

# A CLASS OF DIFFERENTIAL INVERSE VARIATIONAL INEQUALITIES IN FINITE DIMENSIONAL SPACES\*

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**Abstract** In this paper, we study a class of differential inverse variational inequality (for short, DIVI) in finite dimensional Euclidean spaces. Firstly, under some suitable assumptions, we obtain linear growth of the solution set for the inverse variational inequalities. Secondly, we prove existence theorems for weak solutions of the DIVI in the weak sense of Carathéodory by using measurable selection lemma. Thirdly, by employing the results from differential inclusions we establish a convergence result on Euler time dependent procedure for solving the DIVI. Finally, we give a numerical experiment to verify the validity of the algorithm.

**Keywords** Differential inverse variational inequality, Carathéodory weak solution, Euler time-stepping procedure, algorithm.

**MSC(2010)** 49J40, 47J20.

## 1. Introduction

In 2008, Pang and Stewart [25] introduced and investigated a new class of differential variational inequalities (DVI) in finite dimensions, which significantly extends these differential equations and open up a broad paradigm for the enhanced modeling of complex engineering system. It also has wide applications in engineering and economics. For example, Melanz et al. [23] studied experimental validation of a differential variational inequality-based approach for handling friction and contact in vehicle/granular-terrain interaction. On the other hand, the notion of a noncooperative Nash equilibrium was extended to a dynamic, continuous-time setting by Friesz [6], and the existence of solutions of the dynamical Nash equilibrium problem can be studied by a DVI approach [5]. In addition, Raghunathan [26] used differential variational inequalities to consider parameter estimation in metabolic flux balance models. For more related results, we refer to [2, 3, 24] and the references therein.

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In 2010, Li et al. [19] introduced and investigated a class of differential mixed variational inequalities (DMVIs) in finite-dimensional Euclidean spaces which generalized the corresponding results of [25]. They proved the existence of Carathéodory weak solutions for the DMVI and established a convergence result on Euler time dependent procedure for solving initial-value DMVI. Wang and Huang [32] introduced and studied differential vector variational inequalities in finite-dimensional spaces. In 2013, Gwinner [7] studied stability of a class of differential variational inequality in a Hilbert space. Chen [4] studied the convergence analysis of regularized time-stepping methods for a class of DVI. For more related results, we refer to [8, 9, 16, 17, 21, 30, 31] and the references therein.

As a generalization of variational inequality (VI) and its related problems which have been studied extensively recently [22, 28, 29, 34, 35], the inverse variational inequality (IVI) was firstly proposed and studied by He and Liu [12] in 2006 due to its broad applications in the market equilibrium problem in economics and normative flow control problems, appearing in transportation and telecommunication [11]. In 2008, Yang [36] discussed the dynamic power price problem, in both the discrete and the evolutionary cases, and characterized the optimal price as a solution of an IVI. In 2010, He [11] studied a class of normative control problem that requires the network equilibrium state to be in a linearly constrained set. They formulated the problem as an inverse variational inequality (IVI) and gave a solution method based on proximal point algorithm (PPA). Barbagallo and Maur [1] provided an optimal control perspective on the dynamic oligopolistic market equilibrium problem by introducing an inverse variational inequality. They gave some existence and regularity results for the equilibrium solution and presented a numerical example illustrating important features of the problem. Some related works for IVIs can be found in [10, 13, 15, 27] (see also the references therein).

Recall IVI in finite-dimensional Euclidean space as follows: find  $x \in R^n$  such that

$$f(x) \in K, \quad \langle x' - f(x), x \rangle \geq 0, \quad \forall x' \in K,$$

where  $f : R^n \rightarrow R^n$  is a mapping,  $K \subset R^n$  is a closed convex subset. Let  $S(K, f)$  denote the solution set of this problem. Specially, if  $f^{-1}$  exists, then it is easy to transform IVI to a variational inequality. Very recently, Li et al. [18] introduced and studied a new differential inverse variational inequality (DIVI) in finite dimensional Euclidean spaces. They gave the existence theorems of Carathéodory weak solutions for DIVIs and gave an application to the time-dependent spatial price equilibrium control problem. They assumed the function of commodity shipments and regulatory taxes can be written as a separation form. And the time-dependent spatial price equilibrium control problem can be transformed into a special DIVI. But the DIVI in [18] is special, it can not solve the time-dependent spatial price equilibrium control problem when the function of commodity shipments and regulatory taxes can not be transformed into the separate forms (see [1, 27]). Therefore, it is important and interesting to study a class of generalized differential inverse variational inequality (DIVI) in finite dimensional Euclidean spaces :

$$\begin{cases} \dot{x}(t) = a(t, x(t)) + b(t, x(t))u(t), \\ u(t) \in S(K, g(t, x(t), \cdot)), \\ x(0) = x^0, \end{cases} \quad (1.1)$$

where  $K \subset R^n$  is a nonempty closed convex subset,  $\Omega \equiv [0, T] \times R^m$ ,  $(a, b) : \Omega \rightarrow R^m \times R^{m \times n}$  are given functions,  $g : [0, T] \times R^m \times R^n \rightarrow R^n$  is a linear function. The time-dependent spatial price equilibrium control problem when the function of commodity shipments and regulatory taxes can not be transformed into the separate forms can be transformed into the DIVI(1.1). It is also easy to see that the DIVI(1.1) includes the differential inverse variational inequality in [18] as a special case. We are interested in finding the time-dependent trajectories  $x(t)$  and  $u(t)$  such that (1.1) holds in the weak sense of Carathéodory for  $t \in [0, T]$ . That means  $x$  is an absolutely continuous function on  $[0, T]$ , that  $u$  is an integrable function on  $[0, T]$ , and that the differential equation need only be satisfied for almost all  $t \in [0, T]$ . Moreover, the membership for  $u(t)$  means that, for any continuous  $\tilde{u} : [0, T] \rightarrow K$ ,

$$\int_0^T \langle \tilde{u}(t) - g(t, x(t), u(t)), u(t) \rangle dt \geq 0, \quad (1.2)$$

which implies that, for almost all  $t \in [0, T]$ ,  $u(t) \in S(K, g(t, x(t), \cdot))$ .

