) = v'(0) = 0, v'(1) = \alpha v'(\eta), \end{cases} \tag{1.1}$$

where $f, g \in C([0,1] \times [0,+\infty) \times [0,+\infty), [0,+\infty))$, $0 < \eta < 1$, $1 < \alpha < \frac{1}{\eta}$, $a, b \in C([0,1], [0,+\infty))$.

Our purpose here is to give the existence of single and multiple positive solutions to the problem (1.1). By a positive solution of (1.1) we understand a function (u, v) which is positive on $0 < t < 1$ and satisfies the differential equation and boundary conditions in (1.1).

Assuming that

(H_1) $a(t), b(t)$ are continuous and do not vanish identically on any subinterval of $[0, 1]$;

(H_2) $f, g : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ are continuous.

2. Preliminary results

In this section we summarize some lemmas which will be used throughout this paper.

Lemma 2.1 (see [11]). *Let X be a Banach space and K be a cone in X. For $r > 0$, define $K_r = \{x \in K : \|x\| \le r\}$. Assume that $T : \overline{K_r} \to K$ is a compact map such that $Tx \ne x$ for $x \in \partial K_r$.*

(i) If $\|x\| \le \|Tx\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 0$;

(ii) If $\|x\| \ge \|Tx\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 1$, where i denotes the fixed point index.

From [15], $(u, v) \in C^3([0,1], [0,+\infty)) \times C^3([0,1], [0,+\infty))$ is a solution of (1.1) if and only if $(u, v) \in C^3([0,1], [0,+\infty)) \times C^3([0,1], [0,+\infty))$ is a solution of the following system:

$$\begin{cases} u(t) = \int_0^1 G(t,s)a(s)f(s,u(s),v(s))ds, \\ v(t) = \int_0^1 G(t,s)b(s)g(s,u(s),v(s))ds, \end{cases} \tag{2.1}$$

where $G(t, s)$ is the Green's function given by:

$$G(t,s) = \frac{1}{2(1-\alpha\eta)} \begin{cases} (2ts - s^2)(1-\alpha\eta) + t^2 s(\alpha - 1), & s \le \min\{\eta, t\}, \\ t^2(1-\alpha\eta) + t^2 s(\alpha - 1), & t \leqslant s \leqslant \eta, \\ (2ts - s^2)(1-\alpha\eta) + t^2(\alpha\eta - s), & \eta \leqslant s \leqslant t, \\ t^2(1-s), & \max\{\eta, t\} \leqslant s. \end{cases} \tag{2.2}$$

The Green's function $G(t,s)$ has the following properties.

Lemma 2.2 (see [12]). *If $0<\eta<1$ and $1<\alpha<\frac{1}{\eta}$. Then for any $(t,s)\in[0,1]\times[0,1]$, $0\leqslant G(t,s)\leqslant\beta(s)$, where $\beta(s)=\frac{1+\alpha}{1-\alpha\eta}s(1-s)$, $s\in[0,1]$.*

Lemma 2.3 (see [12]). *If $0<\eta<1$ and $1<\alpha<\frac{1}{\eta}$. Then for any $(t,s)\in[\frac{\eta}{\alpha},\eta]\times[0,1]$, $G(t,s)\geqslant k\beta(s)$, where $0<k=\frac{\eta^2}{2\alpha^2(1+\alpha)}\min\{\alpha-1,1\}<1$ and $\beta(s)$ is given as in Lemma 2.2.*

3. Main results

For convenience, we set

$$f_0=\lim_{(u,v)\to(0^+,0^+)}\frac{f(t,u,v)}{u+v},\ f_\infty=\lim_{(u,v)\to(+\infty,+\infty)}\frac{f(t,u,v)}{u+v},$$
$$g_0=\lim_{(u,v)\to(0^+,0^+)}\frac{g(t,u,v)}{u+v},\ g_\infty=\lim_{(u,v)\to(+\infty,+\infty)}\frac{g(t,u,v)}{u+v},$$

and

$$l_1=\int_0^1\beta(s)a(s)ds,\ l_2=k^2\int_{\frac{\eta}{\alpha}}^{\eta}\beta(s)a(s)ds,$$
$$l_3=\int_0^1\beta(s)b(s)ds,\ l_4=k^2\int_{\frac{\eta}{\alpha}}^{\eta}\beta(s)b(s)ds,$$
$$\tau_1=\max\{\frac{3}{4l_2},\frac{3}{4l_4}\},\ h=\max\{l_1,l_3\},\ l^{'}=\min\{l_2,l_4\}.$$

