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NEW EXACT SOLUTIONS OF A GENERALISED BOUSSINESQ EQUATION WITH DAMPING TERM AND A SYSTEM OF VARIANT BOUSSINESQ EQUATIONS VIA DOUBLE REDUCTION THEORY

Justina Ebele Okeke†, Rivendra Narain
and Keshlan Sathasiva Govinder

Abstract The conservation laws of a generalised Boussinesq (GB) equation with damping term are derived via the partial Noether approach. The derived conserved vectors are adjusted to satisfy the divergence condition. We use the definition of the association of symmetries of partial differential equations with conservation laws and the relationship between symmetries and conservation laws to find a double reduction of the equation. As a result, several new exact solutions are obtained. A similar analysis is performed for a system of variant Boussinesq (VB) equations.

Keywords Double reduction theory, conservation laws, partial Noether approach, Lie symmetry method, associated symmetry.

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1. Introduction

Nonlinear evolution equations (NLEEs) are widely used as models to describe numerous physical behaviours in various fields of life science especially in natural, applied and social science. Therefore, finding exact solutions of NLEEs is not only important but it is also necessary since they can provide much physical information and more insight into the real features of the models. Over the past few decades, a wide range of effective methods for finding exact solutions to NLEEs has been developed. Such methods include the homogeneous balance method [32], the ansatz method [13], the extended tanh method [9], the Jacobi elliptic function method [34], the projective Riccati equation method [21], the direct method [6, 7, 20], the sine-cosine method [33] and the Lie group method [25] (Here, hidden symmetries have played an interesting role [1, 2].). Although considerable work has been done over the years on the subject of finding exact solutions to NLEEs, there is no unique method that can be used to tackle all types of NLEEs.

Lie symmetries and conservation laws are important tools for understanding the behaviour of physical systems and for finding solutions of many problems in

†the corresponding author. Email address: okeke.justina@gmail.com (J. E. Okeke)
School of Mathematics, Statistics and Computer Science, University of KwaZulu–Natal, Private Bag X54001, 4000 Durban, South Africa

mathematical physics. Different methods for obtaining conservation laws of partial differential equations (PDEs) have been developed. Noether's theorem provides an elegant and constructive way of finding conservation laws for a system of partial differential equations which has a Lagrangian formulation [4]. The central problem in calculus of variations is the determination of a Lagrangian, so that the differential equation is then regarded as the Euler-Lagrange equation. This is regarded as the inverse problem in the calculus of variations [8, 25]. There are methods to obtain conservation laws without making use of a Lagrangian. The direct method introduced by Laplace [3, 18] is used to construct conserved quantities. The multiplier approach involves writing a conservation law in characteristic form, where the characteristics are the multipliers of the differential equations [30]. A recent method for constructing conservation laws without the use of a Lagrangian was provided by Ibragimov [14]. A more systematic way of constructing conservation laws for a system of PDEs without the existence of Lagrangians is via the partial Noether approach, introduced by Kara and Mahomed [16]. It works like the Noether approach for differential equations with or without a Lagrangian. Narain and Kara used the partial Noether's method to compute conservation laws for a class of nonlinear PDEs with a mixed derivative term (a term involving derivatives of more than one independent variables) [22]. The resultant conserved vectors do not satisfy the divergence relationship due to the presence of a mixed derivative term. A number of extra terms arise that need to be adjusted to satisfy the total divergence of the computed conserved vectors. These terms are necessary as they contribute to the trivial part of conserved vector and may guarantee the notion of association between conserved vectors and symmetries [3, 15].

It is well known that a conservation law is associated with a Noether symmetry admitted by a differential equation. Recently, this idea of associating conservation laws with Noether symmetries was extended to Lie-Bäcklund symmetries [15] and non-local symmetries [29]. The association of symmetry with a conserved vector leads to the development of the double reduction theory for PDEs with two independent variables [28]. The fundamental basis of this method is that when a symmetry is associated with a conserved vector, a double reduction transformation exists. PDEs of order n with two independent variables are reduced to ODEs of order $n-1$, which are generally easier to solve. Hence, the association of symmetry with the conserved vector firstly reduces the number of independent variables and secondly reduces the order of the differential equation. It is worth noting that the double reduction theory yields a new way of finding invariants and exact solutions of PDEs which may not be obtained using classical symmetry analysis [23].

In this paper, we consider two systems of NLEEs found in mathematical physics. The first one is the generalised Boussinesq (GB) equation with damping term [19,35]

$$u_{tt} + 2\rho u_{xxt} + \beta u_{xxxx} + \gamma (u^n)_{xx} = 0, \tag{1.1}$$

where ρ, β, γ are constants and n is a nonzero real number. Equation (1.1) is widely used as a model to describe natural phenomena in many scientific fields such as plasma waves, solid physics and fluid mechanics [35]. It is to be noted that when $\rho = 0,\ \beta = -1,\ \gamma = 1$, and $n = 3$, in (1.1) we obtain the modified Boussinesq equation [11]. The modified Boussinesq equation is used as a model to describe the temporal evolution of nonlinear finite amplitude waves on a density front in a rotating fluid. Exact travelling wave solutions for (1.1) were studied in [19] using the extended tanh method [9]. Yan *et al.* [35] investigated the solitary wave solutions of the equation (1.1) for $n = 3$ using the direct method [6, 7, 20].

The second system is the variant Boussinesq (VB) equations [27, 31]

$$\begin{aligned} u_t + (uv)_x + v_{xxx} &= 0, \\ v_t + u_x + vv_x &= 0, \end{aligned} \tag{1.2}$$

described as a model for water waves, where $v(t, x)$ represents the velocity and $u(t, x)$ represents the total depth. Solitary wave solutions and multi-solitary wave solutions of the system (1.2) were obtained in [37] using the homogeneous balance method [32]. Fu *et al.* [10] examined the system (1.2) for periodic wave solutions using the ansatz method [13]. Conservation laws for the system were derived by Naz *et al.* [24] by increasing the order of the equation and using Noether's approach.

