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FINITE ELEMENT ALGORITHM BASED ON HIGH-ORDER TIME APPROXIMATION FOR TIME FRACTIONAL CONVECTION-DIFFUSION EQUATION*

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Abstract In this paper, finite element method with high-order approximation for time fractional derivative is considered and discussed to find the numerical solution of time fractional convection-diffusion equation. Some lemmas are introduced and proved, further the stability and error estimates are discussed and analyzed, respectively. The convergence result $O(h^{r+1} + \tau^{3-\alpha})$ can be derived, which illustrates that time convergence rate is higher than the order $(2-\alpha)$ derived by $L1$-approximation. Finally, to validate our theoretical results, some computing data are provided.

Keywords Time fractional convection-diffusion equation, high-order approximation, finite element method, error estimates.

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1. Introduction

The mathematical models on fractional differential equations [44] have been concerned by many researchers in several fields of sciences and engineering, which include fractional diffusion equations [3,9,15–21,23,24,31,33,36,41,43,45–47], fractional water wave model [37], fractional Cable equations [25,40], fractional wave equations [11–13,29,32,42], fractional Maxwell equations [26], fractional fourth-order model [6,19] and so on. In this article, we consider the following time fractional convection-diffusion equation with initial and boundary conditions

$$\begin{cases} D_{0,t}^{\alpha}u(x,t) = \dfrac{\partial^2 u(x,t)}{\partial x^2} - \dfrac{\partial u(x,t)}{\partial x} + f(x,t), (x,t) \in \Omega \times J, \\ u(x_L,t) = u(x_R,t) = 0, t \in \bar{J}, \\ u(x,0) = u_0(x), x \in \Omega. \end{cases} \tag{1.1}$$

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In equation (1.1), $\Omega = [x_L, x_R]$ is spatial domain, $J = (0, T]$ is the time interval with $0 < T < \infty$. $u_0(x)$ and $f(x,t)$ are given functions, $D^\alpha_{0,t}u(x,t)$ is Caputo fractional derivative operator defined by

$$D^\alpha_{0,t}u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\tau)}{\partial \tau} \frac{d\tau}{(t-\tau)^\alpha}, \tag{1.2}$$

where $0 < \alpha < 1$ and $\Gamma(.)$ is Gamma function.

Fractional convection-diffusion equations, which can be derived from dozens of practical problems, such as global weather production, oil reservoir simulations and transport of mass and energy and so forth, have been analytically and numerically solved by some researchers. In [1], Atangana and Kilicman solved space-time fractional advection dispersion equation by analytical methods. In [4], Cui solved numerically a fractional convection-diffusion equation by developing a high-order compact exponential scheme. In [30], Meerschaert and Tadjeran solved numerically spatial fractional advection-dispersion equations by finite difference methods. In [10], Gao and Sun solved the advection-diffusion equations by presenting a compact difference scheme. In [3], Wang and Wang solved fractional advection-diffusion problems by applying a fast characteristic difference algorithm. In [48], Zhang et al. considered a temporal variable fractional mobile-immobile advection-dispersion equation. In [7], Ervin and Roop developed the variational formulation for the stationary advection dispersion equation with fractional derivative. In [34], Shen et al. solved the variable-order fractional advection-diffusion problem by considering a characteristic difference scheme. In [27], Liu et al. studied finite difference method to solve space-time fractional diffusion equation with advection term. In [35], Su et al. developed a characteristic difference algorithm to find the numerical solution for two-sided space fractional convection-diffusion problems. Zheng et al [49] considered fractional advection diffusion equation in space by finite element method. In [14], Hejazi et al. solved space advection-dispersion problems by finite volume method. In [5], Chen and Deng arrived at the numerical solution of 2D convection diffusion equation covering two-sided spatial fractional derivatives. In [50], Zhao et al. found finite element solution of space fractional advection-dispersion equations in 2D. In [2], Bhrawy and Baleanu discussed a spectral collocation method to get numerical solution of spatial fractional advection diffusion equations. In [38], Wang and Wang considered a high-order compact scheme combined with an ADI algorithm for 2D fractional convection-subdiffusion equations in time. In [22], Li et al. developed some high-order numerical algorithm for solving advection-diffusion equations with Caputo-type fractional derivatives. Feng et al. [8] presented some high-order numerical algorithms for advection-dispersion equations with Riesz space fractional derivative. In [39], Wang et al. developed a mixed finite element scheme with second-order approximation in time for nonlinear time fractional convection diffusion problem. In [51], Zhuang et al. looked for the numerical solutions of nonlinear variable-order fractional advection diffusion by some numerical methods.

In addition to the above mentioned numerical methods for fractional differential equations, there are a lot of other numerical methods. Recently, Li et al. [28] established a high-order approximation formula to Caputo fractional derivative, and gave an application based on finite difference method. However, finite element algorithm based on the high-order time approximation [28] for time fractional convection-diffusion equation has not been reported so far.

The purpose of our article is to investigate finite element algorithm combined with the high-order time approximation for time fractional convection-diffusion equation. As is known to us all, the convergence result with $O(\tau^{2-\alpha}+h^{r+1})$ can be derived by combining $L1$-approximation with finite difference/element method. Here, the convergence result with $O(\tau^{3-\alpha}+h^{r+1})$ is obtained by applying the new high-order time approximation for time fractional derivative with finite element method. In this article, some important lemmas on the boundedness of coefficients are derived to arrive at the numerical theories based on finite element method. The stability of finite element scheme with high-order approximation formula is proved and some error estimates are derived. Furthermore, some numerical results are shown to verify our theoretical analysis.

The structure of this paper is as follows: In Section 2, the finite element discrete scheme based on high-order approximation is formulated. The stability based on the fully discrete scheme, and some important lemmas are proved in Section 3. In Section 4, the error estimates of our scheme are proved. In Section 5, the numerical example is given to test the correctness of theoretical results. In Section 6, some conclusions are given.

2. Discrete scheme

For the discretization for time-fractional derivative, let $0=t_0<t_1<t_2<\cdots<t_M=T$ be a given partition of the time interval $[0,T]$ with step length $\tau=T/M$ and nodes $t_k=k\tau$, in which M is some positive integer. At $t=t_j$, we write $u(t_j)$ as u^j.

Lemma 2.1. *Based on the discussion in [28], the time fractional derivative (1.2) at $t=t_k$, for any $\alpha\in(0,1)$, can be written by*

$$\begin{aligned} D_{0,t}^{\alpha}u(x,t_k)=&\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{k-1}(w_{1,k-j}(u^{j+1}-u^{j-1})\\ &+w_{2,k-j}(u^{j+1}-2u^{j}+u^{j-1}))+r^{k}, \end{aligned}\tag{2.1}$$

where

$$\begin{aligned} w_{1,k-j}&=\frac{2-\alpha}{2}[(k-j)^{1-\alpha}-(k-j-1)^{1-\alpha}],\\ w_{2,k-j}&=(k-j)^{2-\alpha}-(k-j-1)^{2-\alpha}-(2-\alpha)(k-j-1)^{1-\alpha}, \end{aligned}\tag{2.2}$$

$j=0,1,...,k-1,k=1,2,...,M$, and r^k is the truncation error in the following form

$$r^k=\frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^{k-1}\int_{t_j}^{t_{j+1}}(t_k-s)^{-\alpha}[-c_u\tau^2+3c_u(s-t_j)^2]ds+O(\tau^3),\tag{2.3}$$

where $c_u=\frac{u^{(3)}(t_j)}{3!}$ is a constant relying only on u.

