Journal of Applied Analysis and Computation
Volume 7, Number 4, November 2017, 1586–1597
Website:http://jaac-online.com/
DOI:10.11948/2017096

EXACT TRAVELLING WAVE SOLUTIONS FOR NONLINEAR SCHRÖDINGER EQUATION WITH VARIABLE COEFFICIENTS*

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Abstract In this paper, two nonlinear Schrödinger equations with variable coefficients in nonlinear optics are investigated. Based on travelling wave transformation and the extended $(\frac{G'}{G})$-expansion method, exact travelling wave solutions to nonlinear Schrödinger equation with time-dependent coefficients are derived successfully, which include bright and dark soliton solutions, triangular function periodic solutions, hyperbolic function solutions and rational function solutions.

Keywords Nonlinear Schrödinger equation, travelling wave solution, extended $(\frac{G'}{G})$-expansion method.

MSC(2010) 35C07, 35Q41.

1. Introduction

Nonlinear evolution equations (NLEEs) are used to describe many nonlinear phenomena which occur in the field of nature science. The constructing of exact solutions to nonlinear evolution equations is an important topic in studying these nonlinear phenomena. During the past decades, much efforts have been made to give exact travelling wave solutions of NLEEs and various powerful approaches have been proposed for obtaining exact travelling wave solutions, such as the inverse scattering method, the homogenous balance method, the tanh-function method, the bilinear method and so on [1, 4–8, 10, 11, 16, 17, 21–23, 27].

Lately, M L Wang et al. [26] presented a new method called the $(\frac{G'}{G})$-expansion method and illustrated that this is a powerful approach to obtaining the analytic solution of NLEEs. The key ideas of the $(\frac{G'}{G})$-expansion method are that the travelling wave solutions of NLEEs can be expressed by a polynomial in $(\frac{G'}{G})$, where $G = G(\zeta)$ satisfies a second order linear ordinary differential equation(ODE). By means of various extended $(\frac{G'}{G})$-expansion method, many researchers have obtained travelling wave solutions of a large of NLEEs [3, 14, 15, 24, 30, 31]. More recently, Malik et al. [18] proposed the extended $(\frac{G'}{G})$-expansion method based on a new assumption. They illustrated the efficiency of this new method to solve the Bogoyavlenskii equation and gave some exact solutions.

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*The authors were supported by National Natural Science Foundation of China (11371221), the National Science Foundation of Shandong Province (ZR2014AM032) and the Science Foundation of Heze University (XY16PY04).

The well-known cubic nonlinear Schrödinger equation(NLSE) which describe the wave dynamics of nonlinear pulses propagation in monomode fiber is in the following form [29]

$$iu_t + \alpha u_{xx} + \gamma |u|^2 u = 0, \tag{1.1}$$

where $u(x,t)$ is a complex-valued function of related system dynamics in nonlinear optics, while x, t are the independent variables. And α, γ are the group velocity dispersion and self-phase modulation parameters, respectively.

In the present paper, we employ the extended $(\frac{G'}{G})$-expansion method to construct more travelling wave solutions for the following nonlinear Schrödinger equation with time-dependent coefficient

$$iu_t + f(t)u_{xx} + g(t)|u|^2 u = 0, \tag{1.2}$$

and the generalized cubic nonlinear Schrödinger equation with variable coefficients [28]

$$iu_t + \frac{1}{2}\beta(t)u_{xx} + \alpha(t)|u|^2 u - iu = 0, \tag{1.3}$$

where $u(x,t)$ is a complex function that represents complex amplitude of the wave form, the variable x represents the normalized propagation distance, and t represents the retarded time. Under some special parameters of $f(t)$ and $g(t)$, Eq.(1.2) can be reduced to some Schrödinger-tpye equations. For example, If $f(t) = \alpha, g(t) = \gamma$, then Eq.(1.2) become Eq.(1.1). Taghizadeh *et al* constructed exact solutions of this model by employing the first integral method [20] . Due to widely application of Eqs.(1.2) and (1.3) in various area of physics, such as nonlinear optics, plasma physics and quantum mechanics [1,2,9,12,13,25], much attention has been paid to Eq.(1.2) and their various generalizations. For more details about this equation, the readers are advised to see reference [19] and references therein.

The rest of the paper is organized as follows. In Section 2, the extended $(\frac{G'}{G})$-expansion method is described. In Section 3, the proposed method and several suitable transformations is used to solve two nonlinear Schrödinger equation with timedependent coecient and more travelling wave solutions are derived. In Section 4, some conclusions and remarks are given.