In this study, the conservation laws of the GB equation (1.1) which are not derived from a variational principle are constructed for the first time using the partial Lagrangian method. Since the GB equation contains an odd order term which consists of a mixed derivative i.e. the derivative w.r.t. both t and x, determining its standard Lagrangian is not possible and thus the Noether approach is not applicable for finding its conservation laws. The partial Noether approach is then used to derive the conservation laws. These conserved vectors constructed by the partial Noether's theorem fails to satisfy the divergence property. A number of extra terms arise because of the odd order term which consists of a mixed derivative. These extra terms contribute to the trivial part of the conserved vector and need to be adjusted to satisfy the divergence property. After construction of conservation laws the solutions of the GB equation are derived by double reduction theory.

A similar analysis is performed for the system of VB equations (1.2) to obtain exact solutions of the system. The paper is organised as follows: In the next Section, the Lie point symmetries of the GB equation and conservation laws are utilised to obtain a double reduction of the equation. As a result, some invariants and exact solutions are obtained. In Section 3, the conservation laws of the GB equation are derived. Section 4 discusses the double reduction and exact solutions of the GB equation while Section 5 deals with the exact solutions of a system of VB equations using double reduction theory. Concluding remarks are presented in Section 6.

2. Lie Symmetries of the GB equation

The Lie point symmetries admitted by (1.1) are generated by a vector field of the form

$$X = \xi^1(t, x, u)\frac{\partial}{\partial t} + \xi^2(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u}. \tag{2.1}$$

The operator X satisfies the Lie symmetry condition [26]

$$X^{[4]} \left[u_{tt} + 2\rho u_{xxt} + \beta u_{xxxx} + \gamma (u^n)_{xx} = 0, \right] \big|_{(1.1)} = 0, \tag{2.2}$$

where $X^{[4]}$ is the fourth prolongation of the operator X defined by

$$\begin{aligned} X^{[4]} =& X + \zeta_t \frac{\partial}{\partial u_t} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_{xt} \frac{\partial}{\partial u_{xt}} + \zeta_{tt} \frac{\partial}{\partial u_{tt}} + \zeta_{xx} \frac{\partial}{\partial u_{xx}} \\ &+ \zeta_{xxx} \frac{\partial}{\partial u_{xxx}} + \zeta_{xxt} \frac{\partial}{\partial u_{xxt}} + \zeta_{xxxx} \frac{\partial}{\partial u_{xxxx}} \end{aligned} \tag{2.3}$$

and the coefficients $\zeta_t,\ \zeta_x,\ \zeta_{tt},\ \zeta_{xx},\ \zeta_{xt},\ \zeta_{xxt},\ \zeta_{xxx}$ and ζ_{xxxx} are given by

$$\begin{aligned}
\zeta_t &= D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2),\\
\zeta_x &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2),\\
\zeta_{xx} &= D_x(\zeta_x - u_{xt} D_x(\xi^1) - u_{xx} D_x(\xi^2),\\
\zeta_{tt} &= D_t(\zeta_t) - u_{tt} D_x(\xi^1) - u_{tx} D_x(\xi^2),\\
\zeta_{xt} &= D_x(\zeta_t) - u_{xt} D_x(\xi^1) - u_{xx} D_x(\xi^2),\\
\zeta_{xxx} &= D_x(\zeta_{xx}) - u_{xxt} D_x(\xi^1) - u_{xxx} D_x(\xi^2),\\
\zeta_{xxt} &= D_t(\zeta_{xx}) - u_{xxt} D_t(\xi^1) - u_{xxx} D_x(\xi^2),\\
\zeta_{xxxx} &= D_x(\zeta_{xxx}) - u_{xxxt} D_x(\xi^1) - u_{xxxx} D_x(\xi^2).
\end{aligned}$$

Here $D_x,\ D_t$ denote the total derivative operators defined by

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} + \cdots,\ D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_t} + \cdots. \tag{2.4}$$

Expansion and separation of (2.2) with respect to the powers of different derivatives of u yields an overdetermined system in the unknown coefficients ξ^1, ξ^2 and η.

Solving the overdetermined system for arbitrary parameters gives two different cases as follows:

Case (1): Provided $\rho\beta\gamma(n-1) \neq 0$, we have the following three Lie point symmetries:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t},\\
X_2 &= \frac{\partial}{\partial x},\\
X_3 &= t\frac{\partial}{\partial t} - \frac{u}{(n-1)}\frac{\partial}{\partial u} + \frac{x}{2}\frac{\partial}{\partial x}.
\end{aligned} \tag{2.5}$$

Case (2): When $\rho\beta\gamma \neq 0,\ n = 1$, we obtain, in addition to X_1 and X_2, another symmetry $X_4 = u\dfrac{\partial}{\partial u}$ and an infinite-dimensional symmetry, $X_5 = F_1(t,x)\dfrac{\partial}{\partial u}$ which is expected as (1.1) is now linear.

3. Conservation laws of the GB equation

A conserved vector corresponding to a conservation law of the GB equation (1.1) is a 2−tuple (T^t, T^x) , such that

$$D_t T^t + D_x T^x = 0 \tag{3.1}$$

along the solutions of the equation.

