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ON A SEMILINEAR DOUBLE FRACTIONAL HEAT EQUATION DRIVEN BY FRACTIONAL BROWNIAN SHEET*

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Abstract In this paper, we consider the stochastic heat equation of the form

$$\frac{\partial u}{\partial t} = (\Delta_\alpha + \Delta_\beta)u + \frac{\partial f}{\partial x}(t, x, u) + \frac{\partial^2 W}{\partial t \partial x},$$

where $1 < \beta < \alpha < 2$, $W(t, x)$ is a fractional Brownian sheet, $\Delta_\theta := -(-\Delta)^{\theta/2}$ denotes the fractional Lapalacian operator and $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a nonlinear measurable function. We introduce the existence, uniqueness and Hölder regularity of the solution. As a related question, we consider also a large deviation principle associated with the above equation with a small perturbation via an equivalence relationship between Laplace principle and large deviation principle.

Keywords Mixed fractional heat equation, fractional Brownian sheet, Hölder regularity, Large deviation principle.

MSC(2010) 60H15, 60F10, 60G22.

1. Introduction

Fractional calculus has attracted lots of attention in several fields including mathematics, physics, chemistry, engineering, hydrology and even finance and social sciences (see Herrmann [17] and Meerschaert-Sikorskii [24]). Fractional diffusion equations are becoming popular in many areas of application. The classical fractional heat equation $\frac{\partial}{\partial t}u = \frac{1}{2}\Delta_\alpha u$ describes heat propagation in homogeneous medium, where $\Delta_\alpha = -(-\Delta)^{\frac{\alpha}{2}}$ is the fractional power of Lapalacian. On the other hand, stochastic heat equation driven by fractional Brownian motion (sheet) is a recent research direction in probability theory and its applications. Many interesting researches can be found in Balan [2], Balan and Tudor [3,4], Bo etc [7], Chen etc [10], Diop and Huang [13], Duncan etc [15], Hu etc [18, 19], Liu and Yan [23], Jiang

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etc [20, 21], Song [29], Tudor [30], Yan and Yu [33, 34] and the references therein. The increasing interest in this class of equations is motivated both by their applications to statistical mechanics, viscoelasticity, heat conduction in materials with memory, electrodynamics with memory and also because they can be employed to approach nonlinear conservation laws (see, for example, Sobczyk [28] and Droniou and Imbert [14]). Therefore, it is of great significance to import the stochastic effects into the investigation of fractional heat equations driven by fractional Brownian sheet. We have known that in recent years, the theory of large deviations for stochastic (partial) differential equations have been become an important field in stochastic analysis (see, for example, Azencott [1], Budhiraja etc [8], Mellali and Mellouk [25], Sritharan and Sundar [27]). In this paper, we are concerned with the following stochastic heat equation and a related Large Deviation Principle:

$$\frac{\partial u(t,x)}{\partial t} = (\Delta_\alpha + \Delta_\beta)u(t,x) + \frac{\partial f}{\partial x}(t,x,u(t,x)) + \frac{\partial^2 W}{\partial t \partial x}, \quad t \in [0,T],\ x \in \mathbb{R} \tag{1.1}$$

with $1 < \beta < \alpha < 2$ and $u(0,x) = u_0(x)$ on $\mathbb{R}$, where $\Delta_\alpha + \Delta_\beta$ is a pseudo differential operator on $\mathbb{R}$ which arises from a family of Lévy processes consisting of the independent sum of a symmetric α-stable process and a symmetric β-stable process, $\dot{W}(t,x)$ is the so-called fractional noise, the nonlinear measurable function $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and the initial-value $u_0(x)$ satisfy the following assumptions:

- Assumption 1. For all $p \geq 2$, we have

  $$\sup_{x\in\mathbb{R}} \mathbb{E}(|u_0(x)|^p) < +\infty \tag{1.2}$$

  and there exists constant $\theta \in (0,1)$ with $p\theta < 1$ such that

  $$\sup_{x\in\mathbb{R}} \mathbb{E}(|u_0(x+y) - u_0(x)|^p) < C_p|y|^{p\theta} \tag{1.3}$$

  for all $x, y \in \mathbb{R}$.

- Assumption 2. For each $T > 0$, there exists a constant $C > 0$ such that

  $$|f(t,x,u)| \leq C(1 + |u|), \tag{1.4}$$
  $$|f(t,x,u) - f(s,y,u')| \leq C(|t-s| + |x-y| + |u-u'|) \tag{1.5}$$

  for all $s, t \in [0,T]$ and $x, y, u, u' \in \mathbb{R}$.

The paper is organized as follows. In Section 2 we collect some preliminaries on the pseudo differential operator $\Delta_\alpha + \Delta_\beta$, some basic results and estimates for Green function and the double-parameter fractional noises. Section 3 is devoted to the proof of Theorem 4.1. Section 4 is devoted to show that the Hölder regularity of $u(t,x)$. In Section 5, as a related question, we consider also a large deviation principle associated with the above equation with a small perturbation. Such SPDEs with a small perturbation can be written as

$$\frac{\partial u^\varepsilon(t,x)}{\partial t} = (\Delta_\alpha + \Delta_\beta)u^\varepsilon(t,x) + \frac{\partial f}{\partial x}(t,x,u^\varepsilon(t,x)) + \sqrt{\varepsilon}\frac{\partial^2 W}{\partial t \partial x}, \quad t \in [0,T],\ x \in \mathbb{R}, \tag{1.6}$$

where $0 < \varepsilon \leq 1$, $(t,x) \in [0,T] \times \mathbb{R}$. We shall let $u^\varepsilon(0,x) = 0$ for all $x \in \mathbb{R}$. By using the weak convergence approach and an equivalence relationship between Laplace principle and large deviation principle, we construct a rate function such that (1.6) satisfies a large deviation principle.

2. Preliminaries

In this section, we briefly recall fractional Brownian sheet and some basic results and estimates for the heat kernel $G_{\alpha,\beta}$. We refer to Chen etc [11, 12], Nualart [26] and the references therein for more details. Throughout of this paper, for simplicity we denote by C a positive constant depending only on the subscripts and its value may not be the same in each occurrence. Moreover, we assume that the notation $F \asymp G$ means that there are positive constants c_1 and c_2 such that

$$c_1 G(x) \le F(x) \le c_2 G(x)$$

in the common domain of definition for F and G.

2.1. Pseudo differential operators $\Delta_\alpha + \Delta_\beta$

Consider a symmetric α-stable motion $X^\alpha = \{X_t^\alpha, t \ge 0\}$ and an independent symmetric β-stable motion $X^\beta = \{X_t^\beta, t \ge 0\}$ with $0 < \beta < \alpha < 2$ on $\mathbb{R}^d$. Then, the process

$$Y_t := X_t^\alpha + X_t^\beta, \quad t \ge 0$$

is a diffusion such that its transition density function $G_{\alpha,\beta}(t,x)$ satisfies

$$\int_{\mathbb{R}^d} G_{\alpha,\beta}(t,x) e^{izx} dx = e^{-t\left(|z|^\alpha + |z|^\beta\right)}$$

for all $t \ge 0$ and $z \in \mathbb{R}^d$, and moreover $G_{\alpha,\beta}(t,x)$ is the fundamental solution of the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = (\Delta_\alpha + \Delta_\beta) u, \\ u(0,x) = \delta_0(x). \end{cases}$$

The transition density function $G_{\alpha,\beta}(t,x)$ is also called the heat kernel of the operator $\Delta_\alpha + \Delta_\beta$. We denote

$$G_{\alpha,\beta}(s,y;t,x) = G_{\alpha,\beta}(t-s, x-y)$$

for all $x, y \in \mathbb{R}^d$ and $s, t \ge 0$. In this paper, we consider only the case $d = 1$. For the heat kernel $G_{\alpha,\beta}$, we have the following estimates (see, for examples, Bass and Levin [6], Chen etc [12] and Kolokoltsov [22]):

$$G_{\alpha,\beta}(s,y;t,x) \asymp \left((t-s)^{-\frac{1}{\alpha}} \wedge (t-s)^{-\frac{1}{\beta}}\right) \wedge \left(\frac{t-s}{|x-y|^{1+\alpha}} + \frac{t-s}{|x-y|^{1+\beta}}\right), \quad (2.1)$$

where $a_1 \wedge a_2 := \min\{a_1, a_2\}$ for $a_1, a_2 \in \mathbb{R}$. We can simplify the representation and get the estimates for $\frac{\partial}{\partial t} G_{\alpha,\beta}(s,y;t,x)$ and $\frac{\partial}{\partial y} G_{\alpha,\beta}(s,y;t,x)$ as follows

(1) when $(t-s)^{-\frac{1}{\alpha}-1} \wedge (t-s)^{-\frac{1}{\beta}-1} < \frac{1}{|x-y|^{1+\alpha}} + \frac{1}{|x-y|^{1+\beta}}$, we have

$$G_{\alpha,\beta}(s,y;t,x) \asymp (t-s)^{-\frac{1}{\alpha}} \wedge (t-s)^{-\frac{1}{\beta}},$$

$$\left|\frac{\partial G_{\alpha,\beta}}{\partial t}(s,y;t,x)\right| \asymp (t-s)^{-\frac{1}{\alpha}-1} \wedge (t-s)^{-\frac{1}{\beta}-1},$$

$$\left|\frac{\partial G_{\alpha,\beta}}{\partial y}(s,y;t,x)\right| \asymp (t-s)^{-\frac{2}{\alpha}} \wedge (t-s)^{-\frac{2}{\beta}};$$

(2) when $(t-s)^{-\frac{1}{\alpha}-1} \wedge (t-s)^{-\frac{1}{\beta}-1} > \frac{1}{|x-y|^{1+\alpha}} + \frac{1}{|x-y|^{1+\beta}}$, we have

$$G_{\alpha,\beta}(s,y;t,x) \asymp \frac{t-s}{|x-y|^{1+\alpha}} + \frac{t-s}{|x-y|^{1+\beta}},$$
$$\left|\frac{\partial G_{\alpha,\beta}}{\partial t}(s,y;t,x)\right| \asymp \frac{1}{|x-y|^{1+\alpha}} + \frac{1}{|x-y|^{1+\beta}},$$
$$\left|\frac{\partial G_{\alpha,\beta}}{\partial y}(s,y;t,x)\right| \asymp \frac{1}{|x-y|}\left(\frac{t-s}{|x-y|^{1+\alpha}} + \frac{t-s}{|x-y|^{1+\beta}}\right)$$

for all $T > t > s > 0$ and $x, y \in \mathbb{R}$.