2. Description of the extended $(\frac{G'}{G})$-expansion method

In this section, we describe the main steps of the extended $(\frac{G'}{G})$-expansion method.

For a given NLEEs

$$F(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \cdots) = 0, \tag{2.1}$$

with two independent variables $X = (x,t)$. To obtain the travelling wave solution of Eq.(2.1), we make use of the generalized wave transformation

$$u(x,t) = u(\zeta), \zeta = \zeta(X).$$

Then Eq.(2.1) can be reduced to the following ODE

$$G(u, u', u'', \cdots) = 0. \tag{2.2}$$

we introduce the solution of Eq.(2.2) as

$$u(\zeta) = a_0(X) + \sum_{i=1}^{N}\Bigg[a_i(X)\Big(\frac{G'}{G}\Big)^i + b_i(X)\Big(\frac{G'}{G}\Big)^{i-1}\sqrt{1+\frac{1}{\mu}\Big(\frac{G'}{G}\Big)^2}\Bigg], \tag{2.3}$$

where $G = G(\zeta)$ satisfies the following second order ODE

$$G''(\zeta) + \mu G(\zeta) = 0, \tag{2.4}$$

where $G' = \frac{\mathrm{d}G(\zeta)}{\mathrm{d}\zeta}$, $G'' = \frac{\mathrm{d}^2G(\zeta)}{\mathrm{d}\zeta^2}$, and $a_0(X), a_i(X), b_i(X)$ $(i = 1, 2, \cdots, N)$ are functions of X to be determined. The integer N in Eq.(2.3) can be determined according to the homogeneous balance principle. We define the degree of $u(\zeta)$ as $D[u(\zeta)] = M$, then

$$D\Big[\frac{d^pu}{d\xi^p}\Big] = M + p, \quad D\Big[u^p\Big(\frac{d^qu}{d\xi^q}\Big)^s\Big] = Mp + (M+q)s.$$

Substituting (2.3) along with (2.4) into Eq.(2.2), collecting all terms with the same power of $(\frac{G'(\zeta)}{G(\zeta)})^i\sqrt{1+\frac{1}{\mu}(\frac{G'}{G})^2}^{\,j}$ $(j = 0, 1, i = 0, 1, 2, \cdots)$ and setting each coefficients to zero, we get a system of algebraic equations for $a_0(X), a_i(X), b_i(X)$.

The solution of Eq.(2.4) are given as

$$(\frac{G'}{G}) = \begin{cases} \sqrt{-\mu}(\frac{A_1\sinh\sqrt{-\mu}\zeta + A_2\cosh\sqrt{-\mu}\zeta}{A_1\cosh\sqrt{-\mu}\zeta + A_2\sinh\sqrt{-\mu}\zeta}), & \mu < 0, \\ \sqrt{\mu}(\frac{A_1\sin\sqrt{\mu}\zeta + A_2\cos\sqrt{\mu}\zeta}{A_2\sin\sqrt{\mu}\zeta - A_1\cos\sqrt{\mu}\zeta}), & \mu > 0, \\ \frac{A_2}{A_1 + A_2\zeta}, & \mu = 0, \end{cases}$$

which can be written in the following simplified form

$$(\frac{G'}{G}) = \begin{cases} \sqrt{-\mu}\tanh(\sqrt{-\mu}\zeta + \zeta_0), & \mu < 0, \ \tanh\zeta_0 = \frac{A_2}{A_1}, \ \mid \frac{A_1}{A_2} \mid > 1, \\ \sqrt{-\mu}\coth(\sqrt{-\mu}\zeta + \zeta_0), & \mu < 0, \ \coth\zeta_0 = \frac{A_2}{A_1}, \ \mid \frac{A_1}{A_2} \mid < 1, \\ \sqrt{\mu}\cot(\sqrt{\mu}\zeta + \zeta_0), & \mu > 0, \ \cot\xi_0 = -\frac{A_2}{A_1}, \\ \frac{A_2}{A_1 + A_2\zeta}, & \mu = 0. \end{cases}$$

By solving the over-determined algebraic system with the help of Mathematica, we obtain the values of a_0, a_i and b_i. Substituting these results into (2.3) and combining with the solution of Eq.(2.4), we obtain some exact solutions of Eq.(2.1).

3. Application

3.1. The exact solutions of Eq.(1.2)

In this section, we will illustrate the extended $(\frac{G'}{G})$-expansion method mentioned in Section 2 to obtain the exact solutions of Eq.(1.2).