The main results of this paper are as follows:

Theorem 3.1. *Assume (H_1), (H_2) and $f_0=g_0=\infty$, $f_\infty=g_\infty=0$ uniformly on $t\in[0,1]$. Then the problem (1.1) has at least one positive solution.*

Theorem 3.2. *Assume that (H_1), (H_2) hold and the following conditions are satisfied:*

(H_3) *there exist two constants $0<\rho_1<\rho_2$ with $\rho_2\geqslant\tau_1\rho_1$ such that*

(i) $f(t,u,v)\geqslant\tau_1(u+v)$, $g(t,u,v)\geqslant\tau_1(u+v)$, $t\in[0,1]$, $0\leqslant u,v\leqslant\rho_1$;

(ii) $f(t,u,v)<\frac{1}{2h}\rho_2$, $g(t,u,v)<\frac{1}{2h}\rho_2$, $t\in[0,1]$, $0\leqslant u,v\leqslant\rho_2$;

(H_4) *$f_\infty=g_\infty=\infty$ uniformly on $t\in[0,1]$.*

Then the problem (1.1) has at least two positive solutions (u_1,v_1) and (u_2,v_2) such that $0<\|(u_1,v_1)\|<\rho_2<\|(u_2,v_2)\|$.

Theorem 3.3. *Assume that (H_1), (H_2) hold and the following conditions are satisfies*

(H_5) *$f_0=g_0=0$, $f_\infty=g_\infty=0$ uniformly on $t\in[0,1]$;*

(H_6) *There exists a constant $\rho_1>0$ such that*

$$f(t,u,v) < \frac{k\rho_1}{2l'},\ g(t,u,v) < \frac{k\rho_1}{2l'},$$

for $t \in [0,1]$ *and* $u+v \in [k\rho_1, \rho_1]$*, where* k *is given as in Lemma 2.3. Then the problem* (1.1) *has at least two positive solutions.*

To prove the above theorems, we need seek some fixed points of the relative nonlinear operators. So we define

$$\begin{cases} A(u,v)(t) = \int_0^1 G(t,s)a(s)f(s,u(s),v(s))ds, \\ B(u,v)(t) = \int_0^1 G(t,s)b(s)g(s,u(s),v(s))ds, \end{cases}$$

and

$$T(u,v)(t) = (A(u,v), B(u,v))(t).$$

Then (2.1) is equivalent to the operator equation $T(u,v) = (u,v)$, so (u,v) is a solution of (1.1) if and only if (u,v) is the fixed point of T. We will work in the usual Banach space $X = C[0,1] \times C[0,1]$, with $\|(u,v)\| = \|u\| + \|v\|$, where $\|u\| = \sup\limits_{t\in[0,1]} |u(t)|$. Let K be the cone defined by

$$K = \{(u,v) \in X : u,v \geqslant 0, \min_{t\in[\frac{\eta}{\alpha},\eta]}(u(t)+v(t)) \geqslant k(\|u\|+\|v\|)\},$$

where k is given as in Lemma 2.3.

Lemma 3.1. $T: K \to K$ *is completely continuous.*

Proof. For $u,v \in K$, by Lemma 2.2,

$$A(u,v)(t) = \int_0^1 G(t,s)a(s)f(s,u(s),v(s))ds \leqslant \int_0^1 \beta(s)a(s)f(s,u(s),v(s))ds.$$

Hence, $\|A(u,v)\| \leqslant \int_0^1 \beta(s)a(s)f(s,u(s),v(s))ds$. For $t \in [\frac{\eta}{\alpha}, \eta]$, by Lemma 2.3,

$$\begin{aligned} A(u,v)(t) &= \int_0^1 G(t,s)a(s)f(s,u(s),v(s))ds \\ &\geqslant k\int_0^1 \beta(s)a(s)f(s,u(s),v(s))ds \\ &\geqslant k\|A(u,v)\|. \end{aligned}$$