Lemma 2.2. *From Ref. [28], we know that the truncation error r^k can be bounded by*

$$|r^k|\le C\tau^{3-\alpha}.\tag{2.4}$$

According to Lemma 2.1, we can get the following semi-discrete scheme

$$\begin{aligned}(\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{k-1}(w_{1,k-j}(u^{j+1}-u^{j-1})+w_{2,k-j}(u^{j+1}-2u^{j}+u^{j-1})),v)\\ +(u_x^k,v_x)-(u^k,v_x)=(f^k,v)+(r^k,v),v\in H_0^1.\end{aligned} \tag{2.5}$$

Then, we obtain fully discrete scheme for $V_h\subset H_0^1$

$$\begin{aligned}(\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{k-1}(w_{1,k-j}(u_h^{j+1}-u_h^{j-1})+w_{2,k-j}(u_h^{j+1}-2u_h^{j}+u_h^{j-1})),v_h)\\ +(u_{hx}^k,v_{hx})-(u_h^k,v_{hx})=(f^k,v_h),v_h\in V_h.\end{aligned} \tag{2.6}$$

Remark 2.1. In (2.5), when $j=0$, we find $u^{j-1}=u^{-1}$ is defined outside of $[0,T]$. Here, we use u^0 to approximate u^{-1}, that is, $u^{-1}=u(0)-\tau u'(0)+\frac{\tau^2}{2}u''(0)+O(\tau^3)$. When $u'=0$ and $u''=0$, then $u^{-1}=u^0+O(\tau^3)$. In this article, we only study the case of $u'=0$ and $u''=0$. For the detailed contents, please see the discussion in Ref. [28].

3. Stability for fully discrete scheme

For analyzing the stability, we need to introduce some lemmas and derive some conclusions.

Lemma 3.1 ([28]). *The coefficients $w_{1,k-j}$ and $w_{2,k-j}(k=1,2,...,N,j=0,1,...,k-1)$ defined by (2.2) for $\alpha\in(0,1)$ satisfy the following properties*

$$(1)\ w_{1,1}=\frac{2-\alpha}{2},w_{2,1}=1, \tag{3.1}$$

$$(2)\ 0<w_{1,k-j+1}<w_{1,k-j}\le\frac{2-\alpha}{2}<1,0<w_{2,k-j+1}<w_{2,k-j}\le 1, \tag{3.2}$$

$$(3)\ 2w_{2,1}-w_{1,2}-w_{2,2}>0, \tag{3.3}$$

$$(4)\ w_{1,3}-w_{1,1}+w_{2,3}-2w_{2,2}+w_{2,1}\begin{cases}<0, if\ \ \alpha\in(0,\alpha_1),\\ \ge 0, if\ \ \alpha\in[\alpha_1,1),\end{cases} \tag{3.4}$$

$$w_{1,k-j+1}-w_{1,k-j-1}+w_{2,k-j+1}-2w_{2,k-j}+w_{2,k-j-1}<0,k-j\ge 3. \tag{3.5}$$

where $\alpha_1\approx 0.37$.

Lemma 3.2. *For the coefficients $w_{1,k-j}$ and $w_{2,k-j}(k=1,2,...N,j=0,1,...,k-1)$ defined by (2.2), we can gain the bounded property for $\alpha\in(0,1)$ as follow*

$$\begin{aligned}&\sum_{j=1}^{k-2}\Big|\frac{w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j}}{w_{1,1}+w_{2,1}}\Big|\\ &+\frac{2w_{2,1}-w_{1,2}-w_{2,2}}{w_{1,1}+w_{2,1}}\le\frac{16}{4-\alpha}.\end{aligned} \tag{3.6}$$

Proof. Dividing $\sum_{j=1}^{k-2}\left|\frac{w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j}}{w_{1,1}+w_{2,1}}\right|$ into two parts by (3.4) and (3.5), we obtain

$$\begin{aligned}
&\sum_{j=1}^{k-2}\left|\frac{w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j}}{w_{1,1}+w_{2,1}}\right| \\
&+\frac{2w_{2,1}-w_{1,2}-w_{2,2}}{w_{1,1}+w_{2,1}} \\
=&\sum_{j=1}^{k-3}\frac{-w_{1,k-j+1}-w_{2,k-j+1}-w_{2,k-j-1}+w_{1,k-j-1}+2w_{2,k-j}}{w_{1,1}+w_{2,1}} \\
&+\frac{|w_{1,3}+w_{2,3}+w_{2,1}-w_{1,1}-2w_{2,2}|}{w_{1,1}+w_{2,1}}+\frac{2w_{2,1}-w_{1,2}-w_{2,2}}{w_{1,1}+w_{2,1}}.
\end{aligned} \tag{3.7}$$

After calculating (3.7), we get immediately

$$\begin{aligned}
&\sum_{j=1}^{k-2}\left|\frac{w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j}}{w_{1,1}+w_{2,1}}\right| \\
&+\frac{2w_{2,1}-w_{1,2}-w_{2,2}}{w_{1,1}+w_{2,1}} \\
=&\frac{w_{1,3}+w_{1,2}-w_{1,k}-w_{1,k-1}+w_{2,k-1}-w_{2,k}+w_{2,3}-w_{2,2}}{w_{1,1}+w_{2,1}} \\
&+\frac{2w_{2,1}-w_{1,2}-w_{2,2}}{w_{1,1}+w_{2,1}}+\frac{|w_{1,3}+w_{2,3}+w_{2,1}-w_{1,1}-2w_{2,2}|}{w_{1,1}+w_{2,1}} \\
=&\frac{w_{1,3}+w_{2,k-1}+w_{2,3}+2w_{2,1}-w_{1,k}-w_{1,k-1}-w_{2,k}-2w_{2,2}}{w_{1,1}+w_{2,1}} \\
&+\frac{|w_{1,3}+w_{2,3}+w_{2,1}-w_{1,1}-2w_{2,2}|}{w_{1,1}+w_{2,1}},
\end{aligned} \tag{3.8}$$

which combines with (3.2) to arrive at the resulting inequality. □

Theorem 3.1. *The following stability holds*

$$\begin{aligned}
\|u_h^k\| \leq & C\Big(\frac{2\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})-\Gamma(3-\alpha)}\|f^k\|_0 \\
&+\left|\frac{w_{2,k}-w_{1,k}}{w_{1,1}+w_{2,1}}\Big(1+\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})-\Gamma(3-\alpha)}\Big)\right|\|u_h^{-1}\|_0 \\
&+\left|\frac{w_{2,k-1}-w_{1,k-1}-2w_{2,k}}{w_{1,1}+w_{2,1}}\Big(1+\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})-\Gamma(3-\alpha)}\Big)\right|\|u_h^0\|_0\Big).
\end{aligned} \tag{3.9}$$

Proof. We rewrite (2.6) as the following equation

$$\begin{aligned}
&\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}(w_{1,1}+w_{2,1})(u_h^k,v_h)+\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}(w_{1,2}+w_{2,2}-2w_{2,1})(u_h^{k-1},v_h) \\
+&\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=1}^{k-2}(w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j})(u_h^j,v_h)
\end{aligned}$$

$$+\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}(w_{2,k-1}-w_{1,k-1}-2w_{2,k})(u_h^0,v_h)+\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}(w_{2,k}-w_{1,k})(u_h^{-1},v_h)$$
$$+(u_{hx}^k,v_{hx})-(u_h^k,v_{hx})=(f^k,v_h). \tag{3.10}$$