Conservation laws of the GB equation via partial Noether's method

Equation (1.1) does not have a standard Lagrangian due to the presence of the odd order term u_{xxt}. Hence, it is not derivable from a variational principle. We investigate the conserved quantities via the partial Noether approach using the partial Lagrangian [16]. This study of the conserved vectors of the equation (1.1) has not been previously conducted. The equation (1.1) possesses a partial Lagrangian

$$L = \frac{1}{2}u_t^2 - \frac{1}{2}\beta u_{xx}^2 + \frac{1}{2}\gamma n u^{n-1} u_x^2. \tag{3.2}$$

The associated partial Euler-Lagrange equation is

$$\frac{\delta L}{\delta u} = 2\rho u_{xxt} + \gamma \frac{1}{2} n(n-1) u^{n-2} u_x^2, \tag{3.3}$$

where $\frac{\delta L}{\delta u}$ is defined by

$$\frac{\delta L}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_t^2 \frac{\partial}{\partial u_{tt}} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_t \frac{\partial}{\partial u_{tx}}, \tag{3.4}$$

and

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u},$$
$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x}.$$

The partial Noether operator is given by

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} \tag{3.5}$$

and satisfies

$$X^{[2]} L + L(D_t \xi^1 + D_x \xi^2) = (\eta - u_t \xi^1 - u_x \xi_2) \frac{\delta L}{\delta u} + B_t^1 + u_t B_u^1 + B_x^2 + u_x B^2 u. \tag{3.6}$$

Separating (3.6), after expansion by the derivatives of u, with the Lagrangian (3.2) yields an overdetermined system. The solution of this system yields the following partial Noether operators and gauge terms:

$$\xi^1 = \xi^2 = 0,\ \eta = c_1 + tc_3 + x(c_2 + tc_4),\ B^1 = u(c_3 + xc_4) + F(t,x),$$
$$B^2 = u^n \gamma (c_2 + tc_4) + G(t,x),\ F_t(t,x) + G_x(t,x) = 0, \tag{3.7}$$

where $c_1,\ c_2,\ c_3,\ c_4$ are constants. Without loss of generality, we set $F(t,x) = G(t,x) = 0$ as $F_t(t,x) + G_x(t,x) = 0$ and obtain the partial Noether operators X_i $(i = 1, 2, \cdots 4)$ of (1.1) presented in Table 1.

Table 1. The partial Noether operators and gauge terms of (1.1)

X_i	operator	gauge function
X_1	$\frac{\partial}{\partial u}$	$B^1 = B^2 = 0$
X_2	$x\frac{\partial}{\partial u}$	$B^1 = B^2 = u^n\gamma$
X_3	$t\frac{\partial}{\partial u}$	$B^1 = u, B^2 = 0$
X_4	$xt\frac{\partial}{\partial u}$	$B^1 = xu, B^2 = \gamma t u^n$

The conserved vectors of (1.1) for the second order partial Lagrangian (3.2) is determined by

$$T^t = B^1 - L\xi^1 - W \frac{\partial L}{\partial u_t},$$
$$T^x = B^2 - L\xi^2 - W \left(\frac{\partial L}{\partial u_x} - D_x \frac{\partial L}{\partial u_{xx}} \right) - D_x(W) \frac{\partial L}{\partial u_{xx}}, \tag{3.8}$$

where $W = \eta - \xi^1 u_t - \xi^2 u_x$ is a multiplier. Thus, by adoption of (3.8), the conserved quantities corresponding to each of the four partial Noether operators in Table 1 are given by

$$\begin{aligned}
&T_1^t = -u_t,\ T_1^x = -\gamma(u^n)_x - \beta u_{xxx},\\
&T_2^t = -xu_t,\ T_2^x = \gamma u^n - x(\gamma(u^n)_x + \beta u_{xxx}) + \beta u_{xx},\\
&T_3^t = u - tu_t,\ T_3^x = -t(\gamma(u^n)_x + \beta u_{xxx}),\\
&T_4^t = ux - xtu_t,\ T_4^x = \gamma tu^n - xt(\gamma(u^n)_x + \beta u_{xxx}) + t\beta u_{xx}.
\end{aligned} \tag{3.9}$$

However, the above conserved vectors (3.9) fail to satisfy the divergence property (3.1) as $D_i T^i \neq 0$ due to the presence of some extra terms. Narain and Kara [22] proved that these extra terms can be absorbed into the conserved vectors T^i to obtain a new conserved vector $\tilde{T}$ that meets the divergence condition $D_i \tilde{T}^i = 0$. Therefore, after making some adjustments, we obtain the following modified conserved vectors and multipliers:

$$\begin{aligned}
&\tilde{T}_1^t = -u_t - 2\rho u_{xx},\ \tilde{T}_1^x = -\gamma(u^n)_x - \beta u_{xxx},\ W_1 = 1,\\
&\tilde{T}_2^t = -x(u_t + 2\rho u_{xx}),\ \tilde{T}_2^x = \gamma u^n - x(\gamma(u^n)_x + \beta u_{xxx}) + \beta u_{xx},\\
&W_2 = x,\\
&\tilde{T}_3^t = u - tu_t,\ \tilde{T}_3^x = -t(\gamma(u^n)_x + \beta u_{xxx} + 2\rho u_{xt}),\ W_3 = t,\\
&\tilde{T}_4^t = ux - xtu_t,\ \tilde{T}_4^x = \gamma tu^n - xt(\gamma(u^n)_x + \beta u_{xxx} + 2\rho u_{xt}) + 2\rho tu_t + t\beta u_{xx},\\
&W_4 = xt.
\end{aligned} \tag{3.10}$$

4. Double reduction and exact solutions of the GB equation

When a PDE of order n with two independent variables, admits a symmetry X that is associated with a conserved vector T, then it can be reduced to an ODE of order $n-1$ [5]. Now, we utilize the relationship between the conservation laws and the Lie point symmetries of equation (1.1) to obtain its doubly reduced equation which is easily solved to find exact solutions.

