Journal of Applied Analysis and Computation
Volume 8, Number 6, December 2018, 1679–1693
Website:http://jaac.ijournal.cn
DOI:10.11948/2018.1679

NEIMARK-SACKER BIFURCATION OF A SEMI-DISCRETE HEMATOPOIESIS MODEL*

Wei Li[1] and Xianyi Li[1,†]

Abstract In this paper, we derive a semi-discrete system for a nonlinear model of blood cell production. The local stability of its fixed points is investigated by employing a key lemma from [23,24]. It is shown that the system can undergo Neimark-Sacker bifurcation. By using the Center Manifold Theorem, bifurcation theory and normal form method, the conditions for the occurrence of Neimark-Sacker bifurcation and the stability of invariant closed curves bifurcated are also derived. The numerical simulations verify our theoretical analysis and exhibit more complex dynamics of this system.

Keywords Semi-discrete blood cell production model, Neimark-Sacker bifurcation, invariant closed curve, center manifold theorem, normal form method.

MSC(2010) 39A11, 37F45, 37G35.

1. Introduction and Preliminaries

To describe the process of blood cells production in the bone marrow, Mackey and Glass [15] proposed the so-called hematopoiesis model

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = -\delta\xi(t) + \frac{p\xi(t-\tau)}{1+\xi^m(t-\tau)}, \quad t \geq 0, \tag{1.1}$$

where p, δ, $m \in (0, +\infty)$, satisfying $0 < \delta < 1$, $p > \delta$, and τ is the time delay between the production of immature cells in the bone marrow and their maturation for release in the circulation blood stream.

Since its introduction in 1977, the hematopoiesis model has gained a lot of attention due to its extensively realistic significance. Recently, there have been extensive contributions on the qualitative behavior of Eq.(1.1), including the existence and attractivity of periodic solutions, oscillation, stability and chaos.

Hale and Sternberg [9] gave interesting and perfect conclusions for the numerical and chaotic problems. Some sufficient and necessary conditions were established by Gopalsamy et al. [7] for the oscillation and global attractivity of all positive solutions to Eq.(1.1). By three kinds of stabilization methods based on conventional feedback, tracking filter, and delayed feedback, Namajtinas et al. [16] provided theoretical and experimental results of stabilizing an unstable steady state in Eq.(1.1).

†Corresponding author. Email address: mathxyli@zust.edu.cn(X. Li)
[1]Institute of Nonlinear Analysis and Department of Big Data Science, School of Science, Zhejiang University of Science and Technology, Hangzhou, Zhejiang 310023, China
*The authors are supported by the Natural Science Foundation of Zhejiang Province(grant: LQ13A010019), NSF of China (grant: 61473340) and NSF of Zhejiang University of Science and Technology (grant: F701108G14).

Gopalsamy et al. [8] suggested new sufficient conditions ensuring the positive equilibrium of Eq.(1.1) to be a global attractor. Kubiaczyk and Saker [10] investigated the oscillation of all positive solutions and the stability of all equilibrium points in Eq.(1.1). The persistence and extinction conditions for Eq.(1.1) with variable coefficients and a nonconstant delay were presented by Berezansky and Braverman [1]. Rost [18] proved the global attractivity of the positive equilibrium for Eq.(1.1). The bubbles behavior of Eq.(1.1) was studied by Krisztin and Liz [11]. Berezansky et al. [2] generalized and unified the existing results on oscillation, stability and chaos. At the same time , some new results for more general versions of Eq.(1.1) were provided, involving the existence, positivity and permanence of solutions, oscillation and global asymptotic stability. In addition, some open problems and topics for further research were stated. For the details, refer to [2] and the references therein. Based on the outlook in [2], Berezansky et al. [3] investigated the permanence, oscillation and stability of the positive equilibrium for non-autonomous equations, which are generalizations of Eq.(1.1), and a linear control was introduced such that stabilizing an unstable positive equilibrium is possible. Kanno and Uchida [12] obtained standard deviation of the probability distribution for finite-time Lyapunov exponents in Eq.(1.1) .

However, most of models in practice cannot be solved completely by analytic techniques. Since discretion models of differential equations can inherit some corresponding properties of original differential equations, and numerical simulations are helpful to understand the dynamics of such systems, regarding scientific computation and real-time simulation, it is quite important to discrete Eq.(1.1). In recent years, there has been a more and more interest in the analysis of discrete systems of Eq.(1.1).

In 2007, by using a new approach better than contraction mapping principle, Wang and Li [22] studied the existence and uniqueness of positive almost periodic solution to the difference equation of Eq.(1.1). Also, some sufficient conditions were established for global attractivity. In [4], for a semidiscretization of Eq.(1.1) with periodic coefficients, Braverman and Saker not only proved the existence and oscillation of a positive periodic solution, but also gave sufficient conditions for the attractivity of this solution. Ding et al. [6] proved that the occurrence of Hopf bifurcation for a discrete version of Eq.(1.1) obtained by Trapezium method, and provided explicit algorithm for determining the direction of the bifurcation and the stability of the bifurcating periodic solutions. Su and Ding [20] studied the discrete system of Eq.(1.1) by appling a nonstandard finite-difference scheme. They analyzed the stability of the fixed point, and obtained the occurrence and direction of Hopf bifurcation at the positive fixed point as the delay increases. In particular, Su et al. [21] discussed Neimark - Sacker and fold bifurcations for the difference equation of Eq.(1.1). Qian [17] considered the global attractivity of periodic solutions of Eq.(1.1) in a discrete form. In [14], by means of Lyapunov exponent spectrum and bifurcation diagrams, Li investigated bifurcation and chaos of the discrete system of Eq.(1.1) formulated by a new method.

In terms of the existing research, even though there have been increasing methods on dynamics of discrete hematopoiesis model, less is known to see that complete and systematic study including bifurcation and other complex dynamics has been conducted. In this paper, we use a semi-discrete scheme to Eq.(1.1), and investigate its dynamics by applying the Center Manifold Theorem, bifurcation theory and numerical simulations, which is motivated by Wang and Li [23].