Setting $v_h=u_h^k$ in (3.10), and using Cauchy inequality, we have

$$\begin{aligned}
&(u_h^k,u_h^k)+\frac{\Gamma(3-\alpha)}{\tau^{-\alpha}(w_{1,1}+w_{2,1})}(u_{hx}^k,u_{hx}^k)=\|u_h^k\|_0^2+\frac{\Gamma(3-\alpha)}{\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|u_{hx}^k\|_0^2\\
\le&\frac{\Gamma(3-\alpha)}{\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|f^k\|_0\|u_h^k\|_0+\left|\frac{w_{2,k}-w_{1,k}}{w_{1,1}+w_{2,1}}\right|\|u_h^{-1}\|_0\|u_h^k\|_0\\
&+\left|\frac{w_{2,k-1}-w_{1,k-1}-2w_{2,k}}{w_{1,1}+w_{2,1}}\right|\|u_h^0\|_0\|u_h^k\|_0+\frac{2w_{2,1}-w_{1,2}-w_{2,2}}{w_{1,1}+w_{2,1}}\|u_h^{k-1}\|_0\|u_h^k\|_0\\
&+\sum_{j=1}^{k-2}\left|\frac{w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j}}{w_{1,1}+w_{2,1}}\right|\|u_h^j\|_0\|u_h^k\|_0\\
&+\frac{\Gamma(3-\alpha)}{\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|u_{hx}^k\|_0\|u_h^k\|_0.
\end{aligned}\tag{3.11}$$

Then, we use Young inequality to obtain

$$\begin{aligned}
&\|u_h^k\|_0^2+\frac{\Gamma(3-\alpha)}{\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|u_{hx}^k\|_0^2\\
\le&\frac{\Gamma(3-\alpha)}{\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|f^k\|_0\|u_h^k\|_0+\left|\frac{w_{2,k}-w_{1,k}}{w_{1,1}+w_{2,1}}\right|\|u_h^{-1}\|_0\|u_h^k\|_0\\
&+\left|\frac{w_{2,k-1}-w_{1,k-1}-2w_{2,k}}{w_{1,1}+w_{2,1}}\right|\|u_h^0\|_0\|u_h^k\|_0+\frac{2w_{2,1}-w_{1,2}-w_{2,2}}{w_{1,1}+w_{2,1}}\|u_h^{k-1}\|_0\|u_h^k\|_0\\
&+\sum_{j=1}^{k-2}\left|\frac{w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j}}{w_{1,1}+w_{2,1}}\right|\|u_h^j\|_0\|u_h^k\|_0\\
&+\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|u_{hx}^k\|_0^2+\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|u_h^k\|_0^2.
\end{aligned}\tag{3.12}$$

Simplifying (3.12), we have

$$\begin{aligned}
&(1-\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})})\|u_h^k\|_0^2+\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|u_{hx}^k\|_0^2\\
\le&\frac{\Gamma(3-\alpha)}{\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|f^k\|_0\|u_h^k\|_0+\left|\frac{w_{2,k}-w_{1,k}}{w_{1,1}+w_{2,1}}\right|\|u_h^{-1}\|_0\|u_h^k\|_0\\
&+\left|\frac{w_{2,k-1}-w_{1,k-1}-2w_{2,k}}{w_{1,1}+w_{2,1}}\right|\|u_h^0\|_0\|u_h^k\|_0+\frac{2w_{2,1}-w_{1,2}-w_{2,2}}{w_{1,1}+w_{2,1}}\|u_h^{k-1}\|_0\|u_h^k\|_0\\
&+\sum_{j=1}^{k-2}\left|\frac{w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j}}{w_{1,1}+w_{2,1}}\right|\|u_h^j\|_0\|u_h^k\|_0.
\end{aligned}\tag{3.13}$$

Multiplying both sides by $\frac{2\tau^{-\alpha}(w_{1,1}+w_{2,1})}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})-\Gamma(3-\alpha)}$, we have

$$\|u_h^k\|_0^2\le\frac{2\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})-\Gamma(3-\alpha)}\|f^k\|_0\|u_h^k\|_0$$

$$
\begin{aligned}
&+\Big|\frac{w_{2,k}-w_{1,k}}{w_{1,1}+w_{2,1}}(1+\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})-\Gamma(3-\alpha)})\Big|\|u_h^{-1}\|_0\|u_h^k\|_0\\
&+\Big|\frac{w_{2,k-1}-w_{1,k-1}-2w_{2,k}}{w_{1,1}+w_{2,1}}(1+\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})-\Gamma(3-\alpha)})\Big|\|u_h^0\|_0\|u_h^k\|_0\\
&+\frac{2w_{2,1}-w_{1,2}-w_{2,2}}{w_{1,1}+w_{2,1}}(1+\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})-\Gamma(3-\alpha)})\|u_h^{k-1}\|_0\|u_h^k\|_0\\
&+\sum_{j=1}^{k-2}\Big|\frac{w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j}}{w_{1,1}+w_{2,1}}\\
&\times(1+\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})-\Gamma(3-\alpha)})\Big|\|u_h^j\|_0\|u_h^k\|_0.
\end{aligned}
\tag{3.14}
$$

Make best use of Gronwall inequality to have

$$
\begin{aligned}
\|u_h^k\|_0\le&(\frac{2\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})-\Gamma(3-\alpha)}\|f^k\|_0\\
&+\Big|\frac{w_{2,k}-w_{1,k}}{w_{1,1}+w_{2,1}}(1+\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})-\Gamma(3-\alpha)})\Big|\|u_h^{-1}\|_0\\
&+\Big|\frac{w_{2,k-1}-w_{1,k-1}-2w_{2,k}}{w_{1,1}+w_{2,1}}(1+\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})-\Gamma(3-\alpha)})\Big|\|u_h^0\|_0)\\
&\times\exp(\sum_{j=1}^{k-2}\Big|\frac{w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j}}{w_{1,1}+w_{2,1}}\\
&\times(1+\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})-\Gamma(3-\alpha)})\Big|\\
&+(1+\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})-\Gamma(3-\alpha)})\frac{2w_{2,1}-w_{1,2}-w_{2,2}}{w_{1,1}+w_{2,1}}).
\end{aligned}
\tag{3.15}
$$

According to the lemma 3.2, we get

$$
\begin{aligned}
\|u_h^k\|_0\le&(\frac{2\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})-\Gamma(3-\alpha)}\|f^k\|_0\\
&+\Big|\frac{w_{2,k}-w_{1,k}}{w_{1,1}+w_{2,1}}(1+\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})-\Gamma(3-\alpha)})\Big|\|u_h^{-1}\|_0\\
&+\Big|\frac{w_{2,k-1}-w_{1,k-1}-2w_{2,k}}{w_{1,1}+w_{2,1}}(1+\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})-\Gamma(3-\alpha)})\Big|\|u_h^0\|_0)\\
&\exp(\frac{32}{4-\alpha}(1+\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})-\Gamma(3-\alpha)})),
\end{aligned}
\tag{3.16}
$$

which indicates the conclusion holds. □

4. Error estimates for fully discrete scheme

Lemma 4.1. *To derive the error estimates, we define an elliptic projection $P_hu\in V_h\subset H_0^1$ of u as the solution of*

$$
a(u-P_hu,v_h)=0,v_h\ \in V_h,
\tag{4.1}
$$

where $a(u,v) = (u_x, v_x)$, which leads to the following estimates

$$\|P_h u - u\|_0 + h\|P_h u_x - u_x\|_0 \le Ch^{r+1}\|u\|_{r+1}. \tag{4.2}$$

Lemma 4.2. *Based on the definition of $w_{1,k-j}$ and $w_{2,k-j}$ in (2.2), we arrive at the bounded property as follow*

$$\begin{aligned}&\Big|\frac{w_{2,k}-w_{1,k}}{28(w_{1,1}+w_{2,1})}\Big| + \Big|\frac{w_{2,k-1}-w_{1,k-1}-2w_{2,k}}{28(w_{1,1}+w_{2,1})}\Big| + \Big|\frac{w_{1,2}+w_{2,2}-2w_{2,1}}{28(w_{1,1}+w_{2,1})}\Big|\\&+\sum_{j=1}^{k-2}\Big|\frac{w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j}}{28(w_{1,1}+w_{2,1})}\Big| \le \frac{1}{4-\alpha}.\end{aligned} \tag{4.3}$$

