SOLVING AN INVERSE PROBLEM FOR A GENERALIZED TIME-DELAYED BURGERS-FISHER EQUATION BY HAAR WAVELET METHOD

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Abstract In this paper, a numerical method consists of combining Haar wavelet method and Tikhonov regularization method to determine unknown boundary condition and unknown nonlinear source term for the generalized time-delayed Burgers-Fisher equation using noisy data is presented. A stable numerical solution is determined for the problem. We also show that the rate of convergence of the method is as exponential $\left(O\left(\frac{1}{2^{J+1}} \right) \right)$, where J is maximal level of resolution of wavelet. Some numerical results are reported to show the efficiency and robustness of the proposed approach for solving the inverse problems.

Keywords Ill-posed inverse problems, Haar wavelet method, Tikhonov regularization method, error estimation, convergence analysis.

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1. Introduction

Inverse problems appear in many important scientific and technological fields. Hence analysis, design implementation and testing of inverse algorithms are also great scientific and technological interest.

Mathematically, the inverse problems belong to the class of problems called the ill-posed problems, i.e. small errors in the measured data can lead to large deviations in the estimated quantities. As a consequence, their solution does not satisfy the general requirement of existence, uniqueness and stability under small changes to the input data. To overcome such difficulties, a variety of techniques for solving inverse problems have been proposed [1, 2, 4–6, 10, 17, 25, 27–29, 34] and among the most versatile methods the following can be mentioned: Tikhonov regularization [31], iterative regularization [2], mollification [23], BFM (Base Function Method) [28], SFDM (Semi Finite Difference Method) [21] and the FSM (Function Specification Method) [4].

Beck and Murio [5] presented a new method that combines the function specification method of Beck with the regularization technique of Tikhonov. Murio and

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Paloschi [22] proposed a combined procedure based on a data filtering interpretation of the mollification method and FSM. Beck et al. [4] compared the FSM, the Tikhonov regularization and the iterative regularization using experimental data.

Wavelet transform or wavelet analysis is a recently developed mathematical tool for many problems. One of the popular families of wavelet is Haar wavelets. Haar functions [15] have been used from 1910 when they were introduced by the Hungarian mathematician, Haar [13]. The Haar transform is one of the earliest of what is known now as a compact, dyadic and orthonormal wavelet transform. The Haar function, being an odd rectangular pulse pair, is the simplest and oldest orthonormal wavelet with compact support. In the mean time, several definitions of the Haar functions and various generalizations have been published and used. They were intended to adopt this concept to some practical applications as well as to extend it in applications to different classes of signals. Haar functions appear very attractive in many applications for example, image coding, edge extraction and binary logic design.

After discretizing the differential equations in a conventional way like the finite difference approximation, wavelets can be used for algebraic manipulations in the system of equations obtained which lead to better condition number of the resulting system.

The previous work [15], in the system analysis via Haar wavelets was led by Chen and Hsiao [7], who first derived a Haar operational matrix for the integrals of the Haar functions vector and put the application for the Haar analysis into the dynamical systems. Then, the pioneer work in state analysis of linear time delayed systems via Haar wavelets was laid down by Hsiao [16], who first proposed a Haar product matrix and a coefficient matrix. Hsiao and Wang proposed a key idea to transform the time-varying function and its product with states into a Haar product matrix. Kalpana and Raja Balachandar [18] presented Haar wavelet based method of analysis for observer design in the generalized state space or singular system of transistor circuits. Also, in [26], Haar basis and Legendre wavelet methods were compared.

In the present work, we consider the following generalized time-delayed Burgers-Fisher equation with Haar wavelet method, to determine **case 1.** $u(x,t)$ and unknown boundary condition $u(0,t)$ and **case 2.** $u(x,t)$ and unknown nonlinear source term $f(u)$.

$$\tau u_{tt} + \left(1 - \tau \frac{df}{du}\right)u_t = u_{xx} - pu^{\frac{l}{m}}u_x + f(u), \qquad f(u) = qu(1 - u^{\frac{l}{m}}), \qquad (1.1)$$

where p, q are constants, l, m are positive integers and τ is a time-delayed constant. Equation (1.1) is an important model, its special types have been applied to describe the forest fire [24], population growth, Neolithic transitions [3,11], the interaction between the reaction mechanism, convection effect and diffusion transport [9,32], etc.

The exact solution of equation (1.1), with the traveling wave solution, obtained by Zhang [33],

$$u(x,t) = u(\xi = x - ct) = \left[\frac{1}{2}\left(1 + \tanh\frac{1}{2}\frac{lp(l+m)(q\tau+1)}{m^2p^2\tau - (l+m)^2}\xi\right)\right]^{\frac{m}{l}},$$

where constants l, m, p, q and τ satisfy $\frac{mp(l+m)(q\tau+1)}{m^2p^2\tau-(l+m)^2} > 0$ and $c = \frac{m^2p^2+q(l+m)^2}{mp(l+m)(q\tau+1)}$ is the soliton velocity.

The organization of this paper is as follows: In Section 2, we introduce Haar wavelets. Partially, in subsection 2.1, function approximation is presented. In Section 3, we formulate and solve an inverse problem for finding unknown boundary condition. In detail, we calculate error estimation of this problem in subsection 3.1. Solution of an inverse problem for finding unknown nonlinear source term will be discussed in Section 4. Convergence analysis of this method is describe in Section 5 and some numerical examples are presented in Section 6. Finally, concluding remarks are given in Section 7.

2. Haar wavelets

For $x \in [0, 1)$, the orthogonal set of Haar wavelet functions are defined by [29],

$$
h_i(x) = \begin{cases} 1, & x \in [\frac{k}{m}, \frac{k+0.5}{m}), \\ -1, & x \in [\frac{k+0.5}{m}, \frac{k+1}{m}), \\ 0, & \text{elsewhere.} \end{cases} \tag{2.1}
$$

Integer $m = 2^j$, $(j = 0, 1, \ldots, J)$ indicates the level of the wavelet; maximal level of resolution is J. $k = 0, 1, \ldots, m - 1$ is the translation parameter. The index i is calculated by $i = m + k + 1$; in the case of minimal values $m = 1$, $k = 0$ we have $i = 2$, the maximal value of i is $i = 2^{J+1} = M$. It is assumed that the value $i = 1$ corresponds to the scaling function for which $h_1 \equiv 1$ in $[0, 1)$.