A Lie point symmetry X of the GB equation (1.1) is associated with its conserved vector (T^t, T^x) if [15]

$$X_i^{[2]} \begin{pmatrix} T_i^t \\ T_i^x \end{pmatrix} + (D_t \xi_t^1 + D_x \xi_x^2) \begin{pmatrix} T_i^t \\ T_i^x \end{pmatrix} - \begin{pmatrix} D_t \xi_t^1 & D_x \xi_t^1 \\ D_t \xi_x^2 & D_x \xi_x^2 \end{pmatrix} \begin{pmatrix} T_i^t \\ T_i^x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.1}$$

In terms of the canonical variables r, s obtained by mapping X to $Y = \frac{\partial}{\partial s}$, the conservation laws can be expressed as [28]

$$D_r T^r + D_s T^s = 0, \tag{4.2}$$

with T^r and T^s given as

$$T^r = \frac{T^t D_t(r) + T^x D_x(r)}{D_t(r)D_x(s) - D_x(r)D_t(s)}, \tag{4.3}$$

$$T^s = \frac{T^t D_t(s) + T^x D_x(s)}{D_t(r)D_x(s) - D_x(r)D_t(s)}. \tag{4.4}$$

For each $X_i,\ i = 1, 2$ of (2.5) and the conserved vector $(\tilde{T_i^t}, \tilde{T_i^x}),\ i = 1, 2, \ldots, 4$ of (3.10) equation (4.1) becomes

$$X_i^{[3]} \begin{pmatrix} \tilde{T^t} \\ \tilde{T^x} \end{pmatrix} + (D_t\xi_t^1 + D_x\xi_x^2) \begin{pmatrix} \tilde{T^t} \\ \tilde{T^x} \end{pmatrix} - \begin{pmatrix} D_t\xi_t^1 & D_x\xi_t^1 \\ D_t\xi_x^2 & D_x\xi_x^2 \end{pmatrix} \begin{pmatrix} \tilde{T^t} \\ \tilde{T^x} \end{pmatrix}$$

$$= X_i^{[3]} \begin{pmatrix} \tilde{T^t} \\ \tilde{T^x} \end{pmatrix} + (0+0) \begin{pmatrix} \tilde{T^t} \\ \tilde{T^x} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{T^t} \\ \tilde{T^x} \end{pmatrix} = 0.$$

Therefore, the symmetry generators X_1 and X_2 are associated with the four conserved quantities. Thus we can perform the double reduction by a combination of the two generators,

$$X = X_1 + cX_2 \tag{4.5}$$

using any one of the four conservation laws. Mapping (4.5) to

$$Y = \frac{\partial}{\partial s} \tag{4.6}$$

yields the canonical coordinates

$$s = t, r = x - ct, w(r) = u. \tag{4.7}$$

The conservation law $\tilde{T} = (\tilde{T^t},\ \tilde{T^x})$ is rewritten as $D_r T^r + D_s T^s = 0$. By using the formulas (4.3) and (4.4), a double reduction by $\tilde{T_1} = (\tilde{T_1^t},\ \tilde{T_1^x})$ results in the reduced conserved form

$$T_1^r = c^2 w_r - 2\rho c w_{rr} + \gamma (w^n)_r + \beta w_{rrr}, \tag{4.8}$$

$$T_1^s = -c w_r + 2\rho w_{rr}. \tag{4.9}$$

Since (4.9) does not depend on s, the reduced conserved vector becomes

$$D_r T^r = 0, \tag{4.10}$$

which implies that

$$c^2 w_r - 2\rho c w_{rr} + \gamma (w^n)_r + \beta w_{rrr} = k, \tag{4.11}$$

where k is a constant. Equation (4.11) is a third order ODE which is a double reduction of the fourth order PDE (1.1). Integrating (4.11) once with respect to r while setting the constant of integration to zero, results in

$$c^2 w - 2\rho c w_r + \gamma w^n + \beta w_{rr} = 0. \tag{4.12}$$

We seek solutions of equation (4.12) by the extended $\left(\frac{G'}{G}\right)$ – expansion method [12]. The method mainly consists of the following steps: Suppose that the solution of (4.12) can be expressed as

$$w(r) = a_0 + \sum_{i=1}^{m} a_i \left(\frac{G'}{G}\right)^i + b_i \left(\frac{G'}{G}\right)^{i-1} \sqrt{\nu\left(1 + \frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}, \tag{4.13}$$

with the new variable $G = G(r)$ satisfying

$$G''(r) + \mu G(r) = 0, \tag{4.14}$$

where $'$ means $\frac{d}{dr}$.

The parameters $a_i,\ b_i\ (i = 1, 2, ..., m)$ and a_0 are constants to be determined, such that $\mu \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (4.12).

Balancing w_{rr} with w^n in (4.12), we obtain an irreducible fraction, $m = \frac{2}{n-1}$, for some n. Therefore we make the following transformation

$$w(r) = h(r)^{\frac{2}{n-1}}, \tag{4.15}$$

and then substitute (4.15) into (4.12) to obtain

$$\begin{aligned}&(n-1)^2(c^2h(r)^2 + \gamma h(r)^4) - 4\rho c(n-1)h(r)h'(r)\\ &+ \beta(2(3-n)h'(r)^2 + 2(n-1)h(r)h''(r)) = 0.\end{aligned} \tag{4.16}$$

Now balancing $h(r)^4$ and $h(r)h''(r)$ we find $m = 1$. Thus, we assume that

$$h(r) = a_0 + a_1\left(\frac{G'}{G}\right) + b_1\sqrt{\nu\left(1 + \frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}. \tag{4.17}$$