The paper is organized in the following way. In Section 2, we present some preliminaries. In Section 3, we show existence theorems of weak solutions for DIVI (1.1) in the sense of Carathéodory by applying a result on differential inclusions involving an upper semicontinuous (u.s.c.) set-valued map with nonempty closed and convex values. In Section 4, we discuss the convergence of Euler time-stepping procedure for solving initial-value DIVI (1.1) under suitable conditions. Finally, we give a numerical experiment to verify the validity of the algorithm.

## 2. Preliminaries

In this section, we recall some preliminaries that shall be used in what follows. For more details, we refer readers to [14, 25, 33].

**Definition 2.1.** A function  $a : \Omega \rightarrow R^n$  (resp.,  $b : \Omega \rightarrow R^{n \times m}$ ) is said to be Lipschitz continuous if there exists a constant  $L_a > 0$  (resp.,  $L_b > 0$ ) such that, for any  $(t_1, x), (t_2, y) \in \Omega$ ,

$$\| a(t_1, x) - a(t_2, y) \| \leq L_a(|t_1 - t_2| + \| x - y \|),$$

$$(\text{resp.}, \| b(t_1, x) - b(t_2, y) \| \leq L_b(|t_1 - t_2| + \| x - y \|)).$$

**Definition 2.2.** A function  $f : R^n \rightarrow R^n$  is said to be para-monotone on a convex set  $K \subset R^n$  if  $f$  is monotone on  $K$ , i.e.,

$$\langle f(v) - f(u), v - u \rangle \geq 0, \quad \forall v, u \in K,$$

and the following plus property holds: for any  $v, u \in K$ ,

$$\langle f(v) - f(u), v - u \rangle = 0 \Rightarrow f(v) = f(u).$$

In the rest of this paper, we assume that the following conditions (A) and (B) hold:

- (A)  $a$  and  $b$  are Lipschitz continuous functions on  $\Omega$  with Lipschitz constants  $L_a > 0$  and  $L_b > 0$ , respectively;

(B)  $b$  are bounded on  $\Omega$  with

$$\sigma_b \equiv \sup_{(t,x) \in \Omega} \|b(t,x)\| < \infty.$$

Let  $\mathbb{F} : \Omega \rightrightarrows R^m$  be a set-valued map defined as follows:

$$\mathbb{F}(t,x) \equiv \{a(t,x) + b(t,x)u : u \in S(K,g(t,x,\cdot))\}. \tag{2.1}$$

**Lemma 2.1.** *Let  $\mathbb{F} : \Omega \rightrightarrows R^m$  be an upper semicontinuous set-valued map with nonempty closed convex values. Suppose that there exists a scalar  $\rho_{\mathbb{F}} > 0$  satisfying*

$$\sup\{\|y\| : y \in \mathbb{F}(t,x)\} \leq \rho_{\mathbb{F}}(1 + \|x\|), \quad \forall (t,x) \in \Omega. \tag{2.2}$$

*Then, for every  $x^0 \in R^n$ , the DI :  $\dot{x} \in \mathbb{F}(t,x), x(0) = x^0$  has a weak solution in the sense of Carathéodory.*

**Lemma 2.2.** *Let  $h : \Omega \times R^m \rightarrow R^m$  be a continuous function and  $U : \Omega \rightrightarrows R^n$  a closed set-valued map such that, for some constant  $\eta_U > 0$ ,*

$$\sup_{u \in U(t,x)} \|u\| \leq \eta_U(1 + \|x\|), \quad \forall (t,x) \in \Omega.$$

*Let  $v : [0,T] \rightarrow R^m$  be a measurable function and  $x : [0,T] \rightarrow R^m$  a continuous function satisfying  $v(t) \in h(t,x(t),U(t,x(t)))$  for almost all  $t \in [0,T]$ . Then there exists a measurable function  $u : [0,T] \rightarrow R^n$  such that  $u(t) \in U(t,x(t))$  and  $v(t) = h(t,x(t),u(t))$  for almost all  $t \in [0,T]$ .*

### 3. Existence of solutions for DIVI (1.1)

**Lemma 3.1.** *Let  $K \subset R^n$  be nonempty closed convex and  $a, b$  satisfy conditions (A) and (B) above,  $g$  be continuous on  $[0,T] \times R^m \times R^n$ . Suppose that there exists a constant  $\rho > 0$  such that, for any  $(t,x) \in \Omega$ ,*

$$\sup\{\|u\| : u \in S(K,g(t,x,\cdot))\} \leq \rho(1 + \|x\|). \tag{3.1}$$

*Then there exists a constant  $\rho^{\mathbb{F}} > 0$  such that (2.2) holds for the map  $\mathbb{F} > 0$  defined by (2.1). Hence  $\mathbb{F}$  is upper semicontinuous and closed-valued on  $\Omega$ .*

**Proof.** Since  $a$  is Lipschitz continuous on  $\Omega$ , we know there exists  $\rho_a > 0$  such that, for all  $(t,x) \in \Omega$ ,

$$\|a(t,x)\| \leq \rho_a(1 + \|x\|).$$

It follows from the above inequality that

$$\begin{aligned} \sup\{\|y\| : y \in \mathbb{F}(t,x)\} &\leq \rho_a(1 + \|x\|) + \sigma_b\rho(1 + \|x\|) \\ &\leq (\rho_a + \sigma_b\rho)(1 + \|x\|). \end{aligned} \tag{3.2}$$

Let  $\rho^{\mathbb{F}} = (\rho_a + \sigma_b\rho)$ , it is easy to see that (2.2) holds. Thus the set-valued map  $\mathbb{F}$  has linear growth.

