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MONOTONICITY OF THE RATIO OF TWO ABELIAN INTEGRALS FOR A CLASS OF SYMMETRIC HYPERELLIPTIC HAMILTONIAN SYSTEMS*

Rasool Kazemi

Abstract In this paper we study the monotonicity of the ratio of two hyperelliptic Abelian integrals $I_0(h) = \oint_{\Gamma_h} ydx$ and $I_1(h) = \oint_{\Gamma_h} xydx$ for which Γ_h is a continuous family of periodic orbits of a Newtonian system with Hamiltonian function of the form $H(x,y) = \frac{1}{2}y^2 \pm \Psi(x)$, where Ψ is an arbitrary even function of degree six.

Keywords Hamiltonian system, hyperelliptic Abelian integral, monotonicity, asymptotic expansion.

MSC(2010) 34C07, 34C08, 37G15, 34M50.

1. Introduction

Consider a perturbed Hamiltonian system

$$\dot{x} = H_y + \varepsilon p(x,y), \qquad \dot{y} = -H_x + \varepsilon q(x,y), \tag{1.1}$$

where p, q and H are polynomials in x, y and ε is a small positive parameter. For $\varepsilon = 0$, the associated unperturbed system is of the form

$$\dot{x} = H_y, \qquad \dot{y} = -H_x. \tag{1.2}$$

Suppose that the Hamiltonian system (1.2) has a family of periodic orbits Γ_h with continuous dependency on parameter $h \in (h_1, h_2)$ defined by $H(x,y) = h$. Then, there exists an Abelian integral, called the first-order Melnikov function, of the form

$$I(h) = \oint_{\Gamma_h} q(x,y)dx - p(x,y)dy, \tag{1.3}$$

which has a key role in the study of bifurcation of limit cycles from system (1.1), since if $I(h)$ is not identically zero, then the number of isolated zeros of $I(h)$ gives an upper bound for the number of limit cycles of (1.1) (see [1,3]). There is a lot of new results on this subject, see for instance [7,8] and references therein.

Now consider the Hamiltonian function $H^{\pm}(x,y) = \frac{1}{2}y^2 \mp \Psi(x)$ where Ψ is an arbitrary even function of degree six. The corresponding Hamiltonian system is of

Email address:r.kazemi@kashanu.ac.ir(R. Kazemi)
Department of Mathematical Sciences, University of Kashan, Km. 6 Ravand road, 87317-53153, Kashan, Iran
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the form

$$\dot{x} = y, \quad \dot{y} = \pm\Psi'(x). \tag{1.4}$$

By Green theorem, one can see that $I_0(h) = \oint_{\Gamma_h} ydx \neq 0$ for any compact component Γ_h of $H^{\pm}(x,y) = h$ except the critical points of $H^{\pm}(x,y)$. So we can define the ratio of two Abelian integrals as $P(h) = \frac{I_1(h)}{I_0(h)}$ where $I_1(h) = \oint_{\Gamma_h} xydx$. By using the excellent criterion given in [6] and inspiring its application in the original paper and [9], in this paper we study the monotonicity of $P(h)$ in some open intervals of h for different level sets of $H^{\pm}(x,y)$ separately. Monotonicity of $P(h)$ is important due to the fact that if in (1.3), we set $q(x,y) = (a+bx)y$ and $p(x,y) = 0$, then $I(h) = aI_0(h) + bI_1(h)$ and the monotonicity of $P(h)$ implies that the Abelian integral $I(h)$ has at most one isolated zero.

This paper is organized as follows. In section 2 we classify all possible states for system (1.4) and give their corresponding phase portraits. In Section 3, using the novel criteria given in [6, Theorem 2.1], we investigate the monotonicity of $P(h)$ as our main results (see Theorems 3.1, 3.2 and 3.3).

2. Classification of system (1.4)

In this section, we will classify all possible states for system (1.4) and determine all topologically different phase portraits of that states. Since we have assumed Ψ to be an even polynomial of degree six, Ψ' is an odd polynomial of degree five and obviously $x = 0$ is one of its roots. So there are three cases for Ψ':

Case I. $x = 0$ is the only real root of Ψ'. In this case either $\Psi' := \Psi_1' = x^5$ or by scaling we may assume $\Psi' := \Psi_2' = x(x^2+\alpha^2)(x^2+1)$, where $\alpha \in \mathbb{R}$. The system (1.4) will be therefor one of the following Hamiltonian systems

$$(X_1^{\pm}) : \begin{cases} \dot{x} = y, \\ \dot{y} = \pm x^5, \end{cases} \qquad (X_2^{\pm}) : \begin{cases} \dot{x} = y, \\ \dot{y} = \pm x(x^2+\alpha^2)(x^2+1), \end{cases}$$

with Hamiltonian functions

$$H_1^{\pm}(x,y) = \frac{1}{2}y^2 \mp \frac{1}{6}x^6,$$

and

$$H_2^{\pm}(x,y) = \frac{1}{2}y^2 \mp (\frac{1}{6}x^6 + \frac{1}{4}(\alpha^2+1)x^4 + \frac{1}{2}\alpha^2x^2),$$

respectively. The origin is the only equilibrium point for any of the above systems. Note that the origin is a nilpotent saddle for (X_1^+), and (X_2^+) only when $\alpha = 0$ (Fig. 1 (a)) and it is a hyperbolic saddle for (X_2^+) when $\alpha \neq 0$ (Fig. 1 (c)). Also the origin is a global nilpotent center for (X_1^-), and (X_2^-) only when $\alpha = 0$ (Fig. 1 (b)) and it is a global elementary center for (X_2^-) when $\alpha \neq 0$ (Fig. 1 (d)).