Let us define the collocation point $x_l = \frac{l-0.5}{M}$, $(l = 1, 2, \ldots, M)$ and discretize the Haar functions $h_i(x)$. In this way we get the coefficient matrix H and the operational matrices of integration P and Q, which are M-square matrices, are defined by the equations

$$
(H)_{il} = (h_i(x_l)), \tag{2.2}
$$

$$
(PH)_{il} = \int_0^{x_l} h_i(x)\, dx, \tag{2.3}
$$

$$
(QH)_{il} = \int_0^{x_l} \int_0^x h_i(s)\, ds\, dx. \tag{2.4}
$$

The elements of the matrices H, P and Q can be evaluated by (2.2), (2.3) and (2.4). For instance when $M = 2, 4$ we have,

$$
H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P_2 = \frac{1}{4} \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \quad Q_2 = \frac{1}{32} \begin{pmatrix} 5 & -4 \\ 4 & -3 \end{pmatrix},
$$

$$
H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad P_4 = \frac{1}{16} \begin{pmatrix} 8 & -4 & -2 & -2 \\ 4 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \quad Q_4 = \frac{1}{128} \begin{pmatrix} 21 & -16 & -4 & -12 \\ 16 & -11 & -4 & -4 \\ 6 & -2 & -3 & 0 \\ 2 & -2 & 0 & -3 \end{pmatrix}.
$$

2.1. Function approximation

Any function $f \in L^2([0,1))$ can be expanded into a Haar series of infinite terms [29],

$$f(x) = c_1 h_1(x) + \sum_{n=2}^{\infty} c_n h_n(x),$$

where the Haar coefficients are determined as

$$c_n = 2^j \int_0^1 f(x) h_n(x) \, dx, \qquad n = 2^j + k + 1, \quad j \geqslant 0, \quad 0 \leqslant k < 2^j,$$

specially $c_1 = \int_0^1 f(x) \, dx$. So

$$f(x) = c_1 h_1(x) + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j - 1} c_{2^j + k + 1} h_{2^j + k + 1}(x).$$

If $f(x)$ is piecewise constant by itself, or may be approximated as piecewise constant during each subinterval, then $f(x)$ will be terminated at finite terms, that is,

$$f(x) \cong c_1 h_1(x) + \sum_{n=2}^{M} c_n h_n(x) = c_1 h_1(x) + \sum_{j=0}^{J} \sum_{k=0}^{2^j - 1} c_{2^j + k + 1} h_{2^j + k + 1}(x) = C_M^T H_M(x),$$

where $'T'$ means transpose and

$$C_M^T = \left(c_1, c_2, \ldots, c_M \right), \qquad H_M(x) = \left(h_1(x), h_2(x), \ldots, h_M(x) \right)^T.$$

3. The inverse problem of finding $(u(x,t), u(0,t))$

Denote $\Omega = \{(x,t) \ : \ 0 < x < 1, \ 0 < t < t_f\}$, consider the following inverse problem for the equation (1.1).

$$\tau u_{tt} + \left(1 - \tau \frac{df}{du} \right) u_t = u_{xx} - p u^{\frac{l}{m}} u_x + f(u), \qquad (x,t) \in \Omega, \tag{3.1a}$$

$$u(x,0) = \varphi(x), \qquad\qquad x \in [0,1], \tag{3.1b}$$

$$u_t(x,0) = \psi(x), \qquad\qquad x \in [0,1], \tag{3.1c}$$

$$u(0,t) = g_1(t), \qquad\qquad t \in [0,t_f], \tag{3.1d}$$

$$u(1,t) = g_2(t), \qquad\qquad t \in [0,t_f], \tag{3.1e}$$

and the overspecified condition

$$u(a,t) = k(t), \qquad\qquad a \in (0,1), \quad t \in [0,t_f], \tag{3.1f}$$

where $\varphi(x)$ and $\psi(x)$ are continuous known function, $g_2(t)$ and $k(t)$ are infinitely differentiable known functions and $t_f > 0$ represents the final time, while the functions $u(x,t)$ and $g_1(t)$ are unknown, which remains to be determined from some interior temperature measurements.

Now, let us divide the interval $[0, t_f]$ into N equal parts of length $\Delta t = \dfrac{t_f}{N}$ and denote $t_s = (s-1)\Delta t$, $s = 1, 2, \ldots, N$. We assume that $\ddot{u}''(x,t)$ can be expanded in terms of Haar wavelets as, [29],

$$\ddot{u}''(x,t) \cong C_M^T H_M(x), \tag{3.2}$$

where $\cdot = \partial/\partial t$ and $' = \partial/\partial x$ and the vector C_M^T is constant in each subinterval $[t_s, t_{s+1}]$, $s = 1, 2, \ldots, N$. Integrating formula (3.2) twice with respect to t from t_s to t, then twice with respect to x from a to x and by using the boundary condition $u(1,t)$ and the overspecified condition $u(a,t)$, we can obtain

$$\dot{u}''(x,t) = (t-t_s)C_M^T H_M(x) + \dot{u}''(x,t_s),$$

$$u''(x,t) = \frac{1}{2}(t-t_s)^2 C_M^T H_M(x) + u''(x,t_s) + (t-t_s)\dot{u}''(x,t_s),$$

$$\dot{u}'(x,t) = (t-t_s)C_M^T\Big[\frac{1}{a-1}PF - \frac{1}{a-1}Q_M H_M(a) + P_M H_M(x)\Big] + \dot{u}'(x,t_s)$$
$$+ \frac{1}{1-a}\Big[g_2'(t) - g_2'(t_s)\Big] - \frac{1}{1-a}\Big[k'(t) - k'(t_s)\Big],$$

$$u'(x,t) = \frac{1}{2}(t-t_s)^2 C_M^T\Big[\frac{1}{a-1}PF - \frac{1}{a-1}Q_M H_M(a) + P_M H_M(x)\Big]$$
$$+ \frac{t-t_s}{1-a}\Big[k'(t_s) - g_2'(t_s)\Big] + u'(x,t_s) + (t-t_s)\dot{u}'(x,t_s)$$
$$+ \frac{1}{1-a}\Big[g_2(t) - g_2(t_s)\Big] + \frac{1}{1-a}\Big[k(t_s) - k(t)\Big],$$

$$\ddot{u}(x,t) = C_M^T\Big[Q_M H_M(x) + \frac{1-x}{a-1}Q_M H_M(a) + \frac{x-a}{a-1}PF\Big] + \frac{1-x}{1-a}k''(t)$$
$$+ \frac{x-a}{1-a}g_2''(t), \tag{3.3}$$

$$\dot{u}(x,t) = (t-t_s)C_M^T\Big[Q_M H_M(x) + \frac{1-x}{a-1}Q_M H_M(a) + \frac{x-a}{a-1}PF\Big]$$
$$+ \frac{1-x}{1-a}\Big[k'(t) - k'(t_s)\Big] + \dot{u}(x,t_s) + \frac{x-a}{1-a}\Big[g_2'(t) - g_2'(t_s)\Big],$$

$$u(x,t) = \frac{1}{2}(t-t_s)^2 C_M^T\Big[Q_M H_M(x) + \frac{1-x}{a-1}Q_M H_M(a) + \frac{x-a}{a-1}PF\Big] + u(x,t_s)$$
$$+ (t-t_s)\dot{u}(x,t_s) + \frac{t-t_s}{1-a}\Big[(x-1)k'(t_s) - (x-a)g_2'(t_s)\Big]$$
$$+ \frac{x-1}{1-a}\Big[k(t_s) - k(t)\Big] + \frac{x-a}{1-a}\Big[g_2(t) - g_2(t_s)\Big], \tag{3.4}$$