Next we prove  $\mathbb{F}$  is upper semicontinuity on  $\Omega$ . Since  $\mathbb{F}$  has linear growth, the upper semicontinuous of  $\mathbb{F}$  holds if  $\mathbb{F}$  is closed. Assume that the sequence  $\{(t_n, x_n)\} \subset \Omega$  converges to some vector  $(t_0, x_0) \in \Omega$  and  $\{a(t_n, x_n) + b(t_n, x_n)u_n\}$

converges to some vector  $z_0 \in R^m$  as  $n \rightarrow \infty$ , where  $u_n \in S(K, g(t_n, x_n, \cdot))$  for every  $n$ , which means that

$$g(t_n, x_n, u_n) \in K \quad (3.3)$$

and

$$\langle \tilde{u} - g(t_n, x_n, u_n), \tilde{u} \rangle \geq 0, \quad \forall \tilde{u} \in K. \quad (3.4)$$

Since  $\|u_n\| \leq \rho(1 + \|x_n\|)$  and  $x_n \rightarrow x_0$ , we get  $\{u_n\}$  is bounded. So  $\{u_n\}$  has a convergent subsequence (denoted it by  $\{u_n\}$  again) with a limit  $u_0 \in R^n$ . Since  $g$  is continuous and  $K$  is closed, by (3.3), one has

$$g(t_n, x_n, u_n) \rightarrow g(t_0, x_0, u_0) \in K.$$

Thus, for any  $\tilde{u} \in K$ ,

$$\langle \tilde{u} - g(t_0, x_0, u_0), \tilde{u} \rangle \geq 0, \quad (3.5)$$

which means that  $u_0 \in S(K, g(t_0, x_0, \cdot))$ . Applying the continuity of  $a$  and  $b$  we know

$$a(t_n, x_n) + b(t_n, x_n)u_n \rightarrow z_0 = a(t_0, x_0) + b(t_0, x_0)u_0 \in \mathbb{F}(t_0, x_0).$$

Therefore,  $\mathbb{F}$  is closed. This completes the proof.  $\square$

**Lemma 3.2.** *Let  $(a, b)$  satisfy conditions (A) and (B) above and  $K \subset R^n$  be nonempty, closed and convex. Let  $g : [0, T] \times R^m \times R^n \rightarrow R^n$  be continuous, linear and para-monotone with the third variable. Suppose that  $S(K, g(t, x, \cdot)) \neq \emptyset$  for all  $(t, x) \in \Omega$ . Then  $S(K, g(t, x, \cdot))$  is closed and convex for all  $(t, x) \in \Omega$ .*

**Proof.** Let  $\{u_n\} \subset S(K, g(t, x, \cdot))$  with  $u_n \rightarrow u_0$ . That means  $g(t, x, u_n) \in K$  and for any  $\tilde{u} \in K$ ,

$$\langle \tilde{u} - g(t, x, u_n), u_n \rangle \geq 0.$$

Since  $K$  is closed,  $g(t, x, \cdot)$  is continuous on  $R^n$ , we know that  $g(t, x, u_0) \in K$  and for any  $\tilde{u} \in K$ ,

$$\langle \tilde{u} - g(t, x, u_0), u_0 \rangle \geq 0. \quad (3.6)$$

This means that  $u_0 \in S(K, g(t, x, \cdot))$  and so  $S(K, g(t, x, \cdot))$  is closed for all  $(t, x) \in \Omega$ . Next we prove that  $S(K, g(t, x, \cdot))$  is convex for all  $(t, x) \in \Omega$ . Let  $u_1, u_2 \in S(K, g(t, x, \cdot))$ . Then

$$g(t, x, u_i) \in K, \quad i = 1, 2 \quad (3.7)$$

and

$$\langle \tilde{u} - g(t, x, u_1), u_1 \rangle \geq 0, \quad \forall \tilde{u} \in K, \quad (3.8)$$

and

$$\langle \tilde{u} - g(t, x, u_2), u_2 \rangle \geq 0, \quad \forall \tilde{u} \in K. \quad (3.9)$$

Applying the convexity of  $K$ , it follows from (3.7) that, for  $\lambda \in (0, 1)$ ,

$$\lambda g(t, x, u_1) + (1 - \lambda)g(t, x, u_2) = g(t, x, \lambda u_1 + (1 - \lambda)u_2) \in K.$$

Letting  $\tilde{u} = g(t, x, u_2)$  in (3.8) and  $\tilde{u} = g(t, x, u_1)$  in (3.9) respectively, one has

$$\langle g(t, x, u_1) - g(t, x, u_2), u_2 - u_1 \rangle \geq 0. \quad (3.10)$$

Since  $g$  is para-monotone with the third variable, it follows from the above inequality that for all  $(t, x) \in \Omega$

$$g(t, x, u_1) = g(t, x, u_2).$$

Applying the inequalities (3.8) again, we get for all  $(t, x) \in \Omega$

$$\langle \tilde{u} - g(t, x, \lambda u_1 + (1 - \lambda)u_2), \lambda u_1 + (1 - \lambda)u_2 \rangle \geq 0, \quad \forall \tilde{u} \in K.$$

Which means that  $\lambda u_1 + (1 - \lambda)u_2 \in S(K, g(t, x, \cdot))$  and so  $S(K, g(t, x, \cdot))$  is convex for all  $(t, x) \in \Omega$ . This completes the proof.  $\square$

**Remark 3.1.** If  $g(t, x, \cdot) = G(t, x) + F(\cdot)$  for all  $(t, x) \in \Omega$ , where  $G$  is Lipschitz continuous, then under the conditions of Lemma 3.2 we can obtain Lemma 2.6 of [18].