First, without loss of generality, we assume $\tau = 1$ in (1.1). In fact, by taking $s = \frac{t}{\tau}$, and letting $\xi(t) = \xi(s\tau) \triangleq \eta(s)$, (1.1) is transformed to

$$\frac{\mathrm{d}\eta}{\mathrm{d}s} = -\delta\tau\eta(s) + \frac{p\tau\eta(s-1)}{1+\eta^m(s-1)}, \quad s \geq 0. \tag{1.2}$$

By resetting p and δ by $\frac{p}{\tau}$ and $\frac{\delta}{\tau}$ respectively, Eq.(1.2) can be rewritten as

$$\frac{\mathrm{d}\eta}{\mathrm{d}s} = -\delta\eta(s) + \frac{p\eta(s-1)}{1+\eta^m(s-1)}, \quad s \geq 0. \tag{1.3}$$

This is just (1.1) in the case of $\tau = 1$.

Suppose $[s]$ denotes the greatest integer not exceeding s. We consider the following semi-discretization version of (1.3)

$$\frac{\mathrm{d}\eta}{\mathrm{d}s} = -\delta\eta([s]) + \frac{p\eta([s-1])}{1+\eta^m([s-1])}, \quad s \neq 0, 1, 2, 3, \ldots. \tag{1.4}$$

Obviously, Eq. (1.4) has piecewise constant arguments, and for $s \in [0, +\infty)$ the solution $\eta(s)$ of Eq. (1.4) possesses the following features:

(i) $\eta(s)$ is continuous on $[0, +\infty)$;

(ii) $\mathrm{d}\eta(s)/\mathrm{d}s$ exists everywhere when $s \in [0, +\infty)$ except for the points $s \in \{0, 1, 2, 3, ...\}$;

(iii) Eq. (1.4) is true in each interval $[k, k+1)$ with $k = 0, 1, 2, 3, \ldots$.

For any $n \in \{0, 1, 2, 3, ...\}$, $s \in [n, n+1)$, we integrate (1.4) on the interval $[n, s]$ and get

$$\eta(s) - \eta(n) = \left(-\delta\eta(n) + \frac{p\eta(n-1)}{1+\eta^m(n-1)}\right)(s-n). \tag{1.5}$$

Letting $s \to (n+1)^-$, Eq. (1.5) becomes

$$\eta(n+1) = (1-\delta)\eta(n) + \frac{p\eta(n-1)}{1+\eta^m(n-1)}, \tag{1.6}$$

which can be viewed as a discrete form of Eq. (1.3) without delay.

Under the transformation

$$\begin{cases} x_n = \eta(n-1), \\ y_n = \eta(n), \end{cases} \tag{1.7}$$

one has

$$\begin{cases} x_{n+1} = y_n, \\ y_{n+1} = (1-\delta)y_n + \dfrac{px_n}{1+x_n^m}, \end{cases} \tag{1.8}$$

which is a discrete system of (1.1), where p, δ, m are defined as in Eq. (1.1), satisfying $0 < \delta < 1$, $p > \delta$, $x_0, y_0 \in (0, +\infty)$.

It is noted that the fixed points of the system (1.8) satisfy

$$\begin{cases} x = y, \\ y = (1-\delta)y + \dfrac{px}{1+x^m}. \end{cases} \tag{1.9}$$

By solving (1.9), we obtain that the system (1.8) has two nonnegative fixed points $O(0,0)$ and $E_+(x_*, y_*)$, where

$$x_* = \left(\frac{p}{\delta} - 1\right)^{1/m}, \qquad y_* = \left(\frac{p}{\delta} - 1\right)^{1/m}. \tag{1.10}$$

This paper aims to investigate the dynamics of the system (1.8). We not only study the stability of the fixed points, but also derive sufficient conditions for the existence of Neimark - Sacker bifurcations by center manifold theory and bifurcation theory, which have not been considered in the existing literature.

In what follows, we recall the definitions of topological types and lemmas of the local stability and bifurcation for a fixed point, needed in the sequel. For more details, the reader can refer to [23, 24].

Definition 1.1. Let $\overline{E}(\overline{x}, \overline{y})$ be a fixed point of the system (1.8) with multipliers λ_1 and λ_2.

(i) A fixed point $\overline{E}(\overline{x}, \overline{y})$ is called sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so a sink is locally asymptotically stable.

(ii) A fixed point $\overline{E}(\overline{x}, \overline{y})$ is called source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, so a source is locally asymptotically unstable.

(iii) A fixed point $\overline{E}(\overline{x}, \overline{y})$ is called saddle if $|\lambda_1| < 1$ and $|\lambda_2| > 1$(or $|\lambda_1| > 1$ and $|\lambda_2| < 1$).

(iv) A fixed point $\overline{E}(\overline{x}, \overline{y})$ is called to be non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

Lemma 1.1. *Let $F(\lambda) = \lambda^2 + B\lambda + C$, where B and C are two real constants. Suppose λ_1 and λ_2 are two roots of $F(\lambda) = 0$. Then the following statements hold.*

(i) *If $F(1) > 0$, then*

(i.1) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $C < 1$*;*

(i.2) $\lambda_1 = -1$ *and* $\lambda_2 \neq -1$ *if and only if* $F(-1) = 0$ *and* $B \neq 2$*;*

(i.3) $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) < 0$*;*

(i.4) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $C > 1$*;*

(i.5) λ_1 *and* λ_2 *are a pair of conjugate complex roots and* $|\lambda_1| = |\lambda_2| = 1$ *if and only if* $-2 < B < 2$ *and* $C = 1$*;*

(i.6) $\lambda_1 = \lambda_2 = -1$ *if and only if* $F(-1) = 0$ *and* $B = 2$.

(ii) *If $F(1) = 0$, namely, 1 is one root of $F(\lambda) = 0$, then the other root λ satisfies $|\lambda| = (<, >)1$ if and only if $|C| = (<, >)1$.*

(iii) *If $F(1) < 0$, then $F(\lambda) = 0$ has one root lying in (1, ∞). Moreover,*

(iii.1) *the other root λ satisfies $\lambda < (=) - 1$ if and only if $F(-1) < (=)0$;*

(iii.2) *the other root λ satisfies $-1 < \lambda < 1$ if and only if $F(-1) > 0$.*

The remainder of the paper is organized as follows. In Section 2, the stability of the fixed points is investigated. In Section 3, we prove sufficient conditions for the occurrence of the Neimark-Sacker bifurcation. In Section 4, we provide numerical simulations to verify the theoretical results and explore some complex dynamics of system (1.8). Finally, a conclusion is established in Section 5.