where H, P and Q are obtained from (2.2)-(2.4) and the vector F is defined as

$$F = \Big[1, \underbrace{0, \ldots, 0}_{(M-1)}\Big]^T.$$

Discretizing the results by assuming $x \to x_l$, $t \to t_{s+1}$, we obtain

$$\dot{u}''(x_l, t_{s+1}) = \Delta t C_M^T H_M(x_l) + \dot{u}''(x_l, t_s),$$

$$u''(x_l, t_{s+1}) = \frac{\Delta t^2}{2}C_M^T H_M(x_l) + u''(x_l, t_s) + \Delta t \dot{u}''(x_l, t_s),$$

$$\dot{u}'(x_l, t_{s+1}) = \Delta t C_M^T \Big[\frac{1}{a-1} PF - \frac{1}{a-1} Q_M H_M(a) + P_M H_M(x_l) \Big] + \dot{u}'(x_l, t_s)$$

$$+ \frac{1}{1-a} \Big[g_2'(t_{s+1}) - g_2'(t_s) \Big] - \frac{1}{1-a} \Big[k'(t_{s+1}) - k'(t_s) \Big],$$

$$u'(x_l, t_{s+1}) = \frac{\Delta t^2}{2} C_M^T \Big[\frac{1}{a-1} PF - \frac{1}{a-1} Q_M H_M(a) + P_M H_M(x_l) \Big] + u'(x_l, t_s)$$

$$+ \Delta t \dot{u}'(x_l, t_s) + \frac{1}{1-a} \Big[g_2(t_{s+1}) - g_2(t_s) \Big] + \frac{1}{1-a} \Big[k(t_s) - k(t_{s+1}) \Big]$$

$$+ \frac{\Delta t}{1-a} \Big[k'(t_s) - g_2'(t_s) \Big],$$

$$\ddot{u}(x_l, t_{s+1}) = C_M^T \Big[Q_M H_M(x_l) + \frac{1-x_l}{a-1} Q_M H_M(a) + \frac{x_l - a}{a-1} PF \Big] + \frac{1-x_l}{1-a} k''(t_{s+1})$$

$$+ \frac{x_l - a}{1-a} g_2''(t_{s+1}),$$

$$\dot{u}(x_l, t_{s+1}) = \Delta t C_M^T \Big[Q_M H_M(x_l) + \frac{1-x_l}{a-1} Q_M H_M(a) + \frac{x_l - a}{a-1} PF \Big] + \dot{u}(x_l, t_s)$$

$$+ \frac{1-x_l}{1-a} \Big[k'(t_{s+1}) - k'(t_s) \Big] + \frac{x_l - a}{1-a} \Big[g_2'(t_{s+1}) - g_2'(t_s) \Big],$$

$$u(x_l, t_{s+1}) = \frac{\Delta t^2}{2} C_M^T \Big[Q_M H_M(x_l) + \frac{1-x_l}{a-1} Q_M H_M(a) + \frac{x_l - a}{a-1} PF \Big] + u(x_l, t_s)$$

$$+ \Delta t \dot{u}(x_l, t_s) + \frac{x_l - 1}{1-a} \Big[k(t_s) - k(t_{s+1}) \Big] + \frac{x_l - a}{1-a} \Big[g_2(t_{s+1}) - g_2(t_s) \Big]$$

$$+ \frac{\Delta t}{1-a} \Big[(x_l - 1) k'(t_s) - (x_l - a) g_2'(t_s) \Big].$$

In the following scheme

$$\tau \ddot{u}(x_l, t_s) = u''(x_l, t_s) - p \Big(u(x_l, t_s) \Big)^{\frac{l}{m}} u'(x_l, t_s) + f \Big(u(x_l, t_s) \Big) - \Big(1 - \tau \frac{df}{du} \Big) \dot{u}(x_l, t_s), \tag{3.5}$$

which leads us from the time layer t_s to t_{s+1} is used, where x_l is collocation point. Substituting (3.3) into (3.5), we obtain

$$\tau C_M^T \Big[Q_M H_M(x_l) + \frac{1-x_l}{a-1} Q_M H_M(a) + \frac{x_l - a}{a-1} PF \Big]$$

$$= u''(x_l, t_s) - p \Big(u(x_l, t_s) \Big)^{\frac{l}{m}} u'(x_l, t_s) + f \Big(u(x_l, t_s) \Big) - \Big(1 - \tau \frac{df}{du} \Big) \dot{u}(x_l, t_s)$$

$$- \tau \Big[\frac{1-x_l}{1-a} k''(t_s) + \frac{x_l - a}{1-a} g_2''(t_s) \Big]. \tag{3.6}$$

From the formula (3.6) the wavelet coefficient C_M^T can be calculated.

In matrix form, the wavelet coefficient C_M^T can be obtained from solving the following matrix equation

$$A\lambda = B. \tag{3.7}$$

The matrix A is ill-conditioned. On the other hand, as $k(t)$ is affected by measurement errors, the estimate of λ by (3.7) will be unstable so that the Tikhonov regularization method must be used to control this measurement errors. The Tikhonov regularized solution ([14, 19, 30, 31]) to the system of linear algebraic equation (3.7) is given by

$$\mathcal{F}_\alpha(\lambda) = \|A\lambda - B\|_2^2 + \alpha \|R^{(0)}\lambda\|_2^2,$$

where the matrix $R^{(0)}$, is given by (see e.g. [20]),

$$R^{(0)} = I_{M \times M} \in \mathbb{R}^{M \times M}.$$

Therefore, we obtain the Tikhonov regularized solution of the regularized equation as

$$\lambda_\alpha = \left[A^T A + \alpha (R^{(0)})^T R^{(0)} \right]^{-1} A^T B.$$

In our computation, we use the generalized cross-validation (GCV) scheme to determine a suitable value of α ([8, 12]).

3.1. Error estimation

In this section, we consider error estimation for $u(0, t) = g_1(t)$. We let $u^*(0, t)$ be estimated value of $u(0, t)$. Now we will prove the following error estimation theorem.