Similarly, $B(u,v)(t) \geqslant k\|B(u,v)\|$ for $u,v \in K, t \in [\frac{\eta}{\alpha}, \eta]$. So we get

$$A(u,v)(t) + B(u,v)(t) \geqslant k(\|A(u,v)\| + \|B(u,v)\|), u,v \in K, t \in [\frac{\eta}{\alpha}, \eta]$$

and in consequence,

$$\min_{t\in[\frac{\eta}{\alpha},\eta]}\{A(u,v)(t) + B(u,v)(t)\} \geqslant k(\|A(u,v)\| + \|B(u,v)\|).$$

Therefore, $T: K \to K$. Note that $G(t,s)$, $f(s,u,v)$, $a(s)$ are continuous, we can easily check that $A: K \to C[0,1]$ is completely continuous by the Ascoli-Arzela theorem. Similarly, $B : K \to C[0,1]$ is completely continuous. So $T : K \to K$ is completely continuous. $\square$

Proof of Theorem 3.1. If $f_0 = g_0 = \infty$, there exists a small number $r_1 > 0$, such that

$$f(t,u,v) \geqslant \tau_1(u+v),\ g(t,u,v) \geqslant \tau_1(u+v)$$

for $0 \leqslant u, v \leqslant r_1$. In addition, we know that τ_1 satisfies $\tau_1 l' > \frac{1}{2}$.

Let $K_{r_1} = \{(u,v) \in K \mid \|(u,v)\| < r_1\}$. Then for $(u,v) \in \partial K_{r_1}$ and $t \in [\frac{\eta}{\alpha}, \eta]$, by Lemma 2.3,

$$\begin{aligned} A(u,v)(t) &= \int_0^1 G(t,s)a(s)f(s,u(s),v(s))ds \geqslant \int_0^1 k\beta(s)a(s)f(s,u(s),v(s))ds \\ &\geqslant \int_0^1 k\beta(s)a(s)\tau_1(u(s)+v(s))ds \geqslant \int_{\frac{\eta}{\alpha}}^{\eta} k\beta(s)a(s)\tau_1(u(s)+v(s))ds \\ &\geqslant \int_{\frac{\eta}{\alpha}}^{\eta} k^2\beta(s)a(s)\tau_1(\|u\|+\|v\|)ds \\ &= \tau_1 l_2(\|u\|+\|v\|) = \tau_1 l_2\|(u,v)\| > \frac{1}{2}\|(u,v)\|. \end{aligned}$$

Hence,

$$\|A(u,v)\| > \frac{1}{2}\|(u,v)\| \ \text{ for } (u,v) \in \partial K_{r_1}.$$

By using the same way,

$$B(u,v)(t) > \frac{1}{2}\|(u,v)\| \ \text{ for } (u,v) \in \partial K_{r_1}.$$

And thus $\|T(u,v)\| = \|A(u,v)\| + \|B(u,v)\| > \|(u,v)\|$, $(u,v) \in \partial K_{r_1}$. From Lemma 2.1,

$$i(T, K_{r_1}, K) = 0. \tag{3.1}$$

If $f_\infty = g_\infty = 0$, there is $R > r_1$ such that

$$f(t,u,v) \leqslant \tau_2(u+v),\ g(t,u,v) \leqslant \tau_2(u+v)$$

for $u + v \geqslant R$, where τ_2 satisfies $\tau_2 h < \frac{1}{2}$.

Take $r_2 > \frac{R}{k}$ and let $K_{r_2} = \{(u,v) \in K \mid \|(u,v)\| < r_2\}$. Then for $(u,v) \in \partial K_{r_2}$, we have $\min\limits_{t \in [\frac{\eta}{\alpha}, \eta]} (u(t)+v(t)) \geqslant k(\|u\|+\|v\|) = kr_2 > R$, and thus

$$\begin{aligned} A(u,v)(t) &= \int_0^1 G(t,s)a(s)f(s,u(s),v(s))ds \leqslant \tau_2 \int_0^1 \beta(s)a(s)(u(s)+v(s))ds \\ &\leqslant \tau_2(\|u\|+\|v\|)\int_0^1 \beta(s)a(s)ds = \tau_2 l_1\|(u,v)\| \\ &\leqslant \tau_2 h\|(u,v)\| < \frac{1}{2}\|(u,v)\|. \end{aligned}$$