2.2. Fractional Brownian noises

For $H_1, H_2 \in (0,1)$, a real-valued fractional Brownian sheet $W = \{W(t,x), (t,x) \in [0,T] \times \mathbb{R}\}$ is a Gaussian random field with $W(0,0) = 0$, $\mathbb{E}W(t,x) = 0$ and the covariance function

$$\mathbb{E}(W(t,x)W(s,y)) = R_{H_1}(s,t)R_{H_2}(x,y)$$

for all $s,t \in [0,T],\ x,y \in \mathbb{R}$, where

$$R_{H_i}(a,b) = \frac{1}{2}\left(|a|^{2H_i} + |b|^{2H_i} - |a-b|^{2H_i}\right),\ i = 1,2;\ a,b \in \mathbb{R}.$$

Let $\mathcal{H}$ be the completion of the linear space $\mathcal{L}$ generated by the indicator functions $\mathbf{1}_{(s,t]\times(x,y]}$ on $[0,T] \times \mathbb{R}$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]\times[0,x]}, \mathbf{1}_{[0,s]\times[0,y]}\rangle_{\mathcal{H}} = R_{H_1}(s,t)R_{H_2}(x,y).$$

The following embedding property follows from Bo etc [7] (see also Jiang etc [21] and Wei [32]).

Proposition 2.1. *For $H > \frac{1}{2}$ we have*

$$L^{\frac{1}{H}}([0,T] \times \mathbb{R}) \subset \mathcal{H}.$$

Define a mapping between $\mathcal{L}$ and the Gaussian space associated with W by

$$g = \mathbf{1}_{[0,t]\times[0,x]} \longmapsto \int_0^T \int_{\mathbb{R}} g(s,x)W(ds,dy) = W(t,x).$$

Then, it is an isometry and it can be extended to $\mathcal{H}$, which is called the Wiener integral of g with respect to W.

Consider the square integrable kernel

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,\ t > s,$$

where $c_H = \left[\frac{H(2H-1)}{\beta(2-2H,H-\frac{1}{2})}\right]$, and it satisfies

$$\frac{\partial K_H}{\partial t}(t,s) = c_H \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}}.$$

Define the linear operator $K^*_{H_1,H_2}$ from $\mathcal{L}$ to $L^2([0,T]\times\mathbb{R})$ as follows:

$$(K^*_{H_1,H_2}\psi)(s,y)=\int_s^T\int_y^\infty \psi(r,z)\frac{\partial K_{H_1}}{\partial r}(r,s)\frac{\partial K_{H_2}}{\partial z}(z,y)dzdr.$$

Then, we have

$$\langle K^*_{H_1,H_2}\psi_1,K^*_{H_1,H_2}\psi_2\rangle_{L^2([0,T]\times\mathbb{R})}=\langle\psi_1,\psi_2\rangle_{\mathcal{H}},$$

which shows that the operator $K^*_{H_1,H_2}$ provides an isometry between the Hilbert spaces $\mathcal{H}$ and $L^2([0,T]\times\mathbb{R})$. Hence, the Gaussian family $B=\{B(t,x),(t,x)\in[0,T]\times\mathbb{R}\}$ defined by

$$B(t,x)=W((K^*_{H_1,H_2})^{-1}1_{[0,t]\times(-\infty,x]})$$

is a space-time noise, and

$$W(t,x)=\int_{[0,t]\times(-\infty,x]}K_{H_1}(s,y)K_{H_2}(s,y)B(ds,dy)$$

for all $t\in[0,T]$ and $x\in\mathbb{R}$. It follows that

$$\int_0^T\int_{\mathbb{R}}\varphi(s,y)W(ds,dy)=\int_0^T\int_{\mathbb{R}}K^*_{H_1,H_2}\varphi(s,y)B(ds,dy).$$

Denote

$$\Theta_H(t,s;x,y)=4H_1H_2(2H_1-1)(2H_2-1)|t-s|^{2H_1-2}|x-y|^{2H_2-2}$$

for any $0\le s<t\le T$ and $x,y\in\mathbb{R}$.

Proposition 2.2. *For $\phi,\varphi\in\mathcal{H}$, we have $\mathbb{E}[W(\phi)]=0$ and*

$$\mathbb{E}[W(\phi)W(\varphi)]=\int_{[0,T]^2}\int_{\mathbb{R}^2}\phi(s,x)\varphi(t,y)\Theta_H(s,t;x,y)dydxdsdt.$$

Proposition 2.3. *If $H\in(\frac{1}{2},1)$ and $\phi,\varphi\in L^{\frac{1}{H}}([a,b])$, then*

$$\int_a^b\int_a^b\phi(x)\varphi(y)|x-y|^{2H-2}dxdy\le C\|\phi\|_{L^{\frac{1}{H}}([a,b])}\|\varphi\|_{L^{\frac{1}{H}}([a,b])}.$$

3. Existence and uniqueness of the solution

Let $(\Omega,\mathcal{F},P)$ be a complete probability space with a usual filtration $(\mathcal{F}_t)_{t\ge0}$ generated by W. For the Cauchy problem (1.1), as usual (see Walsh [31]) we say that the stochastic field $u:[0,T]\times\mathbb{R}\times\Omega\longmapsto\mathbb{R}$ is a mild solution to (1.1) if,

$$\begin{aligned}u(t,x)=&\int_{\mathbb{R}}G_{\alpha,\beta}(0,y;t,x)u_0(y)dy+\int_0^t\int_{\mathbb{R}}G_{\alpha,\beta}(s,y;t,x)W(ds,dy)\\&+\int_0^t\int_{\mathbb{R}}\frac{\partial G_{\alpha,\beta}}{\partial y}(s,y;t,x)f(s,y,u(s,y))dyds\end{aligned}\tag{3.1}$$

for all $t\ge0$ and $x\in\mathbb{R}$.

Theorem 3.1. *Under Assumptions 1 and 2, the equation* (3.1) *admits a unique solution* $u = \{u(t,x), (t,x) \in [0,T] \times \mathbb{R}\}$ *such that*

$$\sup_{t\in[0,T],x\in\mathbb{R}} \mathbb{E}|u(t,x)|^p < +\infty$$

for all $1 < \frac{1+\alpha^2}{2\alpha} < \beta < \alpha < 2$ *and* $p \geq 2$.

Proof. We shall use the Picard's approximation to prove the theorem. Define

$$\begin{aligned} u^{(0)}(t,x) &= \int_{\mathbb{R}} G_{\alpha,\beta}(0,y;t,x)u_0(y)dy, \\ u^{(n+1)}(t,x) &= u^{(0)}(t,x) + \int_0^t \int_{\mathbb{R}} G_{\alpha,\beta}(s,y;t,x)W(dy,ds) \\ &\quad + \int_0^t \int_{\mathbb{R}} \frac{\partial G_{\alpha,\beta}}{\partial y}(s,y;t,x)f(s,y,u^{(n)}(s,y))dyds \end{aligned} \tag{3.2}$$

for all $t \geq 0,\ x \in \mathbb{R}$ and $n \in \mathbb{N} = \{0,1,2,\ldots\}$.

Step I. We claim that

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T],x\in\mathbb{R}} \mathbb{E}|u^{(n)}(t,x)|^p < +\infty.$$

By Hölder's inequality and Assumption 1, we have

$$\begin{aligned} \mathbb{E}|u^{(0)}(t,x)|^p &\leq \mathbb{E}\left(\left(\int_{\mathbb{R}} |G_{\alpha,\beta}(0,y;t,x)|dy\right)^{p-1} \int_{\mathbb{R}} |G_{\alpha,\beta}(0,y;t,x)||u_0(y)|^p dy\right) \\ &\leq \sup_{x\in\mathbb{R}} \mathbb{E}|u_0(x)|^p \left(\int_{\mathbb{R}} |G_{\alpha,\beta}(0,y;t,x)|dy\right)^p \end{aligned} \tag{3.3}$$

for all $t \geq 0$, $x \in \mathbb{R}$, $p \geq 2$ and $n \in \mathbb{N} = \{0,1,2,\ldots\}$. It follows from the estimates (2.1) of Green function that

$$\sup_{t\in[0,T],x\in\mathbb{R}} \int_{\mathbb{R}} |G_{\alpha,\beta}(0,y;t,x)|dy < +\infty,$$

which shows that $\sup_{t\in[0,T],x\in\mathbb{R}} \mathbb{E}|u^{(0)}(t,x)|^p < +\infty$. On the other hand, for each $n \geq 1$ and $p \geq 2$ we denote

$$\begin{aligned} M_p^{(n)}(t,x) &:= \mathbb{E}\left|\int_0^t \int_{\mathbb{R}} G_{\alpha,\beta}(s,y;t,x)W(dy,ds)\right|^p, \\ N_p^{(n)}(t,x) &:= \mathbb{E}\left|\int_0^t \int_{\mathbb{R}} \frac{\partial G_{\alpha,\beta}}{\partial y}(s,y;t,x)f(s,y,u^{(n)}(s,y))dyds\right|^p. \end{aligned}$$

By (3.2) it follows that

$$\mathbb{E}|u^{(n+1)}(t,x)|^p \leq C\left(\mathbb{E}\left|u^{(0)}(t,x)\right|^p + M_p^{(n)}(t,x) + N_p^{(n)}(t,x)\right) \tag{3.4}$$

for all $t \geq 0$, $x \in \mathbb{R}$, $p \geq 2$ and $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$. We need to estimate $M_p^{(n)}(t,x)$ and $N_p^{(n)}(t,x)$. Denote

$$D_x^{s,t,\lambda} = \left\{ y \in \mathbb{R} \,\middle|\, (t-s)^{-\frac{1}{\lambda}-1} < \frac{1}{|x-y|^{1+\alpha}} + \frac{1}{|x-y|^{1+\beta}} \right\}$$

for all $0 < s < t < T$ and $\lambda \in \{\alpha, \beta\}$. Clearly, we have

$$\int_{D_x^{s,t,\alpha}} (t-s)^{-\frac{1}{\alpha H_2}} \, dy \leq C(t-s)^{-\frac{1}{\alpha H_2}+\frac{1}{\alpha}}$$

and

$$\int_{\overline{D_x^{s,t,\alpha}}} \left(\frac{t-s}{|x-y|^{1+\alpha}} + \frac{t-s}{|x-y|^{1+\beta}} \right)^{\frac{1}{H_2}} dy \leq C(t-s)^{\frac{1}{H_2} - \frac{(1+\alpha)^2}{\alpha(1+\beta)H_2} + \frac{1+\alpha}{\alpha(1+\beta)}}$$