Proof. Noticing the lemma 3.2, we have

$$\begin{aligned}&\Big|\frac{w_{2,k}-w_{1,k}}{28(w_{1,1}+w_{2,1})}\Big| + \Big|\frac{w_{2,k-1}-w_{1,k-1}-2w_{2,k}}{28(w_{1,1}+w_{2,1})}\Big| + \Big|\frac{w_{1,2}+w_{2,2}-2w_{2,1}}{28(w_{1,1}+w_{2,1})}\Big|\\&+\sum_{j=1}^{k-2}\Big|\frac{w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j}}{28(w_{1,1}+w_{2,1})}\Big|\\ \le&\Big|\frac{w_{2,k}-w_{1,k}}{28(w_{1,1}+w_{2,1})}\Big| + \Big|\frac{w_{2,k-1}-w_{1,k-1}-2w_{2,k}}{28(w_{1,1}+w_{2,1})}\Big| + \frac{4}{7(4-\alpha)}.\end{aligned} \tag{4.4}$$

According to (3.2), we get the following formula

$$\begin{aligned}&\Big|\frac{w_{2,k}-w_{1,k}}{28(w_{1,1}+w_{2,1})}\Big| + \Big|\frac{w_{2,k-1}-w_{1,k-1}-2w_{2,k}}{28(w_{1,1}+w_{2,1})}\Big| + \Big|\frac{w_{1,2}+w_{2,2}-2w_{2,1}}{28(w_{1,1}+w_{2,1})}\Big|\\&+\sum_{j=1}^{k-2}\Big|\frac{w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j}}{28(w_{1,1}+w_{2,1})}\Big|\\ \le&\frac{3}{28(w_{1,1}+w_{2,1})} + \frac{4}{7(4-\alpha)}\\ \le&\frac{1}{4-\alpha},\end{aligned} \tag{4.5}$$

which tells the resulting inequality holds. □

Theorem 4.1. *For semi-discrete solution u^n in (2.5) and finite element solution u_h^n in (2.6) with $P_h u_0 = u_h^0$, there exists a constant $C > 0$ which does not rely on space-time parameters h and τ such that*

$$\|u^n - u_h^n\|_0 \le C(h^{r+1} + \tau^{3-\alpha}). \tag{4.6}$$

Proof. Subtract (2.6) from (2.5) to get the following error equation

$$\begin{aligned}&\Big(\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{k-1}(w_{1,k-j}(\theta^{j+1}-\theta^{j-1}) + w_{2,k-j}(\theta^{j+1}-2\theta^j+\theta^{j-1})), v_h\Big)\\&+(u_x^k, v_{hx}) - (P_h u_x^k, v_{hx}) + (\theta_x^k, v_{hx}) - (\rho^k, v_{hx}) - (\theta^k, v_{hx})\\ =&-\Big(\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{k-1}(w_{1,k-j}(\rho^{j+1}-\rho^{j-1})\\&+w_{2,k-j}(\rho^{j+1}-2\rho^j+\rho^{j-1})), v_h\Big) + (r^k, v_h), v_h \in V_h,\end{aligned} \tag{4.7}$$

where $u - u_h = (u - P_h u) + (P_h u - u_h) = \rho + \theta$ and $P_h u(0) = u_h(0)$.
Applying to the projection lemma 4.1, we get

$$
\begin{aligned}
&(\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{k-1}(w_{1,k-j}(\theta^{j+1}-\theta^{j-1}) + w_{2,k-j}(\theta^{j+1}-2\theta^{j}+\theta^{j-1})), v_h) \\
&+ (\theta_x^k, v_{hx}) - (\theta^k, v_{hx}) = -(\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{k-1}(w_{1,k-j}(\rho^{j+1}-\rho^{j-1}) \\
&+ w_{2,k-j}(\rho^{j+1}-2\rho^{j}+\rho^{j-1})), v_h) + (\rho^k, v_{hx}) + (r^k, v_h), v_h \in V_h. \qquad (4.8)
\end{aligned}
$$

For the need of estimate, we rewrite (4.8) as

$$
\begin{aligned}
&\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}(w_{1,1}+w_{2,1})(\theta^k, v_h) + \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}(w_{1,2}+w_{2,2}-2w_{2,1})(\theta^{k-1}, v_h) \\
&+ \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=1}^{k-2}(w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j})(\theta^j, v_h) \\
&+ \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}(w_{2,k-1}+w_{1,k-1}-2w_{2,k})(\theta^0, v_h) + \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}(w_{2,k}+w_{1,k})(\theta^{-1}, v_h) \\
&+ (\theta_x^k, v_{hx}) - (\theta^k, v_{hx}) \\
=&(r^k, v_h) - \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}(w_{1,1}+w_{2,1})(\rho^k, v_h) \\
&- \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}(w_{1,2}+w_{2,2}-2w_{2,1})(\rho^{k-1}, v_h) \\
&- \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=1}^{k-2}(w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j})(\rho^j, v_h) \\
&- \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}(w_{2,k-1}-w_{1,k-1}-2w_{2,k})(\rho^0, v_h) \\
&- \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}(w_{2,k}-w_{1,k})(\rho^{-1}, v_h) + (\rho^k, v_{hx}). \qquad (4.9)
\end{aligned}
$$

Setting $v_h = \theta^k$ in (4.9), and applying Cauchy inequality, we get

$$
\begin{aligned}
&\|\theta^k\|_0^2 + \frac{\Gamma(3-\alpha)}{\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|\theta_x^k\|_0^2 \\
\leq&\left|\frac{w_{2,k}-w_{1,k}}{w_{1,1}+w_{2,1}}\right|\|\theta^{-1}\|_0\|\theta^k\|_0 + \left|\frac{w_{2,k-1}-w_{1,k-1}-2w_{2,k}}{w_{1,1}+w_{2,1}}\right|\|\theta^0\|_0\|\theta^k\|_0 \\
&+ \sum_{j=1}^{k-2}\left|\frac{w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j}}{w_{1,1}+w_{2,1}}\right|\|\theta^j\|_0\|\theta^k\|_0 \\
&+ \left|\frac{w_{1,2}+w_{2,2}-2w_{2,1}}{w_{1,1}+w_{2,1}}\right|\|\theta^{k-1}\|_0\|\theta^k\|_0 \\
&+ \frac{\Gamma(3-\alpha)}{\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|\theta^k\|_0\|\theta_x^k\|_0 + \left|\frac{w_{2,k}-w_{1,k}}{w_{1,1}+w_{2,1}}\right|\|\rho^{-1}\|_0\|\theta^k\|_0 \\
&+ \left|\frac{w_{2,k-1}-w_{1,k-1}-2w_{2,k}}{w_{1,1}+w_{2,1}}\right|\|\rho^0\|_0\|\theta^k\|_0
\end{aligned}
$$

$$+\sum_{j=1}^{k-2}\Big|\frac{w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j}}{w_{1,1}+w_{2,1}}\Big|\|\rho^j\|_0\|\theta^k\|_0$$
$$+\Big|\frac{w_{1,2}+w_{2,2}-2w_{2,1}}{w_{1,1}+w_{2,1}}\Big|\|\rho^{k-1}\|_0\|\theta^k\|_0+\|\rho^k\|_0\|\theta^k\|_0$$
$$+\frac{\Gamma(3-\alpha)}{\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|\rho^k\|_0\|\theta^k_x\|_0+\frac{\Gamma(3-\alpha)}{\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|r^k\|_0\|\theta^k\|_0. \tag{4.10}$$