Substituting (4.17) into (4.16) and using (4.14), collecting all terms with the same powers of $\left(\frac{G'}{G}\right)^k$ and $\left(\frac{G'}{G}\right)^k\sqrt{\nu\left(1 + \frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}$ together, and equating each coefficient of them to zero, yield a set of algebraic equations for a_0, a_1, b_1 and μ:

$(i)\ \left(\frac{G'}{G}\right)^0$: $a_0^2(n-1)^2\left(6b_1^2\gamma v + c^2\right) - 2\beta a_1^2\mu^2(n-3) + 4a_1a_0\rho c\mu(n-1) + a_0^4\gamma(n-1)^2 + b_1^1(n-1)v\left(2\beta\mu + b_1^2\gamma(n-1)v + c^2(n-1)\right) = 0,$

$(ii)\ \left(\frac{G'}{G}\right)^1$: $2(n-1)\left(a_1a_0\left(2\beta\mu + 6b_1^2\gamma(n-1)v + c^2(n-1)\right) + 2\rho c\left(a_1^2\mu + b_1^2v\right) + 2a_1a_0^3\gamma(n-1)\right) = 0,$

$(iii)\ \left(\frac{G'}{G}\right)^2$: $\frac{1}{\mu}a_1^2\mu\left(8\beta\mu + 6b_1^2\gamma(n-1)^2v + c^2(n-1)^2\right) + 6a_0^2\gamma(n-1)^2\left(a_1^2\mu + b_1^2v\right) + 4a_1a_0\rho c\mu(n-1) + b_1^2v\left(2b_1^2\gamma(n-1)^2v + 4\beta\mu n + c^2(n-1)^2\right) = 0,$

$$(iv)\left(\frac{G'}{G}\right)^3:\ (n-1)\left(a_0a_1\left(\beta\mu+a_1^2\gamma\mu(n-1)+3b_1^2\gamma(n-1)v\right)+\rho c\left(a_1^2\mu+b_1^2v\right)\right)=0,$$

$$(v)\left(\frac{G'}{G}\right)^4:\ \frac{1}{\mu^2}2a_1^2\mu\left(3b_1^2\gamma(n-1)^2v+\beta\mu(n+1)\right)+a_1^4\gamma\mu^2(n-1)^2+b_1^2v\left(b_1^2\gamma(n-1)^2v+2\beta\mu(n+1)\right)=0,$$

$$(vi)\left(\frac{G'}{G}\right)^0\sqrt{\nu\left(1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}:2b_1(n-1)\left(2a_1\rho c\mu+a_0\left(\beta\mu+2b_1^2\gamma(n-1)v+c^2(n-1)\right)+2a_0^3\gamma(n-1)\right)=0,$$

$$(vii)\left(\frac{G'}{G}\right)^1\sqrt{\nu\left(1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}:2b_1\left(a_1\left(2b_1^2\gamma(n-1)^2v+\beta\mu(n+3)+c^2(n-1)^2\right)+2a_0\rho c(n-1)+6a_1a_0^2\gamma(n-1)^2\right)=0,$$

$$(viii)\left(\frac{G'}{G}\right)^2\sqrt{\nu\left(1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}:\ b_1(n-1)\left(2a_1\rho c\mu+a_0\left(\beta\mu+3a_1^2\gamma\mu(n-1)+b_1^2\gamma(n-1)v\right)\right)=0,$$

$$(ix)\left(\frac{G'}{G}\right)^3\sqrt{\nu\left(1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}:a_1b_1\left(a_1^2\gamma\mu(n-1)^2+b_1^2\gamma(n-1)^2v+\beta\mu(n+1)\right)=0.$$

Solving the resultant algebraic equations, we obtain the following results:

Case 1:

$$b_1=0,\ a_0=\pm\sqrt{\frac{-c^2}{4\gamma}},\ a_1=\pm\frac{4(n+1)\rho}{(n-1)(n+3)}\sqrt{\frac{-1}{\gamma}},\ \mu=-\frac{(n+3)^2(n-1)^2c^2}{64(n+1)^2\rho^2},$$
$$\beta=\frac{8\rho^2(n+1)}{(n+3)^2}. \tag{4.18}$$

Since $\mu<0$, from equations (4.15), (4.17) and (4.18), when $\beta=\frac{8\rho^2(n+1)}{(n+3)^2}$ the GB equation (1.1) has the following solution:

$$u_1=\left(\pm\sqrt{\frac{-c^2}{4\gamma}}\left(1\pm\frac{A\cosh(\sqrt{-\mu}(x-ct))+B\sinh(\sqrt{-\mu}(x-ct))}{B\cosh(\sqrt{-\mu}(x-ct))+A\sinh(\sqrt{-\mu}(x-ct))}\right)\right)^{\frac{2}{n-1}}, \tag{4.19}$$

where $A,\ B$ are arbitrary constants.

Case 2:

$$a_0=\pm\sqrt{\frac{-c^2}{4\gamma}},\ a_1=\pm\frac{2(n+1)\rho}{(n-1)(n+3)}\sqrt{\frac{-1}{\gamma}},\ b_1=\pm\sqrt{\frac{c^2}{4\nu\gamma}},\mu=-\frac{(n+3)^2(n-1)^2c^2}{16(n+1)^2\rho^2},$$
$$\beta=\frac{8\rho^2(n+1)}{(n+3)^2}. \tag{4.20}$$

This case leads to the following solution

$$u_2=\left(\pm\sqrt{\frac{c^2}{4\gamma}}\left(i\pm i\frac{A\cosh(\sqrt{-\mu}(x-ct))+B\sinh(\sqrt{-\mu}(x-ct))}{B\cosh(\sqrt{-\mu}(x-ct))+A\sinh(\sqrt{-\mu}(x-ct))}\right.\right.$$

$$+\sqrt{1-\left(\frac{A\cosh(\sqrt{-\mu}(x-ct))+B\sinh(\sqrt{-\mu}(x-ct))}{B\cosh(\sqrt{-\mu}(x-ct))+A\sinh(\sqrt{-\mu}(x-ct))}\right)^2}\Bigg)\Bigg)^{\frac{2}{n-1}}. \quad (4.21)$$