Since $u = u(x,t)$ is a complex function, we introduce travelling wave transformation in the following form

$$u(x,t) = v(x,t)\exp(i\eta(x,t)), \tag{3.1}$$

where $v(x,t)$ and $\eta(x,t)$ are amplitude and phase functions respectively.

Substituting the wave transformation (3.1) into Eq.(1.2), we have

$$v_t + 2f(t)v_x\eta_x + f(t)v\eta_{xx} = 0, \tag{3.2}$$

and

$$-\eta_t v + f(t)v_{xx} - f(t)\eta_x^2 v + g(t)v^3 = 0. \tag{3.3}$$

Considering the homogeneous balance between v^3 and v_{xx} in Eq.(3.3), we get $N = 1$. In order to search for exact solutions, we assume that Eqs.(3.2) and (3.3) have the following formal solutions

$$v(x,t) = v(\xi) = a_0 + a_1(\frac{G'}{G}) + b_1\sqrt{1 + \frac{1}{\mu}\Big(\frac{G'}{G}\Big)^2}, \tag{3.4}$$

$$\xi = p(t)x + q(t), \quad \eta(x,t) = \alpha(t)x^2 + \beta(t)x + \gamma(t), \tag{3.5}$$

where $G = G(\xi)$ satisfies Eq.(2.4). Substituting (3.4) along with (2.4) into Eqs.(3.2) and (3.3), the left hand side of Eqs.(3.2)-(3.3) is converted into a polynomial of $x^i(\frac{G'}{G})^j\sqrt{1+\frac{1}{\mu}(\frac{G'}{G})^2}^k$ $(k = 0,1)$. Collecting the coefficient of power of $x^i(\frac{G'}{G})^j\sqrt{1+\frac{1}{\mu}(\frac{G'}{G})^2}^k$ $(k = 0,1)$ and setting each coefficient to zero yields a set of algebraic system.

$$\begin{gathered}
-b_1\gamma'(t) + \mu b_1 f(t)p^2(t) - b_1 f(t)\beta^2(t) + 3a_0^2 b_1 g(t) + b_1^3 g(t) = 0,\\
-a_0\gamma'(t) - a_0 f(t)\beta^2(t) + a_0^3 g(t) + 3a_0 b_1^2 g(t) = 0,\\
-a_1\gamma'(t) + 2a_1\mu f(t)p^2(t) - a_1 f(t)\beta^2(t) + 3a_0^2 a_1 g(t) + 3a_1 b_1^2 g(t) = 0,\\
-a_0\beta'(t) - 4a_0\alpha(t)\beta(t)f(t) = 0,\\
-a_1\beta'(t) - 4a_1\alpha(t)\beta(t)f(t) = 0,\\
-b_1\beta'(t) - 4b_1\alpha(t)\beta(t)f(t) = 0,\\
-a_0\alpha'(t) - 4a_0 f(t)\alpha^2(t) = 0,\\
-a_1\alpha'(t) - 4a_1 f(t)\alpha^2(t) = 0,\\
-b_1\alpha'(t) - 4b_1 f(t)\alpha^2(t) = 0,\\
3a_0a_1^2 + \tfrac{3}{\mu}a_0 b_1^2 = 0,\\
2b_1 f(t)p^2(t) + \tfrac{1}{\mu}b_1^3 g(t) + 3a_1^2 b_1 g(t) = 0,\\
2a_1 f(t)p^2(t) + a_1^3 g(t) + \tfrac{3}{\mu}a_1 b_1^2 g(t) = 0.
\end{gathered}$$

Solving the above algebraic system with the help of Mathematica, we get the following three cases.

Case 1.

$$\begin{gathered}
a_0 = 0, \quad b_1 = 0, \quad a_1 = a_1, \quad \beta(t) = c_1\alpha(t), \quad p(t) = c_2\alpha(t),\\
q(t) = c_1c_2 2\alpha(t) + c_3, \quad g(t) = -\tfrac{2c_2^2 f(t)\alpha^2(t)}{a_1^2},\\
\gamma(t) = \tfrac{c_1^2 - 2\mu c_2^2}{4}\alpha(t),
\end{gathered}$$

where $c_i(i=1,2,3,4,5)$ are arbitrary constants, $c_2 \neq 0$ and $\alpha(t)$ is given by

$$\alpha(t)\left[4\int f(t)dt + c_5\right] = 1.$$

In this case, the exact solution of Eq.(1.2) has the form

$$u(x,t) = \begin{cases} \sqrt{-\mu}a_1(\frac{A_1\sinh\sqrt{-\mu}\xi + A_2\cosh\sqrt{-\mu}\xi}{A_1\cosh\sqrt{-\mu}\xi + A_2\sinh\sqrt{-\mu}\xi}) \times \exp[i\eta(x,t)], & \mu < 0, \\ \sqrt{\mu}a_1(\frac{A_2\cos\sqrt{\mu}\xi - A_1\sin\sqrt{\mu}\xi}{A_1\cos\sqrt{\mu}\xi + A_2\sin\sqrt{\mu}\xi}) \times \exp[i\eta(x,t)], & \mu > 0, \\ \frac{A_2}{A_1 + A_2\zeta}, & \mu = 0. \end{cases}$$