**Lemma 3.3.** *Let  $(a, b)$  satisfy conditions (A) and (B) above,  $K \subset R^n$  be a nonempty, closed and convex set,  $g : [0, T] \times R^m \times R^n \rightarrow R^n$  be continuous and para-monotone with the third variable. Suppose that  $S(K, g(t, x, \cdot)) \neq \emptyset$  for all  $(t, x) \in \Omega$  and a constant  $\rho > 0$  exists such that (3.1) holds for all  $(t, x) \in \Omega$ . Then DIVI(1.1) has weak solutions in the sense of Carathéodory.*

**Proof.** Similar the proof of Proposition 6.1 of [25], applying Lemmas 2.1, 2.2, 3.1 and 3.2, we can obtain that DIVI (1.1) has weak solutions in the sense of Carathéodory.  $\square$

**Remark 3.2.** Under the conditions of Lemma 3.3 we can obtain Lemma 2.7 of [18].

**Lemma 3.4.** *Let  $K \subset R^n$  be a nonempty compact convex set,  $g : [0, T] \times R^m \times R^n \rightarrow R^n$  be continuous and para-monotone with the third variable. Suppose that for all  $(t, x) \in \Omega$ ,  $g_{tx}(\cdot) = g(t, x, \cdot)$  is linear, injective and surjective on  $R^n$ , and there exists  $M > 0$  such that  $\|g_{tx}^{-1}\| \leq M$ . Then  $S(K, g(t, x, \cdot))$  is a singleton for all  $(t, x) \in \Omega$  and there exists  $\rho > 0$  such that (3.1) holds for all  $(t, x) \in \Omega$ .*

**Proof.** Let  $P(u) = g_{tx}^{-1}(u) = y_u$ . Then for  $(t, x) \in \Omega$ , we get

$$\langle Pu_1 - Pu_2, u_1 - u_2 \rangle = \langle y_{u_1} - y_{u_2}, g_{tx}(y_{u_1}) - g_{tx}(y_{u_2}) \rangle.$$

Since  $g : [0, T] \times R^m \times R^n \rightarrow R^n$  is monotone with the third variable, we know  $P$  is monotone on  $R^n$ . Then by Theorem 8.1 in [20] we know the variational inequality  $VI(K, P)$  has solutions. Which means that there exists  $u \in K$  such that

$$\langle \hat{u} - u, P(u) \rangle \geq 0, \quad \forall \hat{u} \in K.$$

Since  $g_{tx}$  is surjective on  $R^n$ , we know there exists  $z_{tx} \in R^n$  such that  $g_{tx}(z_{tx}) = g(t, x, z_{tx}) = u \in K$  and

$$\langle \hat{u} - g(t, x, z_{tx}), z_{tx} \rangle \geq 0, \quad \forall \hat{u} \in K,$$

which means that  $z_{tx} \in S(K, g(t, x, \cdot))$  and so  $S(K, g(t, x, \cdot))$  is nonempty for all  $(t, x) \in \Omega$ .

On the other hand, for all  $(t, x) \in \Omega$  and  $u_1, u_2 \in S(K, g(t, x, \cdot))$ , we have

$$g(t, x, u_1) \in K, \quad g(t, x, u_2) \in K,$$

and

$$\begin{aligned}\langle \tilde{u} - g(t, x, u_1), u_1 \rangle &\geq 0, \quad \forall \tilde{u} \in K, \\ \langle \tilde{u} - g(t, x, u_2), u_2 \rangle &\geq 0, \quad \forall \tilde{u} \in K.\end{aligned}$$

Letting  $\tilde{u} = g(t, x, u_2)$  and  $\tilde{u} = g(t, x, u_1)$  respectively in the above inequalities we get

$$\langle g(t, x, u_1) - g(t, x, u_2), u_1 - u_2 \rangle = 0.$$

Since  $g_{tx}(\cdot)$  is para-monotone, we know for all  $(t, x) \in \Omega$ ,

$$g(t, x, u_1) = g(t, x, u_2).$$

Since  $g_{tx}(\cdot)$  is injective, we obtain  $u_1 = u_2$ . Then it is easy to see that there exists a constant  $\rho > 0$  such that (3.1) holds for all  $(t, x) \in \Omega$ .  $\square$

**Theorem 3.1.** *Let  $(a, b)$  satisfy conditions (A) and (B) above,  $K \subset R^n$  be a nonempty compact convex set,  $g : [0, T] \times R^m \times R^n \rightarrow R^n$  be continuous and para-monotone with the third variable. Suppose that for all  $(t, x) \in \Omega$ ,  $g_{tx}(\cdot) = g(t, x, \cdot)$  is linear, injective and surjective on  $R^n$ , and there exists  $M > 0$  such that  $\|g_{tx}^{-1}\| \leq M$ . Then DIVI(1.1) has weak solutions in the sense of Carathéodory.*

**Proof.** By Lemmas 3.4 we know  $S(R^n, g(t, x, \cdot))$  is a singleton for all  $(t, x) \in \Omega$  and there exists  $\rho > 0$  such that (3.1) holds for all  $(t, x) \in \Omega$ . By Lemma 3.3, we can know that DIVI (1.1) has weak solutions in the sense of Carathéodory.  $\square$