It is also easily to prove that

$$B(u,v)(t) < \frac{1}{2}\|(u,v)\| \ \text{ for } (u,v) \in \partial K_{r_2}.$$

So $\|T(u,v)\| = \|A(u,v)\| + \|B(u,v)\| < \|(u,v)\|$ for $(u,v) \in \partial K_{r_2}$. By using Lemma 2.1, we obtain

$$i(T, K_{r_2}, K) = 1. \tag{3.2}$$

Note that $r_1 < r_2$, and from the additivity of the fixed point index and (3.1), (3.2), we have

$$i(T, K_{r_2} \backslash \overline{K_{r_1}}, K) = i(T, K_{r_2}, K) - i(T, K_{r_1}, K) = 1.$$

Therefore, T has a fixed point (u,v) in $K_{r_2} \backslash \overline{K_{r_1}}$. Evidently, (u,v) is a positive solution for problem (1.1) with $r_1 < \|(u,v)\| < r_2$. □

Proof of Theorem 3.2. Let $K_{\rho_1} = \{(u,v) \in K \mid \|(u,v)\| < \rho_1\}$. Then for $(u,v) \in \partial K_{\rho_1}$, and $t \in [\frac{\eta}{\alpha}, \eta]$, from Lemma 2.3 and (H_3), We have

$$\begin{aligned}
A(u,v)(t) &= \int_0^1 G(t,s)a(s)f(s,u(s),v(s))ds \\
&\geqslant \int_0^1 k\beta(s)a(s)f(s,u(s),v(s))ds \\
&\geqslant \int_0^1 k\beta(s)a(s)\tau_1(u(s)+v(s))ds \\
&\geqslant \int_{\frac{\eta}{\alpha}}^{\eta} k\beta(s)a(s)\tau_1(u(s)+v(s))ds \\
&\geqslant k^2\tau_1 \int_{\frac{\eta}{\alpha}}^{\eta} \beta(s)a(s)\|u+v\|ds \\
&= k^2\tau_1\|u+v\| \int_{\frac{\eta}{\alpha}}^{\eta} \beta(s)a(s)ds \\
&= \tau_1 l_2\|u+v\| \geqslant \frac{3}{4}\|u+v\| > \frac{1}{2}\|u+v\|.
\end{aligned}$$

Hence,

$$\|A(u,v)\| > \frac{1}{2}\|u+v\| = \frac{1}{2}\|(u,v)\| \ \text{ for } (u,v) \in \partial K_{\rho_1}.$$

Similarly, we can prove that

$$\|B(u,v)\| > \frac{1}{2}\|(u,v)\| \ \text{ for } (u,v) \in \partial K_{\rho_1}.$$

Therefore, $\|T(u,v)\| = \|A(u,v)\| + \|B(u,v)\| > \|(u,v)\|$ for $(u,v) \in \partial K_{\rho_1}$. By Lemma 2.1, we obtain

$$i(T, K_{\rho_1}, K) = 0. \tag{3.3}$$

Next, Let $K_{\rho_2} = \{(u,v) \in K| \mid \|(u,v)\| < \rho_2\}$. Then for $(u,v) \in \partial K_{\rho_2}$, form Lemma 2.2

$$\begin{aligned} A(u,v)(t) &= \int_0^1 G(t,s)a(s)f(s,u(s),v(s))ds \\ &< \int_0^1 \beta(s)a(s)\frac{1}{2h}\rho_2 ds \\ &= \frac{1}{2h}\rho_2 \int_0^1 \beta(s)a(s)ds \\ &\leqslant \frac{1}{2}\rho_2 = \frac{1}{2}\|(u,v)\|. \end{aligned}$$

Hence,

$$\|A(u,v)\| < \frac{1}{2}\|(u,v)\| \ \text{ for } (u,v) \in \partial K_{\rho_2}.$$

By the similar way, we can prove

$$\|B(u,v)\| < \frac{1}{2}\|(u,v)\| \ \text{ for } (u,v) \in \partial K_{\rho_2}.$$