Case II. $x = 0$ is not the only real root of Ψ' which has exactly five real roots. In this case with no loss of generality we assume that $x = 1$ is the largest real

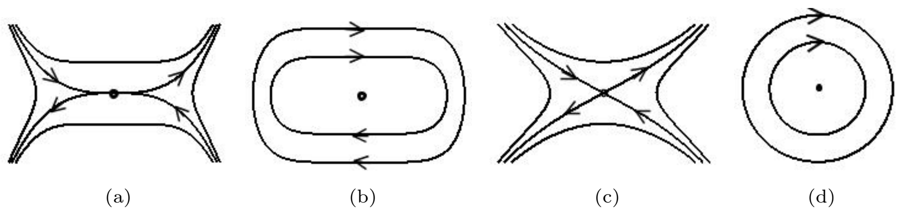

Figure 1. Phase portraits of systems $(X_1^{\pm})$ and $(X_2^{\pm})$.

root of Ψ'. So $\Psi' := \Psi_3' = x(x^2 - \alpha^2)(x^2 - 1)$, where $\alpha \in [0, 1]$. The system (1.4) is therefore of the form

$$(X_3^{\pm}) : \begin{cases} \dot{x} = y, \\ \dot{y} = \pm x(x^2 - \alpha^2)(x^2 - 1), \end{cases}$$

with Hamiltonian functions

$$H_3^{\pm}(x, y) = \frac{1}{2}y^2 \mp (\frac{1}{6}x^6 - \frac{1}{4}(\alpha^2 + 1)x^4 + \frac{1}{2}\alpha^2 x^2).$$

We first classify the integral curves of Hamiltonian system (X_3^-). For $\alpha = 0$, system (X_3^-) has a nilpotent saddle at $(0, 0)$ and two centers at $(\pm 1, 0)$. If $0 < \alpha < 1$, then $(0, 0)$ and $(\pm 1, 0)$ are centers and $(\pm\alpha, 0)$ are hyperbolic saddles of system (X_3^-). Finally for $\alpha = 1$, system (X_3^-) has a center at $(0, 0)$ and two cusps at $(\pm 1, 0)$. The different phase portraits of system (X_3^-) are shown in Fig. 2.

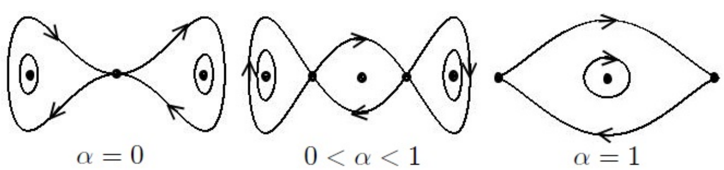

Figure 2. Phase portraits of system (X_3^-) for various $0 \leq \alpha \leq 1$.

Now we classify the integral curves of Hamiltonian system (X_3^+). For $\alpha = 0$, system (X_3^+) has a nilpotent center at $(0, 0)$ and two hyperbolic saddles at $(\pm 1, 0)$. If $0 < \alpha < 1$, then $(0, 0)$ and $(\pm 1, 0)$ are hyperbolic saddles and $(\pm\alpha, 0)$ are centers of system (X_3^+). Note that for $0 < \alpha < \frac{\sqrt{3}}{3}$, $\alpha = \frac{\sqrt{3}}{3}$ and $\frac{\sqrt{3}}{3} < \alpha < 1$, we have $H_3^+(0, 0) < H_3^+(\pm 1, 0)$, $H_3^+(0, 0) = H_3^+(\pm 1, 0)$ and $H_3^+(0, 0) > H_3^+(\pm 1, 0)$, respectively. Therefore, three saddle points $(-1, 0)$, $(0, 0)$ and $(1, 0)$ are on the same level curve of H_3^+ for $\alpha = \frac{\sqrt{3}}{3}$. Finally for $\alpha = 1$, system (X_3^+) has a saddle at $(0, 0)$ and two cusps at $(\pm 1, 0)$. The different phase portraits of system (X_3^+) are shown in Fig. 3.

Case III. $x = 0$ is not the only real root of Ψ' which has exactly three real roots. By scaling we may assume $\Psi' := \Psi_4' = x(x^2 + \alpha^2)(x^2 - 1)$ where $\alpha \in \mathbb{R} \setminus \{0\}$.

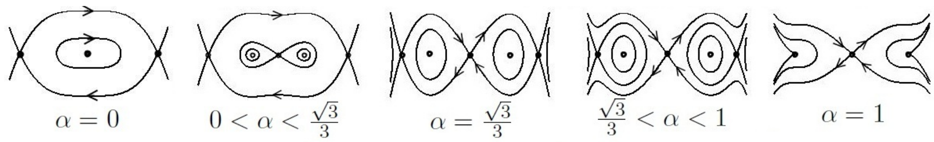

Figure 3. Phase portraits of system (X_3^+) for various $0 \le \alpha \le 1$.

Therefore system (1.4) is of the form

$$(X_4^{\pm}): \begin{cases} \dot{x} = y, \\ \dot{y} = \pm x(x^2+\alpha^2)(x^2-1), \end{cases} \qquad \alpha \neq 0,$$

with Hamiltonian functions

$$H_4^{\pm}(x,y) = \frac{1}{2}y^2 \mp (\frac{1}{6}x^6 + \frac{1}{4}(\alpha^2-1)x^4 - \frac{1}{2}\alpha^2x^2).$$

Note that both of the above systems have three equilibrium points at $(0,0)$ and $(\pm1,0)$. For system (X_4^-), the origin is a hyperbolic saddle and $(\pm1,0)$ are elementary centers (Fig. 4 (a)), but for system (X_4^+), the origin is an elementary center and $(\pm1,0)$ are hyperbolic saddles (Fig. 4 (b)).