for all $0 < s < t < T$ and $t - s < 1$, which imply that

$$\begin{aligned}
&\|G_{\alpha,\beta}(s,\cdot;t,x)\|_{L^{\frac{1}{H_2}}(\mathbb{R})} \\
&= \left(\int_{\mathbb{R}} G_{\alpha,\beta}(s,y;t,x)^{\frac{1}{H_2}} \, dy \right)^{H_2} \\
&\leq C \left(\int_{D_x^{s,t,\alpha}} (t-s)^{-\frac{1}{\alpha H_2}} \, dy + \int_{\overline{D_x^{s,t,\alpha}}} \left(\frac{t-s}{|x-y|^{1+\alpha}} + \frac{t-s}{|x-y|^{1+\beta}} \right)^{\frac{1}{H_2}} dy \right)^{H_2} \qquad (3.5) \\
&\leq C \left((t-s)^{-\frac{1}{\alpha H_2}+\frac{1}{\alpha}} + (t-s)^{\frac{1+\alpha}{\alpha(1+\beta)} - \frac{(1+\alpha)^2}{\alpha(1+\beta)H_2} + \frac{1}{H_2}} \right)^{H_2} \\
&\leq C(t-s)^{1 - \frac{(1+\alpha)^2}{\alpha(1+\beta)} + \frac{H_2(1+\alpha)}{\alpha(1+\beta)}}
\end{aligned}$$

for all $0 < s < t < T$, $t - s < 1$ and $x \in \mathbb{R}$. On the other hand, when $0 < s < t < T$ and $t - s > 1$, we have

$$\int_{D_x^{s,t,\beta}} (t-s)^{-\frac{1}{\beta H_2}} \, dy \leq C(t-s)^{-\frac{1}{\beta H_2}+\frac{1+\beta}{\beta(1+\alpha)}}$$

and

$$\int_{\overline{D_x^{s,t,\beta}}} \left(\frac{t-s}{|x-y|^{1+\alpha}} + \frac{t-s}{|x-y|^{1+\beta}} \right)^{\frac{1}{H_2}} dy \leq C(t-s)^{\frac{1}{H_2} - \frac{(1+\beta)^2}{\beta(1+\alpha)H_2} + \frac{1+\beta}{\beta(1+\alpha)}}$$

for all $0 < s < t < T$, $t - s \geq 1$ and $x \in \mathbb{R}$, which show that

$$\begin{aligned}
&\|G_{\alpha,\beta}(s,\cdot;t,x)\|_{L^{\frac{1}{H_2}}(\mathbb{R})} \\
&= \left(\int_{\mathbb{R}} G_{\alpha,\beta}(s,y;t,x)^{\frac{1}{H_2}} \, dy \right)^{H_2} \\
&\leq C \left(\int_{D_x^{s,t,\beta}} (t-s)^{-\frac{1}{\alpha H_2}} \, dy + \int_{\overline{D_x^{s,t,\beta}}} \left(\frac{t-s}{|x-y|^{1+\alpha}} + \frac{t-s}{|x-y|^{1+\beta}} \right)^{\frac{1}{H_2}} dy \right)^{H_2} \qquad (3.6) \\
&\leq C \left((t-s)^{-\frac{1}{\beta H_2}+\frac{1+\beta}{\beta(1+\alpha)}} + (t-s)^{\frac{1}{H_2} + \frac{1+\beta}{\beta(1+\alpha)} - \frac{(1+\beta)^2}{\beta(1+\alpha)H_2}} \right)^{H_2} \\
&\leq C(t-s)^{1 - \frac{(1+\beta)^2}{\beta(1+\alpha)} + \frac{H_2(1+\beta)}{\beta(1+\alpha)}}
\end{aligned}$$

for all $0 < s < t < T$, $t-s \geq 1$ and $x \in \mathbb{R}$. It follow from (3.5), (3.6), Proposition 2.1 and Proposition 2.2 that

$$\begin{aligned} M_p^{(n)}(t,x) &= \mathbb{E}\left|\int_0^t \int_{\mathbb{R}} G_{\alpha,\beta}(r,z;t,x)W(dr,dz)\right|^p \\ &\leq C\left(\int_{[0,t]^2} dr_1 dr_2 \int_{\mathbb{R}^2} G_{\alpha,\beta}(r_1,z_1;t,x)\Theta(r_1,r_2;z_1,z_2)G_{\alpha,\beta}(r_2,z_2;t,x)dz_1dz_2\right)^{\frac{p}{2}} \\ &= C\left(\int_{[0,t]^2} |r_1-r_2|^{2H_1-2}dr_1dr_2 \right. \\ &\qquad \left. \times \int_{\mathbb{R}^2} |z_1-z_2|^{2H_2-2}G_{\alpha,\beta}(r_1,z_1;t,x)G_{\alpha,\beta}(r_2,z_2;t,x)dz_1dz_2\right)^{\frac{p}{2}} \\ &\leq C\left(\int_{[0,t]^2} |r_1-r_2|^{2H_1-2}\|G_{\alpha,\beta}(r_1,\cdot;t,x)\|_{L^{\frac{1}{H_2}}(\mathbb{R})}\|G_{\alpha,\beta}(r_2,\cdot;t,x)\|_{L^{\frac{1}{H_2}}(\mathbb{R})}dr_1dr_2\right)^{\frac{p}{2}} \\ &\leq C\left(\int_0^T \left(\|G_{\alpha,\beta}(r,\cdot;t,x)\|_{L^{\frac{1}{H_2}}(\mathbb{R})}\right)^{\frac{1}{H_1}} dr\right)^{pH_1} \leq C < +\infty \end{aligned} \tag{3.7}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}$. Similarly, by Hölder inequality we get

$$\begin{aligned} N_p^{(n)}(t,x) &\leq C\left(\int_0^t \int_{\mathbb{R}} \left|\frac{\partial G_{\alpha,\beta}}{\partial y}(s,y;t,x)\right| dyds\right)^{p-1} \\ &\quad \times \mathbb{E}\left(\int_0^t \int_{\mathbb{R}} \left|f(s,y,u^{(n)}(s,y))\right|^p \cdot \left|\frac{\partial G_{\alpha,\beta}}{\partial y}(s,y;t,x)\right| dyds\right) \\ &\leq C\mathbb{E}\left(\int_0^t \int_{\mathbb{R}} \left(1+|u^{(n)}(s,y)|^p\right) \cdot \left|\frac{\partial G_{\alpha,\beta}}{\partial y}(s,y;t,x)\right| dyds\right) \\ &\leq C\int_0^t \left(1+\sup_{y\in\mathbb{R}}\mathbb{E}|u^{(n)}(s,y)|^p\right)\left(\int_{\mathbb{R}}\left|\frac{\partial G_{\alpha,\beta}}{\partial y}(s,y;t,x)\right| dy\right) ds \end{aligned} \tag{3.8}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}$. Combining this with (3.7), we have that

$$\sup_{x\in\mathbb{R}}\mathbb{E}|u^{(n+1)}(t,x)|^p \leq C + \int_0^t \left(1+\sup_{y\in\mathbb{R}}\mathbb{E}|u^{(n)}(s,y)|^p\right) g_x(t-s)ds \tag{3.9}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}$, where

$$g_x(t-s) = \int_{\mathbb{R}} \left|\frac{\partial G_{\alpha,\beta}}{\partial y}(s,y;t,x)\right| dy.$$

By some elementary calculations, one can show that

$$\begin{aligned} &\int_{\mathbb{R}} \left|\frac{\partial G_{\alpha,\beta}}{\partial y}(s,y;t,x)\right| dy \\ &\leq \int_{D_x^{s,t,\alpha}} (t-s)^{-\frac{2}{\alpha}}dy + \int_{\overline{D_x^{s,t,\alpha}}} \frac{1}{|x-y|}\left(\frac{1}{|x-y|^{1+\alpha}} + \frac{1}{|x-y|^{1+\beta}}\right)dy \\ &\leq C\left((t-s)^{-\frac{1}{\alpha}} + (t-s)^{1-\frac{(1+\alpha)^2}{\alpha(1+\beta)}}\right) \leq C(t-s)^{1-\frac{(1+\alpha)^2}{\alpha(1+\beta)}} \end{aligned} \tag{3.10}$$

for all $x \in \mathbb{R}$ and $0 < t - s < 1$, and

$$\begin{aligned}
&\int_{\mathbb{R}} \left| \frac{\partial G_{\alpha,\beta}}{\partial y}(s,y;t,x) \right| dy \\
\leq &\int_{D_x^{s,t,\beta}} (t-s)^{-\frac{2}{\beta}} dy + \int_{\overline{D_x^{s,t,\beta}}} \frac{1}{|x-y|} \left(\frac{1}{|x-y|^{1+\alpha}} + \frac{1}{|x-y|^{1+\beta}} \right) dy \qquad (3.11) \\
\leq &C \left((t-s)^{-\frac{2}{\beta} + \frac{1+\beta}{\beta(1+\alpha)}} + (t-s)^{1-\frac{(1+\beta)^2}{\beta(1+\alpha)}} \right) \leq (t-s)^{-\frac{2}{\beta} + \frac{1+\beta}{\beta(1+\alpha)}}
\end{aligned}$$

for all $x \in \mathbb{R}$ and $t - s \geq 1$. Thus, we get the estimate

$$\sup_{x \in \mathbb{R}} \mathbb{E}|u^{(n+1)}(t,x)|^p \leq C + \int_0^t \left(1 + \sup_{y \in \mathbb{R}} \mathbb{E}|u^{(n)}(s,y)|^p \right) (t-s)^{1-\frac{(1+\alpha)^2}{\alpha(1+\beta)}} ds, \quad n \geq 0$$

for all $t \in [0,T]$ by (3.9) since

$$1 - \frac{(1+\alpha)^2}{\alpha(1+\beta)} \leq -\frac{2}{\beta} + \frac{1+\beta}{\beta(1+\alpha)},$$

when $2 > \alpha > \beta > 1$. This shows that

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0,T], x \in \mathbb{R}} \mathbb{E}|u^{(n)}(t,x)|^p < +\infty$$

for $\beta > \frac{1+\alpha^2}{2\alpha}$ by Gronwall's lemma.