And thus $\|T(u,v)\| = \|A(u,v)\| + \|B(u,v)\| < \|(u,v)\|$, for $(u,v) \in \partial K_{\rho_2}$. Using Lemma 2.1, we get

$$i(T, K_{\rho_2}, K) = 1. \tag{3.4}$$

From (H_4), $f_\infty = g_\infty = \infty$, there exists $R > \rho_2$ such that

$$f(t,u,v) \geqslant \tau_2(u+v),\ g(t,u,v) \geqslant \tau_2(u+v)$$

for $u + v \geqslant R$, where τ_2 satisfies $\tau_2 l' > \frac{1}{2}$.

Take $\rho_3 \geqslant \frac{R}{k}$ and Let $K_{\rho_3} = \{(u,v) \in K| \mid \|(u,v)\| < \rho_3\}$. Then for $(u,v) \in \partial K_{\rho_3}$, we get $\min\limits_{t\in[\frac{\eta}{\alpha},\eta]}(u(t)+v(t)) \geqslant k(\|u\|+\|v\|) = k\rho_3 \geqslant R$, and thus

$$\begin{aligned} A(u,v)(t) &= \int_0^1 G(t,s)a(s)f(s,u(s),v(s))ds \geqslant \int_0^1 k\beta(s)a(s)f(s,u(s),v(s))ds \\ &\geqslant \int_{\frac{\eta}{\alpha}}^{\eta} k\beta(s)a(s)f(s,u(s),v(s))ds \geqslant \int_{\frac{\eta}{\alpha}}^{\eta} k\beta(s)a(s)\tau_2(u(s)+v(s))ds \\ &\geqslant \int_{\frac{\eta}{\alpha}}^{\eta} k^2\beta(s)a(s)\tau_2\|u+v\|ds = \tau_2 k^2 \int_{\frac{\eta}{\alpha}}^{\eta} a(s)\beta(s)ds(\|u\|+\|v\|) \\ &= \tau_2 l_2\|(u,v)\| > \frac{1}{2}\|(u,v)\|. \end{aligned}$$

Hence,

$$\|A(u,v)\| > \frac{1}{2}\|(u,v)\| \ \text{ for } (u,v) \in \partial K_{\rho_3}.$$

Using the same proof, we get

$$\|B(u,v)\| > \frac{1}{2}\|(u,v)\| \ \text{ for } (u,v) \in \partial K_{\rho_3}.$$

Further, we have $\|T(u,v)\| = \|A(u,v)\| + \|B(u,v)\| > \|(u,v)\|$ for $(u,v) \in \partial K_{\rho_3}$. From Lemma 2.1, we obtain

$$i(T, K_{\rho_3}, K) = 0. \tag{3.5}$$

Because $\rho_1 < \rho_2 < \rho_3$, from (3.3)-(3.5) and the additivity of the fixed point index, we have

$$\begin{aligned} i(T, K_{\rho_3} \backslash \overline{K_{\rho_2}}, K) &= i(T, K_{\rho_3}, K) - i(T, K_{\rho_2}, K) = -1, \\ i(T, K_{\rho_2} \backslash \overline{K_{\rho_1}}, K) &= i(T, K_{\rho_2}, K) - i(T, K_{\rho_1}, K) = 1. \end{aligned}$$

Therefore, we can conclude that T has two points (u_1, v_1) and (u_2, v_2) with $(u_1, v_1) \in K_{\rho_2} \backslash \overline{K_{\rho_1}}$, $(u_2, v_2) \in K_{\rho_3} \backslash \overline{K_{\rho_2}}$. That is, these are the positive solutions for the problem (1.1) which satisfy $0 < \|(u_1, v_1)\| < \rho_2 < \|(u_2, v_2)\|$. □

Proof of Theorem 3.3. Firstly, since $f_0 = g_0 = 0$, there exists $r_1 \in (0, \rho)$ such that

$$f(t,u,v) \leqslant \tau_1^{'}(u+v), \;\; g(t,u,v) \leqslant \tau_1^{'}(u+v)$$

for $0 < u, v \leqslant r_1$, where $\tau_1^{'}$ satisfies $\tau_1^{'} h < \frac{1}{2}$.