We make use of Young inequality to obtain

$$\|\theta^k\|_0^2+\frac{\Gamma(3-\alpha)}{\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|\theta^k_x\|_0^2$$
$$\le\Big|\frac{14(w_{2,k}-w_{1,k})}{w_{1,1}+w_{2,1}}\Big|\|\theta^{-1}\|_0^2+\Big|\frac{w_{2,k}-w_{1,k}}{56(w_{1,1}+w_{2,1})}\Big|\|\theta^k\|_0^2$$
$$+\Big|\frac{14(w_{2,k-1}-w_{1,k-1}-2w_{2,k})}{w_{1,1}+w_{2,1}}\Big|\|\theta^0\|_0^2+\Big|\frac{w_{2,k-1}-w_{1,k-1}-2w_{2,k}}{56(w_{1,1}+w_{2,1})}\Big|\|\theta^k\|_0^2$$
$$+\sum_{j=1}^{k-2}\Big|\frac{14(w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j})}{w_{1,1}+w_{2,1}}\Big|\|\theta^j\|_0^2$$
$$+\sum_{j=1}^{k-2}\Big|\frac{w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j}}{56(w_{1,1}+w_{2,1})}\Big|\|\theta^k\|_0^2$$
$$+\Big|\frac{14(w_{1,2}+w_{2,2}-2w_{2,1})}{w_{1,1}+w_{2,1}}\Big|\|\theta^{k-1}\|_0^2+\Big|\frac{w_{1,2}+w_{2,2}-2w_{2,1}}{56(w_{1,1}+w_{2,1})}\Big|\|\theta^k\|_0^2$$
$$+\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|\theta^k\|_0^2+\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|\theta^k_x\|_0^2+\Big|\frac{14(w_{2,k}-w_{1,k})}{w_{1,1}+w_{2,1}}\Big|\|\rho^{-1}\|_0^2$$
$$+\Big|\frac{w_{2,k}-w_{1,k}}{56(w_{1,1}+w_{2,1})}\Big|\|\theta^k\|_0^2+\Big|\frac{14(w_{2,k-1}-w_{1,k-1}-2w_{2,k})}{w_{1,1}+w_{2,1}}\Big|\|\rho^0\|_0^2$$
$$+\Big|\frac{w_{2,k-1}-w_{1,k-1}-2w_{2,k}}{56(w_{1,1}+w_{2,1})}\Big|\|\theta^k\|_0^2$$
$$+\sum_{j=1}^{k-2}\Big|\frac{14(w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j})}{w_{1,1}+w_{2,1}}\Big|\|\rho^j\|_0^2$$
$$+\sum_{j=1}^{k-2}\Big|\frac{w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j}}{56(w_{1,1}+w_{2,1})}\Big|\|\theta^k\|_0^2$$
$$+\Big|\frac{14(w_{1,2}+w_{2,2}-2w_{2,1})}{w_{1,1}+w_{2,1}}\Big|\|\rho^{k-1}\|_0^2+\Big|\frac{w_{1,2}+w_{2,2}-2w_{2,1}}{56(w_{1,1}+w_{2,1})}\Big|\|\theta^k\|_0^2+\frac{1}{2}\|\rho^k\|_0^2$$
$$+\frac{1}{2}\|\theta^k\|_0^2+\frac{\Gamma(3-\alpha)}{\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|\rho^k\|_0^2+\frac{\Gamma(3-\alpha)}{4\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|\theta^k_x\|_0^2$$
$$+\frac{\Gamma(3-\alpha)}{\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|r^k\|_0^2+\frac{\Gamma(3-\alpha)}{4\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|\theta^k\|_0^2. \tag{4.11}$$

Applying lemma 4.2 to (4.11), we get

$$\|\theta^k\|_0^2+\frac{\Gamma(3-\alpha)}{4\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|\theta^k_x\|_0^2$$

$$
\begin{aligned}
\leq&(\frac{1}{4-\alpha}+\frac{1}{2}+\frac{3\Gamma(3-\alpha)}{4\tau^{-\alpha}(w_{1,1}+w_{2,1})})\|\theta^k\|_0^2+\Big|\frac{14(w_{2,k}-w_{1,k})}{w_{1,1}+w_{2,1}}\Big|\|\theta^{-1}\|_0^2\\
&+\Big|\frac{14(w_{2,k-1}-w_{1,k-1}-2w_{2,k})}{w_{1,1}+w_{2,1}}\Big|\|\theta^0\|_0^2+\Big|\frac{14(w_{1,2}+w_{2,2}-2w_{2,1})}{w_{1,1}+w_{2,1}}\Big|\|\theta^{k-1}\|_0^2\\
&+\sum_{j=1}^{k-2}\Big|\frac{14(w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j})}{w_{1,1}+w_{2,1}}\Big|\|\theta^j\|_0^2\\
&+\Big|\frac{14(w_{2,k}-w_{1,k})}{w_{1,1}+w_{2,1}}\Big|\|\rho^{-1}\|_0^2+\Big|\frac{14(w_{2,k-1}-w_{1,k-1}-2w_{2,k})}{w_{1,1}+w_{2,1}}\Big|\|\rho^0\|_0^2\\
&+\Big|\frac{14(w_{1,2}+w_{2,2}-2w_{2,1})}{w_{1,1}+w_{2,1}}\Big|\|\rho^{k-1}\|_0^2\\
&+\sum_{j=1}^{k-2}\Big|\frac{14(w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j})}{w_{1,1}+w_{2,1}}\Big|\|\rho^j\|_0^2\\
&+\frac{1}{2}\|\rho^k\|_0^2+\frac{\Gamma(3-\alpha)}{\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|\rho^k\|_0^2+\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|r^k\|_0^2. \qquad (4.12)
\end{aligned}
$$

Rewrite (4.12) to gain

$$
\begin{aligned}
&(\frac{1}{2}-\frac{1}{4-\alpha}-\frac{3\Gamma(3-\alpha)}{4\tau^{-\alpha}(w_{1,1}+w_{2,1})})\|\theta^k\|_0^2+\frac{\Gamma(3-\alpha)}{4\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|\theta_x^k\|_0^2\\
\leq&\Big|\frac{14(w_{2,k}-w_{1,k})}{w_{1,1}+w_{2,1}}\Big|\|\theta^{-1}\|_0^2+\Big|\frac{14(w_{2,k-1}-w_{1,k-1}-2w_{2,k})}{w_{1,1}+w_{2,1}}\Big|\|\theta^0\|_0^2\\
&+\Big|\frac{14(w_{1,2}+w_{2,2}-2w_{2,1})}{w_{1,1}+w_{2,1}}\Big|\|\theta^{k-1}\|_0^2\\
&+\sum_{j=1}^{k-2}\Big|\frac{14(w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j})}{w_{1,1}+w_{2,1}}\Big|\|\theta^j\|_0^2\\
&+\Big|\frac{14(w_{2,k}-w_{1,k})}{w_{1,1}+w_{2,1}}\Big|\|\rho^{-1}\|_0^2+\Big|\frac{14(w_{2,k-1}-w_{1,k-1}-2w_{2,k})}{w_{1,1}+w_{2,1}}\Big|\|\rho^0\|_0^2\\
&+\Big|\frac{14(w_{1,2}+w_{2,2}-2w_{2,1})}{w_{1,1}+w_{2,1}}\Big|\|\rho^{k-1}\|_0^2\\
&+\sum_{j=1}^{k-2}\Big|\frac{14(w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j})}{w_{1,1}+w_{2,1}}\Big|\|\rho^j\|_0^2\\
&+\frac{1}{2}\|\rho^k\|_0^2+\frac{\Gamma(3-\alpha)}{\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|\rho^k\|_0^2+\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|r^k\|_0^2. \qquad (4.13)
\end{aligned}
$$