2. Stability of a fixed point

In this section, we determine the local stability of every fixed point of the system (1.8).

The Jacobian matrix of the system (1.8) at a fixed point $\overline{E}(\overline{x},\overline{y})$ is

$$J=\begin{pmatrix} 0 & 1 \\ \\ \dfrac{p+p(1-m)\overline{x}^m}{(1+\overline{x}^m)^2} & 1-\delta \end{pmatrix}. \tag{2.1}$$

The characteristic equation of J is

$$\lambda^2-\mathrm{Tr}(J)\lambda+\mathrm{Det}(J)=0, \tag{2.2}$$

where $\mathrm{Tr}(J)$ and $\mathrm{Det}(J)$ are the trace and determinant of (2.1) respectively, namely,

$$\mathrm{Tr}(J)=1-\delta \tag{2.3}$$

and

$$\mathrm{Det}(J)=-\frac{p+p(1-m)\overline{x}^m}{(1+\overline{x}^m)^2}. \tag{2.4}$$

Now, we give the results for the stability of the fixed points $O(0,0)$ and $E_+(x_*,y_*)$ in Theorem 2.1 and Theorem 2.2 respectively.

Theorem 2.1. *The following statements about the fixed point $O(0,0)$ of the system (1.8) are true:*

(i) *When $\delta<p<2-\delta$, $O(0,0)$ is a saddle.*

(ii) *When $p=2-\delta$, $O(0,0)$ is non-hyperbolic.*

(iii) *When $p>2-\delta$, $O(0,0)$ is a source.*

Proof. The Jacobian matrix J of the system (1.8) at $O(0,0)$ is given by

$$J(O)=\begin{pmatrix} 0 & 1 \\ \\ p & 1-\delta \end{pmatrix}. \tag{2.5}$$

The characteristic equation of (2.5) can be formulated as

$$F(\lambda)=\lambda^2-(1-\delta)\lambda-p=0. \tag{2.6}$$

It is obvious that

$$F(1)=\delta-p<0, \tag{2.7}$$

and

$$F(-1)=2-\delta-p. \tag{2.8}$$

When $\delta<p<2-\delta$, $F(-1)>0$. By Lemma 1.1 (iii.2), Eq. (2.6) has two eigenvalues λ_1 and λ_2 with $|\lambda_1|>1$ and $|\lambda_2|<1$, so $O(0,0)$ is a saddle.

When $p=2-\delta$, $F(-1)=0$. By Lemma 1.1 (iii.1), Eq. (2.6) has two eigenvalues λ_1 and λ_2 with $|\lambda_1|>1$ and $|\lambda_2|=1$, so $O(0,0)$ is non-hyperbolic.

When $p>2-\delta$, $F(-1)<0$. By Lemma 1.1 (iii.1), Eq. (2.6) has two eigenvalues λ_1 and λ_2 with $|\lambda_1|>1$ and $|\lambda_2|>1$, so $O(0,0)$ is a source.

□

Theorem 2.2. *$E_+(x_*, y_*)$ is the unique positive fixed point of the system (1.8), where $x_* = y_* = (\frac{p}{\delta} - 1)^{1/m}$. Let $\frac{p(1+\delta)}{\delta(p-\delta)} \triangleq m_0$, then the following statements about $E_+(x_*, y_*)$ are true:*

(i) *E_+ is a sink if $m < m_0$, at this time E_+ is locally asymptotically stable;*

(ii) *E_+ is a source if $m > m_0$, then E_+ is unstable;*

(iii) *E_+ is non-hyperbolic if $m = m_0$, and the system (1.8) may undergo a Neimark-Sacker bifurcation.*

Proof. The Jacobian matrix J of the system (1.8) at E_+ is given by

$$J(E_+) = \begin{pmatrix} 0 & 1 \\ \dfrac{\delta[(1-m)p + \delta m]}{p} & 1-\delta \end{pmatrix}. \tag{2.9}$$

We can express the characteristic equation of (2.9) as

$$F(\lambda) = \lambda^2 + B\lambda + C = 0, \tag{2.10}$$

where $B = -(1-\delta)$, $C = -\dfrac{\delta}{p}[(1-m)p + \delta m]$.

By computing we get

$$F(1) = \delta m\left(1 - \frac{\delta}{p}\right) > 0 \tag{2.11}$$

and

$$F(-1) = 2(1-\delta) + \delta m\left(1 - \frac{\delta}{p}\right) > 0. \tag{2.12}$$

When $m < \dfrac{p(1+\delta)}{\delta(p-\delta)} \triangleq m_0$, $m\delta(p-\delta) < p + p\delta$, $-\delta[p(1-m) - \delta m] < p$, then $C = -\dfrac{\delta}{p}[p(1-m) - \delta m] < 1$. By Lemma 1.1 (i.1), Eq.(2.10) has two eigenvalues λ_1 and λ_2 with $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so E_+ is a sink.

When $m > \dfrac{p(1+\delta)}{\delta(p-\delta)} \triangleq m_0$, $m\delta(p-\delta) > p + p\delta$, $-\delta[p(1-m) - \delta m] > p$, then $C > 1$. Therefore, Lemma 1.1 (i.4) tells us that Eq.(2.10) has two eigenvalues λ_1 and λ_2 with $|\lambda_1| > 1$ and $|\lambda_2| > 1$, and hence E_+ is a source.

When $m = m_0$, $m\delta(p-\delta) = p + p\delta$, $-\delta[p(1-m) - \delta m] = p$, then $C = 1$ and $B = -(1-\delta) \in (-1, 0)$. By Lemma 1.1(i.5), Eq.(2.10) has a pair of conjugate complex roots λ_1 and λ_2 with$|\lambda_1| = |\lambda_2| = 1$, so E_+ is non-hyperbolic . □

3. Neimark-Sacker bifurcation analysis

In this section, by using the Center Manifold Theorem and bifurcation theory in [5, 13, 19, 25], we investigate the Neimark-Sacker bifurcation of the system (1.8) at $E_+(x_*, y_*)$.