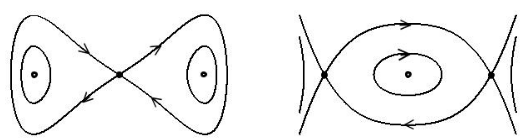

Figure 4. Phase portraits of systems $(X_4^{\pm})$.

3. Monotonicity of $P(h)$ for $H_i^{\pm}(x,y) = h$

In the following we study the monotonicity of $P(h)$ where Γ_h is a compact component of $H_i^{\pm}(x,y) = h$ surrounding a unique center of system $X_i^{\pm}$ for $i = 1, \cdots, 4$. We begin our study by noticing that:

Remark 3.1. If $(0,0)$ is a center of (1.4), then the closed orbit Γ_h of system (1.4) surrounding the origin is symmetric, and the orientation of Γ_h is clockwise. Thus $I_0(h) = \oint_{\Gamma_h} y\,dx < 0$ and $I_1(h) = \oint_{\Gamma_h} xy\,dx \equiv 0$, which imply $P(h) \equiv 0$ for $h > 0$.

It should be noted that systems (X_1^+) and (X_2^+) don't have any center. Also the only center of systems (X_1^-) and (X_2^-) is $(0,0)$ and by Remark 3.1, it is a trivial case. Now we consider the monotonicity of $P(h)$, when Γ_h is a continuous family of ovals surrounding one of the centers $(\pm1,0)$ of system (X_3^-). Our result is as follows.

Theorem 3.1. *Let $0 \le \alpha < 1$ and suppose that Γ_h is a continuous family of ovals surrounding one of two centers $(\pm1,0)$ of system (X_3^-). Then the function $P(h)$ is monotone for $h \in (0, \frac{1}{12}\alpha^4(3-\alpha^2))$.*

Proof. Due to the symmetry property of system (X_3^-), we only need to state the proof for Γ_h being a continuous family of ovals surrounding $(1,0)$. Note that for $0 \le \alpha < 1$, the Hamiltonian function

$$H_3^-(x,y) = \frac{1}{2}y^2 + \frac{1}{6}x^6 - \frac{1}{4}(\alpha^2+1)x^4 + \frac{1}{2}\alpha^2 x^2 = \frac{1}{2}y^2 + \Psi_3(x),$$

has a local minimum at $(1,0)$ and a continuous family of ovals Γ_h surrounding the center $(1,0)$. The period annulus Γ_h is bounded by a homoclinic loop connecting the saddle point $(\alpha,0)$ (nilpotent saddle (0,0) when $\alpha = 0$). The projection of this period annulus on the x-axis is the interval $(\alpha, \frac{1}{2}\sqrt{2(3-\alpha^2)})$ and $\Psi_3(\alpha) = \Psi_3(\frac{1}{2}\sqrt{2(3-\alpha^2)}) = \frac{1}{12}\alpha^4(3-\alpha^2)$. Since $\Psi_3'(x)(x-1) > 0$ for every $x \in (\alpha, \frac{1}{2}\sqrt{2(3-\alpha^2)})$, the maps

$$\Psi_3 : (\alpha, 1) \to (\frac{1}{4}(\alpha^2 - \frac{1}{3}), \frac{1}{12}\alpha^4(3-\alpha^2))$$

and

$$\Psi_3 : (1, \frac{1}{2}\sqrt{2(3-\alpha^2)}) \to (\frac{1}{4}(\alpha^2 - \frac{1}{3}), \frac{1}{12}\alpha^4(3-\alpha^2))$$

are strictly monotone and they have analytic inverse functions, respectively denoted by $\mu(h)$ and $\nu(h)$. Thus $\Psi_3(\mu(h)) = \Psi_3(\nu(h))$ and $\alpha < \mu(h) < 1 < \nu(h) < \frac{1}{2}\sqrt{2(3-\alpha^2)}$ for every $h \in (\frac{1}{4}(\alpha^2 - \frac{1}{3}), \frac{1}{12}\alpha^4(3-\alpha^2))$. Let

$$U(h) = \mu(h) + \nu(h), \quad s(h) = \frac{\mu(h)+\nu(h)}{2} \quad \text{and} \quad r(h) = \frac{\nu(h)-\mu(h)}{2}.$$

So $U(h)$ is an analytic function on $(\frac{1}{4}(\alpha^2-\frac{1}{3}), \frac{1}{12}\alpha^4(3-\alpha^2))$. The equality $\Psi_3(\mu(h)) = \Psi_3(\nu(h))$ implies that

$$\int_1^{\mu(h)} x(x^2-\alpha^2)(x^2-1)dx = \int_1^{\nu(h)} x(x^2-\alpha^2)(x^2-1)dx.$$

Applying two change of variables in the above integrals ($x \mapsto 1-x$ in the left integral and $x \mapsto 1+x$ in the right one), we obtain

$$\int_0^{1-\mu(h)} x(x-1)(x-2)((x-1)^2-\alpha^2)dx = \int_0^{\nu(h)-1} x(x+1)(x+2)((x+1)^2-\alpha^2)dx.$$