When $\mu < 0, \tanh\xi_0 = \frac{A_2}{A_1}, \mid \frac{A_1}{A_2} \mid > 1$, the dark solitary wave solution of Eq.(1.2) can be expressed by

$$u_{1,1}(x,t) = \sqrt{-\mu}a_1\tanh(\sqrt{-\mu}\xi + \xi_0) \times \exp[i\eta(x,t)]. \tag{3.6}$$

When $\mu < 0, \coth\xi_0 = \frac{A_2}{A_1}, \mid \frac{A_1}{A_2} \mid < 1$, the hyperbolic function solution of Eq.(1.2) can be written as

$$u_{1,2}(x,t) = \sqrt{-\mu}a_1\coth(\sqrt{-\mu}\xi + \xi_0) \times \exp[i\eta(x,t)]. \tag{3.7}$$

When $\mu > 0, \tan\xi_0 = \frac{A_1}{A_2}$, the triangular periodic wave solution of Eq.(1.2) is given by

$$u_{1,3}(x,t) = \sqrt{\mu}a_1\cot(\sqrt{\mu}\xi + \xi_0) \times \exp[i\eta(x,t)], \tag{3.8}$$

where $\eta(x,t) = \alpha(t)(x^2 + c_1x + \frac{c_1^2 - 2\mu c_2^2}{4})$.

Case 2.

$$a_0 = 0, \quad a_1 = 0, \quad b_1 = b_1, \quad \beta(t) = c_1\alpha(t), \quad p(t) = c_2\alpha(t),$$
$$q(t) = \frac{c_1c_2}{2}\alpha(t) + c_3,$$
$$g(t) = -\frac{2\mu c_2^2 f(t)\alpha^2(t)}{b_1^2}, \quad \gamma(t) = \frac{c_1^2 + \mu c_2^2}{4}\alpha(t),$$

where $\alpha(t)$ satisfies the constraint condition

$$\alpha(t)\left[4\int f(t)dt + c_5\right] = 1.$$

In this case, the exact solution of Eq.(1.2) has the form

$$u(x,t) = \begin{cases} b_1\sqrt{1 - (\frac{A_1\sinh\sqrt{-\mu}\xi + A_2\cosh\sqrt{-\mu}\xi}{A_1\cosh\sqrt{-\mu}\xi + A_2\sinh\sqrt{-\mu}\xi})^2} & \\ \times\exp[i\eta(x,t)], & \mu < 0, \\ b_1\sqrt{1 + (\frac{A_2\cos\sqrt{\mu}\xi - A_1\sin\sqrt{\mu}\xi}{A_1\cos\sqrt{\mu}\xi + A_2\sin\sqrt{\mu}\xi})^2} & \\ \times\exp[i\eta(x,t)], & \mu > 0. \end{cases}$$

When $\mu < 0, \tanh\xi_0 = \frac{A_2}{A_1}, \mid \frac{A_1}{A_2} \mid > 1$, the bright solitary wave solution of Eq.(1.2) can be expressed by

$$u_{2,1}(x,t) = b_1\mathrm{sech}(\sqrt{-\mu}\xi + \xi_0) \times \exp[i\eta(x,t)]. \tag{3.9}$$

When $\mu < 0, \coth\xi_0 = \frac{A_2}{A_1}, \mid \frac{A_1}{A_2} \mid < 1$, the singular solution of Eq.(1.2) becomes

$$u_{2,2}(x,t) = ib_1\mathrm{csch}(\sqrt{-\mu}\xi + \xi_0) \times \exp[i\eta(x,t)]. \tag{3.10}$$

When $\mu > 0, \tan\xi_0 = \frac{A_1}{A_2}$, the triangular periodic solution of Eq.(1.2) is given by

$$u_{2,3}(x,t) = b_1 \csc(\sqrt{\mu}\xi + \xi_0) \times \exp[i\eta(x,t)], \tag{3.11}$$

where $\eta(x,t) = \alpha(t)(x^2 + c_1x + \frac{c_1^2+\mu c_2^2}{4})$.