Case 3:

$$a_0=a_1=\rho=0,\ b_1=\pm\sqrt{\frac{-(n+1)c^2}{2\nu\gamma}},\ \mu=\frac{(n-1)^2c^2}{4\beta}. \quad (4.22)$$

From (4.22), the solutions of (1.1) are as follows:

$$u_3=\left(\frac{-(n+1)c^2}{2\gamma}\left(1-\left(\frac{A\cosh(\sqrt{-\mu}(x-ct))+B\sinh(\sqrt{-\mu}(x-ct))}{B\cosh(\sqrt{-\mu}(x-ct))+A\sinh(\sqrt{-\mu}(x-ct))}\right)^2\right)\right)^{\frac{1}{n-1}},$$

$$\beta<0, \quad (4.23)$$

$$u_4=\left(\frac{-(n+1)c^2}{2\gamma}\left(1+\left(\frac{A\cos(\sqrt{\mu}(x-ct))-B\sin(\sqrt{\mu}(x-ct))}{B\cos(\sqrt{\mu}(x-ct))+A\sin(\sqrt{\mu}(x-ct))}\right)^2\right)\right)^{\frac{1}{n-1}},$$

$$\beta>0. \quad (4.24)$$

Remark: These results are a generalisation of those covered in [19, 35]. In particular, the cases $A=0,\ B\neq 0$ and $A\neq 0,\ B=0$, with $2\rho=\alpha,\ \mu=-\frac{(n+3)^2(n-1)^2c^2}{4(n+1)^2\alpha^2}$ in (4.19)–(4.24), contain the results of Chen *et al.* [19], who applied the extended-tanh method developed by Fan [9] to explore some exact solutions of the GB (1.1) equation.

Further, if $n=3,\ A=0,\ B\neq 0,\ 2\rho=\alpha,\ \mu=-\frac{(n+3)^2(n-1)^2c^2}{4(n+1)^2\alpha^2}$, then (4.19) becomes

$$u_1=\pm\sqrt{\frac{-c^2}{4\gamma}}\left(1\pm\tanh(\sqrt{-\mu}(x-ct))\right). \quad (4.25)$$

This is a form of solitary wave solution of the GB equation (1.1) obtained by Yan *et al.* [35], who used both the direct method by Clarkson and Kruskal [6, 7] and the improved direct method by Lou [20].

5. Double reduction and exact solutions of a system of VB equations

The conservation laws of the system (1.2) are given by [24]

$$(T_1^t, T_1^x)=(v, u+\frac{1}{2}v^2), \quad (5.1)$$

$$(T_2^t, T_2^x)=(v, uv+v_{xx}), \quad (5.2)$$

$$(T_3^t, T_3^x)=(uv, \frac{1}{2}u^2+uv^2-\frac{1}{2}v_x^2+vv_{xx}), \quad (5.3)$$

with the corresponding multipliers

$$Q_1 = [0, 1],$$
$$Q_2 = [1, 0],$$
$$Q_3 = [v, u].$$

We apply the double reduction to the conserved vector T^3 in equation (5.3) to investigate exact solutions of the system. Equation (1.2) admits the four Lie point symmetries

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}, \\ X_4 &= \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - \frac{1}{2}v\frac{\partial}{\partial v}. \end{aligned} \tag{5.4}$$

It can be easily shown that X_1 and X_2 of (5.4) are associated with the conserved vector $T_3 = (T_3^t, T_3^x)$ in equation (5.3). We consider a linear combination $X = X_1 + cX_2$ and transform this generator to its canonical form via

$$r = x - ct, s = t, q(r) = v, w(r) = u. \tag{5.5}$$

The components of the reduced conserved form are given by

$$T_r^3 = cwq - \frac{1}{2}w^2 - wq^2 + \frac{1}{2}q_r^2 - qq_{rr}, \tag{5.6}$$

$$T_s^3 = -wq, \tag{5.7}$$

where the reduced conserved form satisfies

$$D_r T_3^r = 0. \tag{5.8}$$

Thus, the double reduced equation is given as

$$cwq - \frac{1}{2}w^2 - wq^2 + \frac{1}{2}q_r^2 - qq_{rr} = k_1, \tag{5.9}$$

where k_1 is a constant. Differentiating (5.9) implicitly with respect to r results in

$$cw_r q + cwq_r - ww_r - w_r q^2 - 2wqq_r - qq_{rrr} = 0. \tag{5.10}$$

Since the multipliers of the conserved vector (5.3) are $q_1 = v$ and $q_2 = u$, we can also obtain a reduced conserved form for the equation

$$v(u_t + vu_x + +v_x u + v_{xxx}) - u(v_t + u_x + vv_x) = 0. \tag{5.11}$$

The above equation, (5.11), in the canonical variables (5.5) is given as

$$cqw_r - q^2 w_r - qq_{rrr} - cwq_r + ww_r = 0. \tag{5.12}$$

Substituting for q_{rrr} from (5.10) into (5.12) yields the first order ODE

$$cq_r - w_r - qq_r = 0. \tag{5.13}$$

Integrating (5.13) with respect to r results in

$$w = cq - \frac{1}{2}q^2 + k_2, \tag{5.14}$$

where k_2 is a constant of integration. The substitution of (5.14) into (5.9) gives

$$qq_{rr} - \frac{1}{2}q_r^2 - \frac{3}{8}q^4 + cq^3 - \frac{1}{2}q^2\left(c^2 - k_2\right) + \frac{1}{2}(k_2^2 + 2k_1) = 0. \tag{5.15}$$