Step II. We prove that $\{u^{(n)}(t,x)\}_{n \in \mathbb{N}}$ converges in $L^p(\Omega)$ for any $p \geq 2$. For $n \geq 2$, we have

$$\begin{aligned}
&\mathbb{E}\left(\left| u^{(n+1)}(t,x) - u^{(n)}(t,x) \right|^p \right) \\
=&\mathbb{E}\left(\left| \int_0^t \int_{\mathbb{R}} \frac{\partial G_{\alpha,\beta}}{\partial y}(s,y;t,x) \left(f(s,y,u^{(n)}(s,y)) - f(s,y,u^{(n-1)}(s,y)) \right) dyds \right|^p \right) \\
\leq &C \int_0^t \mathbb{E}\left| u^{(n)}(s,y) - u^{(n-1)}(s,y) \right|^p ds \int_{\mathbb{R}} \left| \frac{\partial G_{\alpha,\beta}}{\partial y}(s,y;t,x) \right| dy \\
\leq &C \int_0^t \sup_{y \in \mathbb{R}} \mathbb{E}\left| u^{(n)}(s,y) - u^{(n-1)}(s,y) \right|^p g_x(t-s) ds
\end{aligned}$$

and

$$\sup_{y \in \mathbb{R}} \mathbb{E}\left| u^{(1)}(s,y) - u^{(0)}(s,y) \right|^p \leq C_p \left(\mathbb{E}\left| u^{(0)}(s,y) \right|^p + \mathbb{E}\left| u^{(1)}(s,y) \right|^p \right) < +\infty.$$

Then Gronwall's lemma yields that

$$\sum_{n \in \mathbb{N}} \sup_{t \in [0,T], x \in \mathbb{R}} \mathbb{E}\left(\left| u^{(n+1)}(t,x) - u^{(n)}(t,x) \right|^p \right) < +\infty.$$

Hence, $\{u^{(n)}(t,x)\}_{n \geq 0}$ is a Cauchy sequence in $L^p(\Omega)$. Define

$$u(t,x) := \lim_{n \to +\infty} u^{(n)}(t,x)$$

in $L^p(\Omega)$. Then we have

$$\sup_{t\in[0,T],x\in\mathbb{R}} \mathbb{E}|u(t,x)|^p < +\infty$$

for each $(t,x) \in [0,T]\times\mathbb{R}$. Taking $n \to +\infty$ in $L^p(\Omega)$ at both sides of (3.2), we see that $\{u(t,x) : (t,x) \in [0,t]\times\mathbb{R}\}$ satisfies (3.1).

Step III. We prove the uniqueness. Let u and $\widehat{u}$ be the two solutions of (3.1) with the same u_0, then we have

$$\begin{aligned}
&\mathbb{E}\left(|u(t,x)-\widehat{u}(t,x)|^p\right)\\
=&\mathbb{E}\left(\left|\int_0^t\int_{\mathbb{R}}\frac{\partial G_{\alpha,\beta}}{\partial y}(s,y;t,x)\left(f(s,y,u(s,y))-f(s,y,\widehat{u}(s,y))\right)dyds\right|^p\right)\\
\le&C\int_0^t \mathbb{E}\left|u(s,y)-\widehat{u}(s,y)\right|^p ds\int_{\mathbb{R}}\left|\frac{\partial G_{\alpha,\beta}}{\partial y}(s,y;t,x)\right|dy\\
\le&C\int_0^t \sup_{y\in\mathbb{R}}\mathbb{E}\left|u(s,y)-\widehat{u}(s,y)\right|^p g_x(t-s)ds
\end{aligned}$$

for all $t\in[0,T]$ and $x\in\mathbb{R}$. Then Gronwall's lemma implies that

$$\sup_{t\in[0,T],x\in\mathbb{R}}\mathbb{E}\left|u(t,x)-\widehat{u}(t,x)\right|^p = 0.$$

Thus, we have completed the proof of the theorem. □

4. Hölder regularity of the solution

In this section we expound and prove the next theorem which gives the Hölder regularity of the unique solution $u = \{u(t,x),(t,x)\in[0,T]\times\mathbb{R}\}$ to (3.1).

Theorem 4.1. *Let $H_1,H_2\in(\frac{1}{2},1)$ and $1<\frac{1+\alpha^2}{2\alpha}<\beta<\alpha<2$. Then, under Assumptions 1 and 2, the solution $u(t,x)$ exists a continuous modification $\widetilde{u}(t,x)$ which is θ_1-Hölder continuous in t with $\theta_1\in(0,\vartheta_1)$ and θ_2-Hölder continuous in x with $\theta_2\in(0,\vartheta_2)$, where*

$$\begin{aligned}
\vartheta_1 &:= \min\left\{\frac{\theta}{\alpha}, 1-\frac{2}{\beta}+\frac{1+\beta}{\beta(1+\alpha)}, \frac{H_2(1+\alpha)-(1+\alpha)^2}{\alpha(1+\beta)}+H_1+1\right\},\\
\vartheta_2 &:= \min\left\{\theta, H_2, 1-\frac{2}{\beta}+\frac{1+\beta}{\beta(1+\alpha)}, \frac{H_2(1+\beta)}{1+\alpha}+\beta H_1-1\right\}.
\end{aligned}$$

In order to show that the theorem we need the following two lemmas.

Lemma 4.1. *There exists a constant $C>0$ such that*

$$\begin{aligned}
&\int_{\mathbb{R}}\left(\left|\frac{\partial G_{\alpha,\beta}}{\partial t}(r,z;t,x)\right|^{\theta_1}|G_{\alpha,\beta}(r,z;t,x)|^{1-\theta_1}\right)^{\frac{1}{H_2}}dz\\
\le&C(t-r)^{\frac{1-\theta_1}{H_2}-\frac{(1+\alpha)^2}{\alpha(1+\beta)H_2}+\frac{1+\alpha}{\alpha(1+\beta)}}
\end{aligned}\tag{4.1}$$

for all $0 < r < t \leq T$, $x \in \mathbb{R}$ and $\theta_1 \in (0,1)$. Moreover, when $0 < \theta_1 < \frac{H_2(1+\alpha)-(1+\alpha)^2}{\alpha(1+\beta)} + H_1 + 1$, we have

$$\int_0^t \left(\int_{\mathbb{R}} \left(\left| \frac{\partial G_{\alpha,\beta}}{\partial t}(r,z;t,x) \right|^{\theta_1} |G_{\alpha,\beta}(r,z;t,x)|^{1-\theta_1} \right)^{\frac{1}{H_2}} dz \right)^{\frac{H_2}{H_1}} dr \leq C \tag{4.2}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}$.

Proof. Denote

$$D_{x,\sigma}^{r,t,\lambda} = \left\{ z \in \mathbb{R} \middle| (t-r)^{-\frac{1}{\lambda}-1} < \frac{2}{|x-z|^{1+\sigma}} \right\}.$$

Recall that

$$D_x^{r,t,\lambda} = \left\{ z \in \mathbb{R} \middle| (t-r)^{-\frac{1}{\lambda}-1} < \frac{1}{|x-z|^{1+\alpha}} + \frac{1}{|x-z|^{1+\beta}} \right\}$$

for all $T > t > r > 0$ and $\lambda, \sigma \in \{\alpha, \beta\}$. We then have that

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\left| \frac{\partial G_{\alpha,\beta}}{\partial t}(r,z;t,x) \right|^{\theta_1} |G_{\alpha,\beta}(r,z;t,x)|^{1-\theta_1} \right)^{\frac{1}{H_2}} dz \\
\leq &\int_{D_x^{r,t,\alpha}} \left| (t-r)^{-\frac{1}{\alpha}-1} \right|^{\frac{\theta_1}{H_2}} |t-r|^{-\frac{1-\theta_1}{\alpha H_2}} dz \\
&+ \int_{\overline{D_x^{r,t,\alpha}}} \left(\frac{1}{|x-z|^{1+\alpha}} + \frac{1}{|x-z|^{1+\beta}} \right)^{\frac{\theta_1}{H_2}} \left| \frac{t-r}{|x-z|^{1+\alpha}} + \frac{t-r}{|x-z|^{1+\beta}} \right|^{\frac{1-\theta_1}{H_2}} dz \\
\equiv &A_{2,1,1,1}^1(t,r,x) + A_{2,1,1,2}^1(t,r,x)
\end{aligned}$$

for $t - r < 1$ and $x \in \mathbb{R}$. Clearly, some elementary calculations can show that

$$\begin{aligned}
A_{2,1,1,1}^1(t,r,x) &= \int_{D_x^{r,t,\alpha}} \left| (t-r)^{-\frac{1}{\alpha}-1} \right|^{\frac{\theta_1}{H_2}} |t-r|^{-\frac{1-\theta_1}{\alpha H_2}} dz \\
&\leq C(t-r)^{-\frac{1+\alpha\theta_1}{\alpha H_2}} \left(\int_{D_{x,\alpha}^{r,t,\alpha}} \mathbf{1}_{\{|x-z|<1\}} dz + \int_{D_{x,\beta}^{r,t,\alpha}} \mathbf{1}_{\{|x-z|>1\}} dz \right) \\
&\leq C(t-r)^{-\frac{1+\alpha\theta_1}{\alpha H_2}+\frac{1}{\alpha}} + C(t-r)^{-\frac{1+\alpha\theta_1}{\alpha H_2}+\frac{1+\alpha}{\alpha(1+\beta)}} \leq C(t-r)^{-\frac{1+\alpha\theta_1}{\alpha H_2}+\frac{1}{\alpha}}
\end{aligned}$$

and

$$\begin{aligned}
A_{2,1,1,2}^1(t,r,x) = &\int_{\overline{D_x^{r,t,\alpha}}} \left(\frac{1}{|x-z|^{1+\alpha}} + \frac{1}{|x-z|^{1+\beta}} \right)^{\frac{\theta_1}{H_2}} \left| \frac{t-r}{|x-z|^{1+\alpha}} + \frac{t-r}{|x-z|^{1+\beta}} \right|^{\frac{1-\theta_1}{H_2}} dz \\
\leq &C(t-r)^{\frac{1-\theta_1}{H_2}} \int_{\overline{D_{x,\beta}^{r,t,\alpha}}} |x-z|^{-\frac{1+\alpha}{H_2}} \mathbf{1}_{\{|x-z|<1\}} dz \\
&+ C(t-r)^{\frac{1-\theta_1}{H_2}} \int_{\overline{D_{x,\alpha}^{r,t,\alpha}}} |x-z|^{-\frac{1+\beta}{H_2}} \mathbf{1}_{\{|x-z|>1\}} dz \\
\leq &C(t-r)^{\frac{1-\theta_1}{H_2}-\frac{(1+\alpha)^2}{\alpha(1+\beta)H_2}+\frac{1+\alpha}{\alpha(1+\beta)}} + C(t-r)^{\frac{1-\theta_1}{H_2}-\frac{1+\beta}{\alpha H_2}+\frac{1}{\alpha}}
\end{aligned}$$

$$\leq C(t-r)^{\frac{1-\theta_1}{H_2}-\frac{(1+\alpha)^2}{\alpha(1+\beta)H_2}+\frac{1+\alpha}{\alpha(1+\beta)}}$$

for $t-r<1$ and $x\in\mathbb{R}$. Similarly, we have also that

$$\begin{aligned}&\int_{\mathbb{R}}\left(\left|\frac{\partial G_{\alpha,\beta}}{\partial t}(r,z;t,x)\right|^{\theta_1}|G_{\alpha,\beta}(r,z;t,x)|^{1-\theta_1}\right)^{\frac{1}{H_2}}dz\\
\leq&\int_{D_x^{r,t,\beta}}\left|(t-r)^{-\frac{1}{\beta}-1}\right|^{\frac{\theta_1}{H_2}}|t-r|^{-\frac{1-\theta_1}{\beta H_2}}dz\\
&+\int_{\overline{D_x^{r,t,\beta}}}\left(\frac{1}{|x-z|^{1+\alpha}}+\frac{1}{|x-z|^{1+\beta}}\right)^{\frac{\theta_1}{H_2}}\left|\frac{t-r}{|x-z|^{1+\alpha}}+\frac{t-r}{|x-z|^{1+\beta}}\right|^{\frac{1-\theta_1}{H_2}}dz\\
\leq&C(t-r)^{-\frac{1+\beta\theta_1}{\beta H_2}+\frac{1+\beta}{\beta(1+\alpha)}}+C(t-r)^{\frac{1-\theta_1}{H_2}-\frac{(1+\beta)^2}{\beta(1+\alpha)H_2}+\frac{1+\beta}{\beta(1+\alpha)}}\end{aligned}$$

for $t-r\geq 1$ and $x\in\mathbb{R}$. Thus, we have introduced (4.1) and (4.2) follows. □