Case 3.

$$a_0 = 0, \quad a_1 = a_1, \quad b_1 = \pm\sqrt{\mu}a_1, \quad \beta(t) = c_1\alpha(t),$$
$$p(t) = c_2\alpha(t), \quad q(t) = \tfrac{c_1c_2}{2}\alpha(t) + c_3,$$
$$g(t) = -\tfrac{c_2^2 f(t)\alpha^2(t)}{2a_1^2}, \quad \gamma(t) = \tfrac{c_1^2 - \frac{3}{2}\mu c_2^2}{4}\alpha(t),$$

where $\alpha(t)$ satisfies the constraint condition

$$\alpha(t)\left[4\int f(t)dt + c_5\right] = 1.$$

Substituting Case 3 into (3.4) and (3.5), and using (3.1) and the general solutions of (2.4), we get two types of exact solutions for equation (1.2).

When $\mu < 0$, the hyperbolic function solution can be expressed by

$$\begin{aligned} u_{3,1}(x,t) = \ & \sqrt{-\mu}a_1\Bigg[\Big(\frac{A_1\sinh\sqrt{-\mu}\xi + A_2\cosh\sqrt{-\mu}\xi}{A_1\cosh\sqrt{-\mu}\xi + A_2\sinh\sqrt{-\mu}\xi}\Big) \\ & \pm i\sqrt{1 - \Big(\frac{A_1\sinh\sqrt{-\mu}\xi + A_2\cosh\sqrt{-\mu}\xi}{A_1\cosh\sqrt{-\mu}\xi + A_2\sinh\sqrt{-\mu}\xi}\Big)^2}\Bigg] \exp[i\eta(x,t)]. \end{aligned} \tag{3.12}$$

If we set $\mu < 0, \tanh\xi_0 = \frac{A_2}{A_1}, \mid \frac{A_1}{A_2} \mid > 1$, then the bright-dark soliton solution of (1.2) are derived

$$\begin{aligned} u(x,t) = \ & \sqrt{-\mu}a_1[\tanh(\sqrt{-\mu}\xi + \xi_0) \\ & \pm i\mathrm{sech}(\sqrt{-\mu}\xi + \xi_0)] \times \exp(i\eta(x,t)). \end{aligned} \tag{3.13}$$

If we set $\mu < 0, \coth\xi_0 = \frac{A_2}{A_1}, \mid \frac{A_1}{A_2} \mid < 1$, then the solution (3.12) becomes the following singular wave solution

$$\begin{aligned} u(x,t) = \ & \sqrt{-\mu}a_1[\coth(\sqrt{-\mu}\xi + \xi_0) \\ & \pm \mathrm{csch}(\sqrt{-\mu}\xi + \xi_0)] \times \exp(i\eta(x,t)). \end{aligned} \tag{3.14}$$

When $\mu > 0$, the triangular periodic solution can be given by

$$\begin{aligned} u_{3,2}(x,t) = \ & \sqrt{\mu}a_1\Bigg[\Big(\frac{A_2\cos\sqrt{\mu}\xi - A_1\sin\sqrt{\mu}\xi}{A_1\cos\sqrt{\mu}\xi + A_2\sin\sqrt{\mu}\xi}\Big) \\ & \pm\sqrt{1 + \Big(\frac{A_2\cos\sqrt{\mu}\xi - A_1\sin\sqrt{\mu}\xi}{A_1\cos\sqrt{\mu}\xi + A_2\sin\sqrt{\mu}\xi}\Big)^2}\Bigg] \times \exp[i\eta(x,t)]. \end{aligned} \tag{3.15}$$

If we set $\mu > 0, \tan\xi_0 = \frac{A_1}{A_2}$, then the solution (3.13) becomes

$$u(x,t) = \sqrt{\mu}a_1[\cot(\sqrt{\mu}\xi + \xi_0) \pm i\csc(\sqrt{\mu}\xi + \xi_0)] \times \exp(i\eta(x,t)),$$

where $\eta(x,t) = \alpha(t)(x^2 + c_1x + \frac{c_1^2 - \frac{3}{2}\mu c_2^2}{4})$.

3.2. The exact solutions of Eq.(1.3)

In this section, we will construct the exact solutions of Eq.(1.3) by using of the extended $(\frac{G'}{G})$-expansion method.