Set

$$S_{E_+} = \{(m, p, \delta) \in (0, +\infty)^3 : 0 < \delta < 1, p > \delta, m = m_0 \triangleq \frac{p(1+\delta)}{\delta(p-\delta)}\}.$$

Theorem 2.2 (iii) shows that the fixed point $E_+(x_*, y_*)$ can undergo a Neimark-Sacker bifurcation when the parameters $(m, p, \delta) \in S_{E_+}$ and m varies in a small neighborhood of m_0.

By arbitrarily taking parameters $(m, p, \delta) \in S_{E_+}$, we consider the system (1.8) as follows

$$\begin{cases} x \to y, \\ y \to (1-\delta)y + \dfrac{px}{1+x^{m_0}}, \end{cases} \tag{3.1}$$

Now we study the stablity of Neimark-Sacker bifurcation according to the ways or methods formulated in [5, 13, 19, 25].

The first step. For convenience, we choose the parameter m as a bifurcation parameter. Giving a small perturbation m^* of m_0, a perturbation of the system (3.1) is described by:

$$\begin{cases} x \to y, \\ y \to (1-\delta)y + \dfrac{px}{1+x^{m_0+m^*}}, \end{cases} \tag{3.2}$$

where $|m^*| \ll 1$.

The second step. Let $u = x - x_*$ and $v = y - y_*$, then the fixed point $E_+(x_*, y_*)$ is transformed into the origin $O(0,0)$, and system (3.2) is rewritten as

$$\begin{cases} u \to v, \\ v \to (1-\delta)v + \dfrac{p(u+x_*)}{1+(u+x_*)^{m_0+m^*}} - \delta y_*. \end{cases} \tag{3.3}$$

The corresponding characteristic equation of the system (3.3) at $(u, v) = (0, 0)$ is

$$\lambda^2 - a(m^*)\lambda + b(m^*) = 0, \tag{3.4}$$

where

$$a(m^*) = 1 - \delta,$$

and

$$b(m^*) = -\frac{p + p(1 - m_0 - m^*)x_*^{m_0+m^*}}{(1 + x_*^{m_0+m^*})^2}.$$

The roots of (3.4) are

$$\lambda_{1,2}(m^*) = \frac{1}{2}\left[a(m^*) \pm \sqrt{4b(m^*) - a^2(m^*)}i\right]. \tag{3.5}$$

Hence

$$|\lambda_{1,2}(m^*)| = \sqrt{b(m^*)} \tag{3.6}$$

and

$$\frac{\mathrm{d}|\lambda_{1,2}|}{\mathrm{d}m^*}\bigg|_{m^*=0} = -\frac{p\{[(1-m_0)x_*^{m_0}\ln x_* - x_*^{m_0}](1+x_*^{m_0}) - 2[1+(1-m_0)x_*^{m_0}]x_*^{m_0}\ln x_*\}}{2(1+x_*^{m_0})^3\sqrt{b(m^*)}}. \tag{3.7}$$

The occurence of Neimark-Sacker bifurcation requires the following conditions

$$(\mathbf{C.1}) \quad \left.\frac{\mathrm{d}|\lambda_{1,2}|}{\mathrm{d}m^*}\right|_{m^*=0} \neq 0;$$

$$(\mathbf{C.2}) \quad \lambda_{1,2}^i \neq 1,\ i = 1, 2, 3, 4.$$

Since $a(m^*)|_{m^*=0} = 1 - \delta$ and $b(m^*)|_{m^*=0} = 1$, we have $\lambda_{1,2} = \frac{1}{2}[(1-\delta) \pm \sqrt{(3-\delta)(1+\delta)}i]$, which obviously satisfy condition(C.2).

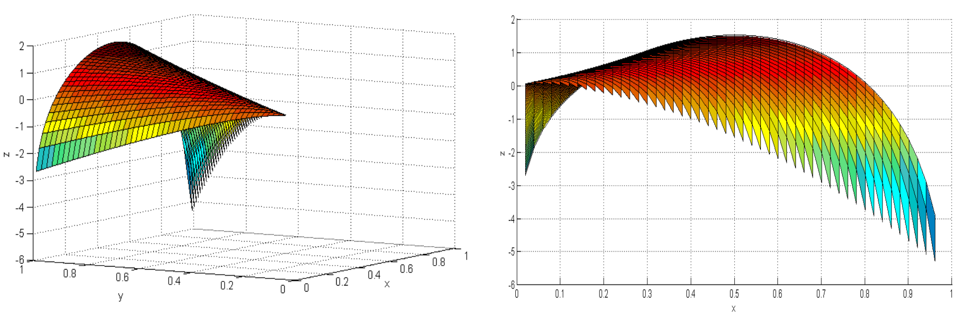
With the expression in (3.7), the condition(C.1) is equivalent to the following condition:

$$(1 + m_0)\ln x_* + (1 - m_0)x_*^{m_0}\ln x_* + x_*^{m_0} + 1 \neq 0. \tag{3.8}$$

Substituting $x_* = \left(\frac{p}{\delta} - 1\right)^{1/m_0}, m_0 = \frac{p(1+\delta)}{\delta(p-\delta)}$ into (3.8), one has

$$(\delta^2 + 2\delta - p)\ln\left(\frac{p}{\delta} - 1\right) + p(1 + \delta) \neq 0. \tag{3.9}$$

However (3.9) is uncertain when $0 < \delta < 1$, $p > \delta$. To demonstrate this, we can regard $(\delta^2 + 2\delta - p)\ln\left(\frac{p}{\delta} - 1\right) + p(1+\delta)$ as a two-variable function $h(\delta, p)$. It is easy to get the following surface graph and projection graph on (x, z) plane of $h(\delta, p)$ by Matlab.



(a) surface graph

(b) projection graph on (x, z) plane

Figure 1. Surface graph and projection graph on (x, z) plane of $h(\delta, p)$

Figure 1(b) shows that there exist some (δ, p)'s such that $h(\delta, p) = 0$. So Condition (3.9) must be provided to ensure condition(C.1) is satisfied.