We use Gronwall inequality to arrive at

$$
\begin{aligned}
&(\frac{1}{2}-\frac{1}{4-\alpha}-\frac{3\Gamma(3-\alpha)}{4\tau^{-\alpha}(w_{1,1}+w_{2,1})})\|\theta^k\|_0^2+\frac{\Gamma(3-\alpha)}{4\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|\theta_x^k\|_0^2\\
\leq&\Big(\Big|\frac{14(w_{2,k}-w_{1,k})}{w_{1,1}+w_{2,1}}\Big|\|\theta^{-1}\|_0^2+\Big|\frac{14(w_{2,k-1}-w_{1,k-1}-2w_{2,k})}{w_{1,1}+w_{2,1}}\Big|\|\theta^0\|_0^2\\
&+\Big|\frac{14(w_{2,k}-w_{1,k})}{w_{1,1}+w_{2,1}}\Big|\|\rho^{-1}\|_0^2+\Big|\frac{14(w_{2,k-1}-w_{1,k-1}-2w_{2,k})}{w_{1,1}+w_{2,1}}\Big|\|\rho^0\|_0^2\\
&+\Big|\frac{14(w_{1,2}+w_{2,2}-2w_{2,1})}{w_{1,1}+w_{2,1}}\Big|\|\rho^{k-1}\|_0^2
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{k-2}\Big|\frac{14(w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j})}{w_{1,1}+w_{2,1}}\Big|\|\rho^j\|_0^2\\
&+\frac{1}{2}\|\rho^k\|_0^2+\frac{\Gamma(3-\alpha)}{\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|\rho^k\|_0^2+\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|r^k\|_0^2\Big)\\
&\times\exp\Big(\Big|\frac{14(w_{1,2}+w_{2,2}-2w_{2,1})}{w_{1,1}+w_{2,1}}\Big|\\
&+\sum_{j=1}^{k-2}\Big|\frac{14(w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j})}{w_{1,1}+w_{2,1}}\Big|\Big)\\
\le&\Big(\Big|\frac{14(w_{2,k}-w_{1,k})}{w_{1,1}+w_{2,1}}\Big|\|\theta^{-1}\|_0^2+\Big|\frac{14(w_{2,k-1}-w_{1,k-1}-2w_{2,k})}{w_{1,1}+w_{2,1}}\Big|\|\theta^0\|_0^2\\
&+\Big|\frac{14(w_{2,k}-w_{1,k})}{w_{1,1}+w_{2,1}}\Big|\|\rho^{-1}\|_0^2+\Big|\frac{14(w_{2,k-1}-w_{1,k-1}-2w_{2,k})}{w_{1,1}+w_{2,1}}\Big|\|\rho^0\|_0^2\\
&+\Big|\frac{14(w_{1,2}+w_{2,2}-2w_{2,1})}{w_{1,1}+w_{2,1}}\Big|\|\rho^{k-1}\|_0^2\\
&+\sum_{j=1}^{k-2}\Big|\frac{14(w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}-w_{1,k-j-1}-2w_{2,k-j})}{w_{1,1}+w_{2,1}}\Big|\|\rho^j\|_0^2\\
&+\frac{1}{2}\|\rho^k\|_0^2+\frac{\Gamma(3-\alpha)}{\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|\rho^k\|_0^2+\frac{\Gamma(3-\alpha)}{2\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|r^k\|_0^2\Big)\exp(\frac{14\times 16}{4-\alpha}).
\end{aligned}
\tag{4.14}
$$

We get the following inequality by using lemma 2.1 and $P_h u(0)=u_h(0)$

$$
\begin{aligned}
&(\frac{1}{2}-\frac{1}{4-\alpha}-\frac{3\Gamma(3-\alpha)}{4\tau^{-\alpha}(w_{1,1}+w_{2,1})})\|\theta^k\|_0^2+\frac{\Gamma(3-\alpha)}{4\tau^{-\alpha}(w_{1,1}+w_{2,1})}\|\theta_x^k\|_0^2\\
\le&\Big(\Big|\frac{14(w_{2,k}-w_{1,k})}{w_{1,1}+w_{2,1}}\Big|+\Big|\frac{14(w_{2,k-1}-w_{1,k-1}-2w_{2,k})}{w_{1,1}+w_{2,1}}\Big|+\Big|\frac{14(w_{1,2}-w_{2,2}-2w_{2,1})}{w_{1,1}+w_{2,1}}\Big|\\
&+\frac{1}{2}+\frac{\Gamma(3-\alpha)}{\tau^{-\alpha}(w_{1,1}+w_{2,1})}\Big)\exp(\frac{226}{4-\alpha})h^{2r+2}\|u\|_{r+1}+C\tau^{6-2\alpha},
\end{aligned}
\tag{4.15}
$$

which is simplified as

$$\|\theta^k\|_0\le C(h^{r+1}+\tau^{3-\alpha}). \tag{4.16}$$

Combine (4.16) with the error estimate for ρ in lemma 4.1, we use triangle inequality to get the conclusion of theorem. □

Remark 4.1. 1) Here, the current approximate scheme can get the order $3-\alpha$ in time, which is higher than the $2-\alpha$ arrived at by $L1$-approximation.
2) When we derive the error estimates of finite element method, for giving the estimate results of θ in norm, we need to use the known error results for the norms of ρ, so we have to introduce some useful lemmas on the coefficients' inequalities, which are not needed in finite difference method.

5. Numerical test

In this section, we consider a numerical example to test our theorem of error analysis. In (1.1), we take the space-time domain $[0,1]\times[0,1]$, the source term $f(x,t)=$

$\frac{\Gamma(6)t^{5-\alpha}}{\Gamma(6-\alpha)}\sin(\pi x)+\pi^2 t^5\sin(\pi x)+\pi t^5\cos(\pi x)$, the initial value $u(x,0)=0$. Then, we easily check that the exact solution is $u(x,t)=t^5\sin(\pi x)$.

In the following contents, we will provide the numerical algorithm to tell ones that the numerical process is how to be implemented and give the calculated data by using Matlab procedure to verify the theoretical results.

5.1. Numerical algorithm

Now, we introduce the process of algorithm based on the basis functions $\{\phi_i(x)\}|_0^n$. In (2.6), letting $u_h^p=\sum\limits_{i=0}^{n}u_i^p\phi_i(x), i=0,1,\cdots,n$, and choosing $v_h=\phi_l(x), l=1,2,\cdots,n-1$, we gain

$$\begin{aligned}
&(\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{k-1}(w_{1,k-j}(\sum_{i=0}^{n}u_i^{j+1}\phi_i(x)-\sum_{i=0}^{n}u_i^{j-1}\phi_i(x))+w_{2,k-j}(\sum_{i=0}^{n}u_i^{j+1}\phi_i(x)\\
&-2\sum_{i=0}^{n}u_i^{j}\phi_i(x)+\sum_{i=0}^{n}u_{j-1}^{k}\phi_i(x)),\phi_l(x))+(\sum_{i=0}^{n}u_i^k\phi_{ix}(x),\phi_{lx}(x))\\
&+(\sum_{i=0}^{n}u_i^k\phi_{ix}(x),\phi_l(x))=(f^k,\phi_l(x)),
\end{aligned}\tag{5.1}$$

where the basis functions $\phi_j(x)$ are taken as follows

$$\phi_j(x)=\begin{cases}\dfrac{x-x_{j-1}}{h}, x\in[x_{j-1},x_j],\\ \dfrac{x_{j+1}-x}{h}, x\in[x_j,x_{j+1}],\ \ j=1,2,\cdots,n-1,\\ 0, \text{elsewhere},\end{cases}$$

$$\phi_0(x)=\begin{cases}\dfrac{x_1-x}{h}, x\in[x_0,x_1],\\ 0, \text{elsewhere},\end{cases}$$

and

$$\phi_n(x)=\begin{cases}\dfrac{x-x_{n-1}}{h}, x\in[x_{n-1},x_n],\\ 0, \text{elsewhere}.\end{cases}$$