Since $u = u(x,t)$ is a complex function, we assume that travelling wave transformation is in the form

$$u(x,t) = v(x,t)\exp(i\theta(x,t)), \tag{3.16}$$

where $v(x,t)$ and $\theta(x,t)$ are amplitude and phase functions respectively. Substituting the wave transformation (3.16) into (1.3) and separating the real and imaginary parts, we have

$$-v\theta_t + \frac{1}{2}\beta(t)(v_{xx} - v\theta_x^2) + \alpha(t)v^3 = 0, \tag{3.17}$$

and

$$v_t + \frac{1}{2}\beta(t)(2\theta_x v_x + v\theta_{xx}) - v = 0. \tag{3.18}$$

Considering the homogeneous balance in Eq.(3.17), we assume that Eqs.(3.17) and (3.18) have the following solutions

$$v(x,t) = v(\xi) = a_0 + a_1\Big(\frac{G'}{G}\Big) + b_1\sqrt{1 + \frac{1}{\mu}\Big(\frac{G'}{G}\Big)^2}, \tag{3.19}$$

$$\xi = p(t)x + q(t), \quad \theta(x,t) = a(t)x^2 + b(t)x + c(t), \tag{3.20}$$

where $G = G(\xi)$ satisfies Eq.(2.4). Substituting (3.19) along with (2.4) into Eqs.(3.17) and (3.18), the left hand side of Eqs.(3.17)-(3.18) is converted into a polynomial of $x^i(\frac{G'}{G})^j\sqrt{1+\frac{1}{\mu}(\frac{G'}{G})^2}^k$ $(k = 0,1)$. Collecting the coefficient of power of $x^i(\frac{G'}{G})^j\sqrt{1+\frac{1}{\mu}(\frac{G'}{G})^2}^k$ $(k = 0,1)$ and setting each coefficient to zero yields a set of algebraic system.

$$
\begin{gathered}
-a_0a'(t) - 2a_0a^2(t)\beta(t) = 0,\\
-a_1a'(t) - 2a_1a^2(t)\beta(t) = 0,\\
-b_1a'(t) - 2b_1a^2(t)\beta(t) = 0,\\
-b_1b'(t) - 2b_1a(t)b(t)\beta(t) = 0,\\
b_1\beta(t)p^2(t) + \tfrac{\alpha(t)b_1^3}{\mu} + 3a_1^2b_1\alpha(t) = 0,\\
a_1\beta(t)p^2(t) + a_1^3\alpha(t) + \tfrac{3a_1b_1^2\alpha(t)}{\mu} = 0,\\
3a_0a_1^2 + \tfrac{3a_0b_1^2}{\mu} = 0,\\
\tfrac{1}{2}b_1\mu\beta(t)p^2(t) + \alpha(t)b_1^3 - b_1c'(t) + 3a_0^2b_1\alpha(t) - \tfrac{1}{2}b_1\beta(t)b^2(t) = 0,\\
-a_1c'(t) + a_1\mu\beta(t)p^2(t) + 3\alpha(t)a_0^2a_1 + 3a_1b_1^2\alpha(t) - \tfrac{1}{2}a_1b^2(t)\beta(t) = 0,\\
p'(t) + 2p(t)a(t)\beta(t) = 0,\\
q'(t) + p(t)\beta(t)b(t) = 0,\\
a(t)\beta(t) = 1.
\end{gathered}
$$

Solving the above algebraic system with the help of Mathematica, we get the following three cases.

Case 1.

$$a_0=0,\quad b_1=0,\quad a_1=a_1,\quad p(t)=c_2a(t),\quad q(t)=\tfrac{1}{2}c_1c_2a(t)+c_3,$$
$$b(t)=c_1a(t),\quad c(t)=\tfrac{1}{4}(c_1^2-2\mu c_2^2)a(t)+c_4,$$
$$a(t)\beta(t)=1,\quad \alpha(t)=-\tfrac{c_2^2\beta(t)a^2(t)}{a_1^2}.$$

where $c_i(i=1,2,3,4,5)$ are arbitrary constants, and $a(t)$ is given by

$$a(t)\left[2\int\beta(t)dt+c_5\right]=1.$$

In this case, the exact solution of Eq.(1.3) has the form

$$u(x,t)=\begin{cases}\sqrt{-\mu}a_1(\frac{A_1\sinh\sqrt{-\mu}\xi+A_2\cosh\sqrt{-\mu}\xi}{A_1\cosh\sqrt{-\mu}\xi+A_2\sinh\sqrt{-\mu}\xi})\times\exp[i\theta(x,t)], & \mu<0,\\ \sqrt{\mu}a_1(\frac{A_2\cos\sqrt{\mu}\xi-A_1\sin\sqrt{\mu}\xi}{A_1\cos\sqrt{\mu}\xi+A_2\sin\sqrt{\mu}\xi})\times\exp[i\theta(x,t)], & \mu>0,\\ \frac{A_2}{A_1+A_2\zeta}, & \mu=0.\end{cases}$$