Equation (5.15) admits the symmetry generator $X = \frac{\partial}{\partial r}$. The reduction using this symmetry through the similarity variables $z = q,\ p(z) = q_r$ leads to the Bernoulli equation

$$p_z p - \frac{1}{2z}p^2 - \frac{3}{8}z^3 + cz^2 - \frac{1}{2}z(c^2 - k_2) + \frac{1}{2z}(k_2^2 + 2k_1) = 0, \tag{5.16}$$

whose general solution is

$$p = \pm\sqrt{\frac{1}{4}z^4 - cz^3 + z^2(c^2 - k_2) + k_3 z + k_2^2 + 2k_1}, \tag{5.17}$$

where k_3 is a constant. Then the corresponding general solution of the ODE (5.15) is given implicitly by

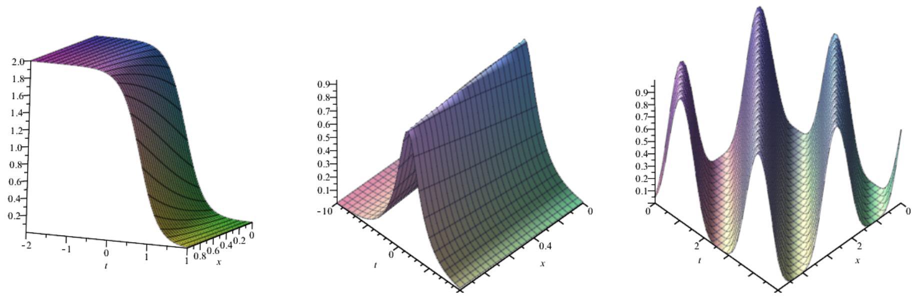
$$\pm\int \frac{dq}{\frac{1}{2}\sqrt{q^4 - 4cq^3 + 4(c^2 - k_2)q^2 + 4k_3 q + 4(k_2^2 + 2k_1)}} = r + k_4, \tag{5.18}$$

where k_4 is also a constant of integration.

6. Discussion

The double reduction theory based on the association of Lie point symmetries and conservation laws was utilised to construct new exact solutions of the GB equation and a system of VB equations. Firstly, the GB equation was considered and the conservation laws were computed via the partial Noether's approach. The derived conserved vectors failed to satisfy the divergence relation due to the presence of the mixed derivative term. The conserved vectors were then adjusted to absorb the extra term. As a result new forms of the conserved vectors satisfying the divergence condition were found. To the best of our knowledge, these conserved vectors have not been reported in the literature.

The importance of these conservation laws was illustrated by finding several exact travelling wave solutions for the GB equation through the application of the double reduction method. The solutions obtained behave as solitary and periodic waves for different values of special parameters involved. The important kink solitary waves, bell shaped solitary waves and periodic travelling waves can be obtained from the solutions (4.19), (4.23) and (4.24) respectively as shown in Figure 1. We have shown that our results were not only a generalisation of the work previously done by some authors but also contain some new exact solutions.



(a) (b) (c)

Figure 1. (a) Kink shaped solitary wave solution for u_1, with $n = 3, c = 2, \gamma = -1, \mu = -1, \rho = \frac{3}{4}, \beta = \frac{1}{2}, A = 1, B = 2$; (b) Bell shaped solitary wave solution of u_3, with $n = 5, c = 1, \gamma = -3, \mu = -1, \rho = 0, \beta = -4, A = 1, B = 2$; (c) Periodic wave solution of u_4, with $n = \frac{1}{2}, c = 1, \gamma = -\frac{3}{4}, \mu = 1, \rho = 0, \beta = \frac{1}{16}, A = B = 1$.

A similar analysis is carried out to obtain new exact solutions for a system of VB equations. We observe that Yasar and Giresunlu [36] undertook a Lie symmetry analysis and produced conservation laws of this system. They used the Lie point symmetries of the system with the aid of the simplest equation method to obtain invariant solutions. In addition they derived the conservation laws of the system using the second order multiplier approach. In contrast, our work produces new exact solutions via the double reduction method. In particular, we found a family of solutions of the VB equations not contained in [36]. We note that the authors of [36] do indicate that a conservation law approach will be part of future work.

We believe that we have shown that the double reduction method is an effective and convenient method which allows us to solve certain complicated nonlinear differential equations in mathematical physics. The new solutions presented in this paper may be used to study disturbance or wave propagation problems in fluid mechanics and space plasma physics.

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References

[1] B. Abraham–Shrauner and K. S. Govinder, *Provenance of type II hidden symmetries from nonlinear partial differential equations*, J. Nonlinear Math. Phys., 2006, 13(4), 612–622.

[2] B. Abraham-Shrauner and K. S. Govinder, *Master partial differential equations for a type II hidden symmetry*, J. Math. Anal. Appl., 2008, 343(1), 525–530.

[3] S. C. Anco, G.W. Bluman, *Direct construction method for conservation laws of partial differential equations part II: general treatment*, Eur. J. Appl. Math., 2002, 13(5), 567–585.

[4] E. Bessel-Hagen, *Über die Erhaltungssätze der Elektrodynamik*, Math. Ann., 1921, 84(3), 258–276.

[5] A. H. Bokhari, A. Y. Al-Dweik, F. D. Zaman, A. H. Kara and F. M. Mahomed, *Generalization of the double reduction theory*, Nonlinear Anal. Real World Appl., 2010, 11(5), 3763–3769.

[6] P. A. Clarkson, *New similarity solutions for the modified Boussinesq equation*, J. Phys. A Math. Gen., 1989, 22(13), 2355–2367.