The third step. Look for the normal form of the system (3.3) when $m^* = 0$. First the system (3.3) is expanded as Taylor series at $(u, v) = (0, 0)$ to the third order as follows:

$$\begin{cases} u \to a_{10}u + a_{01}v + a_{20}u^2 + a_{11}uv + a_{02}v^2 + a_{30}u^3 \\ \qquad +a_{21}u^2v + a_{12}uv^2 + a_{03}v^3 + O((\sqrt{|u|^2 + |v|^2})^4), \\ v \to b_{10}u + b_{01}v + b_{20}u^2 + b_{11}uv + b_{02}v^2 + b_{30}u^3 \\ \qquad +b_{21}u^2v + b_{12}uv^2 + b_{03}v^3 + O((\sqrt{|u|^2 + |v|^2})^4), \end{cases} \tag{3.10}$$

where

$$a_{ij} = \frac{1}{i!j!}\cdot\frac{\partial^{i+j}f(u,v)}{\partial^i u\partial^j v}\bigg|_{(0,0)},\ i,j=0,1,2,3,\ f(u,v)=v,$$

$$b_{ij} = \frac{1}{i!j!}\cdot\frac{\partial^{i+j}g(u,v)}{\partial^i u\partial^j v}\bigg|_{(0,0)},\ i,j=0,1,2,3,\ g(u,v)=(1-\delta)v+\frac{p(u+x_*)}{1+(u+x_*)^{m_0}}-\delta y_*.$$

Thereout one has

$$a_{01}=1, a_{10}=a_{20}=a_{11}=a_{02}=a_{21}=a_{12}=a_{03}=a_{30}=0,$$

$$b_{10}=-1, b_{01}=1-\delta, \quad b_{11}=b_{02}=b_{12}=b_{21}=b_{03}=0,$$

$$b_{20}=\frac{-pm_0x_*^{m_0-1}[m_0+1+(1-m_0)x_*^{m_0}]}{2(1+x_*^{m_0})^3},$$

$$b_{30}=\frac{pm_0x_*^{m_0-2}[1-{m_0}^2+2(1+2{m_0}^2)x_*^{m_0}+(1-{m_0}^2)x_*^{2m_0}]}{6(1+x_*^{m_0})^4}.$$

Let

$$J(E_+)=\begin{pmatrix} a_{10} & a_{01} \\ b_{10} & b_{01}\end{pmatrix}, \quad \text{namely,} \quad J(E_+)=\begin{pmatrix} 0 & 1 \\ -1 & 1-\delta\end{pmatrix}.$$

Then the eigenvalues of the matrix $J(E_+)$ are

$$\lambda_1=\frac{1-\delta+\sqrt{3+2\delta-\delta^2}i}{2} \quad \text{and} \quad \lambda_2=\frac{1-\delta-\sqrt{3+2\delta-\delta^2}i}{2}.$$

Take an invertible matrix

$$T=\begin{pmatrix} 0 & 1 \\ \frac{\sqrt{3+2\delta-\delta^2}}{2} & \frac{1-\delta}{2}\end{pmatrix}, \quad \text{then} \quad T^{-1}=\begin{pmatrix} \frac{\delta-1}{\sqrt{3+2\delta-\delta^2}} & \frac{2}{\sqrt{3+2\delta-\delta^2}} \\ 1 & 0\end{pmatrix}.$$

Finally using translation

$$(u,v)^T=T(X,Y)^T,$$

we transform the system (3.10) into the following normal form

$$\begin{cases} X \to \frac{1-\delta}{2}X-\frac{\sqrt{3+2\delta-\delta^2}}{2}Y+F(X,Y)+O((\sqrt{|X|^2+|Y|^2})^4), \\ Y \to \frac{\sqrt{3+2\delta-\delta^2}}{2}X+\frac{1-\delta}{2}Y+G(X,Y)+O((\sqrt{|X|^2+|Y|^2})^4), \end{cases} \tag{3.11}$$

where

$$F(X,Y)=\frac{2}{\sqrt{3+2\delta-\delta^2}}(b_{20}Y^2+b_{30}Y^3)$$

and

$$G(X,Y)=0.$$

Furthermore

$$F_{XX}|_{(0,0)} = F_{XY}|_{(0,0)} = F_{XXX}|_{(0,0)} = F_{XXY}|_{(0,0)} = F_{XYY}|_{(0,0)} = 0,$$
$$F_{YY}|_{(0,0)} = \frac{4b_{20}}{\sqrt{3+2\delta-\delta^2}}, \quad F_{YYY}|_{(0,0)} = \frac{12b_{30}}{\sqrt{3+2\delta-\delta^2}},$$
$$G_{XX}|_{(0,0)} = G_{XY}|_{(0,0)} = G_{YY}|_{(0,0)} = 0,$$
$$G_{XXX}|_{(0,0)} = G_{XXY}|_{(0,0)} = G_{XYY}|_{(0,0)} = G_{YYY}|_{(0,0)} = 0.$$

The fourth step. In order to determine the stability of the invariant curve bifurcated from Nemark-Sacker bifurcation of the system (3.11), one requires that the following discriminating quantity a^* is not zero(see [5, 13, 19, 25]):

$$a^* = -Re\left[\frac{(1-2\lambda)\overline{\lambda}^2}{1-\lambda}L_{11}L_{20}\right] - \frac{1}{2}|L_{11}|^2 - |L_{02}|^2 + Re(\overline{\lambda}L_{21}), \tag{3.12}$$

where

$$\begin{aligned}
L_{20} &= \frac{1}{8}[(F_{XX} - F_{YY} + 2G_{XY}) + i(G_{XX} - G_{YY} - 2F_{XY})]\\
L_{11} &= \frac{1}{4}[(F_{XX} + F_{YY}) + i(G_{XX} + G_{YY})],\\
L_{02} &= \frac{1}{8}[(F_{XX} - F_{YY} - 2G_{XY}) + i(G_{XX} - G_{YY} + 2F_{XY})],\\
L_{21} &= \frac{1}{16}[(F_{XXX}+F_{XYY}+G_{XXY}+G_{YYY})+i(G_{XXX}+G_{XYY}-F_{XXY}-F_{YYY})].
\end{aligned} \tag{3.13}$$

By calculation we get

$$\begin{aligned}
L_{20} &= \frac{-b_{20}}{2\sqrt{3+2\delta-\delta^2}}, \quad L_{11} = \frac{b_{20}}{\sqrt{3+2\delta-\delta^2}},\\
L_{02} &= \frac{-b_{20}}{2\sqrt{3+2\delta-\delta^2}}, \quad L_{21} = \frac{-3b_{30}}{4\sqrt{3+2\delta-\delta^2}}i.
\end{aligned} \tag{3.14}$$