Lemma 4.2. *There exists a constant $C>0$ such that*

$$\int_{\mathbb{R}}\left(\left|\frac{\partial G_{\alpha,\beta}}{\partial x}(r,z;t,x)\right|^{\theta_2}\cdot|G_{\alpha,\beta}(r,z;t,x)|^{1-\theta_2}\right)^{\frac{1}{H_2}}dz\leq C(t-r)^{-\frac{(1+\theta_2)}{\beta H_2}+\frac{1+\beta}{\beta(1+\alpha)}} \tag{4.3}$$

and

$$\int_0^T\left(\int_{\mathbb{R}}\left(\left|\frac{\partial G_{\alpha,\beta}}{\partial x}(r,z;t,x)\right|_2^{\theta}|G_{\alpha,\beta}(r,z;t,x)|^{1-\theta_2}\right)^{\frac{1}{H_2}}dz\right)^{\frac{H_2}{H_1}}dr\leq C \tag{4.4}$$

for all $t\in[0,T]$, $0<\theta_2<\frac{H_2(1+\beta)}{1+\alpha}+\beta H_1-1$, and $x\in\mathbb{R}$.

Proof. When $t-r<1$, we have that

$$\begin{aligned}&\int_{\mathbb{R}}\left(\left|\frac{\partial G_{\alpha,\beta}}{\partial x}(r,z;t,x)\right|^{\theta_2}|G_{\alpha,\beta}(r,z;t,x)|^{1-\theta_2}\right)^{\frac{1}{H_2}}dz\\
\leq&C\int_{D_x^{r,t,\alpha}}\left|(t-r)^{-\frac{2}{\alpha}}\right|^{\frac{\theta_2}{H_2}}\left|(t-r)^{-\frac{1}{\alpha}}\right|^{\frac{1-\theta_2}{H_2}}dz\\
&+\int_{\overline{D_x^{r,t,\alpha}}}\left|\frac{1}{|x-z|}\left(\frac{1}{|x-z|^{1+\alpha}}+\frac{1}{|x-z|^{1+\beta}}\right)\right|^{\frac{\theta_2}{H_2}}\left|\frac{t-r}{|x-z|^{1+\alpha}}+\frac{t-r}{|x-z|^{1+\beta}}\right|^{\frac{1-\theta_2}{H_2}}dz\\
\leq&C\int_{D_x^{r,t,\alpha}}(t-r)^{-\frac{1+\theta_2}{\alpha H_2}}dz+C(t-r)^{\frac{1}{H_2}}\\
&\times\int_{\overline{D_x^{r,t,\alpha}}}\left|\frac{1}{|x-z|^{2+\alpha}}+\frac{1}{|x-z|^{2+\beta}}\right|^{\frac{\theta_2}{H_2}}\left|\frac{1}{|x-z|^{1+\alpha}}+\frac{1}{|x-z|^{1+\beta}}\right|^{\frac{1-\theta_2}{H_2}}dz\\
\equiv&A^2_{2,1,1}(t,r,x)+A^2_{2,1,2}(t,r,x)\end{aligned}$$

for all $x\in\mathbb{R}$. Some elementary calculations can show that

$$A^2_{2,1,1}(t,r,x)=C(t-r)^{-\frac{1+\theta_2}{\alpha H_2}}\left(\int_{D_{x,\alpha}^{r,t,\alpha}}\mathbf{1}_{\{|x-z|<1\}}dz+\int_{D_{x,\beta}^{r,t,\alpha}}\mathbf{1}_{\{|x-z|>1\}}dz\right)$$

$$\leq C(t-r)^{-\frac{1+\theta_2}{\alpha H_2}+\frac{1}{\alpha}} + C(t-r)^{-\frac{1+\theta_2}{\alpha H_2}+\frac{1+\alpha}{\alpha(1+\beta)}} \leq C(t-r)^{-\frac{1+\theta_2}{\alpha H_2}+\frac{1}{\alpha}}$$

and

$$\begin{aligned}
A^2_{2,1,2}(t,r,x) &= C(t-r)^{\frac{1}{H_2}} \int_{\overline{D^{r,t,\alpha}_{x,\beta}}} |x-z|^{-\frac{(2+\beta)\theta_2}{H_2}-\frac{(1+\beta)(1-\theta_2)}{H_2}} \mathbf{1}_{\{|x-z|<1\}} dz \\
&\quad + C(t-r)^{\frac{1}{H_2}} \int_{\overline{D^{r,t,\alpha}_{x,\alpha}}} |x-z|^{-\frac{(2+\alpha)\theta_2}{H_2}-\frac{(1+\beta)(1-\theta_2)}{H_2}} \mathbf{1}_{\{|x-z|>1\}} dz \\
&\leq C(t-r)^{\frac{1}{H_2}-\frac{(1+\alpha)(2+\beta)\theta_2}{\alpha(1+\beta)H_2}-\frac{(1+\alpha)(1-\theta_2)}{\alpha H_2}+\frac{1+\alpha}{\alpha(1+\beta)}} \\
&\qquad + C(t-r)^{\frac{1}{H_2}-\frac{(2+\alpha)\theta_2}{\alpha H_2}-\frac{(1+\beta)(1-\theta_2)}{\alpha H_2}+\frac{1}{\alpha}} \\
&\leq C(t-r)^{\frac{1}{H_2}-\frac{(1+\alpha)(2+\beta)\theta_2}{\alpha(1+\beta)H_2}-\frac{(1+\alpha)(1-\theta_2)}{\alpha H_2}+\frac{1+\alpha}{\alpha(1+\beta)}}
\end{aligned}$$

for $t-r<1$, $x\in\mathbb{R}$ and $\theta<H_2$. Similarly, when $t-r>1$, we have also that

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\left|\frac{\partial G_{\alpha,\beta}}{\partial x}(r,z;t,x)\right|^{\theta_2} |G_{\alpha,\beta}(r,z;t,x)|^{1-\theta_2} \right)^{\frac{1}{H_2}} dz \\
\leq& C\int_{D^{r,t,\beta}_x} \left|(t-r)^{-\frac{2}{\beta}}\right|^{\frac{\theta_2}{H_2}} \left|(t-r)^{-\frac{1}{\beta}}\right|^{\frac{1-\theta_2}{H_2}} dz \\
&+ \int_{\overline{D^{r,t,\beta}_x}} \left|\frac{t-r}{|x-z|}\left(\frac{1}{|x-z|^{1+\alpha}}+\frac{1}{|x-z|^{1+\beta}}\right)\right|^{\frac{\theta_2}{H_2}} \left|\frac{1}{|x-z|^{1+\alpha}}+\frac{1}{|x-z|^{1+\beta}}\right|^{\frac{1-\theta_2}{H_2}} dz \\
\leq& C\int_{D^{r,t,\beta}_x} (t-r)^{-\frac{1+\theta_2}{\beta H_2}} dz + C(t-r)^{\frac{1}{H_2}} \int_{\overline{D^{r,t,\beta}_x}} \left|\frac{1}{|x-z|^{2+\alpha}}+\frac{1}{|x-z|^{2+\beta}}\right|^{\frac{\theta_2}{H_2}} \\
&\times \left|\frac{1}{|x-z|^{1+\alpha}}+\frac{1}{|x-z|^{1+\beta}}\right|^{\frac{1-\theta_2}{H_2}} dz \\
\leq& C(t-r)^{-\frac{1+\theta_2}{\beta H_2}+\frac{1+\beta}{\beta(1+\alpha)}} + C(t-r)^{\frac{1}{H_2}-\frac{(1+\beta)(2+\beta)\theta_2}{\beta(1+\alpha)H_2}-\frac{(1+\beta)^2(1-\theta_2)}{\beta(1+\alpha)H_2}+\frac{1+\beta}{\beta(1+\alpha)}}
\end{aligned}$$

for all $x\in\mathbb{R}$, $0<\theta_2<\frac{H_2(1+\beta)}{1+\alpha}+\beta H_1-1$ and $T>t>r>0$. Thus, we have obtained the estimates (4.3) and (4.4). □

Proof of Theorem 4.1. We shall divide the proof into two steps.

Step1. We first consider the temporal case.

$$\begin{aligned}
A^1_1(t,s,x) &:= \int_{\mathbb{R}} (G_{\alpha,\beta}(0,y;t,x) - G_{\alpha,\beta}(0,y;s,x)) u_0(y) dy, \\
A^1_2(t,s,x) &:= \int_0^t \int_{\mathbb{R}} G_{\alpha,\beta}(r,y;t,x) W(dy,dr) - \int_0^s \int_{\mathbb{R}} G_{\alpha,\beta}(r,y;s,x) W(dy,dr), \\
A^1_3(t,s,x) &:= \int_0^t \int_{\mathbb{R}} \frac{\partial G_{\alpha,\beta}}{\partial y}(r,y;t,x) f(r,y,u(r,y)) dy dr \\
&\quad - \int_0^s \int_{\mathbb{R}} \frac{\partial G_{\alpha,\beta}}{\partial y}(r,y;t,x) f(r,y,u(r,y)) dy dr
\end{aligned}$$

for all $x\in\mathbb{R}$ and $0\leq s<t\leq T$. Then, we have

$$|\widetilde{u}(t,x)-\widetilde{u}(s,x)| \leq |A^1_1(t,s,x)| + |A^1_2(t,s,x)| + |A^1_3(t,s,x)|$$

for all $x \in \mathbb{R}$ and $0 \le s < t \le T$. By Hölder's inequality, the semigroup property and (3.2), we have

$$
\begin{aligned}
&\mathbb{E}|A_1^1(t,s,x)|^p \\
=&\mathbb{E}\bigg|\int_{\mathbb{R}^2}(G_{\alpha,\beta}(0,z;t-s,y)G_{\alpha,\beta}(0,y;s,x))u_0(z)dydz \\
&-\int_{\mathbb{R}}G_{\alpha,\beta}(0,y;s,x)u_0(y)dy\bigg|^p \\
=&\mathbb{E}\bigg|\int_{\mathbb{R}}G_{\alpha,\beta}(0,z;t-s,0)dz\int_{\mathbb{R}}G_{\alpha,\beta}(0,y;s,x)(u_0(y-z)-u_0(y))dy\bigg|^p \\
\le& C\int_{\mathbb{R}}|G_{\alpha,\beta}(0,z;t-s,0)|dz\int_{\mathbb{R}}G_{\alpha,\beta}(0,y;s,x)\mathbb{E}|u_0(y-z)-u_0(y)|^p dy \\
\le& C\int_{\mathbb{R}}|G_{\alpha,\beta}(0,y;s,x)|dy\int_{\mathbb{R}}|G_{\alpha,\beta}(0,z;t-s,0)||z|^{p\theta}dz.
\end{aligned}
$$

We need to estimate $\int_{\mathbb{R}}|G_{\alpha,\beta}(0,y;s,x)|dy$ and $\int_{\mathbb{R}}|G_{\alpha,\beta}(0,z;t-s,0)||z|^{p\theta}dz$.