Since the integrand of the right integral is greater than the integrand of the left one, we deduce $\nu(h)-1 < 1-\mu(h)$, so $s(h) < 1$. Now for any $h \in (\frac{1}{4}(\alpha^2-\frac{1}{3}), \frac{1}{12}\alpha^4(3-\alpha^2))$ define

$$g(t) = \Psi_3(s(h)+t) - \Psi_3(s(h)-t), \quad t \in (0, r(h)). \tag{3.1}$$

From (3.1), it is obvious that $g(t)$ is a polynomial of degree 5. Furthermore, $t = 0$ and $t = \pm r(h)$ are three roots of $g(t)$. Direct computation shows that for $t \in (0, r(h))$

$$g(t) = 2s(h)t[t^4 + (\frac{10}{3}s(h)^2 - \alpha^2 - 1)t^2 + (s(h)^2-\alpha^2)(s(h)^2-1)]. \tag{3.2}$$

Since $(s(h)^2-\alpha^2)(s(h)^2-1) < 0$ we deduce that $g(t)$ dose not have any other real root. This implies that $g(t) < 0$ for all $t \in (0, r(h))$. Now we can prove that $U(h)$

is monotone for $h \in (\frac{1}{4}(\alpha^2 - \frac{1}{3}), \frac{1}{12}\alpha^4(3-\alpha^2))$. By way of contradiction suppose that there exist h_1 and h_2 such that $\frac{1}{4}(\alpha^2 - \frac{1}{3}) < h_1 < h_2 < \frac{1}{2}\alpha^4(3-\alpha^2)$ and $U(h_1) = U(h_2)$ which implies that $s(h_1) = s(h_2)$. It is clear that

$$0 < r(h_1) = \frac{\nu(h_1) - \mu(h_1)}{2} < \frac{\nu(h_2) - \mu(h_2)}{2} = r(h_2).$$

Setting $h = h_2$ in $g(t)$ leads to

$$g(t) = \Psi_3(s(h_2) + t) - \Psi_3(s(h_2) - t) < 0, \qquad t \in (0, r(h_2)). \tag{3.3}$$

Since $0 < r(h_1) < r(h_2)$, calculating $g(t)$ in (3.3) at $t = r(h_1)$ yields

$$\begin{aligned} g(r(h_1)) &= \Psi_3(s(h_2) + r(h_1)) - \Psi_3(s(h_2) - r(h_1)) \\ &= \Psi_3(s(h_1) + r(h_1)) - \Psi_3(s(h_1) - r(h_1)) \\ &= \Psi_3(\nu(h_1)) - \Psi_3(\mu(h_1)) = h_1 - h_1 = 0. \end{aligned}$$

This contradicts $g(t) < 0$ for all $t \in (0, r(h_2))$. So $U(h)$ is monotone in $(\frac{1}{4}(\alpha^2 - \frac{1}{3}), \frac{1}{2}\alpha^4(3-\alpha^2))$ and by Theorem 2.1 in [6], $P(h)$ is monotone in $(\frac{1}{4}(\alpha^2 - \frac{1}{3}), \frac{1}{2}\alpha^4(3-\alpha^2))$. □

Now we study the monotonicity of $P(h)$ when Γ_h is a compact component of $H_3^+(x,y) = h$ surrounding a unique center of system (X_3^+). Note that if $\alpha = 0$, then $(0,0)$ is a center, and as already mentioned in Remark 3.1, $P(h)$ is always zero for $h \in (0, \frac{1}{12})$. Also for $\alpha = 1$, the system (X_3^+) dose not have any center. So we consider the monotonicity of $P(h)$, when Γ_h is a continuous family of ovals surrounding one of two centers $(\pm\alpha, 0)$ of system (X_3^+) for $0 < \alpha < 1$. Our result is as follows.

Theorem 3.2. *Let $0 < \alpha < 1$ and suppose that Γ_h is a continuous family of ovals surrounding one of two centers $(\pm\alpha, 0)$ of system (X_3^+). Then*

(i) if $0 < \alpha < \frac{\sqrt{3}}{3}$, then $P(h)$ is monotone in $(\frac{1}{12}\alpha^4(\alpha^2 - 3), 0)$;

(ii) if $\alpha = \frac{\sqrt{3}}{3}$, then $P(h)$ is monotone in $(-\frac{2}{81}, 0)$;

(iii) if $\frac{\sqrt{3}}{3} < \alpha < \sqrt{\frac{3}{7}}$, then $P(h)$ is not monotone in $(\frac{1}{12}\alpha^4(\alpha^2 - 3), \frac{1}{12}(3\alpha^2 - 1))$;

(iv) if $\sqrt{\frac{3}{7}} \le \alpha < 1$, then $P(h)$ is monotone in $(\frac{1}{12}\alpha^4(\alpha^2 - 3), \frac{1}{12}(3\alpha^2 - 1))$.

Proof. By symmetry property of system (X_3^+), we only state the proof in the case where Γ_h is a continuous family of ovals surrounding $(\alpha, 0)$. Also we use the notation $\mu(h)$, $\nu(h)$, $U(h)$, $s(h)$ and $r(h)$ as defined in the proof of Theorem 3.1.