[7] P. A. Clarkson, *Dimensional reductions and exact solutions of a generalized nonlinear Schrodinger equation*, Nonlinearity 1992, 5(2), 453–472.

[8] J. Douglas, *Solutions of the inverse problem of the calculus of variations*, J.Amer. Math Soc., 1941, 50(1), 71–128.

[9] E. Fan, *Extended tanh-function method and its applications to nonlinear equations*, Phys. Lett. A, 2000, 277(4), 212–218.

[10] Z. Fu, S. Liu and S. Liu, *New transformations and new approach to find exact solutions to nonlinear equations*, Phys. Lett. A, 2002, 299(5), 507–512.

[11] M. Ghil and N. Paldor, *A model equation for nonlinear wavelength selection and amplitude evolution of frontal waves*, J . Nonlinear Sci., 1994, 4(1), 471–496.

[12] R. K. Gupta and A. Bansal, *Painlevé analysis, Lie symmetries and invariant solutions of potential Kadomstev-Petviashvili equation with time dependent coefficients*, Appl. Math. Comput., 2013, 219(10), 5290–5302.

[13] J. L. Hu, *A new method for finding exact traveling wave solutions to nonlinear partial differential equations*, Phys. Lett. A, 2001, 286(2), 175–179.

[14] N. H. Ibragimov, *A new conservation theorem*, J. Math. Anal. Appl., 2007, 333(1), 311–328.

[15] A. H. Kara and F. M. Mahomed, *Relationship between symmetries and conservation laws*, Int. J. Theor. Phys., 2000, 39(1), 23–40.

[16] A. H. Kara and F. M. Mahomed, *Noether-type symmetries and conservation laws via partial Lagrangians*, Nonlinear Dynam., 2006, 45(3), 367–383.

[17] B. Kolev, *Poisson brackets in hydrodynamics*, Discrete Contin. Dyn. Syst., 2007, 19(3), 555–574.

[18] P. S. Laplace, *Celestial Mechanics (English translation)*, Chelsea, New York, 1966.

[19] B. Li, Y. Chen and H. Q. Zhang, *Explicit exact solutions for some nonlinear partial differential equations with nonlinear terms of any order*, Czech J. Phys., 2003, 53(4), 283–295.

[20] S. Y. Lou, *Nonclassical symmetry reductions for the dispersive wave equations in shallow water*, J. Math. Phys., 1992, 33(12), 4300–4305.

[21] D. C. Lu and B. J. Hong, *New exact solutions for the (2 + 1)-dimensional generalized Broer–Kaup system*, Appl. Math. Comput., 2008, 199(2), 572–580.

[22] R. Narain and A. H. Kara, *On the redefinition of variational and 'partial' variational conservation laws in a class of non linear PDEs with mixed derivatives*, Math. Comput. Appl., 2010, 15(4), 732–741.

[23] R. Naz, M. D. Khan and I. Naeem, *Conservation laws and exact solutions of a class of non linear regularised long wave equations via double reduction theory and Lie symmetries*, Commun. Nonlinear Sci. Numer. Simulat., 2013, 18(4), 826–834.

[24] R. Naz, F. Mahomed, T. Hayat, Conservation laws for third-order variant Boussinesq system, Appl. Math. Lett., 2010, 23(8), 883–886.

[25] E. Noether, *Invariant variation problems*, Transp. Theory. Stat. Phys., 1971, 1(3), 186–207.

[26] P. J. Olver, *Application of Lie groups to differential equations*, Springer-Verlag, New York, 1993.

[27] R. L. Sachs, *On the integrable variant of the Boussinesq system: Painlevé property, rational solutions, a related many-body system*, and equivalence with the AKNS hierarchy, Physica D, 1988, 30(1–2), 1–27.

[28] A. Sjöberg, *On double reductions from symmetries and conservation laws*, Nonlinear Anal. Real World Appl., 2009, 10(6), 3472–3477.

[29] A. Sjöberg and F. Mahomed, *The association of non-local symmetries with conservation laws: applications to the heat and Burgers equations*, Appl. Math. Comput., 2005, 168(2), 1098–1108.

[30] H. Steudel, *Uber die zuordnung zwischen invarianzeigenschaften und erhaltungssatzen*, Z. Naturforsch A, 1962, 17(2), 129–132.

[31] M. Wang, *Solitary wave solutions for variant Boussinesq equations*, Phys. Lett. A, 1995, 199(3–4), 169–172.

[32] M. Wang, Y. Zhou and Z. Li, *Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics*, Phys. Lett. A, 1996, 216(1–5), 67–75.

[33] M. Wazwaz, *The tanh and sine–cosine method for compact and noncompact solutions of nonlinear Klein-Gordon equation*, Appl. Math. Comput., 2005, 167(2), 1179–1195.

[34] Z. Y. Yan, *Abundant families of Jacobi elliptic functions of the (2 + 1)–dimensional integrable Davey–Stewartson–type equation via a new method*, Chaos Solitons Fractals, 2003, 18(2), 299–309.

[35] Z. Y. Yan, F. D. Xie and H. Q. Zhang, *Symmetry reductions, integrability and solitary wave solutions to high–order modified Boussinesq equations with damping term*, Commun. Theor. Phys., 2001, 36(1), 1–6.

[36] E. Yasar and BI.B. Giresunlu, *Lie symmetry reductions, exact solutions and conservation laws of the third order variant Boussinesq system*, Acta Phys. Pol., 2015, 128(3), 243–255.

[37] J. F. Zhang, *Multi-solitary wave solutions for variant Boussinesq equations and Kupershmidt equations*, Appl. Math. Mech., 2000, 21(2), 171–175.