Let $K_{r_1} = \{(u,v) \in K \mid \|(u,v)\| < r_1\}$. Then for any $(u,v) \in \partial K_{r_1}$, by using the same calculation in the proof of Theorem 3.2, we have

$$\begin{aligned} A(u,v)(t) &= \int_0^1 G(t,s)a(s)f(s,u(s),v(s))ds \\ &\leqslant \int_0^1 \beta(s)a(s)f(s,u(s),v(s))ds \\ &\leqslant \tau_1 \int_0^1 \beta(s)a(s)(u(s)+v(s))ds \\ &\leqslant \tau_1 \int_0^1 \beta(s)a(s)\|u+v\|ds \\ &= \tau_1 l_1 \|(u,v)\| \\ &\leqslant \tau_1 h \|(u,v)\| < \frac{1}{2}\|(u,v)\|, \end{aligned}$$

and thus

$$\|A(u,v)\| < \frac{1}{2}\|(u,v)\| \;\text{ for } (u,v) \in \partial K_{r_1}.$$

Similarly, we can get

$$\|B(u,v)\| < \frac{1}{2}\|(u,v)\| \;\text{ for } (u,v) \in \partial K_{r_1}.$$

Further, $\|T(u,v)\| = \|A(u,v)\| + \|B(u,v)\| < \|(u,v)\|$ for $(u,v) \in \partial K_{r_1}$. Therefore, by Lemma 2.1,

$$i(T, K_{r_1}, K) = 1. \tag{3.6}$$

Secondly, in view of $f_\infty = g_\infty = 0$, there exists $R > \rho_1$, such that

$$f(t,u,v) \leqslant \tau_2(u+v),\ \ g(t,u,v) \leqslant \tau_2(u+v)$$

for $(u+v) \geqslant R$, where $\tau_2 > 0$ with $\tau_2 h < \frac{1}{4}$.

We divide the proof into two cases: f is bounded and f is unbounded.
Case (i), suppose f is bounded, which implies that there exists $M_1 > 0$ such that $f(t,u,v) \leqslant M_1$ for all $u,v \in [0,+\infty)$.

Now choose $r_2 > \max\{2hM_1, R\}$, so that for $(u,v) \in K$ with $\|(u,v)\| = r_2$, we have

$$\begin{aligned} A(u,v)(t) &= \int_0^1 G(t,s)a(s)f(s,u(s),v(s))ds \\ &\leqslant M_1 \int_0^1 \beta(s)a(s)ds \\ &= l_1 M_1 \leqslant M_1 h < \frac{r_2}{2} = \frac{1}{2}\|(u,v)\|. \end{aligned}$$

Case (ii), suppose f is unbounded. Then because $f : [0,\infty)\times[0,\infty) \to (0,\infty)$ are continuous, we know that there is $r_2 > \max\{\rho_1, \frac{R}{k}\}$ such that $f(t,u,v) \leqslant f(t,r_2,r_2)$ for $0 < u,v \leqslant r_2$. Then for $(u,v) \in K$ with $\|(u,v)\| = r_2$, we have

$$\begin{aligned} A(u,v)(t) &= \int_0^1 G(t,s)a(s)f(s,u(s),v(s))ds \\ &\leqslant \int_0^1 \beta(s)a(s)f(t,r_2,r_2)ds \\ &\leqslant \int_0^1 \beta(s)a(s)\tau_2(r_2+r_2)ds \\ &= \tau_2 l_1 2r_2 = 2\tau_2 l_1\|(u,v)\| \\ &\leqslant 2\tau_2 h\|(u,v)\| < \frac{1}{2}\|(u,v)\|. \end{aligned}$$