(i) By Theorem 2.1 in [6], it suffices to prove $U'(h) < 0$ for $h \in (h_\alpha, 0)$, where $h_\alpha = \frac{1}{12}\alpha^4(\alpha^3 - 3)$. First we prove that $U'(h_\alpha) < 0$. For this purpose, since $-\Psi_3(\nu(h)) = h$,

$$\sqrt{h - h_\alpha} = (\nu(h) - \alpha)\sqrt{\alpha^2(1-\alpha^2) - \frac{1}{3}\alpha(7\alpha^2 - 3)(\nu(h) - \alpha) + O((\nu(h) - \alpha)^2)}.$$

So when $0 < h - h_\alpha \ll 1$, one has

$$\nu(h) - \alpha = \frac{1}{\sqrt{\alpha^2(1-\alpha^2)}}\sqrt{h - h_\alpha} + \frac{7\alpha^2 - 3}{6\alpha^3(1-\alpha^2)^2}(h - h_\alpha) + O((h - h_\alpha)^{\frac{3}{2}}).$$

Also the equality $-\Psi_3(\mu(h)) = h$ and similar computations show that

$$\mu(h) - \alpha = -\frac{1}{\sqrt{\alpha^2(1-\alpha^2)}}\sqrt{h-h_\alpha} + \frac{7\alpha^2-3}{6\alpha^3(1-\alpha^2)^2}(h-h_\alpha) + O((h-h_\alpha)^{\frac{3}{2}}).$$

Therefore,

$$U(h) = \mu(h) + \nu(h) = 2\alpha + \frac{7\alpha^2-3}{3\alpha^3(1-\alpha^2)^2}(h-h_\alpha) + O((h-h_\alpha)^{\frac{3}{2}}),$$

and

$$U'(h_\alpha) = \lim_{h\to h_\alpha^+}(\mu'(h)+\nu'(h)) = \frac{7\alpha^2-3}{3\alpha^3(1-\alpha^2)^2} < 0.$$

Now we show that $U'(0)<0$. For this, we note that $-\Psi_3(\mu(h))=h$ and $-\Psi_3(\nu(h))=h$. Therefore, $-\mu'(h)\Psi_3'(\mu(h)) = 1$ and $-\nu'(h)\Psi_3'(\nu(h)) = 1$. So

$$\begin{aligned}\lim_{h\to 0^-} U'(h) &= \lim_{h\to 0^-}(\mu'(h)+\nu'(h)) \\ &= -\lim_{h\to 0^-}\left(\frac{1}{\Psi_3(\mu(h))} + \frac{1}{\Psi_3(\nu(h))}\right) \\ &= -\lim_{h\to 0^-}\left(\frac{1}{\mu(h)(\mu(h)^2-\alpha^2)(\mu(h)^2-1)} + \frac{1}{\nu(h)(\nu(h)^2-\alpha^2)(\nu(h)^2-1)}\right) \\ &= -\infty,\end{aligned}$$

since as $h\to 0^-$, one has $\mu(h)\to 0^+$ and $\nu(h)\to \bar{x}^-$, where

$$\bar{x} = \frac{1}{2}\sqrt{3(\alpha^2+1) - \sqrt{3(1-3\alpha^2)(3-\alpha^2)}},$$

is the intersection point of double homoclinic loop with the positive x-axis and $\alpha < \bar{x} < 1$.

Finally we show that $U'(h) < 0$ for $h \in (h_\alpha, 0)$. By way of contradiction, if $U'(h)$ has zeros in $h \in (h_\alpha, 0)$, then it has at least two zeros, since $U'(h)$ has the same sign at the end points of $(h_\alpha, 0)$. By taking $U(0) = \bar{x} < 2\alpha = U(h_\alpha)$ into account, one deduces that for some U_0, $U(h) = U_0$ has at least three distinct solution, namely $h_1 < h_2 < h_3$. If we set $\tau(h) := \mu(h)\nu(h)$, then clearly $\tau(h) = \frac{U^2(h)}{4} - r^2(h)$. By taking $U(h_1) = U(h_2) = U(h_3)$ and $0 < r(h_1) < r(h_2) < r(h_3)$ into account we deduce that $\tau(h_1)$, $\tau(h_2)$ and $\tau(h_3)$ are pairwise distinct. Also the relations $U(h) = \mu(h) + \nu(h)$ and $\tau(h) = \mu(h)\nu(h)$ imply that

$$\mu(h) = \frac{U(h) - \sqrt{U^2(h) - 4\tau(h)}}{2}, \quad \nu(h) = \frac{U(h) + \sqrt{U^2(h) - 4\tau(h)}}{2}.$$

By substituting in $\Psi_3(\mu(h)) = \Psi_3(\nu(h))$ we get

$$6\,\tau^2(h) + \left(-8\,U^2(h) + 6\,\alpha^2 + 6\right)\tau(h) + 2\,U^4(h) - 3\,\alpha^2 U^2(h) - 3\,U^2(h) + 6\,\alpha^2 = 0. \tag{3.4}$$

If we take $U(h) = U_0$, then $\tau(h_1)$, $\tau(h_2)$ and $\tau(h_3)$ should be satisfied in the equation (3.4). But the equation (3.4) is of degree two and has at most two zeros and this is a contradiction. So $U'(h) < 0$ for $h \in (h_\alpha, 0)$.

(ii) We set $\alpha = \frac{\sqrt{3}}{3}$ in system (X_3^+) and move the center $(\frac{\sqrt{3}}{3}, 0)$ to the origin by change of variables $x = X + \frac{\sqrt{3}}{3}$ and $y = Y$. Accordingly system (X_3^+) transforms to

$$\begin{aligned}
\dot{X} &= Y,\\
\dot{Y} &= X(X + \frac{\sqrt{3}}{3})(X + \frac{2\sqrt{3}}{3})(X + \frac{\sqrt{3}}{3} + 1)(X + \frac{\sqrt{3}}{3} - 1),
\end{aligned}$$

with Hamiltonian function $H_1(X, Y) = \frac{1}{2}Y^2 + A(X)$, where

$$A(x) = -\frac{1}{6}x^6 - \frac{\sqrt{3}}{3}x^5 - \frac{1}{2}x^4 + \frac{2\sqrt{3}}{27}x^3 + \frac{2}{9}x^2.$$