Particularly, the dark solitary wave solution of Eq.(1.3) can be expressed by

$$u_1(x,t)=\sqrt{-\mu}a_1\tanh(\sqrt{-\mu}\xi+\xi_0)\times\exp\left\{ia(x,t)\left[x^2+c_1x+\frac{1}{4}(c_1^2-2\mu c_2^2)\right]\right\}.\quad(3.21)$$

The hyperbolic function solution of Eq.(1.3) can be written as

$$u_2(x,t)=\sqrt{-\mu}a_1\coth(\sqrt{-\mu}\xi+\xi_0)\times\exp\left\{ia(x,t)\left[x^2+c_1x+\frac{1}{4}(c_1^2-2\mu c_2^2)\right]\right\}.\quad(3.22)$$

The triangular periodic wave solution of Eq.(1.3) is given by

$$u_3(x,t)=\sqrt{\mu}a_1\cot(\sqrt{\mu}\xi+\xi_0)\times\exp\left\{ia(x,t)\left[x^2+c_1x+\frac{1}{4}(c_1^2-2\mu c_2^2)\right]\right\}.\quad(3.23)$$

Case 2.

$$a_0=0,\quad a_1=0,\quad b_1=b_1,\quad p(t)=c_2a(t),\quad b(t)=c_1a(t),$$
$$q(t)=\tfrac{1}{2}c_1c_2a(t)+c_3,\quad a(t)\beta(t)=1,$$
$$c(t)=\tfrac{1}{4}(c_1^2+\mu c_2^2)a(t)+c_4,$$
$$\alpha(t)=-\tfrac{\mu c_2^2\beta(t)a^2(t)}{b_1^2},$$

where $a(t)$ satisfies the constraint condition

$$a(t)\left[2\int\beta(t)dt+c_5\right]=1.$$

In this case, the exact solution of Eq.(1.3) has the form

$$u(x,t)=\begin{cases}b_1\sqrt{1-(\frac{A_1\sinh\sqrt{-\mu}\xi+A_2\cosh\sqrt{-\mu}\xi}{A_1\cosh\sqrt{-\mu}\xi+A_2\sinh\sqrt{-\mu}\xi})^2}\\ \times\exp[i\theta(x,t)], & \mu<0,\\ b_1\sqrt{1+(\frac{A_2\cos\sqrt{\mu}\xi-A_1\sin\sqrt{\mu}\xi}{A_1\cos\sqrt{\mu}\xi+A_2\sin\sqrt{\mu}\xi})^2}\\ \times\exp[i\theta(x,t)], & \mu>0.\end{cases}$$

The bright solitary wave solution of Eq.(1.3) can be expressed by

$$u_4(x,t) = b_1 \text{sech}(\sqrt{-\mu}\xi + \xi_0) \times \exp\Big\{ia(x,t)\Big[x^2 + c_1 x + \frac{1}{4}(c_1^2 + \mu c_2^2)\Big]\Big\}. \quad (3.24)$$

The singular solution of Eq.(1.3) becomes

$$u_5(x,t) = ib_1 \text{csch}(\sqrt{-\mu}\xi + \xi_0) \times \exp\Big\{ia(x,t)\Big[x^2 + c_1 x + \frac{1}{4}(c_1^2 + \mu c_2^2)\Big]\Big\}. \quad (3.25)$$

The triangular periodic solution of Eq.(1.3) is given by

$$u_6(x,t) = b_1 \csc(\sqrt{\mu}\xi + \xi_0) \times \exp\Big\{ia(x,t)\Big[x^2 + c_1 x + \frac{1}{4}(c_1^2 + \mu c_2^2)\Big]\Big\}. \quad (3.26)$$

Case 3.

$$\begin{gathered} a_0 = 0, \quad a_1 = a_1, \quad b_1 = b_1, \quad p(t) = c_2 a(t), \\ q(t) = \tfrac{1}{2} c_1 c_2 a(t) + c_3, \quad b(t) = c_1 a(t), \\ c(t) = \tfrac{1}{8}(c_1^2 - 2\mu c_2^2) a(t) + c_4, \\ a(t)\beta(t) = 1, \quad \alpha(t) = \tfrac{c_1^2 \beta(t) a^2(t)}{4b_1^2}, \end{gathered}$$

where $a(t)$ satisfies the constraint condition

$$a(t)\Big[2\int \beta(t)dt + c_5\Big] = 1.$$

In this case, we get two types of exact solutions for equation (1.3).