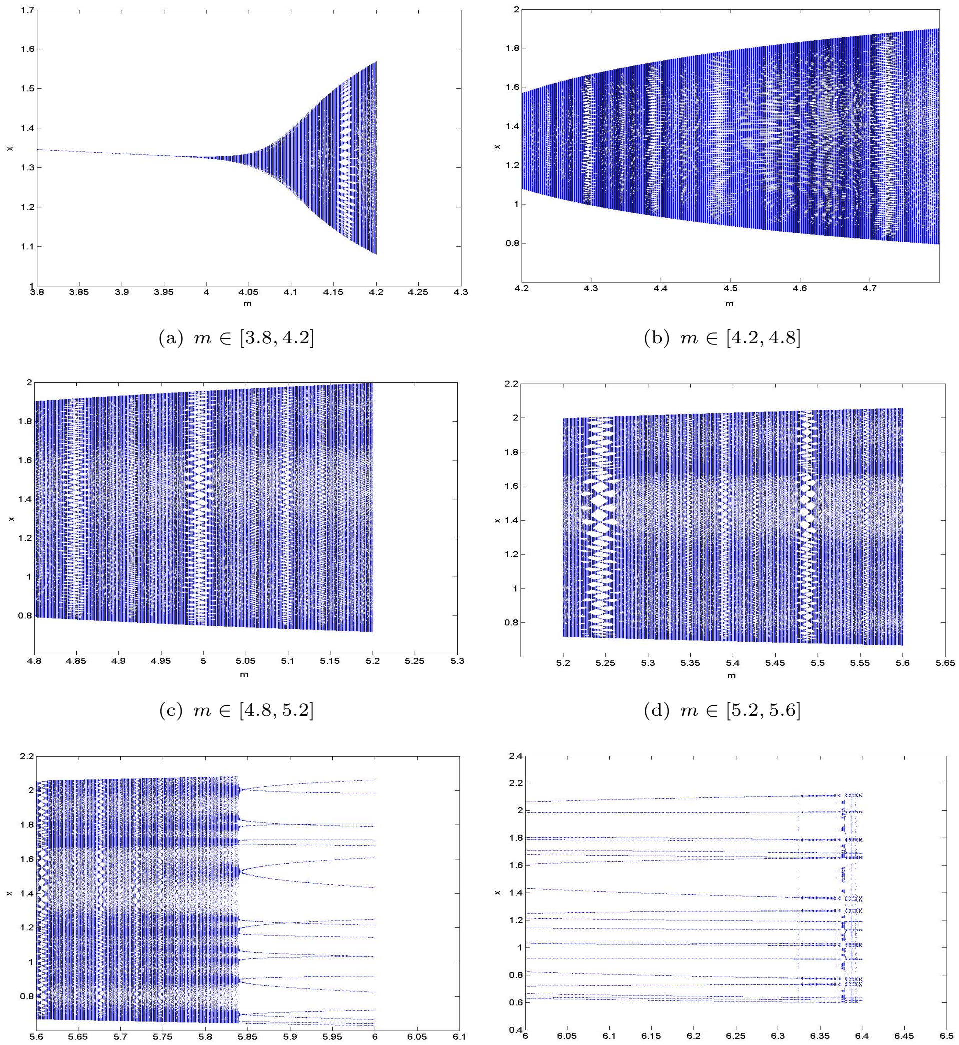
Thus

$$\begin{aligned}
a^* =& \frac{b_{20}^2(\delta^3 - 7\delta - 6)}{4(\delta+1)(3+2\delta-\delta^2)} - \frac{3b_{30}}{8}\\
=& \frac{\delta^3 - 7\delta - 6}{4(\delta+1)(3+2\delta-\delta^2)}\left(\frac{-pm_0x_*^{m_0-1}[m_0+1+(1-m_0)x_*^{m_0}]}{(1+x_*^{m_0})^3}\right)^2\\
&- \frac{3pm_0x_*^{m_0-2}[1-{m_0}^2+2(1+2{m_0}^2)x_*^{m_0}+(1-{m_0}^2)x_*^{2m_0}]}{8(1+x_*^{m_0})^4}.
\end{aligned} \tag{3.15}$$

Summarizing the above analysis, we obtain the following theorem.

Theorem 3.1. *Assume that* $(m, p, \delta) \in S_{E_+}$, x_*, y_* *and* a^* *are described as* (1.10) *and* (3.15), *respectively. If* $a^* \neq 0$ *and* (3.9) *holds, then the system* (1.8) *undergoes* *a Neimark-Sacker bifurcation at the fixed point $E_+(x_*, y_*)$ when the parameter m^* varies in the small neighborhood of origin. Moreover, if $a^* < 0$ (resp., $a^* > 0$), then an attracting (resp., repelling) invariant closed curve bifurcates from the fixed point for $m^* > 0$ (resp., $m^* < 0$).*

4. Numerical simulation



(a) $m \in [3.8, 4.2]$

(b) $m \in [4.2, 4.8]$

(c) $m \in [4.8, 5.2]$

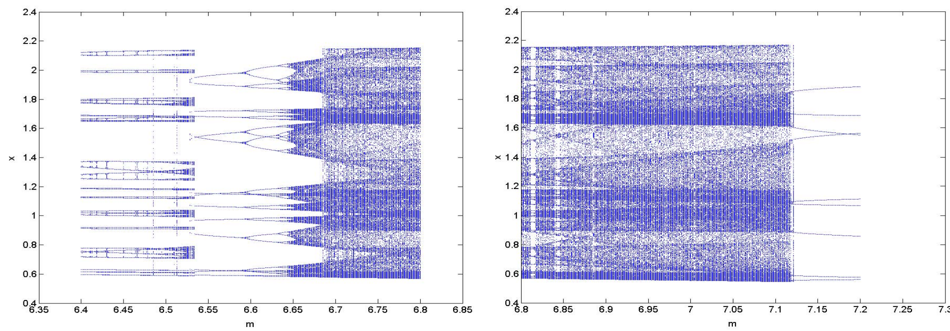
(d) $m \in [5.2, 5.6]$

(e) $m \in [5.6, 6.0]$

(f) $m \in [6.0, 6.4]$

Figure 2. Bifurcation of the system (1.8) in (m, x) plane for $\delta = 0.5, p = 2$.