Clearly, when $0 < s < 1$ and $t - s < 1$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}}|G_{\alpha,\beta}(0,y;s,x)|dy \\
\le& C\int_{D_x^{s,0,\alpha}}s^{-\frac{1}{\alpha}}dy+\int_{\overline{D_x^{s,0,\alpha}}}\left(\frac{s}{|x-y|^{1+\alpha}}+\frac{s}{|x-y|^{1+\beta}}\right)dy \\
\le& C\int_{D_{x,\beta}^{s,0,\alpha}}s^{-\frac{1}{\alpha}}\mathbf{1}_{\{|x-y|>1\}}dy+C\int_{D_{x,\alpha}^{s,0,\alpha}}s^{-\frac{1}{\alpha}}\mathbf{1}_{\{|x-y|<1\}}dy \\
&+C\int_{\overline{D_{x,\beta}^{s,0,\alpha}}}\left(\frac{s}{|x-y|^{1+\alpha}}+\frac{s}{|x-y|^{1+\beta}}\right)\mathbf{1}_{\{|x-y|<1\}}dy \\
&+C\int_{\overline{D_{x,\alpha}^{s,0,\alpha}}}\left(\frac{s}{|x-y|^{1+\alpha}}+\frac{s}{|x-y|^{1+\beta}}\right)\mathbf{1}_{\{|x-y|>1\}}dy \\
\le& C(s^{-\frac{1}{\alpha}+\frac{1+\alpha}{\alpha(1+\beta)}}+1)+C(s^{1-\frac{\beta}{\alpha}}+s^{1-\frac{1+\alpha}{1+\beta}})\le Cs^{1-\frac{1+\alpha}{1+\beta}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}}|G_{\alpha,\beta}(0,z;t-s,0)||z|^{p\theta}dz \\
=&C\int_{D_0^{s,t,\alpha}}(t-s)^{-\frac{1}{\alpha}}|z|^{p\theta}dz+C\int_{\overline{D_0^{s,t,\alpha}}}\left(\frac{t-s}{|z|^{1+\alpha}}+\frac{t-s}{|z|^{1+\beta}}\right)|z|^{p\theta}dz \\
\le& C\int_{D_{0,\alpha}^{s,t,\alpha}}(t-s)^{-\frac{1}{\alpha}}|z|^{p\theta}\mathbf{1}_{\{|z|<1\}}dz+C\int_{D_{0,\beta}^{s,t,\alpha}}(t-s)^{-\frac{1}{\alpha}}|z|^{p\vartheta}\mathbf{1}_{\{|z|>1\}}dz \\
&+C\int_{\overline{D_{0,\beta}^{s,t,\alpha}}}(t-s)\frac{|z|^{p\theta}}{|z|^{1+\alpha}}\mathbf{1}_{\{|z|<1\}}dz+C\int_{\overline{D_{0,\alpha}^{s,t,\alpha}}}(t-s)\frac{|z|^{p\theta}}{|z|^{1+\beta}}\mathbf{1}_{\{|z|>1\}}dz \\
\le& C(t-s)^{\frac{p\theta}{\alpha}}+C(t-s)^{1+\frac{p\theta-\beta}{\alpha}}+C(t-s)^{1+\frac{(1+\alpha)p\theta-(1+\alpha)^2+C(1+\alpha)}{\alpha(1+\beta)}} \\
&+C(t-s)^{-\frac{1}{\alpha}+C\frac{(1+\alpha)p\theta+(1+\alpha)}{\alpha(1+\beta)}} \\
\le& C(t-s)^{\frac{p\theta}{\alpha}}
\end{aligned}
$$

for all $0 < s < t < T$, $x \in \mathbb{R}$ and $p\theta < \alpha$.

On the other hand, when $s > 1$ and $t - s > 1$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}} |G_{\alpha,\beta}(0,y;s,x)|dy \\
&\le C\int_{D_x^{s,0,\beta}} s^{-\frac{1}{\beta}}dy + \int_{\overline{D_x^{s,0,\beta}}} \left(\frac{s}{|x-y|^{1+\alpha}} + \frac{s}{|x-y|^{1+\beta}}\right)dy \\
&\le C\int_{D_{x,\beta}^{s,0,\beta}} s^{-\frac{1}{\beta}}\mathbf{1}_{\{|x-y|>1\}}dy + C\int_{D_{x,\alpha}^{s,0,\beta}} s^{-\frac{1}{\beta}}\mathbf{1}_{\{|x-y|<1\}}dy \\
&\quad + C\int_{\overline{D_{x,\alpha}^{s,0,\beta}}} \frac{s}{|x-y|^{1+\beta}}\mathbf{1}_{\{|x-y|>1\}}dy \\
&\le C + Cs^{1-\frac{(1+\beta)^2}{\beta(1+\alpha)}+\frac{1+\beta}{\beta(1+\alpha)}} + Cs^{-\frac{1}{\beta}+\frac{1+\beta}{\beta(1+\alpha)}} \le Cs^{-\frac{1}{\beta}+\frac{1+\beta}{\beta(1+\alpha)}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}} |G_{\alpha,\beta}(0,z;t-s,0)||z|^{p\theta}dz \\
&= C\int_{D_z^{s,t,\beta}} (t-s)^{-\frac{1}{\beta}}|z|^{p\theta}dz + C\int_{\overline{D_z^{s,t,\beta}}} \left(\frac{t-s}{|z|^{1+\alpha}} + \frac{t-s}{|z|^{1+\beta}}\right)|z|^{p\theta}dz \\
&\le C\int_{D_{\alpha,z}^{s,t,\beta}} (t-s)^{-\frac{1}{\beta}}|z|^{p\theta}\mathbf{1}_{\{|z|<1\}}dz + C\int_{D_{\beta,z}^{s,t,\beta}} (t-s)^{-\frac{1}{\beta}}|z|^{p\theta}\mathbf{1}_{\{|z|>1\}}dz \\
&\quad + C\int_{\overline{D_{\alpha,z}^{s,t,\beta}}} \frac{t-s}{|z|^{1+\beta}}|z|^{p\theta}\mathbf{1}_{\{|z|>1\}}dz \\
&\le C\left((t-s)^{-\frac{1}{\beta}+\frac{(1+\beta)(p\theta+1)}{\beta(1+\alpha)}} + (t-s)^{\frac{p\theta}{\beta}} + (t-s)^{1+\frac{(1+\beta)(p\theta+1)}{\beta(1+\alpha)}-\frac{(1+\beta)^2}{\beta(1+\alpha)}}\right) \\
&\le C(t-s)^{\frac{p\theta}{\beta}}
\end{aligned}
$$

for all $0 < s < t < T$, $x \in \mathbb{R}$ and $p\theta < \beta$. Then, we get

$$
\mathbb{E}|A_1^1(t,s,x)|^p \le C(t-s)^{\frac{p\theta}{\alpha}} \tag{4.5}
$$

for all $0 < s < t < T$, $x \in \mathbb{R}$ and $p\theta < \beta$.

Next we estimate the term $A_2^1(t,s,x)$. Denote

$$
\begin{aligned}
A_{2,1}^1(t,s,x) &:= \int_0^s \int_{\mathbb{R}} (G_{\alpha,\beta}(r,z;t,x) - G_{\alpha,\beta}(r,z;s,x))W(dz,dr), \\
A_{2,2}^1(t,s,x) &:= \int_s^t \int_{\mathbb{R}} G_{\alpha,\beta}(r,z;t,x)W(dz,dr)
\end{aligned}
$$

for $0 < s < t < T$, $x \in \mathbb{R}$. We then have

$$
|A_2^1(t,s,x)| \le |A_{2,1}^1(t,s,x)| + |A_{2,2}^1(t,s,x)|
$$

for $0 < s < t < T$, $x \in \mathbb{R}$. Moreover, for every $\theta_1 \in (0,1)$ we let

$$
A_{2,1,1}^1(t,s,x) := \||G_{\alpha,\beta}(\cdot,\cdot;t,x) - G_{\alpha,\beta}(\cdot,\cdot;s,x)|^{\theta_1}|G_{\alpha,\beta}(\cdot,\cdot;t,x)|^{1-\theta_1}\|_{\mathcal{H}}^2,
$$

$$A^1_{2,1,2}(t,s,x) := \| |G_{\alpha,\beta}(\cdot,\cdot;t,x) - G_{\alpha,\beta}(\cdot,\cdot;s,x)|^{\theta_1} |G_{\alpha,\beta}(\cdot,\cdot;s,x)|^{1-\theta_1}\|^2_{\mathcal{H}}$$

for $0 < s < t < T$, $x \in \mathbb{R}$. Then, we have

$$\begin{aligned}
&\mathbb{E}|A^1_{2,1}(t,s,x)|^p \\
&\le C\|G_{\alpha,\beta}(\cdot,\cdot;t,x) - G_{\alpha,\beta}(\cdot,\cdot;s,x)\|^p_{\mathcal{H}} \\
&= C\left(\| |G_{\alpha,\beta}(\cdot,\cdot;t,x) - G_{\alpha,\beta}(\cdot,\cdot;s,x)|^{\theta_1} |G_{\alpha,\beta}(\cdot,\cdot;t,x) - G_{\alpha,\beta}(\cdot,\cdot;s,x)|^{1-\theta_1}\|^2_{\mathcal{H}}\right)^{\frac{p}{2}} \\
&\le C\left(|A^1_{2,1,1}(t,s,x)| + |A^1_{2,1,2}(t,s,x)|\right)^{\frac{p}{2}}
\end{aligned}$$