**Lemma 3.5.** *Let  $g(t, x, \cdot) : R^n \rightarrow R^n$  is para-monotone and continuous on  $R^n$ . Suppose that there exists  $v_0 \in R^n$  such that for all  $(t, x) \in \Omega$ .*

$$\liminf_{\|v_0\| \rightarrow \infty} \frac{\langle g(t, x, v_0), v_0 \rangle}{\|v_0\|^2} > 0. \quad (3.11)$$

*Then  $S(R^n, g(t, x, \cdot))$  is nonempty and there exists  $\rho > 0$  such that (3.1) holds for all  $(t, x) \in \Omega$ .*

**Proof.** By Lemma 4.1 in [12], we know that in order to prove  $S(R^n, g(t, x, \cdot)) \neq \emptyset$ , we only need to show that there exists  $v \in R^{2n}$  such that

$$\langle \tilde{v} - v, P(v) \rangle \geq 0 \quad \forall \tilde{v} \in R^{2n},$$

where

$$v = \begin{pmatrix} u \\ y \end{pmatrix},$$

and

$$P(v) = \begin{pmatrix} g(t, x, u) - y \\ u \end{pmatrix}.$$

Since  $g(t, x, \cdot) : R^n \rightarrow R^n$  is para-monotone on  $R^n$ , we can deduce that

$$\begin{aligned}& \langle P(v_1) - P(v_2), v_1 - v_2 \rangle \\ &= \langle g(t, x, u_1) - g(t, x, u_2) + y_2 - y_1, u_1 - u_2 \rangle + \langle y_1 - y_2, u_1 - u_2 \rangle \\ &= \langle g(t, x, u_1) - g(t, x, u_2), u_1 - u_2 \rangle \\ &\geq 0.\end{aligned} \quad (3.12)$$

Which means that  $P$  is monotone on  $R^{2n}$ . By Theorem 3.2 in [19], we know there exists  $v \in R^{2n}$  such that

$$\langle \tilde{v} - v, P(v) \rangle \geq 0 \quad \forall \tilde{v} \in R^{2n}.$$

So  $S(R^n, g(t, x, \cdot))$  is nonempty. It follows from Lemma 3.2 that  $S(K, g(t, x, \cdot))$  is closed and convex set for all  $(t, x) \in \Omega$ .

Next, we prove the second assertion. Suppose to the contrary, there exist  $\{t_k, x_k\} \subset \Omega$  and  $\{u_k\} \subset R^n$  such that, for any  $\tilde{u} \in R^n$ ,

$$\langle \tilde{u} - g(t_k, x_k, u_k), u_k \rangle \geq 0 \tag{3.13}$$

and  $\|u_k\| \geq k(1 + \|x_k\|)$ . So we know  $\{u_k\}$  is unbounded. Letting  $\tilde{u} = v_0$ , in (3.13), we have

$$\langle \tilde{u} - g(t_k, x_k, u_k), u_k \rangle \geq 0, \tag{3.14}$$

and so

$$\frac{\langle v_0 - g(t_k, x_k, u_k), u_k \rangle}{\|u_k\|^2} \geq 0.$$

which contradicts (3.15). This shows that there exists a constant  $\rho > 0$  such that (3.1) holds for all  $(t, x) \in \Omega$ .  $\square$

Combining Lemma 3.5 and Lemma 3.3 we can easily obtain the following Theorem:

**Theorem 3.2.** *Let  $(a, b)$  satisfy conditions (A) and (B) above,  $g : [0, T] \times R^m \times R^n \rightarrow R^n$  be continuous and para-monotone with the third variable. Suppose that there exists  $v_0 \in R^n$  such that for all  $(t, x) \in \Omega$ ,*

$$\liminf_{\|u\| \rightarrow \infty} \frac{\langle g(t, x, u), u \rangle}{\|u\|^2} > 0. \tag{3.15}$$

*Then DIVI(1.1) has weak solutions in the sense of Carathéodory.*

**Proof.** By Lemmas 3.2 and 3.5 we know  $S(R^n, g(t, x, \cdot))$  is nonempty, closed and convex, and there exists  $\rho > 0$  such that (3.1) holds for any  $(t, x) \in \Omega$ . By Lemma 3.3, we can obtain that DIVI (1.1) has weak solutions in the sense of Carathéodory.  $\square$

## 4. Computational methods for DIVI (1.1)

**Lemma 4.1.** *Let  $(a, b)$  satisfy conditions (A) and (B) above,  $g : [0, T] \times R^m \times R^n \rightarrow R^n$  be continuous and para-monotone with the third variable. Assume that for any  $x \in C([0, T]; R^n)$  and  $u \in L^2([0, T]; R^n)$ ,  $g(t, x(t), u(t)) \in K$  for  $t \in [0, T]$ . If for any continuous function  $P : [0, T] \rightarrow K$ ,*

$$\int_0^T \{ \langle P(t) - g(t, x(t), u(t)), u(t) \rangle \} dt \geq 0, \tag{4.1}$$

*then  $u(t) \in S(K, g(t, x(t), \cdot))$  for almost all  $t \in [0, T]$ .*

**Proof.** We assume that the contrary holds. Then there exists a subset  $E \subset [0, T]$  with  $\hat{m}(E) > 0$  (where  $\hat{m}(E)$  denotes the Lebesgue measure of  $E$ ) such that, for any  $t \in E$ ,

$$u(t) \notin S(K, g(t, x(t), \cdot)).$$

By Lusin theorem, we know that there exists a closed subset  $E_1$  of  $E$  with  $m(E_1) > 0$  such that  $u(t)$  is continuous on  $E_1$ . And so there exists a  $P_0 \in K$  such that, for almost all  $t \in E_1$ ,

$$\int_{E_1} \{ \langle P_0 - g(t, x(t), u(t)), u(t) \rangle \} dt < 0.$$

Let

$$F_0(t) = \begin{cases} P_0, & t \in E_1, \\ g(t, x(t), u(t)), & t \in [0, T] \setminus E_1. \end{cases} \tag{4.2}$$