By computing for (5.1), we arrive at

$$\begin{aligned}
&\sum_{j=1}^{k-2}\frac{h\tau^{-\alpha}}{6\Gamma(3-\alpha)}(w_{1,k-j+1}+w_{2,k-j+1}+w_{2,k-j-1}\\
&-w_{1,k-j-1}-2w_{2,k-j})(u_{i-1}^j+4u_i^j+u_{i+1}^j)\\
&+\frac{h\tau^{-\alpha}}{6\Gamma(3-\alpha)}(w_{1,2}+w_{2,2}-2w_{2,1})(u_{i-1}^{k-1}+4u_i^{k-1}+u_{i+1}^{k-1})\\
&+[(\frac{h(4-\alpha)\tau^{-\alpha}}{12\Gamma(3-\alpha)}-\frac{2+h}{2h^2})u_{i-1}^k+(\frac{h(4-\alpha)\tau^{-\alpha}}{3\Gamma(3-\alpha)}+\frac{2}{h^2})u_i^k\\
&+(\frac{h(4-\alpha)\tau^{-\alpha}}{12\Gamma(3-\alpha)}-\frac{2-h}{2h^2})u_{i+1}^k]=(f^k,\phi_l(x)).
\end{aligned}\tag{5.2}$$

Now, we rewrite (5.2) into matrix equation to obtain

$$\mathbf{C}\mathbf{u}^k = \mathbf{g}^k - \sum_{j=1}^{k-2} \mathbf{A}\mathbf{u}^j - \mathbf{B}\mathbf{u}^{k-1}, \tag{5.3}$$

in which $\mathbf{u}^p = (u_1^p, u_2^p, ..., u_{n-1}^p)^T$,

$$\mathbf{A} = \begin{pmatrix} \frac{2h\tau^{-\alpha}}{3\Gamma(3-\alpha)}a_j & \frac{h\tau^{-\alpha}}{6\Gamma(3-\alpha)}a_j & & & 0 \\ \frac{h\tau^{-\alpha}}{6\Gamma(3-\alpha)}a_j & \frac{2h\tau^{-\alpha}}{3\Gamma(3-\alpha)}a_j & \frac{h\tau^{-\alpha}}{6\Gamma(3-\alpha)}a_j & & \\ & \frac{h\tau^{-\alpha}}{6\Gamma(3-\alpha)}a_j & \frac{2h\tau^{-\alpha}}{3\Gamma(3-\alpha)}a_j & \ddots & \\ & & \ddots & \ddots & \frac{h\tau^{-\alpha}}{6\Gamma(3-\alpha)}a_j \\ 0 & & & \frac{h\tau^{-\alpha}}{6\Gamma(3-\alpha)}a_j & \frac{2h\tau^{-\alpha}}{3\Gamma(3-\alpha)}a_j \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} \frac{2h\tau^{-\alpha}}{3\Gamma(3-\alpha)}b & \frac{h\tau^{-\alpha}}{6\Gamma(3-\alpha)}b & & & 0 \\ \frac{h\tau^{-\alpha}}{6\Gamma(3-\alpha)}b & \frac{2h\tau^{-\alpha}}{3\Gamma(3-\alpha)}b & \frac{h\tau^{-\alpha}}{6\Gamma(3-\alpha)}b & & \\ & \frac{h\tau^{-\alpha}}{6\Gamma(3-\alpha)}b & \frac{2h\tau^{-\alpha}}{3\Gamma(3-\alpha)}b & \ddots & \\ & & \ddots & \ddots & \frac{h\tau^{-\alpha}}{6\Gamma(3-\alpha)}b \\ 0 & & & \frac{h\tau^{-\alpha}}{6\Gamma(3-\alpha)}b & \frac{2h\tau^{-\alpha}}{3\Gamma(3-\alpha)}b \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} 4d + \frac{2}{h^2} & d - \frac{2-h}{2h^2} & & & 0 \\ d - \frac{2+h}{2h^2} & 4d + \frac{2}{h^2} & d - \frac{2-h}{2h^2} & & \\ & d - \frac{2+h}{2h^2} & 4d + \frac{2}{h^2} & \ddots & \\ & & \ddots & \ddots & d - \frac{2-h}{2h^2} \\ 0 & & & d - \frac{2+h}{2h^2} & 4d + \frac{2}{h^2} \end{pmatrix},$$

$$\mathbf{g}^k = \begin{pmatrix} (f^k, \phi_1(x)) \\ (f^k, \phi_2(x)) \\ \vdots \\ (f^k, \phi_{n-2}(x)) \\ (f^k, \phi_{n-1}(x)) \end{pmatrix} = \begin{pmatrix} q_k(1) \\ q_k(2) \\ \vdots \\ q_k(n-2) \\ q_k(n-1) \end{pmatrix},$$

where $a_j = w_{1,k-j+1} + w_{2,k-j+1} + w_{2,k-j-1} - w_{1,k-j-1} - 2w_{2,k-j}$, $b = w_{1,2} + w_{2,2} - 2w_{2,1}$, $d = \frac{h(4-\alpha)\tau^{-\alpha}}{12\Gamma(3-\alpha)}$, and $q_k(m) = \frac{\Gamma(6)ht_k^{5-\alpha}(2\sin mh\pi - \sin(m-1)h\pi - \sin(m+1)h\pi)}{\Gamma(6-\alpha)\pi^2h^2} + \frac{ht_k^5(2\cos mh\pi - \cos(m-1)h\pi - \cos(m+1)h\pi)}{\pi h^2}$.

By the iterative computation for (5.3), we can get a unique numerical solution of u.

5.2. Numerical results

In Table 1, for different fractional parameters $\alpha = 0.1, 0.3, 0.5, 0.7, 0.9$, we take changed time meshes $\tau = 1/10,\ 1/20,\ 1/40$ with the fixed space step $h = 1/800$, and get time convergence rate, which is a little lower than the result with $O(\tau^{3-\alpha})$ obtained from the theorem. The phenomenon for the lower time convergence rate is similar to that computed by the numerical methods in Ref. [28]. However, the convergence order arrived at by the current method is much higher than the result with $O(\tau^{2-\alpha})$ calculated by the $L1$ approximation [21].

Table 1. Temporal convergence results in L^2-norm with fixed $h = 1/800$ and different α

α	$\tau_1 = 1/10$	$\tau_2 = 1/20$	$\tau_3 = 1/40$	Rate $(\frac{\tau_1}{\tau_2})$	Rate $(\frac{\tau_2}{\tau_3})$
0.1	7.8044E-05	1.2676E-05	2.0304E-06	2.6222	2.6423
0.3	4.0692E-04	7.2309E-05	1.2294E-05	2.4925	2.5562
0.5	1.1910E-03	2.3619E-04	4.4689E-05	2.3342	2.4020
0.7	2.9394E-03	6.5829E-04	1.4081E-04	2.1587	2.2250
0.9	6.6479E-03	1.6945E-03	4.1320E-04	1.9720	2.0359

In Table 2, based on the same choice for fractional parameters α to above, we choose changed space step $h = 1/10$, $1/20$, $1/40$ and the fixed time mesh $\tau = 1/400$, and obtain the space convergence order, which approximate theoretical second-order convergence rate with index $r = 1$.