In this section, to verify our theoretical results and reveal some new dynamical behaviors in the system (1.8), we present the bifurcation diagrams, phase portraits and Lyapunov exponents for specific parameter values by Matlab software.



(g) $m \in [6.4, 6.8]$ (h) $m \in [6.8, 7.2]$

Figure 2. Bifurcation of the system (1.8) in (m, x) plane for $\delta = 0.5, p = 2$. (con't)

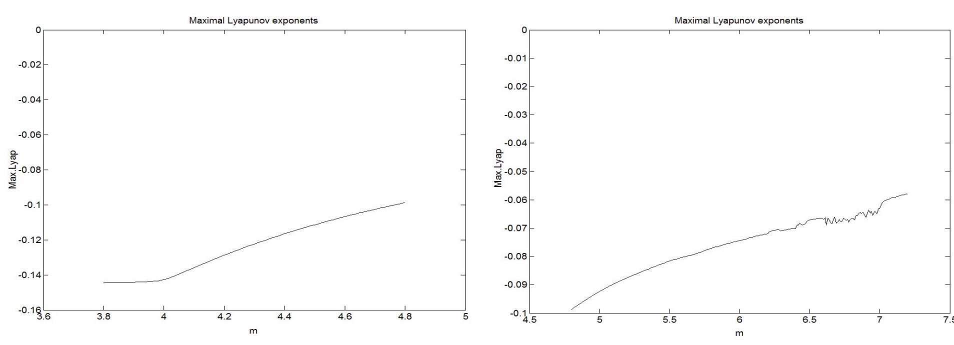
Based on the theoretical analysis, we choose the parameters $\delta = 0.5, p = 2$, the initial values $(x_0, y_0) = (0.1,\ 0.1)$, and assume that these values are fixed in each simulation. It is easy to get the unique positive fixed point $E_+(3^{1/m}, 3^{1/m})$ and $m_0 = 4$. Since m is chosen as a bifurcation parameter, we let m vary in the interval $[3.8,\ 7.2]$.

Now the bifurcation diagrams in the (m, x) plane are given in Figure 2.

In (1.8), $x_{n+1} = y_n$. So, the bifurcation diagrams for x and y are naturally the same. And hence for the bifurcation diagrams for y, refer to the ones for x in Figures 2 and omitted here.

Figure 2(a) shows that $E_+(3^{1/m}, 3^{1/m})$ is stable for $m < 4$, and the Neimark-Sacker bifurcation occurs at the fixed point (1.32, 1.32) when $m = 4$, whose multipliers are $\lambda_{1,2} = \frac{1}{4} \pm \frac{\sqrt{15}}{4} i$ with $|\lambda_{1,2}| = 1$. Meanwhile $E_+(3^{1/m}, 3^{1/m})$ becomes unstable when $m > 4$. This is in accordance with the results in Theorem 2.2. Figure 2 also displays some interesting dynamics as m increases.

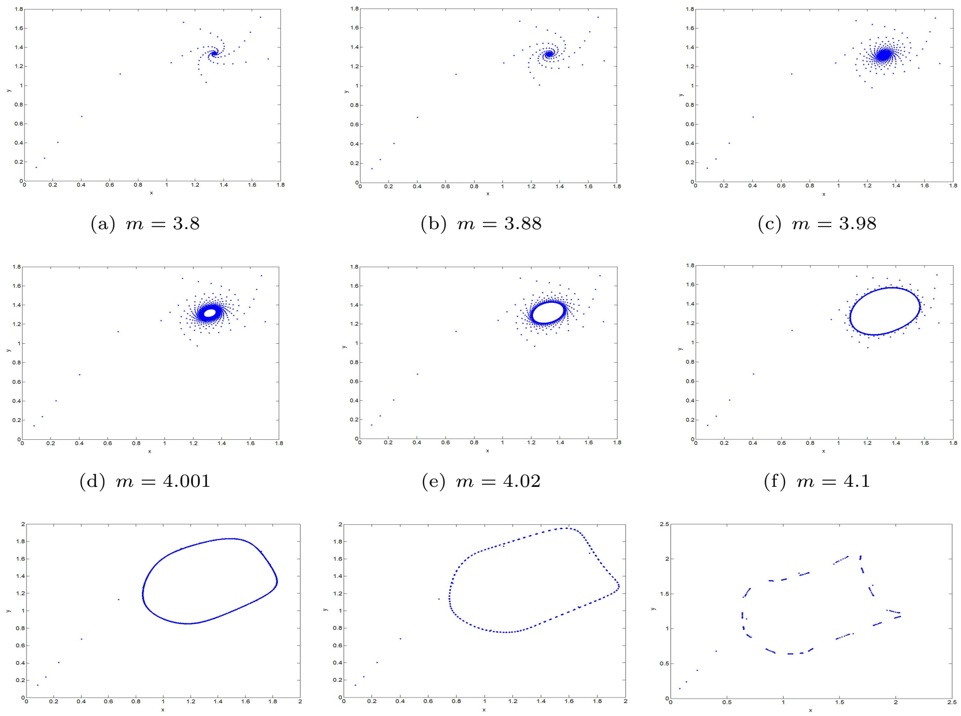
Figure 3 depicts the corresponding maximum Lyapunov exponents, in which one can easily see that the maximal Lyapunov exponents are always negative for the parameter $m \in (3.8,\ 7.2)$, that is to say, chaos doesn't occur .



(a) $m \in [3.8, 4.8]$ (b) $m \in [4.8, 7.2]$

Figure 3. Maximal Lyapunov exponent for the system (1.8) with $\delta = 0.5, p = 2$.

In the following, various phase portraits are plotted for $\delta = 0.5, p = 2$ and different m in Figure 4.



(a) $m = 3.8$ (b) $m = 3.88$ (c) $m = 3.98$

(d) $m = 4.001$ (e) $m = 4.02$ (f) $m = 4.1$

(g) $m = 4.5$ (h) $m = 4.9$ (i) $m = 5.8$

Figure 4. Phase portraits for the system (1.8) with $\delta = 0.5, p = 2$ and different m.