Theorem 3.1. *If* $g_1, g_2, k \in C^1([0, 1])$ *then*

$$\left| u(0, t) - u^*(0, t) \right| \le 2\Delta t L \left[\frac{\Delta t}{4L} |\beta| + \frac{2}{1 - a} \right],$$

where L *and* β *are constants.*

Proof. According to equation (3.4), we can obtain

$$u^*(0, t) = \frac{(t - t_s)^2}{2} C_M^T \left[\frac{1}{a - 1} Q_M H_M(a) - \frac{a}{a - 1} PF \right] + u(0, t_s) + (t - t_s) \dot{u}(0, t_s)$$
$$+ \frac{t - t_s}{1 - a} \left[a g_2'(t_s) - k'(t_s) \right] + \frac{1}{1 - a} \left[k(t) - k(t_s) \right] - \frac{a}{1 - a} \left[g_2(t) - g_2(t_s) \right]$$
$$= \frac{(t - t_s)^2}{2} C_M^T \left[\frac{1}{a - 1} Q_M H_M(a) - \frac{a}{a - 1} PF \right] + g_1(t_s) + (t - t_s) g_1'(t_s)$$
$$+ \frac{t - t_s}{1 - a} \left[a g_2'(t_s) - k'(t_s) \right] + \frac{1}{1 - a} \left[k(t) - k(t_s) \right] - \frac{a}{1 - a} \left[g_2(t) - g_2(t_s) \right].$$

Now

$$\left| u(0, t) - u^*(0, t) \right| = \left| g_1(t) - \frac{(t - t_s)^2}{2} C_M^T \left[\frac{1}{a - 1} Q_M H_M(a) - \frac{a}{a - 1} PF \right] - g_1(t_s) \right.$$
$$- (t - t_s) g_1'(t_s) - \frac{t - t_s}{1 - a} \left[a g_2'(t_s) - k'(t_s) \right]$$
$$\left. - \frac{1}{1 - a} \left[k(t) - k(t_s) \right] + \frac{a}{1 - a} \left[g_2(t) - g_2(t_s) \right] \right|.$$

So

$$\left| u(0, t) - u^*(0, t) \right| \le \frac{(t - t_s)^2}{2} \left| \beta \right| + \left| g_1(t) - g_1(t_s) \right| + (t - t_s) \left| g_1'(t_s) \right| + \frac{1}{1 - a} \left| k(t) - k(t_s) \right|$$
$$+ \frac{(t - t_s)}{1 - a} \left| k'(t_s) \right| + \frac{a}{1 - a} \left| g_2(t) - g_2(t_s) \right| + \frac{a(t - t_s)}{1 - a} \left| g_2'(t_s) \right|,$$

where $\beta = C_M^T \left[\frac{1}{a - 1} Q_M H_M(a) - \frac{a}{a - 1} PF \right]$ is a real number. Using mean value theorem of derivatives,

$$\exists\, t_s \le \xi < t,$$

such that

$$\left|u(0,t)-u^*(0,t)\right|\leq\frac{(t-t_s)^2}{2}|\beta|+(t-t_s)|g_1'(\xi)|+(t-t_s)|g_1'(t_s)|+\frac{(t-t_s)}{1-a}|k'(\xi)|$$

$$+\frac{(t-t_s)}{1-a}|k'(t_s)|+\frac{a(t-t_s)}{1-a}|g_2'(\xi)|+\frac{a(t-t_s)}{1-a}|g_2'(t_s)|$$

$$\leq\frac{(t-t_s)^2}{2}|\beta|+(t-t_s)L_1+(t-t_s)L_1+\frac{(t-t_s)}{1-a}L_2+\frac{(t-t_s)}{1-a}L_2$$

$$+\frac{a(t-t_s)}{1-a}L_3+\frac{a(t-t_s)}{1-a}L_3$$

$$=\frac{(t-t_s)^2}{2}|\beta|+2(t-t_s)L_1+\frac{2(t-t_s)}{1-a}L_2+\frac{2a(t-t_s)}{1-a}L_3$$

$$\leq\frac{\Delta t^2}{2}|\beta|+2\Delta t L_1+\frac{2\Delta t}{1-a}L_2+\frac{2a\Delta t}{1-a}L_3.$$

Put $L=\max\{L_1,L_2,L_3\}$, so we have

$$\left|u(0,t)-u^*(0,t)\right|\leq 2\Delta t L\left[\frac{\Delta t}{4L}|\beta|+\frac{2}{1-a}\right].$$

$$\square$$

4. The inverse problem of finding $(u(x,t),f(u))$

In this part, we consider the following inverse problem for the equation (1.1).

$$\tau u_{tt}+\left(1-\tau\frac{df}{du}\right)u_t=u_{xx}-pu^{\frac{l}{m}}u_x+f(u),\qquad (x,t)\in\Omega, \tag{4.1a}$$

$$u(x,0)=\varphi(x),\qquad\qquad x\in[0,1], \tag{4.1b}$$

$$u_t(x,0)=\psi(x),\qquad\qquad x\in[0,1], \tag{4.1c}$$

$$u(0,t)=g_1(t),\qquad\qquad t\in[0,t_f], \tag{4.1d}$$

$$u(1,t)=g_2(t),\qquad\qquad t\in[0,t_f], \tag{4.1e}$$

and the overspecified condition

$$u(a,t)=k(t),\qquad\qquad a\in(0,1),\quad t\in[0,t_f], \tag{4.1f}$$

where $\varphi(x)$ and $\psi(x)$ are continuous known function, $g_1(t)$, $g_2(t)$ and $k(t)$ are infinitely differentiable known functions and $t_f>0$ represents the final time, while the functions $u(x,t)$ and $f(u)$ are unknown, which remains to be determined from some interior temperature measurements.

Now, as we saw in section 3, assume that $\ddot{u}''(x,t)$ can be expanded in terms of Haar wavelets as,

$$\ddot{u}''(x,t)\cong C_M^T H_M(x). \tag{4.2}$$

Integrating formula (4.2) twice with respect to t from t_s to t, then twice with respect

to x from 0 to x and by using the boundary conditions $u(0,t)$ and $u(1,t)$, we obtain

$$\dot{u}''(x,t) = (t-t_s)C_M^T H_M(x) + \dot{u}''(x,t_s),$$

$$u''(x,t) = \frac{1}{2}(t-t_s)^2 C_M^T H_M(x) + u''(x,t_s) + (t-t_s)\dot{u}''(x,t_s),$$

$$\dot{u}'(x,t) = (t-t_s)C_M^T\Big[P_M H_M(x) - PF\Big] + \dot{u}'(x,t_s) + \Big[g_2'(t) - g_2'(t_s)\Big]$$
$$- \Big[g_1'(t) - g_1'(t_s)\Big],$$

$$u'(x,t) = \frac{1}{2}(t-t_s)^2 C_M^T\Big[P_M H_M(x) - PF\Big] + \Big[g_2(t) - g_2(t_s)\Big] + \Big[g_1(t_s) - g_1(t)\Big]$$
$$+ u'(x,t_s) + (t-t_s)\dot{u}'(x,t_s) + (t-t_s)\Big[g_1'(t_s) - g_2'(t_s)\Big],$$

$$\ddot{u}(x,t) = C_M^T\Big[Q_M H_M(x) - xPF\Big] + (1-x)g_1''(t) + xg_2''(t), \tag{4.3}$$

$$\dot{u}(x,t) = (t-t_s)C_M^T\Big[Q_M H_M(x) - xPF\Big] + (1-x)[g_1'(t) - g_1'(t_s)]$$
$$+ x[g_2'(t) - g_2'(t_s)] + \dot{u}(x,t_s),$$

$$u(x,t) = \frac{1}{2}(t-t_s)^2 C_M^T\Big[Q_M H_M(x) - xPF\Big] + u(x,t_s) + (t-t_s)\dot{u}(x,t_s)$$
$$+ (x-1)\Big[g_1(t_s) - g_1(t)\Big] + x\Big[g_2(t) - g_2(t_s)\Big]$$
$$+ (t-t_s)\Big[(x-1)g_1'(t_s) - xg_2'(t_s)\Big]. \tag{4.4}$$