for any $\theta_1 \in (0,1)$. Using (3.2), Proposition 2.3, Lemma 4.1 and the mean-value theorem, one see that there is an ξ between s and t such that

$$\begin{aligned}
|A^1_{2,1,1}(t,s,x)| &= \left\| \left|\frac{\partial G_{\alpha,\beta}}{\partial t}(\cdot,\cdot;\xi,x)\right|^{\theta_1} |t-s|^{\theta_1} |G_{\alpha,\beta}(\cdot,\cdot;t,x)|^{1-\theta_1}\right\|^2_{\mathcal{H}} \\
&= |t-s|^{2\theta_1}\int_{[0,t]^2} dr_1 dr_2 \int_{\mathbb{R}^2}\left|\frac{\partial G_{\alpha,\beta}}{\partial t}(r_1,z_1;\xi,x)\right|^{\theta_1} |G_{\alpha,\beta}(r_1,z_1;t,x)|^{1-\theta_1} \\
&\qquad \times \Theta_H(r_1,r_2;z_1,z_2)\left|\frac{\partial G_{\alpha,\beta}}{\partial t}(r_2,z_2;\xi,x)\right|^{\theta_1} |G_{\alpha,\beta}(r_2,z_2;t,x)|^{1-\theta_1} dz_1 dz_2 \\
&\le C|t-s|^{2\theta_1}\left(\int_0^T\left(\int_{\mathbb{R}}\left(\left|\frac{\partial G_{\alpha,\beta}}{\partial t}(r,z;t,x)\right|^{\theta_1}|G_{\alpha,\beta}(r,z;t,x)|^{1-\theta_1}\right)^{\frac{1}{H_2}} dz\right)^{\frac{H_2}{H_1}} dr\right)^{2H_1} \\
&\le C|t-s|^{2\theta_1}
\end{aligned}$$

for all $0 < \theta_1 < \frac{H_2(1+\alpha)-(1+\alpha)^2}{\alpha(1+\beta)} + H_1 + 1$.

Similarly, one can prove that

$$|A^1_{2,1,2}(t,s,x)| \le C|t-s|^{2\theta_1}.$$

It follows that

$$\mathbb{E}|A^1_{2,1}(t,s,x)|^p \le C|t-s|^{p\theta_1} \tag{4.6}$$

for all $\theta_1 \in (0,\vartheta_1)$. On the other hand, by (3.5), (3.6) and (3.7), we have

$$\begin{aligned}
\mathbb{E}|A^1_{2,2}(t,s,x)|^p &= \mathbb{E}\left\|\int_s^t\int_{\mathbb{R}} G_{\alpha,\beta}(r,z;t,x)W(dr,dz)\right\|^p \\
&\le C\left(\int_{[s,t]^2}\int_{\mathbb{R}^2} G_{\alpha,\beta}(r_1,z_1;t,x)\Theta(r_1,r_2;z_1,z_2)G_{\alpha,\beta}(r_2,z_2;t,x)dz_1dz_2dr_1dr_2\right)^{\frac{p}{2}} \\
&= C\Big(\int_{[s,t]^2}|r_1-r_2|^{2H_1-2}\int_{\mathbb{R}^2}|z_1-z_2|^{2H_2-2} \\
&\qquad \times G_{\alpha,\beta}(r_1,z_1;t,x)G_{\alpha,\beta}(r_2,z_2;t,x)dz_1dz_2dr_1dr_2\Big)^{\frac{p}{2}} \\
&\le C\left(\int_{[s,t]^2}|r_1-r_2|^{2H_1-2}\|G_{\alpha,\beta}(r_1,\cdot;t,x)\|_{L^{\frac{1}{H_2}}(\mathbb{R})}\|G_{\alpha,\beta}(r_2,\cdot;t,x)\|_{L^{\frac{1}{H_2}}(\mathbb{R})}dr_1dr_2\right)^{\frac{p}{2}}
\end{aligned}$$

$$\leq C\left(\int_s^t \left(\|G_{\alpha,\beta}(r,\cdot;t,x)\|_{L^{\frac{1}{H_2}}(\mathbb{R})}\right)^{\frac{1}{H_1}} dr\right)^{pH_1}$$
$$\leq C|t-s|^{p\left(H_1+1+\frac{H_2(1+\alpha)}{\alpha(1+\beta)}-\frac{(1+\alpha)^2}{\alpha(1+\beta)}\right)} \tag{4.7}$$

for $0 < s < t < T$, $x \in \mathbb{R}$.

Combining this with (4.6), we get

$$\mathbb{E}|A_2^1(t,s,x)|^p \leq C|t-s|^{p\theta_1} \tag{4.8}$$

for $\theta_1 \in (0, \vartheta_1)$.

Finally, we consider $A_3^1(t,s,x)$.

$$\begin{aligned}
&|A_3^1(t,s,x)| \\
\leq &\int_0^s \int_{\mathbb{R}} \frac{\partial G_{\alpha,\beta}}{\partial y}(r,y;s,x)(f(r+t-s,y,\widetilde{u}(r+t-s,y)) - f(r,y,\widetilde{u}(r,y)))dydr \\
&+ \int_0^{t-s} \int_{\mathbb{R}} \frac{\partial G_{\alpha,\beta}}{\partial y}(r,y;t,x)f(r,y,\widetilde{u}(r,y))dydr \\
\equiv &|A_{3,1}^1(t,s,x)| + |A_{3,2}^1(t,s,x)|
\end{aligned} \tag{4.9}$$

for all $t > s > 0$. By Hölder inequality, Assumption 2, Theorem 3.1 and (3.8), we have

$$\begin{aligned}
&\mathbb{E}|A_{3,1}^1(t,s,x)|^p \\
\leq &C\left|\int_0^s \int_{\mathbb{R}} \frac{\partial G_{\alpha,\beta}}{\partial y}(r,y;s,x)\mathbb{E}|f(r+t-s,y,\widetilde{u}(r+t-s,y))\right. \\
&\left.\quad -f(r,y,\widetilde{u}(r,y))|^p dydr \left(\int_0^s \int_{\mathbb{R}} \frac{\partial G_{\alpha,\beta}}{\partial y}(r,y;s,x)dydr\right)^{p-1}\right| \\
\leq &C\left(|t-s|^{p\left(1-\frac{2}{\beta}+\frac{1+\beta}{\beta(1+\alpha)}\right)} + \int_0^s \sup_{y\in\mathbb{R}} \mathbb{E}|\widetilde{u}(r+t-s,y) - \widetilde{u}(r,y)|^p dr\right)
\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
&\mathbb{E}|A_{3,2}^1(t,s,x)|^p \\
\leq &C\int_0^{t-s} \int_{\mathbb{R}} \frac{\partial G_{\alpha,\beta}}{\partial y}(r,y;t,x)\mathbb{E}|f(r,y,\widetilde{u}(r,y))|^p dydr \\
&\times\left(\int_0^{t-s} \int_{\mathbb{R}} \frac{\partial G_{\alpha,\beta}}{\partial y}(r,y;s,x)dydr\right)^{p-1} \\
\leq &C|t-s|^{p\left(1-\frac{2}{\beta}+\frac{1+\beta}{\beta(1+\alpha)}\right)}\left(1+\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}|\widetilde{u}(t,x)|^p\right) \\
\leq &C|t-s|^{p\left(1-\frac{2}{\beta}+\frac{1+\beta}{\beta(1+\alpha)}\right)}.
\end{aligned} \tag{4.11}$$

By (4.9), (4.10) and (4.11), then we get

$$\begin{aligned}
&\mathbb{E}|A_3^1(t,s,x)|^p \\
\leq &C\left(|t-s|^{p\left(1-\frac{2}{\beta}+\frac{1+\beta}{\beta(1+\alpha)}\right)} + \int_0^s \sup_{y\in\mathbb{R}} \mathbb{E}|\widetilde{u}(r+t-s,y) - \widetilde{u}(r,y)|^p dr\right)
\end{aligned} \tag{4.12}$$

for all $T > t > s > 0$ and $x \in \mathbb{R}$.

Thus, we have obtained the desired estimate

$$\begin{aligned}\mathbb{E}|\widetilde{u}(t,x)-\widetilde{u}(s,x)|^p &\le C(\mathbb{E}|A_1^1(t,s,x)|^p+\mathbb{E}|A_2^1(t,s,x)|^p+\mathbb{E}|A_3^1(t,s,x)|^p)\\ &\le C\Big(|t-s|^{\frac{p\theta}{\alpha}}+|t-s|^{p\theta_1}+|t-s|^{p\left(1-\frac{2}{\beta}+\frac{1+\beta}{\beta(1+\alpha)}\right)}\\ &\quad+\int_0^s\sup_{y\in\mathbb{R}}\mathbb{E}|\widetilde{u}(r+t-s,y)-\widetilde{u}(r,y)|^p dr\Big)\end{aligned}$$

for all $T > t > s > 0$ and $x \in \mathbb{R}$, which implies that

$$\mathbb{E}|\widetilde{u}(t,x)-\widetilde{u}(s,x)|^p \le C\Big(|t-s|^{p\eta}+\int_0^s\sup_{y\in\mathbb{R}}\mathbb{E}|\widetilde{u}(r+t-s,y)-\widetilde{u}(r,y)|^p dr\Big)$$

for all $T > t > s > 0$ and $x \in \mathbb{R}$ by taking $\eta \in \min\{\frac{\theta}{\alpha}, \theta_1, 1-\frac{2}{\beta}+\frac{1+\beta}{\beta(1+\alpha)}\}$. This gives that the Hölder continuity in time variables t by Gronwall's inequality.