Hence, in either case, we may always set $K_{r_2} = \{(u,v) \in K : \|(u,v)\| < r_2\}$, such that

$$\|A(u,v)\| < \frac{1}{2}\|(u,v)\| \ \text{ for } (u,v) \in \partial K_{r_2}.$$

Similarly, we can prove

$$\|B(u,v)\| < \frac{1}{2}\|(u,v)\| \ \text{ for } (u,v) \in \partial K_{r_2}.$$

Consequently, $\|T(u,v)\| = \|A(u,v)\| + \|B(u,v)\| < \|(u,v)\|$ for $(u,v) \in \partial K_{r_2}$. Therefore, by Lemma 2.1 implies that

$$i(T, K_{r_2}, K) = 1. \tag{3.7}$$

Finally, let $K_{\rho_1} = \{(u,v) \in K : \|(u,v)\| < \rho_1\}$. Since $(u,v) \in \partial K_{\rho_1} \subset K$, $\min\limits_{t\in[\frac{\eta}{\alpha},\eta]}(u(t)+v(t)) \geqslant k(\|u\|+\|v\|) = k\rho_1$. Hence, for any $(u,v) \in \partial K\rho_1$, by (H_6) we have

$$A(u,v)(t) \geqslant \int_{\frac{\eta}{\alpha}}^{\eta} G(t,s)a(s)f(s,u(s),v(s))ds$$

$$
\begin{aligned}
&\geqslant k\int_{\frac{\eta}{\alpha}}^{\eta}\beta(s)a(s)f(s,u(s),v(s))ds\\
&> \frac{k\rho_1}{2l'}\int_{\frac{\eta}{\alpha}}^{\eta}ka(s)\beta(s)ds\\
&= \frac{l_2\rho_1}{2l'} = \frac{l_2}{2l'}\|(u,v)\| \geqslant \frac{1}{2}\|(u,v)\|.
\end{aligned}
$$

So we can get

$$\|A(u,v)\| > \frac{1}{2}\|(u,v)\| \quad \text{for } (u,v)\in\partial K_{\rho_1}.$$

Similarly, we can prove

$$\|B(u,v)\| > \frac{1}{2}\|(u,v)\| \quad \text{for } (u,v)\in\partial K_{\rho_1}.$$

So we can get

$$\|T(u,v)\| = \|A(u,v)\| + \|B(u,v)\| > \|(u,v)\| \quad \text{for } (u,v)\in\partial K_{\rho_1}.$$

Hence, Lemma 2.1 shows that

$$i(T, K_{\rho_1}, K) = 0. \tag{3.8}$$

Note that $r_1 < \rho_1 < r_2$, from (3.6)-(3.8) we have

$$i(T, K_{\rho_1}\backslash\overline{K_{r_1}}, K) = i(T, K_{\rho_1}, K) - i(T, K_{r_1}, K) = -1,$$

and

$$i(T, K_{r_2}\backslash\overline{K_{\rho_2}}, K) = i(T, K_{r_2}, K) - i(T, K_{\rho_1}, K) = 1.$$

This shows that T has two fixed points, and consequently, the problem (1.1) has two positive solutions. This completes the proof. □

4. Examples

Example 4.1. Consider the following third-order differential system:

$$
\begin{cases}
-u'''(t) = (1+2t^3+3t^3)\sqrt{u(t)+v(t)},\ t\in(0,1),\\
-v'''(t) = (2+t^5+t^6)\sqrt[3]{u(t)+v(t)},\ t\in(0,1),\\
u(0)=u'(0)=0,\ u'(1)=2u'(\tfrac{1}{4}),\\
v(0)=v'(0)=0,\ v'(1)=2v'(\tfrac{1}{4}).
\end{cases} \tag{4.1}
$$

In this example, $a(t) = 1+2t^3+3t^3$, $b(t) = 2+t^5+t^6$, $\alpha = 2$, $\eta = \frac{1}{4}$. In addition, $f(t,u,v) = \sqrt{u+v}$, $g(t,u,v) = \sqrt[3]{u+v}$. Evidently, (H_1), (H_2) hold and

$$f_0 = \lim_{(u,v)\to(0^+,0^+)}\frac{f(t,u,v)}{u+v} = \lim_{(u,v)\to(0^+,0^+)}\frac{1}{\sqrt{u+v}} = +\infty,$$

$$f_\infty = \lim_{(u,v)\to(+\infty,+\infty)} \frac{f(t,u,v)}{u+v} = \lim_{(u,v)\to(+\infty,+\infty)} \frac{1}{(u+v)^{\frac{2}{3}}} = 0,$$
$$g_0 = \lim_{(u,v)\to(0^+,0^+)} \frac{g(t,u,v)}{u+v} = \lim_{(u,v)\to(0^+,0^+)} \frac{1}{(u+v)^{\frac{2}{3}}} = +\infty,$$
$$g_\infty = \lim_{(u,v)\to(+\infty,+\infty)} \frac{g(t,u,v)}{u+v} = \lim_{(u,v)\to(+\infty,+\infty)} \frac{1}{(u+v)^{\frac{2}{3}}} = 0.$$

Hence, the conditions of Theorem 3.1 are satisfied. So the problem (4.1) has at least one positive solution.