Then it is clear that  $F_0(t)$  is an integrable function on  $[0, T]$ . Since the space of continuous functions  $C([0, T]; R^n)$  is dense in  $L^1([0, T]; R^n)$ , we can approximate  $F_0(t) \in L^1([0, T]; R^n)$  by continuous function. Thus, there exists a continuous function  $\bar{F}(t) : [0, T] \rightarrow K$  such that

$$\int_0^T \{ \langle \bar{F}(t) - g(t, x(t), u(t)), u(t) \rangle \} dt < 0,$$

which contradicts with (4.1). This completes the proof. □

Now we discuss the convergence for a weak solution of the initial-value DI-VI(1.1). Let us choose an equidistant grid  $0 = t_0 < t_1 < \dots < t_N = T$ , with stepsize  $h = \frac{T}{N}$ . Setting  $x^{h,0} = x^*$ , we compute iterates

$$\{x^{h,1}, x^{h,2}, \dots, x^{h,N}\} \subset R^m, \quad \{u^{h,1}, u^{h,2}, \dots, u^{h,N}\} \subset R^n \tag{4.3}$$

as follows: for  $i = 0, 1, \dots, N_h$ ,

$$\begin{cases} x^{h,i+1} = x^{h,i} + h [a(t_{h,i}, x^{h,i}) + b(t_{h,i}, x^{h,i})u^{h,i+1}], \\ u^{h,i+1} \in S(K, g(t_{h,i}, x^{h,i}, \cdot)) \end{cases} \tag{4.4}$$

where  $N_h = \frac{T}{h} - 1$ .

Let the set-valued map  $\mathbb{F}$  be defined by(2.1). From(4.4), we have

$$x^{h,i+1} \in x^{h,i}\mathbb{F}(t_{h,i}, x^{h,i}).$$

Let  $\hat{x}^h(\cdot)$  be the continuous piecewise linear interpolant of the family  $\{x^{h,i+1}\}$  and  $\hat{u}^h(\cdot)$  be the constant piecewise interpolant of the family  $\{u^{h,i+1}\}$ , i.e.,

$$\begin{cases} \hat{x}^h(t) = x^{h,i} + \frac{t-t_i}{h}(x^{h,i+1} - x^{h,i}), & \forall t \in [t_{h,i}, t_{h,i+1}], \\ \hat{u}^h(t) = u^{h,i+1}, & \forall t \in (t_{h,i}, t_{h,i+1}] \end{cases} \tag{4.5}$$

for  $i = 0, 1, \dots, N_h$ . We denote, by  $L^2([0, T]; R^n)$ , the set of all measurable functions  $u : [0, T] \rightarrow R^n$  satisfying  $\int_0^T \|u(t)\|^2 dt < \infty$ , in which the inner product is defined as

$$\langle u, v \rangle = \int_0^T \langle u(t), v(t) \rangle dt, \quad \forall u, v \in L^2([0, T]; R^n).$$

**Theorem 4.1.** *Let  $(a, b)$  satisfy conditions (A) and (B) above,  $K \subset R^n$  be a nonempty closed and convex set,  $g : [0, T] \times R^m \times R^n \rightarrow R^n$  be Lipschitz continuous on  $[0, T] \times R^m \times R^n$  and strong continuous on  $L^2([0, T], K)$  with respect to the third variable. Suppose that  $S(K, g(t, x, \cdot)) \neq \emptyset$ , moreover there exists a constant  $\rho > 0$  such that (3.1) holds. Then every sequence pair  $\{(\hat{x}^h, \hat{u}^h)\}$  defined by (4.5) has a subsequence pair  $\{(\hat{x}^{h_v}, \hat{u}^{h_v})\}$  such that  $\hat{x}^{h_v} \rightarrow \tilde{x}$  uniformly in  $[0, T]$  and  $\hat{u}^{h_v} \rightarrow \tilde{u}$  weakly in  $L^2([0, T]; R^n)$ , as  $v \rightarrow \infty$ . And  $(\tilde{x}, \tilde{u})$  is a Carathéodory weak solution of the initial-value DIVI(1.1).*

**Proof.** By Lemma 3.1 and formula 4.5, we have there exists  $\rho_u > 0$  such that

$$\|u^{h,i+1}\| \leq \rho_u(1 + \|x^{h,i}\|)$$

and

$$\|x^{h,i+1} - x^{h,i}\| \leq h\rho_{\mathbb{F}}(1 + \|x^{h,i}\|). \tag{4.6}$$

So

$$\begin{aligned} \|x^{h,i+1}\| &\leq h\rho_{\mathbb{F}} + (h\rho_{\mathbb{F}} + 1)\|x^{h,i}\| \\ &\leq (1 + h\rho_{\mathbb{F}})^{i+1}\|x^0\| + h\rho_{\mathbb{F}} \sum_{j=0}^i (1 + h\rho_{\mathbb{F}})^j \\ &\leq (1 + h\rho_{\mathbb{F}})^{i+1}\|x^0\| + h\rho_{\mathbb{F}} \frac{(1 + h\rho_{\mathbb{F}})^{i+1} - 1}{h\rho_{\mathbb{F}}} \\ &\leq e^{h(i+1)\rho_{\mathbb{F}}}\|x^0\| + e^{T\rho_{\mathbb{F}}} - 1. \end{aligned} \tag{4.7}$$