Discretizing the results by assuming $x \to x_l$, $t \to t_{s+1}$, one can obtain

$$\dot{u}''(x_l,t_{s+1}) = \Delta t\, C_M^T H_M(x_l) + \dot{u}''(x_l,t_s),$$

$$u''(x_l,t_{s+1}) = \frac{\Delta t^2}{2}C_M^T H_M(x_l) + u''(x_l,t_s) + \Delta t\, \dot{u}''(x_l,t_s),$$

$$\dot{u}'(x_l,t_{s+1}) = \Delta t\, C_M^T\Big[P_M H_M(x_l) - PF\Big] + \dot{u}'(x_l,t_s) + \Big[g_2'(t_{s+1}) - g_2'(t_s)\Big]$$
$$- \Big[g_1'(t_{s+1}) - g_1'(t_s)\Big],$$

$$u'(x_l,t_{s+1}) = \frac{\Delta t^2}{2}C_M^T\Big[P_M H_M(x_l) - PF\Big] + u'(x_l,t_s) + \Delta t\, \dot{u}'(x_l,t_s)$$
$$+ \Delta t\Big[g_1'(t_s) - g_2'(t_s)\Big] + \Big[g_2(t_{s+1}) - g_2(t_s)\Big] + \Big[g_1(t_s) - g_1(t_{s+1})\Big],$$

$$\ddot{u}(x_l,t_{s+1}) = C_M^T\Big[Q_M H_M(x_l) - x_l PF\Big] + (1-x_l)g_1''(t_{s+1}) + x_l g_2''(t_{s+1}),$$

$$\dot{u}(x_l,t_{s+1}) = \Delta t\, C_M^T\Big[Q_M H_M(x_l) - x_l PF\Big] + \dot{u}(x_l,t_s) + (1-x_l)[g_1'(t_{s+1}) - g_1'(t_s)]$$
$$+ x_l[g_2'(t_{s+1}) - g_2'(t_s)],$$

$$u(x_l,t_{s+1}) = \frac{\Delta t^2}{2}C_M^T\Big[Q_M H_M(x_l) - x_l PF\Big] + (x_l - 1)\Big[g_1(t_s) - g_1(t_{s+1})\Big] + u(x_l,t_s)$$
$$+ x_l\Big[g_2(t_{s+1}) - g_2(t_s)\Big] + \Delta t\, \dot{u}(x_l,t_s) + \Delta t\Big[(x_l - 1)g_1'(t_s) - x_l g_2'(t_s)\Big].$$

In the following scheme

$$\tau\ddot{u}(x_l,t_s) = u''(x_l,t_s) - p\Big(u(x_l,t_s)\Big)^{\frac{l}{m}}u'(x_l,t_s) + f\Big(u(x_l,t_s)\Big) - \Big(1 - \tau\frac{df}{du}\Big)\dot{u}(x_l,t_s),$$
$$\tag{4.5}$$

which leads us from the time layer t_s to t_{s+1} is used, where x_l is collocation point. Substituting (4.3) into (4.5), we obtain

$$\tau C_M^T \Big[Q_M H_M(x_l) - x_l PF \Big] = u''(x_l, t_s) - p\Big(u(x_l, t_s)\Big)^{\frac{l}{m}} u'(x_l, t_s) + f\Big(u(x_l, t_s)\Big)$$
$$- \Big(1 - \tau \frac{df}{du}\Big) \dot{u}(x_l, t_s) - \tau \Big[(1 - x_l)g_1''(t_s) + x_l g_2''(t_s)\Big].$$
$$(4.6)$$

Remark 4.1. In this work, the polynomial form proposed for the unknown $f(u)$ before performing the inverse calculation. Therefore $f(u)$ approximated as

$$f(u) = u(a_1 + a_2 u + \cdots + a_n u^{n-1}), \tag{4.7}$$

where $\{a_1, a_2, \ldots, a_n\}$ are constants which remain to be determined simultaneously by least-squares method.

4.1. Least-Squares Minimization Technique

To minimize the sum of the squares of the deviations between $u^*(a, t)$ (calculated by (4.4)) and $k(t)$, we use least-squares method. The error in the estimates $E(a_1, a_2, \ldots, a_n)$ can be expressed as

$$E(a_1, a_2, \ldots, a_n) = \sum_{s=1}^{N} \Big(u^*(a, t_{s+1}) - k(t_{s+1})\Big)^2, \tag{4.8}$$

which must be minimized. The estimated values of a_i are determined until the value of $E(a_1, a_2, \ldots, a_n)$ is minimum. The computational procedure for estimating unknown coefficients a_i, differentiation of $E(a_1, a_2, \ldots, a_n)$ with respect to $a_1, a_2, \ldots, a_n$ will be performed. Thus the linear system corresponding to coefficients a_i of Equation (4.8) can be expressed as

$$\Lambda \Theta = \Pi. \tag{4.9}$$

The matrix Λ is ill-conditioned. On the other hand, as $k(t)$ is affected by measurement errors, the estimate of Θ by (4.9) will be unstable so that the Tikhonov regularization method must be used to control this measurement errors (see section 3).

5. Convergence analysis of Haar wavelet method

In this part, we assume that $f_J(x)$ is an approximation of $f(x)$ as follows (subsection 2.1),

$$f_J(x) \cong c_1 h_1(x) + \sum_{j=0}^{J} \sum_{k=0}^{2^j - 1} c_{2^j + k + 1} h_{2^j + k + 1}(x),$$

then the corresponding error is defined as,

$$e_J(x) = f(x) - f_J(x).$$

So we have

$$e_J(x) = \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} c_{2^j+k+1} h_{2^j+k+1}(x).$$

Now we will prove the following convergence theorem.