Example 4.2. Consider the following third-order differential system:

$$\begin{cases} -u^{'''}(t) = t^2 f(t,u,v),\ t\in(0,1), \\ -v^{'''}(t) = tg(t,u,v),\ t\in(0,1), \\ u(0)=u'(0)=0,\ u'(1)=\frac{3}{2}u'(\frac{1}{2}), \\ v(0)=v'(0)=0,\ v'(1)=\frac{3}{2}v'(\frac{1}{2}), \end{cases} \tag{4.2}$$

where

$$f(t,u,v) = \begin{cases} 8.6\times10^4(u+v), 0\leqslant u,v\leqslant\frac{1}{3}, \\ \frac{86}{15}\times10^4, \frac{1}{3}\leqslant u,v\leqslant 10^5, \\ \frac{43}{3}\times10^{-7}(u+v)^2, u,v\geqslant 10^5, \end{cases}$$

$$g(t,u,v) = \begin{cases} 8.6\times10^4(u+v), 0\leqslant u,v\leqslant\frac{1}{3}, \\ \frac{86}{15}\times10^4, \frac{1}{3}\leqslant u,v\leqslant 10^5, \\ \frac{43}{6}\times10^{-10}(u+v)^3, u,v\geqslant 10^5. \end{cases}$$

In this example, $a(t)=t^2$, $b(t)=t$, $\alpha=\frac{3}{2}$, $\eta=\frac{1}{2}$. Evidently, $(H_1),(H_2)$ holds and

$$k=\frac{1}{90},\ \beta(s)=10(s-s^2),\ l_1=\int_0^1 10(s-s^2)s^2ds=\frac{1}{2},$$
$$l_2=(\frac{1}{90})^2\int_{\frac{1}{3}}^{\frac{1}{2}} 10(s^3-s^4)ds=\frac{1}{810}\times\frac{553}{77760},$$
$$l_3=\int_0^1 10(s-s^2)sds=\frac{5}{6}, l_4=(\frac{1}{90})^2\int_{\frac{1}{3}}^{\frac{1}{2}} 10(s^2-s^3)ds=\frac{1}{810}\times\frac{87}{5184},$$
$$\tau_1=\frac{47239200}{553}\approx 8.54\times10^4,\ h=\frac{5}{6},\ l^{'}=\frac{553}{62985600}\approx 8.78\times10^{-6}.$$

Take $\rho_1=\frac{1}{3}$, $\rho_2=10^5$. Then $\rho_1<\rho_2$ and
(i) when $0\leqslant u,v\leqslant\rho_1$, $f(t,u,v)=8.6\times10^4(u+v)\geqslant\tau_1(u+v)$;
(ii) when $0\leqslant u,v\leqslant\rho_2$,

$$f(t,u,v)\leqslant\frac{86}{15}\times10^4=\frac{1}{2}\times\frac{172}{15\times10}\times10^5<\frac{1}{2}\times\frac{6}{5}\rho_2=\frac{1}{2h}\rho_2;$$

(*iii*)

$$f_\infty = \lim_{(u,v)\to(+\infty,+\infty)} \frac{\frac{43}{6}\times 10^{-10}(u+v)^3}{u+v} = \lim_{(u,v)\to(+\infty,+\infty)} \frac{43}{6}\times 10^{-10}(u+v)^2 = +\infty.$$

By using the same way, we can prove that $g(t,u,v)$ has the same properties. So the conditions of Theorem 3.2 are satisfied. Therefore, the problem (4.2) has at least two positive solutions (u_1,v_1) and (u_2,v_2) with $0 < \|(u_1,v_1)\| < 10^5 < \|(u_2,v_2)\|$.

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