That means that there exist constants  $c_{0,x}, c_{1,x}, c_{1,u}$  and  $h_1 > 0$  such that, for any  $h \in (0, h_1]$  and any  $i = 0, 1, \dots, N_h$ ,

$$\begin{cases} \|x^{h,i+1}\| \leq c_{0,x} + c_{1,x}\|x^0\|, \\ \|u^{h,i+1}\| \leq c_{0,u} + c_{1,u}\|x^0\|. \end{cases} \tag{4.8}$$

By (4.8) and (4.6), we deduce that, for  $h > 0$  sufficiently small, there exists  $L_{x_0} > 0$ , which is independent of  $h$ , such that

$$\|x^{h,i+1} - x^{h,i}\| \leq L_{x_0}h, \quad i = 0, 1, \dots, N_h. \tag{4.9}$$

It follows from (4.5) that

$$\begin{aligned} \hat{x}^h(t_1) - \hat{x}^h(t_2) &= \frac{t_1 - t_2}{h} \|t_1 - t_2\| (x^{h,i+1} - x^{h,i}) \\ &\leq L_{x_0} \|t_1 - t_2\|, \end{aligned} \tag{4.10}$$

which means that  $\hat{x}^h$  is Lipschitz continuous on  $[0, T]$ , and the Lipschitz constant is independent of  $h$ . Thus, there exists an  $h_0 > 0$  such that the family of functions  $\{\hat{x}^h\}(h \in (0, h_0])$  is an equicontinuous family of functions. Letting

$$\|\hat{x}^h\|_{L^\infty} = \sup_{t \in [0, T]} \|\hat{x}^h(t)\|,$$

it follows from (4.5) and (4.8) that  $\{\hat{x}^h\}$  is uniformly bounded. By using the Arzelá-Ascoli theorem, there exists a sequence  $\{h_v\} \downarrow 0$  such that  $\{\hat{x}^{h_v}\}$  converges uniformly to  $\tilde{x}$  on  $[0, T]$ . Thus, from (3.1) and (4.8), we know that the iterates  $\{u^{h,i+1}\}$  is

uniformly bounded in the  $L^\infty$  norm on  $[0, T]$  and so  $\{\hat{u}^h\}$  is uniformly bounded in the  $L^\infty$  norm on  $[0, T]$ . Since  $L^2([0, T]; R^n)$  is a reflexive Banach space, it is easy to know that there exists a sequence  $\{h_v\} \downarrow 0$  such that  $\hat{u}^{h_v} \rightarrow \tilde{u}$  weakly in  $L^2[0, T]$ .

Now we show that  $(\tilde{x}, \tilde{u})$  is a Carathéodory weak solution of DIVI(1.1).

(I) We first prove that  $\tilde{x}(0) = x^0$ . In fact, since  $\hat{x}^h(0) = x^0$  for all  $h > 0$  sufficiently small and  $\hat{x}^{h_v} \rightarrow \tilde{x}$  uniformly as  $v \rightarrow \infty$ , we know that  $\tilde{x}(0) = x^0$ .

(II) We next show that, for almost all  $t \in [0, T]$ ,

$$\tilde{u}(t) \in S(K, g(t, \tilde{x}(t), \cdot)).$$

In fact, applying the strong continuity of  $g$ , we know that  $\{g(t, \hat{x}^{h_v}, \hat{u}^{h_v})\}$  converges to  $g(t, \tilde{x}, \tilde{u})$ . Consequently, for any continuous function  $\bar{F} : [0, T] \rightarrow K$ , one has

$$\limsup_{v \rightarrow \infty} \int_0^T \langle \bar{F}(t) - g(t, \hat{x}^{h_v}(t), \hat{u}^{h_v}), \hat{u}^{h_v} \rangle dt \leq \int_0^T \langle \bar{F}(t) - g(t, \tilde{x}(t), \tilde{u}), \tilde{u}(t) \rangle dt.$$

On the other hand, since

$$\hat{x}^h(t) = x^{h,i} + \frac{t - t_i}{h}(x^{h,i+1} - x^{h,i}), \quad \forall t \in [t_{h,i}, t_{h,i+1}],$$

we have

$$\|\hat{x}^h - x^{h,i+1}\| = \left\| \frac{t - t_i - h}{h}(x^{h,i+1} - x^{h,i}) \right\|.$$

and so

$$\begin{aligned} & \left\| \sum_{i=0}^{N_h} \int_{t_{h,i}}^{t_{h,i+1}} \langle g(t_{h,i+1}, x^{h,i+1}, u^{h,i+1}) - g(t, \hat{x}^h, u^{h,i+1}), u^{h,i+1} \rangle dt \right\| \\ & \leq Nh[L_g(h + L_{x^0}h)]\|u^h\|_{L^\infty}. \end{aligned} \tag{4.11}$$

It follows from (4.11) that

$$\begin{aligned} & \int_0^T \{ \langle \bar{F}(t) - g(t, \hat{x}^h, \hat{u}^h), \hat{u}^h \rangle \} dt \\ & = \sum_{i=0}^{N_h} \int_{t_{h,i}}^{t_{h,i+1}} \langle \bar{F}(t) - g(t, \hat{x}^h(t), u^{h,i+1}), u^{h,i+1} \rangle dt \\ & = \sum_{i=0}^{N_h} \int_{t_{h,i}}^{t_{h,i+1}} \langle \bar{F}(t) - g(t_{h,i}, x^{h,i}, u^{h,i+1}), u^{h,i+1} \rangle dt \\ & \quad + \sum_{i=0}^{N_h} \int_{t_{h,i}}^{t_{h,i+1}} \langle g(t_{h,i}, x^{h,i}, u^{h,i+1}) - g(t, \hat{x}^h, u^{h,i+1}), u^{h,i+1} \rangle dt \\ & \geq h \sum_{i=0}^{N_h} \frac{1}{h} \int_{t_{h,i}}^{t_{h,i+1}} \langle \bar{F}(t) - g(t_{h,i+1}, x^{h,i+1}, u^{h,i+1}), u^{h,i+1} \rangle dt \\ & \quad - Nh[L_g(h + L_{x^0}h)]\|u^h\|_{L^\infty}. \end{aligned} \tag{4.12}$$