Then $H_1(0, 0) = 0$, $H_1(-\frac{\sqrt{3}}{3}, 0) = H_1(1-\frac{\sqrt{3}}{3}, 0) = \frac{2}{81}$ and the ovals Γ_h of $H_3^+(x, y) = h$ with $h \in (-\frac{2}{81}, 0)$ will be mapped to ovals γ_l of $H_1(X, Y) = l$ with $l \in (0, \frac{2}{81})$. So

$$\begin{aligned}
I_0(h) &= \oint_{\Gamma_h} ydx = \oint_{\gamma_l} YdX = J_{10}(l),\\
I_1(h) &= \oint_{\Gamma_h} xydx = \oint_{\gamma_l} (X + \frac{\sqrt{3}}{3})YdX\\
&= \oint_{\gamma_l} XYdX + \frac{\sqrt{3}}{3}\oint_{\gamma_l} YdX := J_{11}(l) + \frac{\sqrt{3}}{3}J_{10}(l).
\end{aligned}$$

Thus, $P(h) = \frac{I_1(h)}{I_0(h)} = \frac{\sqrt{3}}{3} + \frac{J_{11}(l)}{J_{10}(l)} = \frac{\sqrt{3}}{3} + Q_1(l)$. Note that for $l \in (0, \frac{2}{81})$, the period annulus γ_l is bounded by a heteroclinic loop connecting two hyperbolic saddles $(-\frac{\sqrt{3}}{3}, 0)$ and $(1 - \frac{\sqrt{3}}{3}, 0)$. Projection of this period annulus on the x-axis is the interval $I = (-\frac{\sqrt{3}}{3}, 1 - \frac{\sqrt{3}}{3})$ and $xA'(x) > 0$ for evry $x \in I \setminus \{0\}$. So there exists an analytic involution $z : (0, 1 - \frac{\sqrt{3}}{3}) \to (-\frac{\sqrt{3}}{3}, 0)$ such that $A(x) = A(z(x))$. We recall that z is named an involution if $zoz = Id$, but $z \neq Id$. By a straightforward calculation and using relations $A(x) = A(z(x))$ and $z(0) = 0$ we deduce that $z(x)$ is implicitly defined by $g(x, z) = \sqrt{3}(x + z) + x^2 + xz + z^2$ and $z'(x) = -\frac{\sqrt{3}+2\,x+z}{\sqrt{3}+2\,z+x}$.

Monotonicity of $P(h)$ on $(-\frac{2}{81}, 0)$ is equivalent to monotonicity of $Q_1(l)$ on $(0, \frac{2}{81})$ and this is equivalent to that $\{J_{10}, J_{11}\}$ be an extended complete Chebyshev system on $(0, \frac{2}{81})$, i.e. any nontrivial linear combination $aJ_{10}(h) + bJ_{11}(h)$ has at most one zero on $(0, \frac{2}{81})$. By Theorem B of [2], the latter is equivalent to $\{\ell_0, \ell_1\}$ being an extended complete Chebyshev system on $(0, 1-\frac{\sqrt{3}}{3})$, where $\ell_i(x) = \frac{x^i}{A'(x)} - \frac{(z(x))^i}{A'(z(x))}$, $i = 0, 1$. For this purpose we must prove two Wronskians $W[\ell_0](x)$ and $W[\ell_0, \ell_1](x)$ is non-vanishing on $(0, 1 - \frac{\sqrt{3}}{3})$. It is obvious that

$$W[\ell_0](x) = \ell_0(x) = \frac{729(x - z(x))k_0(x, z(x))}{A'(x)A'(z(x))},$$

where

$$\begin{aligned}
k_0(x, z) = {}& 9(x^4 + z^4 + xz(x^2 + zx + z^2)) + 15\sqrt{3}(x^3 + z^3 + xz(x + z))\\
&+ 18(x^2 + zx + z^2) - 2\sqrt{3}(x + z) - 4,
\end{aligned}$$

and (x,z) satisfies $q(x,z)=0$ for $-\frac{\sqrt{3}}{3}<z<0<x<1-\frac{\sqrt{3}}{3}$. We eliminate z between $q(x,z)=0$ and $k_0(x,z)=0$ by using Maple 17 and calculate the resultant of $q(x,z)$ and $k_0(x,z)$ with respect to z. Hence

$$Res(q,k_0,z)=9^7(x+\frac{\sqrt{3}}{3})^2(x+\frac{\sqrt{3}}{3}-1)^2(x+\frac{\sqrt{3}}{3}+1)^2\neq 0,\quad \text{for}\quad x\in(0,1-\frac{\sqrt{3}}{3}).$$

Therefore, $q(x,z)$ and $k_0(x,z)$ have no common roots and so $W[\ell_0](x)\neq 0$ on $(0,1-\frac{\sqrt{3}}{3})$.