Table 2. Spatial convergence results for u with fixed $\tau = 1/400$ and different α

α	$h_1 = 1/10$	$h_2 = 1/20$	$h_3 = 1/40$	Rate $(\frac{h_1}{h_2})$	Rate $(\frac{h_2}{h_3})$
0.1	7.7909E-04	1.9561E-04	4.8957E-05	1.9938	1.9984
0.3	9.6256E-04	2.4161E-04	6.0479E-05	1.9942	1.9982
0.5	1.2040E-03	3.0213E-04	7.5705E-05	1.9946	1.9967
0.7	1.5097E-03	3.7905E-04	9.5368E-05	1.9938	1.9908
0.9	1.8854E-03	4.7495E-04	1.2132E-04	1.9890	1.9690

For checking the errors at different time points $t = 0, 0.1, \cdots, 0.9, 1$, we give the calculated data in Tables 3-5. In Table 3, taking the fixed spatial mesh parameter $h = 1/1000$ and fractional parameter $\alpha = 0.1$, we list the error results in L^2-norm, from which ones can see that the numerical results are convergent at each time point with changed time step length $\tau = 1/20, 1/40, 1/80$ and the L^2-errors gradually becomes greater with the increased time. Similarly, in Tables 4-5, we also give the error results for the cases $\alpha = 0.5$ and $\alpha = 0.9$, which also have the similar behaviors as the ones for the case $\alpha = 0.1$.

In Figures 1-4, we give the contour plots of error $u - u_h$ based on different space-time parameter pairs $(h, \tau) = (\frac{1}{40}, \frac{1}{20})$, $(\frac{1}{100}, \frac{1}{50})$, $(\frac{1}{400}, \frac{1}{200})$, $(\frac{1}{800}, \frac{1}{400})$. It is easy to see from these contour plots that the magnitude of errors changes from 10^{-4} to 10^{-7}, which tells ones that the studied method in this paper is convergent.

From these data analyzed in above contents, ones can know that our method can get better approximation results.

6. Some concluding remarks

In this paper, we study finite element method combined with the high-order time approximation scheme presented in Ref. [28] to solve numerically fractional convection-

Table 3. Errors in L^2-norm with $h = 1/1000$, $\alpha = 0.1$

t	$\tau_1 = 1/20$	$\tau_2 = 1/40$	$\tau_3 = 1/80$
0	0	0	0
0.1	3.6995E-08	9.0269E-09	1.6917E-09
0.2	2.7175E-07	5.0925E-08	8.4424E-09
0.3	7.6294E-07	1.3165E-07	2.1026E-08
0.4	1.5323E-06	2.5401E-07	4.0036E-08
0.5	2.5942E-06	4.2025E-07	6.6211E-08
0.6	3.9598E-06	6.3247E-07	1.0057E-07
0.7	5.6387E-06	8.9292E-07	1.4451E-07
0.8	7.6392E-06	1.2042E-06	1.9992E-07
0.9	9.9693E-06	1.5692E-06	2.6927E-07
1	1.2637E-05	1.9914E-06	3.5572E-07

Table 4. Errors in L^2-norm with $h = 1/1000$, $\alpha = 0.5$

t	$\tau_1 = 1/20$	$\tau_2 = 1/40$	$\tau_3 = 1/80$
0	0	0	0
0.1	6.8719E-07	2.0498E-07	4.7597E-08
0.2	5.1809E-06	1.1994E-06	2.4448E-07
0.3	1.4683E-05	3.1169E-06	6.0905E-07
0.4	2.9485E-05	6.0002E-06	1.1483E-06
0.5	4.9753E-05	9.8743E-06	1.8670E-06
0.6	7.5597E-05	1.4757E-05	2.7690E-06
0.7	1.0710E-04	2.0662E-05	3.8581E-06
0.8	1.4432E-04	2.7601E-05	5.1383E-06
0.9	1.8731E-04	3.5585E-05	6.6140E-06
1	2.3613E-04	4.4624E-05	8.2900E-06

Table 5. Errors in L^2-norm with $h = 1/1000$, $\alpha = 0.9$

t	$\tau_1 = 1/20$	$\tau_2 = 1/40$	$\tau_3 = 1/80$
0	0	0	0
0.1	3.1297E-06	1.0833E-06	3.2019E-07
0.2	2.7580E-05	8.1814E-06	2.1605E-06
0.3	8.7546E-05	2.4010E-05	6.0709E-06
0.4	1.8806E-04	4.9409E-05	1.2226E-05
0.5	3.3082E-04	8.4698E-05	2.0696E-05
0.6	5.1651E-04	1.3003E-04	3.1516E-05
0.7	7.4551E-04	1.8549E-04	4.4708E-05
0.8	1.0181E-03	2.5113E-04	6.0285E-05
0.9	1.3343E-03	3.2699E-04	7.8260E-05
1	1.6944E-03	4.1310E-04	9.8643E-05

diffusion equation. For the purpose of deriving the theoretical results on stability and error estimates, we give and prove some lemmas. We do some detailed analysis

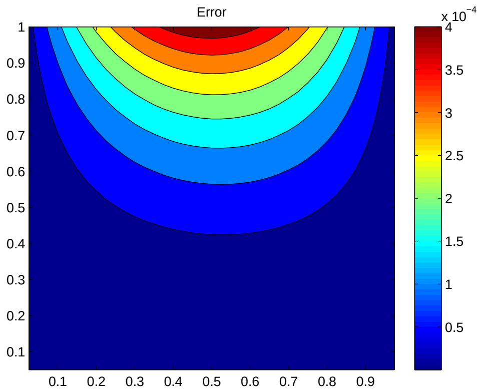

Figure 1. The contour plot of error $u - u_h$ $h = \frac{1}{40}, \tau = \frac{1}{20}$

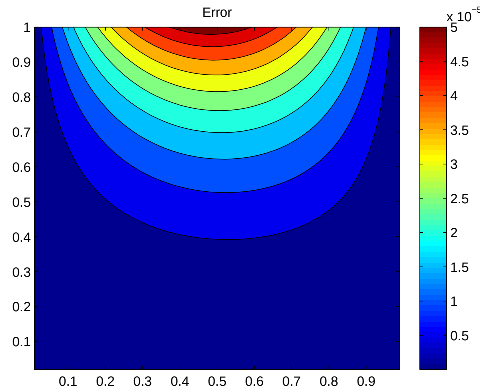

Figure 2. The contour plot of error $u - u_h$ $h = \frac{1}{100}, \tau = \frac{1}{50}$

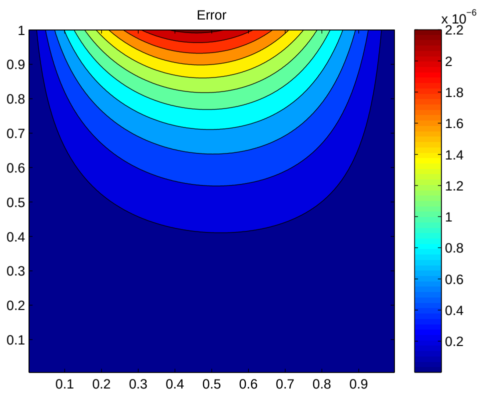

Figure 3. The contour plot of error $u - u_h$ with $h = \frac{1}{400}, \tau = \frac{1}{200}$

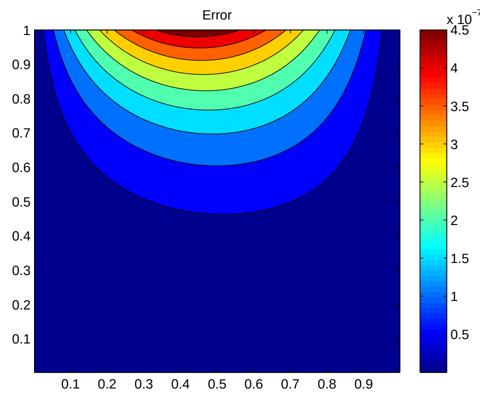

Figure 4. The contour plot of error $u - u_h$ with $h = \frac{1}{800}, \tau = \frac{1}{400}$

for stability and error estimates. Finally, by providing some numerical calculations, we test and verify our theory.

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