When $\mu < 0$, the hyperbolic function solution can be expressed by

$$\begin{aligned} u_7(x,t) = & \sqrt{-\mu}a_1 \Bigg[\Big(\frac{A_1 \sinh\sqrt{-\mu}\xi + A_2 \cosh\sqrt{-\mu}\xi}{A_1 \cosh\sqrt{-\mu}\xi + A_2 \sinh\sqrt{-\mu}\xi}\Big) \\ & \pm i\sqrt{1 - \Big(\frac{A_1 \sinh\sqrt{-\mu}\xi + A_2 \cosh\sqrt{-\mu}\xi}{A_1 \cosh\sqrt{-\mu}\xi + A_2 \sinh\sqrt{-\mu}\xi}\Big)^2}\,\Bigg] \exp[i\theta(x,t)]. \end{aligned} \quad (3.27)$$

If we set $\mu < 0, \tanh \xi_0 = \frac{A_2}{A_1}, \mid \frac{A_1}{A_2} \mid > 1$, then the bright-dark soliton solution of (1.3) are derived

$$\begin{aligned} u(x,t) = & \sqrt{-\mu}a_1[\tanh(\sqrt{-\mu}\xi + \xi_0) \\ & \pm i\text{sech}(\sqrt{-\mu}\xi + \xi_0)] \times \exp(i\theta(x,t)). \end{aligned} \quad (3.28)$$

If we set $\mu < 0, \coth \xi_0 = \frac{A_2}{A_1}, \mid \frac{A_1}{A_2} \mid < 1$, then the solution (3.27) becomes the following singular wave solution

$$\begin{aligned} u(x,t) = & \sqrt{-\mu}a_1[\coth(\sqrt{-\mu}\xi + \xi_0) \\ & \pm \text{csch}(\sqrt{-\mu}\xi + \xi_0)] \times \exp(i\theta(x,t)). \end{aligned} \quad (3.29)$$

When $\mu > 0$, the triangular periodic solution can be given by

$$\begin{aligned} u_8(x,t) = & \sqrt{\mu}a_1 \Bigg[\Big(\frac{A_2 \cos\sqrt{\mu}\xi - A_1 \sin\sqrt{\mu}\xi}{A_1 \cos\sqrt{\mu}\xi + A_2 \sin\sqrt{\mu}\xi}\Big) \\ & \pm\sqrt{1 + \Big(\frac{A_2 \cos\sqrt{\mu}\xi - A_1 \sin\sqrt{\mu}\xi}{A_1 \cos\sqrt{\mu}\xi + A_2 \sin\sqrt{\mu}\xi}\Big)^2}\,\Bigg] \times \exp[i\theta(x,t)]. \end{aligned} \quad (3.30)$$

If we set $\mu > 0, \tan\xi_0 = \frac{A_1}{A_2}$, then the solution (3.30) becomes

$$u(x,t) = \sqrt{\mu}a_1[\cot(\sqrt{\mu}\xi + \xi_0) \pm i\csc(\sqrt{\mu}\xi + \xi_0)] \times \exp(i\theta(x,t)).$$

where $\theta(x,t) = a(t)[x^2 + c_1x + \frac{c_1^2-2\mu c_2^2}{8}a(t) + c_4],\ \xi = c_2a(t)x + \frac{1}{2}c_1c_2a(t) + c_3$.

4. Discussion and Conclusion

In summary, the extended $(\frac{G'}{G})$-expansion method is developed to construct exact solutions of nonlinear evolution equations with variable coefficients. The key of the used methodology is the idea to transform nonlinear evolution equation with variable coefficients to a system of nonlinear algebraic. By means of solutions to the auxiliary equation, we derive numerous exact travelling wave solutions of nonlinear Schrödinger equations with variable coefficients under constraint condition, which include triangular function periodic solutions, hyperbolic function solutions and rational function solutions. Particularly, when the parameters take some special values, the bright soliton solutions, the dark soliton solutions and combined soliton solutions are obtained. These results may be helpful for an explanation of some practical problems in nonlinear optics. The paper shows that the combination of the extended $(\frac{G'}{G})$-expansion method and several suitable transformations provide a powerful tool for finding exact solutions of the nonlinear evolution equations(NLEEs) with time-dependent coefficients and also can be used to solve other NLEEs with variable coefficients in mathematical physics.

Remark 4.1. All the solutions obtained in this paper for Eqs.(1.2) and (1.3) have been checked by Maple software.

Acknowledgements

The author would like to express her thanks to the referees for careful reading of the manuscript and their helpful suggestions.

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