Theorem 5.1. *Suppose that $f(x)$ satisfies the Lipschitz condition on $[0,1]$, that is,*

$$\exists \, \kappa > 0, \ \forall \, x,y \in [0,1] \ : \ |f(x) - f(y)| \le \kappa|x-y|. \tag{5.1}$$

Then the Haar wavelet method will be convergent in the sense that $e_J(x)$ goes to zero as M goes to infinity. Moreover, the convergence is of order exponential, that is,

$$\|e_J\|_2 = O\left(\frac{1}{M}\right).$$

Proof.   We have

$$\|e_J\|_2^2 = \int_0^1 \left(\sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} c_{2^j+k+1} h_{2^j+k+1}(x) \right)^2 \mathrm{d}x$$

$$= \int_0^1 \left(\sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} c_{2^j+k+1} h_{2^j+k+1}(x) \right) \left(\sum_{m=J+1}^{\infty} \sum_{n=0}^{2^m-1} c_{2^m+n+1} h_{2^m+n+1}(x) \right) \mathrm{d}x$$

$$= \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} \sum_{m=J+1}^{\infty} \sum_{n=0}^{2^m-1} c_{2^j+k+1} c_{2^m+n+1} \left(\int_0^1 h_{2^j+k+1}(x) h_{2^m+n+1}(x) \mathrm{d}x \right).$$

The orthonormality of the sequence $h_i(x)$ on $[0,1)$ implies that

$$\int_0^1 h_l(x) h_{l'}(x) \ \mathrm{d}x = \begin{cases} \dfrac{1}{2^j} & l = l', \\ 0 & l \ne l', \end{cases}$$

so

$$\|e_J\|_2^2 = \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} \frac{1}{2^j} c_{2^j+k+1}^2.$$

Since $c_{2^j+k+1} = 2^j \int_0^1 f(x) h_{2^j+k+1}(x) \mathrm{d}x$, according to (2.1), we can write

$$c_{2^j+k+1} = 2^j \left(\int_{\frac{k}{2^j}}^{\frac{k+0.5}{2^j}} f(x) \mathrm{d}x - \int_{\frac{k+0.5}{2^j}}^{\frac{k+1}{2^j}} f(x) \mathrm{d}x \right).$$

Now, using the mean value theorem, we can conclude

$$\exists \, x_1 \in \left[\frac{k}{2^j}, \frac{k+0.5}{2^j} \right], \quad x_2 \in \left[\frac{k+0.5}{2^j}, \frac{k+1}{2^j} \right],$$

such that

$$\int_{\frac{k}{2^j}}^{\frac{k+0.5}{2^j}} f(x) \mathrm{d}x = \frac{1}{2^{j+1}} f(x_1), \qquad \int_{\frac{k+0.5}{2^j}}^{\frac{k+1}{2^j}} f(x) \mathrm{d}x = \frac{1}{2^{j+1}} f(x_2).$$

Thus, we can compute c_{2^j+k} as follows

$$c_{2^j+k+1} = 2^j \left(\frac{1}{2^{j+1}} f(x_1) - \frac{1}{2^{j+1}} f(x_2) \right) = \frac{1}{2} \Big(f(x_1) - f(x_2) \Big) \leq \frac{\kappa}{2} \Big(x_1 - x_2 \Big) \leq \frac{\kappa}{2^{j+1}}.$$

The first inequality is obtained with regard to relation (5.1). On the other hand, we have

$$\|e_J\|_2^2 = \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} \frac{1}{2^j} \, c_{2^j+k}^2 \leq \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} \frac{1}{2^j} \, \frac{\kappa^2}{2^{2j+2}} = \frac{\kappa^2}{4} \sum_{j=J+1}^{\infty} 4^{-j} = \frac{\kappa^2}{3} 4^{-J-1}.$$

Since $M = 2^{J+1}$, we obtain

$$\|e_J\|_2^2 \leq \frac{\kappa^2}{3} M^{-2},$$

and so

$$\|e_J\|_2 \leq \frac{\kappa}{M\sqrt{3}}.$$

Therefore, the Haar wavelet method will be convergent, i.e.

$$\lim_{J \to \infty} e_J(x) = 0.$$

Moreover, the convergence is of order exponential, that is,

$$\|e_J\|_2 = O\left(\frac{1}{2^{J+1}} \right) = O\left(\frac{1}{M} \right).$$

$\square$

6. Numerical experiments and discussion

In this section, we are going to demonstrate numerically, some of results for unknown boundary condition and unknown nonlinear source term, in the two inverse problems (3.1) and (4.1). The main aim here is to show the applicability of the present method, described in Section 3 and 4 for solving the inverse problems (3.1) and (4.1). As we know, the inverse problems are ill-posed and therefore it is necessary to investigate the stability of the present method by giving a test problem. The proposed method is written in the MATLAB 7.14 (R2012a) and is tested on a personal computer with intel(R) core(TM)2 Duo CPU and 4GB RAM.

Remark 6.1. In an inverse problem there are two sources of error in the estimation. The first source is the unavoidable bias deviation (deterministic error). The second source of error is the variance due to the amplification of measurement errors (stochastic error). The global effect of deterministic and stochastic errors is considered in the mean squared error or total error [6].

Therefore, we compare exact and approximate solutions by considering total error S defined by

$$S = \left[\frac{1}{N-1} \sum_{i=1}^{N} \Big(\Phi_i - \Phi_i^* \Big)^2 \right]^{\frac{1}{2}},$$

where N, Φ and Φ^* are the number of estimated values, the exact values and the estimated values, respectively.

Example 6.1. In this example we solve problem (3.1) satisfying,

$$2u_{tt} + \left(1 - 2\frac{df}{du}\right)u_t = u_{xx} - 3u^{\frac{2}{3}}u_x + f(u), \qquad f(u) = u(1 - u^{\frac{2}{3}}),$$

with given data

$$u(x,0) = \left[\frac{1}{2}\left(1 + \tanh\frac{45x}{137}\right)\right]^{\frac{3}{2}}, \qquad\qquad x \in [0,1],$$

$$u_t(x,0) = \frac{3}{2}\left[\frac{1}{2}\left(1 + \tanh\frac{45x}{137}\right)\right]^{\frac{1}{2}}\left[\frac{53}{411}\left(\tanh^2\frac{45x}{137} - 1\right)\right], \qquad x \in [0,1],$$

$$u(1,t) = \left[\frac{1}{2}\left(1 - \tanh(\frac{106t}{411} - \frac{45}{137}))\right]^{\frac{3}{2}}, \qquad\qquad t \in [0,t_f],$$

$$u(0.1,t) = \left[\frac{1}{2}\left(1 - \tanh(\frac{106t}{411} - \frac{9}{274}))\right]^{\frac{3}{2}}, \qquad\qquad t \in [0,t_f].$$

The exact solution of this problem is

$$u(x,t) = \left[\frac{1}{2}\left(1 - \tanh(\frac{106t}{411} - \frac{45x}{137}))\right]^{\frac{3}{2}}, \qquad (x,t) \in \Omega,$$

$$u(0,t) = \left[\frac{1}{2}\left(1 - \tanh\frac{106t}{411}\right)\right]^{\frac{3}{2}}, \qquad\qquad t \in [0,t_f].$$

Table 1. The comparison between the exact and numerical solutions for $u(0.7,t)$ with the noisy data.