Now the convexity of  $K$  shows that

$$\frac{1}{h} \int_{t_{h,i}}^{t_{h,i+1}} \bar{F}(t) dt \in K.$$

Since  $u^{h,i+1} \in S(K, g(t_{h,i+1}, x^{h,i+1}, \cdot))$ , one has

$$h \sum_{i=0}^{N_h} \frac{1}{h} \int_{t_{h,i}}^{t_{h,i+1}} \langle \bar{F}(t) - g(t_{h,i+1}, x^{h,i+1}, u^{h,i+1}), u^{h,i+1} \rangle dt \geq 0,$$

and so it follows from (4.12) that, for all continuous functions  $\bar{F} : [0, T] \rightarrow K$ ,

$$\int_0^T \langle \bar{F}(t) - g(t, \hat{x}^h(t), \hat{u}^h(t)), \hat{u}^h(t) \rangle dt \geq 0.$$

From Lemma 4.1, it is easy to see that, for almost all  $t \in [0, T]$ ,

$$\tilde{u}(t) \in S(K, G(t, x(t)) + F(\cdot), \varphi).$$

(III) Similar to the proof of Theorem 7.1 in [25], we can show that, for any  $0 \leq s \leq t \leq T$ ,

$$\tilde{x}(t) - \tilde{x}(s) = \int_s^t [a(\tau, \tilde{x}(\tau)) + b(\tau, \tilde{x}(\tau))\tilde{u}(\tau)] d\tau.$$

From (I)-(III) discussed above, we know that  $(\tilde{x}, \tilde{u})$  is a Carathéodory weak solution of DIVI(1.1). This completes the proof.  $\square$

## 5. Numerical Experiment

In this section, we provide an example to verify the validity of algorithm introduced in Sect. 4.

Let

$$\begin{aligned} a(t, x(t)) &= 2t + 3x(t), \\ b(t, x(t)) &= t \sin(x(t)), \\ g(t, x(t), u(t)) &= tx(t) - u(t). \end{aligned} \tag{5.1}$$

For each  $t \in [0, 4]$ ,

$$\begin{aligned} \dot{x}(t) &= 2t + 3x(t) + t \sin(x(t))u(t), \quad \forall t \in [0, 4], \\ \langle v - [tx(t) - u(t)], u(t) \rangle &\geq 0, \quad \forall v \in [0, 5], \\ x(0) &= 0. \end{aligned} \tag{5.2}$$

Algorithm: Step 0: It begins with the division of the time interval  $[0, 4]$ ,

$$t^{h,0} = 0 < t^{h,1} = 0.05 < t^{h,2} = 0.1 < \dots < t^{h,80} = 4,$$

with each of length  $h = 0.05$ .

Step 1: Let  $x^{h,0} = 0$ . Compute  $u = u^{h,1}$  which satisfies the following variational inequality,

$$\langle v - [t^{h,0}x^{h,0} - u], u \rangle \geq 0, \quad \forall v \in [0, 5].$$

Step 2: Compute

$$x^{h,i+1} = x^{h,i} + \frac{1}{20} [a(t_{h,i}, x^{h,i}) + b(t_{h,i}, x^{h,i})u^{h,i+1}].$$

Compute  $u = u^{h,i+2}$ , which satisfies the following variational inequality,

$$\langle v - [t^{h,i+1}x^{h,i+1} - u], u \rangle \geq 0, \quad \forall v \in [0, 5].$$

By the recursion, for  $i = 0, 1, \dots, 79$ , the numerical results are shown in Figure 1.

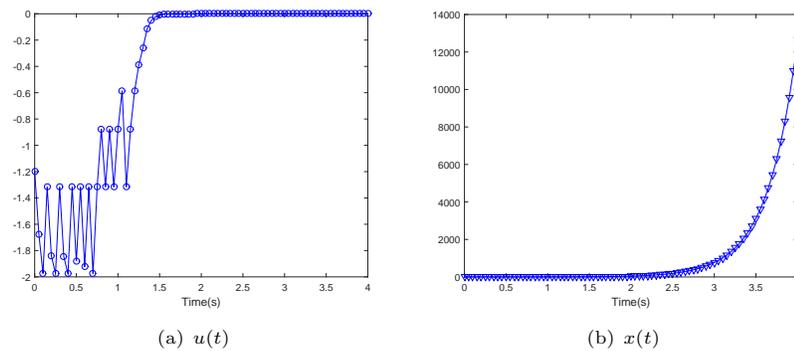


Figure 1. Trajectory of  $x(t)$  and  $u(t)$ .

## 6. Conclusion

The purpose of this article is to study a generalized differential inverse variational inequality in finite dimensional spaces, which includes the differential inverse variational in [18] as a special case. We proved the linear growth of the solution set for the differential inverse variational inequality under various conditions and the existence theorems concerned with the Carathéodory weak solutions for the differential inverse variational inequality in finite dimensional spaces, and extended some results in [18].

Moreover, by employing the results from differential inclusions we establish a convergence result on Euler time dependent procedure for solving initial-value differential inverse variational inequalities. We also gave a numerical experiment to verify the validity of the algorithm.

On the other hand, from [18] we know it is suitable and convenient to employ the differential inverse variational inequality to the study concerned with the ordinary differential equation whose right hand function is parameterized by an algebraic variable that is required to be a solution of the time dependent spatial price equilibrium control problem. So it is important to use the results of this paper to consider time dependent spatial price equilibrium control problem.

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