Also

$$W[\ell_0,\ell_1](x)=\det\begin{pmatrix}\ell_0(x) & \ell_1(x)\\ \ell_0'(x) & \ell_1'(x)\end{pmatrix}=-\frac{531441(x-z(x))^3k_1(x,z(x))}{A'(x)A'(z(x))(2z(x)+x+\sqrt{3})},$$

where

$$\begin{aligned}k_1(x,z)&=24+6597\sqrt{3}x^4z+48\sqrt{3}x^2z+6750\sqrt{3}x^4z^3+2133\sqrt{3}xz^6\\&+6597\sqrt{3}xz^4+4536\sqrt{3}x^2z^5+11736\sqrt{3}x^3z^2+2133\sqrt{3}x^6z\\&+4536\sqrt{3}x^5z^2+11736\sqrt{3}x^2z^3+6750\sqrt{3}x^3z^4+48\sqrt{3}xz^2\\&+621\sqrt{3}z^7+2007\sqrt{3}z^5-44\sqrt{3}z-156\sqrt{3}z^3+621\sqrt{3}x^7\\&-156\sqrt{3}x^3-44\sqrt{3}x+2007\sqrt{3}x^5+5904\,x^3z+8694\,x^2z^2\\&+5904\,xz^3+23652\,x^3z^3+18873\,x^2z^4+9423\,xz^5-756\,xz\\&+2106\,x^3z^5+1215\,x^2z^6+567\,xz^7+9423\,x^5z+18873\,x^4z^2\\&+567\,x^7z+1215\,x^6z^2+2106\,x^5z^3+2430\,x^4z^4+162\,x^8\\&+1728\,z^4+2808\,x^6-516\,z^2+2808\,z^6-516\,x^2+162\,z^8+1728\,x^4.\end{aligned}$$

By computing the resultant of $q(x,z)$ and $2z+x+\sqrt{3}$ with respect to z we have

$$Res(q,2z+x+\sqrt{3},z)=(3x-\sqrt{3})(x+\sqrt{3})\neq 0,\quad \text{for}\quad x\in(0,1-\frac{\sqrt{3}}{3}).$$

So $W[\ell_0,\ell_1](x)$ is well defined for $-\frac{\sqrt{3}}{3}<z<0<x<1-\frac{\sqrt{3}}{3}$. Also

$$Res(q,k_1,z)=3^7(x+\frac{\sqrt{3}}{3})^4(x+\frac{\sqrt{3}}{3}-1)^4(x+\frac{\sqrt{3}}{3}+1)^4(3\,x^4+4\sqrt{3}x^3-2\,x^2-4\sqrt{3}x+6).$$

By applying Sturm's Theorem to $3\,x^4+4\sqrt{3}x^3-2\,x^2-4\sqrt{3}x+6$, we find that it is non-zero on $(0,1-\frac{\sqrt{3}}{3})$ and this implies that $Res(q,k_1,z)\neq 0$ for all $x\in(0,1-\frac{\sqrt{3}}{3})$. Thus, $k_1(x,z)$ and $q(x,z)$ have no common roots and therefore $W[\ell_0,\ell_1](x)\neq 0$ for all $x\in(0,1-\frac{\sqrt{3}}{3})$.

(iii) For $\frac{\sqrt{3}}{3}<\alpha<\sqrt{\frac{3}{7}}$, we compute the asymptotic expansion of $I_0(h)$ and $I_1(h)$ at the end points of $(\frac{1}{12}\alpha^4(\alpha^2-3),\frac{1}{12}(3\alpha^2-1))$ i.e. near the center $(\alpha,0)$ and the homoclinic loop connecting saddle point $(1,0)$, respectively. By applying change of variables $x=X+\alpha$ and $y=Y$, the center $(\alpha,0)$ is moved to the origin. Thus system (X_3^+) will be transform to

$$\begin{aligned}\dot{X}&=Y,\\ \dot{Y}&=X(X+\alpha)(X+2\alpha)(X+\alpha+1)(X+\alpha-1),\end{aligned}$$

with the Hamiltonian function

$$H_2(X,Y) = \frac{1}{2}Y^2 + \alpha^2(1-\alpha^2)X^2 - \frac{\alpha(7\alpha^2-3)}{3}X^3 + \frac{(1-9\alpha^2)}{4}X^4 - \alpha X^5 - \frac{1}{6}X^6.$$

Then $H_2(0,0)=0$, $H_2(1-\alpha,0)=\frac{1}{12}(1-\alpha^2)^3$ and the ovals Γ_h of $H_3^+(x,y)=h$ with $h\in(\frac{1}{12}\alpha^4(\alpha^2-3),\frac{1}{12}(3\alpha^2-1))$ can be mapped to ovals γ_l of $H_2(X,Y)=l$ with $l\in(0,\frac{1}{12}(1-\alpha^2)^3)$. So

$$I_0(h) = \oint_{\Gamma_h} ydx = \oint_{\gamma_l} YdX = J_{20}(l),$$
$$I_1(h) = \oint_{\Gamma_h} xydx = \oint_{\gamma_l}(X+\alpha)YdX = \oint_{\gamma_l} XYdX + \alpha\oint_{\gamma_l} YdX = J_{21}(l) + \alpha J_{20}(l).$$

Thus $P(h)=\frac{I_1(h)}{I_0(h)}=\alpha+\frac{J_{21}(l)}{J_{20}(l)}=\alpha+Q_2(l)$ and $P'(h)=Q_2'(l)$ in the corresponding intervals of h and l. Now we compute asymptotic expansion of $J_{20}(l)$ and $J_{21}(l)$ for $0<l\ll 1$. For this purpose we apply a change of variable in a small neighborhood of the origin as follows:

$$x = X(\alpha^2(1-\alpha^2) - \frac{1}{3}\alpha(7\alpha^2-3)X + \frac{1}{4}(1-9\alpha^2)X^2 - \alpha X^3 - \frac{1}{6}X^4)^{\frac{1}{2}}, \quad y = Y.$$

The inverse of the above transformation is $X=F(x)$ and $Y=y$ where

$$F(x) = \frac{1}{\alpha\sqrt{1-\alpha^2}}x + \frac{7\alpha^2-3}{6\alpha^3(\alpha^2-1)^2}x^2 + O(x^3).$$