Figures 4(a)and (b) show that fixed point E_+ is a stable attractor when $m = 3.8$ and $m = 3.88$. Figure 4(c) depicts the dynamics of the system (1.8) before the occurrence of Neimark-Sacker bifurcation when $m = 3.98$, while Figure 4(d) demonstrates the dynamics of the system (1.8) after the occurrence of Neimark-Sacker bifurcation when $m = 4.001$. Comparing Figure 4(c) and Figure 4(d), we find that the fixed point E_+ becomes unstable as the parameter m goes through the bifurcation value $m_0 = 4$. In Figure 4, subfigures (e)-(f) show that increasing the parameter m leads to instablity of the fixed point E_+ and the creation of invariant closed curve around E_+. This agrees with our Theorem 3.1. As m continues to increase, it is observed that the dynamics of the fixed point E_+ becomes simple from subfigures(g)-(i), which can be considered as the absence of chaos.

5. Conclusion

In this paper, we discuss the dynamics of a semi-discrete system (1.8) derived for a nonlinear model of blood cell production with time delay τ. Under the given parametric conditions, the system (1.8) has two nonnegative equilibria $O(0,0)$ and $E_+(x_*, y_*)$. First combining the associated characteristic equation with Lemma 1.1, the stability of its equilibrium points has been investigated. We find that the positive equilibrium $E_+(x_*, y_*)$ is asymptotically stable when $m < m_0$ and unstable when $m > m_0$, where $m_0 \triangleq \frac{p(1+\delta)}{\delta(p-\delta)}$. Moreover, when m crosses through the critical value m_0, the system can undergo a Neimark-Saker bifurcation at $E_+(x_*, y_*)$. Then the Neimark-Saker bifurcation for the system (1.8) at $E_+(x_*, y_*)$ has been analysed in theory by choosing $m = m_0$ as a bifurcation parameter. Finally we provide numerical simulations,which not only confirm the theoretical analysis results but also exhibit some new properties in the system (1.8).

References

[1] L. Berezansky and E. Braverman, *Mackey-Glass equation with variable coefficients*, Comput. Math. Appl., 2006, 51(1), 1–16.

[2] L. Berezansky, E. Braverman and L. Idels, *The Mackey-Glass model of respiratory dynamics: Review and new results*, Nonlinear Analysis, 2012, 75(16), 6034–6052.

[3] L. Berezansky, E. Braverman and L. Idels, *Mackey-Glass model of hematopoiesis with non-monotone feedback: Stability, oscillation and control*, Applied Mathematics and Computation, 2013, 219(11), 6268–6283.

[4] E. Braverman and S. H. Saker, *Permanence, oscillation and attractivity of the discrete hematopoiesis model with variable coefficients*, Nonlinear Analysis, 2007, 67(10), 2955–2965.

[5] J. Carr, *Application of Center Manifold Theorem*, Springer-Verlag, New York, 1981.

[6] X. H. Ding, D. J. Fan and M. Z. Liu, *Stability and bifurcation of a numerical discretization Mackey-Glass system*, Chaos, Solitons and Fractals, 2007, 34(2), 383–393.

[7] K. Gopalsamy, M. R. S. Kulenovic and G. Ladas, *Oscillation and global attractivity in models of hematopoiesis*, J. Dyn. Diff. Eqns., 1990, 2(2), 117–132.

[8] K. Gopalsamy, S. I. Trofimchuk and N. R. Bantsur, *A note on global attractivity in models of hematopoiesis*, Ukrainian Mathematical Journal, 1998, 50(1), 5–12.

[9] J. Hale and N. Sternberg, *Onset of chaos in differential delay equations*,J. Comput. Phys., 1988, 77(1), 221–239.

[10] I. Kubiaczyk and S. H. Saker, *Oscillation and Stability in Nonlinear Delay Differential Equations of Population Dynamics*, Mathematical and Computer Modelling, 2002, 35(3), 295–301.

[11] T. Krisztin and E. Liz, *Bubbles for a class of delay differential equations*, Qual. Theory Dyn. Syst., 2011, 10(2), 169–196.

[12] K. Kanno and A. Uchida, *Finite-time Lyapunov exponents in time-delayed nonlinear dynamical systems*, Phys. Rev. E, 2014, 89(3), 032918.

[13] Y. A. Kuzenetsov, *Elements of Applied Bifurcation Theory*, 2nd Ed., Springer-Verlag, New York, 1998.

[14] L. Li, *Bifurcation and chaos in a discrete physiological control system*, Applied Mathematics and Computation, 2015, 252(252), 397–404.

[15] M. C. Mackey and L. Glass, *Oscillation and chaos in physiological control systems*, Science, 1977, 197(4300), 287–289.

[16] A. Namajunas, K. Pyragas and A. Tamasevicius, *Stabilization of an unstable steady state in a Mackey-Glass system*, Physics Letters A, 1995, 204(3–4), 255–262.

[17] C. Qian, *Global attractivity of periodic solutions in a higher order difference equation*, Appl. Math. Lett., 2013, 26(5), 578–583.

[18] G. Rost, *On the global attractivity controversy for a delay model of hematopoiesis*, Appl. Math. Comput., 2007, 190(1), 846–850.

[19] C. Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos*, 2nd Ed., Boca Raton, London, New York, 1999.

[20] H. Su and X. Ding, *Dynamics of a nonstandard finite-difference scheme for Mackey-Glass system*, J. Math. Anal. Appl., 2008, 344(2), 932–941.

[21] H. Su, X. Ding and W. Li, *Numerical bifurcation control of Mackey-Glass system*, Appl. Math. Model, 2011, 35(7), 3460–3472.

[22] X. Wang and Z. Li, *Globally dynamical behaviors for a class of nonlinear functional difference equation with almost periodic coefficients*, Appl. Math.Comput., 2007, 190(2), 1116–1124.

[23] C. Wang and X.Y. Li, *Stability and Neimark-Sacker bifurcation of a semi-discrete population model*, Journal of Applied Analysis and Computation, 2014, 4(4), 419–435.

[24] C. Wang and X.Y. Li. *Further investigations into the stability and bifurcation of a discrete predator-prey model*, J. Math. Anal. Appl., 2015, 422(2), 920–939.

[25] S. Winggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Springer-Verlag, New York, 2003.