| t	$u(0.7,t)_{Exact}$	$u^*(0.7,t)_{Haar}$	$|u(0.7,t) - u^*(0.7,t)|$
0.01	0.478483	0.478483	$2.190927e - 08$
0.02	0.477045	0.477046	$6.512387e - 08$
0.1	0.465541	0.465541	$3.772212e - 07$
0.11	0.464102	0.464103	$4.121232e - 07$
0.5	0.408242	0.408244	$1.686209e - 06$
0.51	0.406822	0.406824	$1.675071e - 06$
0.8	0.366141	0.366135	$6.175630e - 06$
0.81	0.364759	0.364752	$6.867010e - 06$
0.9	0.352393	0.352378	$1.507527e - 05$
0.91	0.351027	0.351011	$1.623431e - 05$
1	0.338823	0.338793	$2.932435e - 05$
S			$8.192168e - 06$

We suppose that $u(x,t)$ and $u(0,t)$ are the exact solutions of the problem (3.1). Also let $u^*(x,t)$ and $u^*(0,t)$ be solutions obtained by applying the given method. Our results obtained for $u(0.7,t)$ and $u(0,t)$ when $M = 4$, $a = 0.1$, $t_f = 1$ and

$\Delta t = 0.01$, with noisy data (input data$+0.001\times$rand(1)) are presented in Tables 1, 2 and Figures 1, 2. Also, to validate the theoretical results of the Theorem 3.1, we consider the performance profile with respect to the error estimated value. To this end, for simplicity, we use ESV to denote the error estimated value $|u(0,t)-u^*(0,t)|$ and $\delta = 2\Delta t L \left[\frac{\Delta t}{4L}|\beta| + \frac{2}{1-a} \right]$. The analysis results are presented in Figure 3.

Table 2. The comparison between the exact and numerical solutions for $u(0,t)$ with the noisy data.

| t	$u(0,t)_{Exact}$	$u^*(0,t)_{Haar}$	$|u(0,t)-u^*(0,t)|$
0.01	0.352187	0.352187	$4.635614e-09$
0.02	0.350821	0.350821	$1.806012e-08$
0.1	0.339967	0.339968	$4.584213e-07$
0.11	0.338619	0.338620	$5.611583e-07$
0.5	0.287771	0.287790	$1.887326e-05$
0.51	0.286517	0.286537	$1.987431e-05$
0.8	0.251349	0.251414	$6.521831e-05$
0.81	0.250180	0.250248	$6.741311e-05$
0.9	0.239801	0.239890	$8.935511e-05$
0.91	0.238663	0.238755	$9.204687e-05$
1	0.228567	0.228686	$1.187297e-04$
S			$4.807710e-05$

|	
Execution Time (second)	41.874612
Condition Number of Matrix A Before Regularization	45.551828
Condition Number of Matrix A After Regularization	1
Regularization Parameter (α)	0.251808
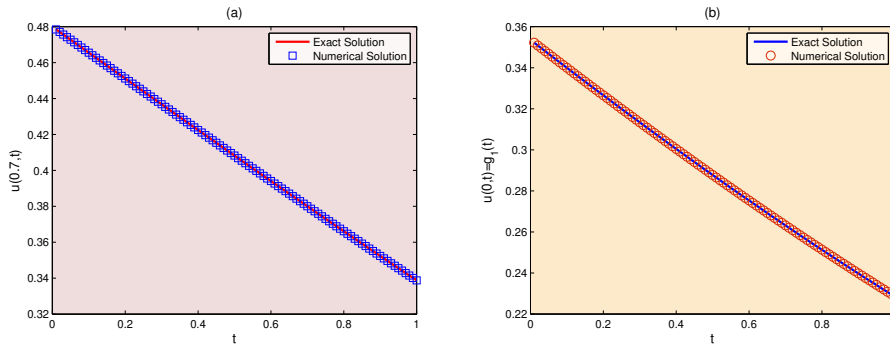
Figure 1. The comparison between the exact and numerical results for (a) $u(0.7,t)$ and (b) $u(0,t)$ in Example 6.1 with the noisy data by using Haar wavelet method.

Example 6.2. In this example we solve problem (4.1) satisfying,

$$2u_{tt} + \left(1 - 2\frac{df}{du}\right)u_t = u_{xx} - 3u^{\frac{2}{3}}u_x + f(u), \qquad f(u) = a_1 u + a_2 u^2,$$
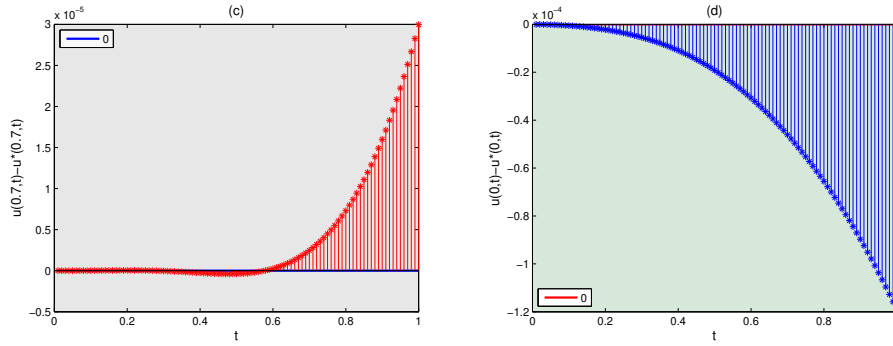
Figure 2. Difference between (c) $u(0.7, t)_{Exact}$ and $u^*(0.7, t)_{Haar}$ and (d) $u(0, t)_{Exact}$ and $u^*(0, t)_{Haar}$ in Example 6.1 with the noisy data.
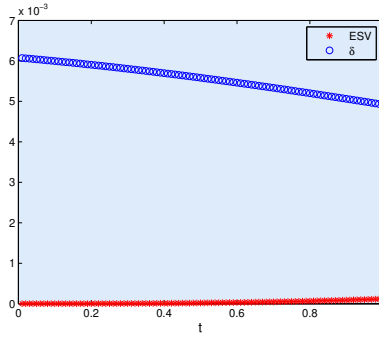
Figure 3. Error estimation performance profile to validate the Theorem 3.1.

with given data

$$u(x, 0) = \left[\frac{1}{2} \left(1 + \tanh \frac{45x}{137} \right) \right]^{\frac{3}{2}}, \qquad\qquad\qquad x \in [0, 1],$$

$$u_t(x, 0) = \frac{3}{2} \left[\frac{1}{2} \left(1 + \tanh \frac{45x}{137} \right) \right]^{\frac{1}{2}} \left[\frac{53}{411} \left(\tanh^2 \frac{45x}{137} - 1 \right) \right], \qquad x \in [0, 1],$$

$$u(0, t) = \left[\frac{1}{2} \left(1 - \tanh \frac{106t}{411} \right) \right]^{\frac{3}{2}}, \qquad\qquad\qquad t \in [0, t_f],$$

$$u(1, t) = \left[\frac{1}{2} \left(1 - \tanh(\frac{106t}{411} - \frac{45}{137}) \right) \right]^{\frac{3}{2}}, \qquad\qquad t \in [0, t_f],$$

$$u(0.1, t) = \left[\frac{1}{2} \left(1 - \tanh(\frac{106t}{411} - \frac{9}{274}) \right) \right]^{\frac{3}{2}}, \qquad\qquad t \in [0, t_f].$$