With this transformation the ovals γ_l have the form $x^2+y^2=l$ with $0<l\ll 1$. Therefore

$$\begin{aligned}
J_{20}(l) &= \oint_{\gamma_l} YdX = -l\int_0^{2\pi}\sin^2(\theta)F'(\sqrt{l}\cos(\theta))d\theta = -\frac{\pi}{\alpha\sqrt{1-\alpha^2}}l + O(l^2),\\
J_{21}(l) &= \oint_{\gamma_l} XYdX = -l\int_0^{2\pi}\sin^2(\theta)F(\sqrt{l}\cos(\theta))F'(\sqrt{l}\cos(\theta))d\theta\\
&= \frac{\pi\sqrt{1-\alpha^2}(7\alpha^2-3)}{8\alpha^4(\alpha^2-1)^3}l^2 + O(l^3).
\end{aligned}$$

A straightforward calculation shows that

$$P'(\frac{\alpha^4(\alpha^2-3)}{12}) = \lim_{l\to 0^+} Q_2'(l) = \frac{7\alpha^2-3}{8\alpha^3(\alpha^2-1)^2} < 0 \quad \text{for} \quad \frac{\sqrt{3}}{3} < \alpha < \sqrt{\frac{3}{7}}. \tag{3.5}$$

For the asymptotic expansion of $I_0(h)$ and $I_1(h)$, we must move $(1,0)$ to the origin. By applying change of variables $x=1-X$ and $y=-Y$, system (X_3^+) transforms to

$$\begin{aligned}
\dot{X} &= Y,\\
\dot{Y} &= -X(X-1)(X-2)(X-\alpha-1)(X+\alpha-1),
\end{aligned}$$

with the Hamiltonian function

$$H_3(X,Y) = \frac{1}{2}Y^2 - \frac{1}{6}X^6 + X^5 - \frac{1}{4}(9-\alpha^2)X^4 + \frac{1}{3}(7-3\alpha^2)X^3 + (\alpha^2-1)X^2.$$

Then $H_3(0,0)=0$, $H_2(\alpha,0)=\frac{1}{12}\alpha^2(\alpha^2+4\,\alpha-3)(\alpha-2)^2$ and the ovals Γ_h of $H_3^+(x,y)=h$ with $h\in(\frac{1}{12}\alpha^4(\alpha^2-3),\frac{1}{12}(3\alpha^2-1))$ can be mapped to ovals γ_l of $H_3(X,Y)=l$ with $l\in(\frac{1}{12}\alpha^2(\alpha^2+4\,\alpha-3)(\alpha-2)^2,0)$. So

$$
\begin{aligned}
I_0(h)&=\oint_{\Gamma_h} ydx=\oint_{\gamma_l} YdX=J_{30}(l),\\
I_1(h)&=\oint_{\Gamma_h} xydx=\oint_{\gamma_l}(1-X)YdX=\oint_{\gamma_l} YdX-\oint_{\gamma_l} XYdX=J_{30}(l)-J_{31}(l).
\end{aligned}
$$

Thus $P(h)=\frac{I_1(h)}{I_0(h)}=1-\frac{J_{31}(l)}{J_{30}(l)}=1-Q_3(l)$ and so $P'(h)=-Q_3'(l)$ in the corresponding intervals of h and l. Now using Corollary 2 of [4], we obtain the following asymptotic expansion of $J_{30}(l)$ and $J_{31}(l)$ for $0<-l\ll 1$:

$$
\begin{aligned}
J_{30}(l)&=\oint_{\gamma_l} YdX=\oint_{\gamma_0} YdX-\frac{1}{\sqrt{2(1-\alpha^2)}}h\ln|h|+O(h),\\
I_{31}(h)&=\oint_{\gamma_l} XYdX=\oint_{\gamma_0} XYdX+O(h).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
P'(\frac{1}{12}(3\alpha^2-1))&=-\lim_{l\to 0^-}\frac{d}{dl}\left(\frac{J_1(l)}{J_0(l)}\right)\\
&=-\lim_{l\to 0^-}\frac{\frac{1}{\sqrt{2(1-\alpha^2)}}\ln|h|\oint_{\gamma_0} XYdX}{(\oint_{\gamma_0} YdX)^2}=\infty>0. \qquad (3.6)
\end{aligned}
$$

By (3.5) and (3.6) it follows that if $\frac{\sqrt{3}}{3}<\alpha<\sqrt{\frac{3}{7}}$ then $P'(h)$ changes sign in the interval $(\frac{1}{12}\alpha^4(\alpha^2-3),\frac{1}{12}(3\alpha^2-1))$ and so $P(h)$ is not monotone.

(iv) The proof of this case is similar to that of Theorem 3.1 and we omit it for the brevity's sake. □

Now it is the time to consider two last cases, i.e. systems $(X_4^{\pm})$. Note that system (X_4^+) has only a center at the origin and by Remark 3.1, $P(h)\equiv 0$. However system (X_4^-) has two centers at $(\pm 1,0)$. So we consider the monotonicity of $P(h)$ when Γ_h is a period annulus surrounding one of the centers $(\pm 1,0)$ of system (X_4^-) in the following theorem.

Theorem 3.3. *Suppose that Γ_h is a continuous family of ovals surrounding one of two centers $(\pm 1,0)$ of system X_4^-. Then the function $P(h)$ is monotone for $h\in(-\frac{1}{12}(1+3\alpha^2),0)$.*

The proof of this theorem is similar to that of Theorem 3.1 and we omit it for the sake of brevity.

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