The exact solution of this problem is

$$u(x, t) = \left[\frac{1}{2} \left(1 - \tanh(\frac{106t}{411} - \frac{45x}{137}) \right) \right]^{\frac{3}{2}}, \qquad\qquad (x, t) \in \Omega,$$

$$f(u) = u(1 - u^{\frac{2}{3}}), \qquad\qquad\qquad (x,t) \in \Omega.$$

By using Least-square Technique, we estimate $\{a_1, a_2\} = \{0.824918, -0.918815\}$. We suppose that $u(x,t)$ and $f(u(x,t))$ are the exact solutions of the problem (4.1). Also let $u^*(x,t)$ and $f^*(u(x,t))$ be solutions obtained by applying the given method. Our results obtained for $u(0.5,t)$ and $f(u(0.5,t))$ when $M = 4$, $a = 0.1$, $t_f = 1$ and $\Delta t = 0.01$, with noisy data (input data+0.001×rand(1)) are presented in Tables 3, 4 and Figures 4, 5.

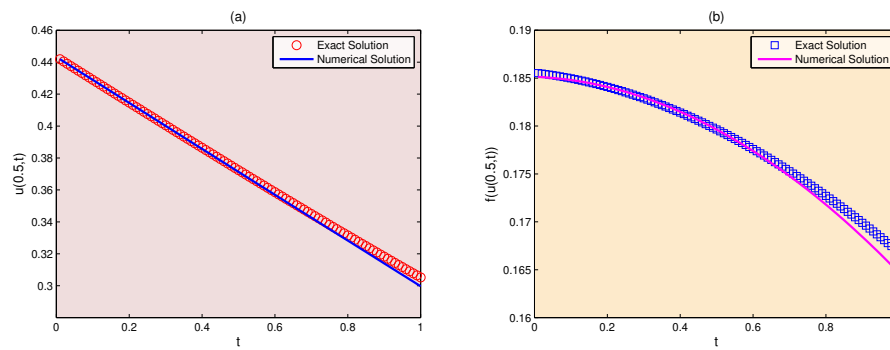**Table 3.** The comparison between the exact and numerical solutions for $u(0.5,t)$ with the noisy data.

t	$u(0.5,t)_{Exact}$	$u^*(0.5,t)_{Haar}$	$|u(0.5,t) - u^*(0.5,t)|$
0.01	0.441864	0.441864	$3.835945e-07$
0.02	0.440429	0.440427	$1.557134e-06$
0.1	0.428963	0.428922	$4.091025e-05$
0.11	0.427532	0.427482	$4.965070e-05$
0.5	0.372420	0.371330	$1.089327e-03$
0.51	0.371030	0.369894	$1.136525e-03$
0.8	0.331480	0.328298	$3.181501e-03$
0.81	0.330145	0.326866	$3.278735e-03$
0.9	0.318224	0.313986	$4.238164e-03$
0.91	0.316911	0.312557	$4.353909e-03$
1	0.305195	0.299724	$5.471442e-03$
S			$2.317585e-03$

**Table 4.** The comparison between the exact and numerical solutions for $f(u(0.5,t))$ with the noisy data.

t	$f(u(0.5,t))$	$f^*(u(0.5,t))$	$|f(u(0.5,t)) - f^*(u(0.5,t))|$
0.01	0.185525	0.185124	$4.007887e-04$
0.02	0.185476	0.185104	$3.722952e-04$
0.1	0.184976	0.184803	$1.736141e-04$
0.11	0.184900	0.184748	$1.524839e-04$
0.5	0.179643	0.179638	$5.082485e-06$
0.51	0.179451	0.179431	$1.959687e-05$
0.8	0.172712	0.171801	$9.116255e-04$
0.81	0.172441	0.171481	$9.602617e-04$
0.9	0.169897	0.168440	$1.456735e-03$
0.91	0.169602	0.168084	$1.518502e-03$
1	0.166850	0.164716	$2.134009e-03$
S			$7.512681e-04$

Execution Time (second)	41.273501
Condition Number of Matrix Λ Before Regularization	29.663759
Condition Number of Matrix Λ After Regularization	1
Regularization Parameter (α)	$2.200734e - 11$

Figure 4. The comparison between the exact and numerical results for (a) $u(0.5, t)$ and (b) $f(u(0.5, t))$ in Example 6.2 with the noisy data by using Haar wavelet method.
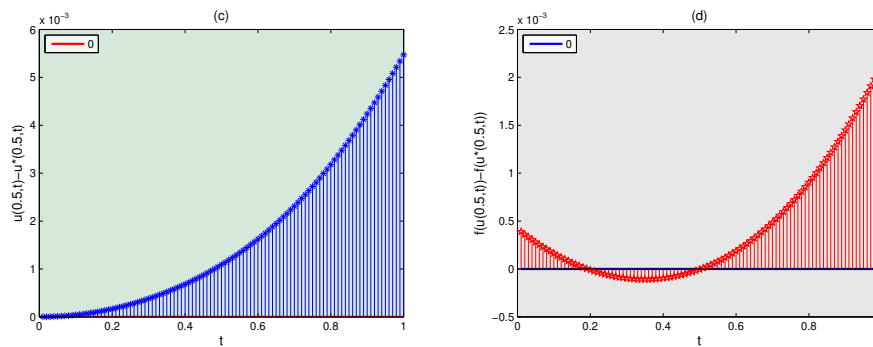
Figure 5. Difference between (c) $u(0.5, t)_{Exact}$ and $u^*(0.5, t)_{Haar}$ and (d) $f(u(0.5, t))_{Exact}$ and $f^*(0.5, t)_{Haar}$ in Example 6.2 with noisy data.

7. Conclusion

The Haar wavelet method has been employed to estimate unknown boundary condition and unknown nonlinear source term, was proposed for problems (3.1) and (4.1). The present study, successfully applies the numerical method to inverse problems. Since the obtained coefficient matrix of these problems is usually ill-conditioned, hence to regularize the resultant ill-conditioned, we have applied the Tikhonov regularization method to obtain a stable numerical approximation to the solution. The convergence rate of the proposed method has been discussed and shown that it is $O(\frac{1}{M})$. The strong point of the method is its easy and simple computation with low-storage space and cost. Moreover, this scheme, does not require extra quest to deal with the nonlinear terms. In general, the reported results show that the promising behavior of the proposed method for solving problems (3.1) and (4.1).

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