Step2. We consider the spatial case. For all $t \in [0,T]$ and $x, y \in \mathbb{R}$, we need to estimate the following expressions:

$$\begin{aligned}A_1^2(t,x,y) &:= \int_{\mathbb{R}}(G_{\alpha,\beta}(0,z;t,x)-G_{\alpha,\beta}(0,z;t,y))u_0(z)dz,\\ A_2^2(t,x,y) &:= \int_0^t\int_{\mathbb{R}}(G_{\alpha,\beta}(r,z;t,x)-G_{\alpha,\beta}(r,z;t,y))W(dz,dr),\\ A_3^2(t,x,y) &:= \int_0^t\int_{\mathbb{R}}\left(\frac{\partial G_{\alpha,\beta}}{\partial z}(r,z;t,x)-\frac{\partial G_{\alpha,\beta}}{\partial z}(r,z;t,y)\right)f(r,z,u(r,z))dzdr.\end{aligned}$$

We have

$$\mathbb{E}|A_1^2(t,x,y)|^p \le \sup_{z\in\mathbb{R}}\mathbb{E}|u_0(z+x-y)-u_0(z)|^p\cdot\left|\int_{\mathbb{R}}G_{\alpha,\beta}(0,z;t,x)dz\right|^p \le |x-y|^{p\theta} \tag{4.13}$$

for all $t \in [0,T]$ and $x, y \in \mathbb{R}$ by Assumption 1. Denote

$$\begin{aligned}A_{2,1}^2(t,x,y) &:= \||G_{\alpha,\beta}(\cdot,\cdot;t,x)-G_{\alpha,\beta}(\cdot,\cdot;t,y)|^{\theta_2}|G_{\alpha,\beta}(\cdot,\cdot;t,x)|^{1-\theta_2}\|_{\mathcal{H}}^p,\\ A_{2,2}^2(t,x,y) &:= \||G_{\alpha,\beta}(\cdot,\cdot;t,x)-G_{\alpha,\beta}(\cdot,\cdot;t,y)|^{\theta_2}|G_{\alpha,\beta}(\cdot,\cdot;t,y)|^{1-\theta_2}\|_{\mathcal{H}}^p\end{aligned}$$

for all $t \in [0,T]$ and $x, y \in \mathbb{R}$. Then, we have

$$\begin{aligned}&\mathbb{E}|A_2^2(t,x,y)|^p\\ =&C\||G_{\alpha,\beta}(\cdot,\cdot;t,x)-G_{\alpha,\beta}(\cdot,\cdot;t,y)|^{\theta_2}|G_{\alpha,\beta}(\cdot,\cdot;t,x)-G_{\alpha,\beta}(\cdot,\cdot;t,y)|^{1-\theta_2}\|_{\mathcal{H}}^p\\ \le&C(A_{2,1}^2(t,x,y)+A_{2,2}^2(t,x,y))^p.\end{aligned}$$

Similar to Step I, by using (3.2), Proposition 2.3, Lemma 4.2 and the mean-value theorem, we have

$$\begin{aligned}&A_{2,1}^2(t,x,y)\\ =&\left\|\left|\frac{\partial G_{\alpha,\beta}}{\partial x}(\cdot,\cdot;t,\xi)\right|^{\theta_2}|x-y|^{\theta_2}|G_{\alpha,\beta}(\cdot,\cdot;t,x)|^{1-\theta_2}\right\|_{\mathcal{H}}\end{aligned}$$

$$\leq C|x-y|^{\theta_2}\left(\int_{[0,T]^2}\int_{\mathbb{R}^2}\left|\frac{\partial G_{\alpha,\beta}}{\partial x}(r_1,z_1;t,\xi)\right|^{\theta_2}|G_{\alpha,\beta}(r_1,z_1;t,x)|^{1-\theta_2}\right.$$
$$\left.\times\Theta_H(r_1,r_2;z_1,z_2)\left|\frac{\partial G_{\alpha,\beta}}{\partial x}(r_2,z_2;t,\xi,)\right|^{\theta_2}|G_{\alpha,\beta}(r_2,z_2;t,x)|^{1-\theta_2}dz_1dz_2dr_1dr_2\right)^{\frac{1}{2}}$$
$$\leq C|x-y|^{p\theta_2}\left(\int_0^T\left(\int_{\mathbb{R}}\left(\left|\frac{\partial G_{\alpha,\beta}}{\partial x}(r,z;t,x)\right|^{\theta_2}|G_{\alpha,\beta}(r,z;t,x)|^{1-\theta_2}\right)^{\frac{1}{H_2}}dz\right)^{\frac{H_2}{H_1}}dr\right)^{H_1}$$
$$\leq C|x-y|^{\theta_2}$$

for all $\theta_2 < H_2$. Similarly, one can also prove

$$A^2_{2,2}(t,x,y)\leq C|x-y|^{\theta_2}$$

for all $\theta_2 < H_2$. It follows that

$$E|A^2_2(t,x,y)|^p\leq C|x-y|^{p\theta_2} \tag{4.14}$$

for all $\theta_2 < H_2$. Finally, we consider the term $|A^2_3(t,x,y)|$. By Hölder inequality, Assumption 1 and (3.8), we have

$$\begin{aligned}
&\mathbb{E}|A^2_3(t,x,y)|^p\\
=&\mathbb{E}\left|\int_0^t\int_{\mathbb{R}}\left(\frac{\partial G_{\alpha,\beta}}{\partial z}(r,z;t,x)-\frac{\partial G_{\alpha,\beta}}{\partial z}(r,z;t,y)\right)f(r,z,\widetilde{u}(r,z))dzdr\right|^p\\
\leq&C\int_0^t\int_{\mathbb{R}}\left|\frac{\partial G_{\alpha,\beta}}{\partial z}(r,z;t,y)\right|\mathbb{E}|f(r,z+x-y,\widetilde{u}(r,z+x-y))\\
&-f(r,z,\widetilde{u}(r,z))|^pdzdr\cdot\left(\int_0^t\int_{\mathbb{R}}\left|\frac{\partial G_{\alpha,\beta}}{\partial z}(r,z;t,y)\right|dzdr\right)^{p-1}\\
\leq&C\int_0^t\int_{\mathbb{R}}\left(|x-y|^{p\left(1-\frac{2}{\beta}+\frac{1+\beta}{\beta(1+\alpha)}\right)}+\sup_{z\in\mathbb{R}}\mathbb{E}|\widetilde{u}(r,z+x-y)-\widetilde{u}(r,z)|^p\right)\\
&\times\left|\frac{\partial G_{\alpha,\beta}}{\partial z}(r,z;t,y)\right|dzdr\\
\leq&C\left(t|x-y|^{p\left(1-\frac{2}{\beta}+\frac{1+\beta}{\beta(1+\alpha)}\right)}+\int_0^t\sup_{z\in\mathbb{R}}\mathbb{E}|\widetilde{u}(r,z+x-y)-\widetilde{u}(r,z)|^pdr\right).
\end{aligned} \tag{4.15}$$

Combining this with (4.13) and (4.14), we have

$$\begin{aligned}
\mathbb{E}|\widetilde{u}(t,x)-\widetilde{u}(s,x)|^p&\leq C(\mathbb{E}|A^2_1(t,x,y)|^p+\mathbb{E}|A^2_2(t,x,y)|^p+\mathbb{E}|A^2_3(t,x,y)|^p)\\
&\leq C\big(|x-y|^{p\theta}+|x-y|^{p\theta_2}+t|x-y|^{p\left(1-\frac{2}{\beta}+\frac{1+\beta}{\beta(1+\alpha)}\right)}\\
&\quad+\int_0^t\sup_{z\in\mathbb{R}}\mathbb{E}|\widetilde{u}(r,z+x-y)-\widetilde{u}(r,z)|^pdr\big)
\end{aligned}$$

for all $0<\theta_2<H_2$, which implies that

$$\mathbb{E}|\widetilde{u}(t,x)-\widetilde{u}(s,x)|^p\leq C\Big(|x-y|^{p\eta}+\int_0^t\sup_{z\in\mathbb{R}}\mathbb{E}|\widetilde{u}(r,z+x-y)-\widetilde{u}(r,z)|^pdr\Big)$$

for all $T > t > s > 0$ and $x, y \in \mathbb{R}$ by taking $\eta \in \min\{\theta, \theta_2, 1 - \frac{2}{\beta} + \frac{1+\beta}{\beta(1+\alpha)}\}$. Thus, we have proved the Hölder continuous in space variables x by Gronwall's inequality. $\square$

5. Large deviation principle

In this section, as a related question, we consider a large deviation principle associated with the equation (1.1) with a small perturbation. Such SPDEs with a small perturbation can be written as

$$\begin{cases} \frac{\partial u^\varepsilon(t,x)}{\partial t} = (\Delta_\alpha + \Delta_\beta) u^\varepsilon(t,x) + \frac{\partial f}{\partial x}(t, x, u^\varepsilon(t,x)) + \sqrt{\varepsilon} \frac{\partial^2 W}{\partial t \partial x}, \quad t \in [0,T],\ x \in \mathbb{R}, \\ u^\varepsilon(0,x) = 0, \quad x \in \mathbb{R}, \end{cases} \tag{5.1}$$

where $0 < \varepsilon \leq 1$, $(t,x) \in [0,T] \times \mathbb{R}$ and $\dot{W}(t,x)$ is a fractional noise with Hurst index $H \in (\frac{1}{2}, 1)$. As an immediate consequence of Theorem 3.1, for every $\varepsilon \in (0,1]$ we see that (5.1) admits a unique solution and this solution is jointly (θ_1, θ_2)-Hölder continuous with $0 < \theta_i \leq \vartheta_i,\ i = 1, 2$, where ϑ_1 and ϑ_2 are given in Theorem 4.1.

Now, we state the precise statement of large deviation principle below. Let $\{X^\varepsilon; \varepsilon > 0\}$ be a family of stochastic variables defined on a probability space $(\Omega, \mathcal{F}, P)$, which takes values in a Polish space $\mathcal{E}$. Recall that a Borel function $I : \mathcal{E} \to [0, \infty]$ is called a rate function on $\mathcal{E}$, if for each $N < \infty$, the level set $\{x \in \mathcal{E} : I(x) \leq N\}$ is a compact subset of $\mathcal{E}$.

Definition 5.1. Let I be a rate function on $\mathcal{E}$. The sequence $\{X^\varepsilon\}$ is said to satisfy the large deviations principle on $\mathcal{E}$ with rate function I, if the following two conditions hold.

(1) For each closed subset F of $\mathcal{E}$,

$$\limsup_{\varepsilon \to 0} \varepsilon \log P(X^\varepsilon \in F) \leq -I(F).$$

(2) For each open subset G of $\mathcal{E}$,

$$\liminf_{\varepsilon \to 0} \varepsilon \log P(X^\varepsilon \in G) \geq -I(G).$$

In this paper, we use the weak convergence approach. In this approach one need to prove the Laplace principle which is equivalent to the large deviation principle (see Dupuis and Ellis [16]).

Definition 5.2. The family of random variables $\{X^\varepsilon\}$ defined on the Polish space $\mathcal{E}$, is said to satisfy the Laplace principle with the rate function I, if

$$\lim_{\varepsilon \to 0} \varepsilon \log E\left(\exp\left(-\frac{1}{\varepsilon} h(X^\varepsilon)\right)\right) = -\inf_{f \in \mathcal{E}} \{h(f) + I(f)\}$$

for any bounded continuous function $h : \mathcal{E} \to \mathbb{R}$.

Let K be a compact subset of $\mathbb{R}$. Denote by $\mathcal{P}_2$ the set of predictable processes belonging to $L^2(\Omega \times [0,T] \times K; \mathbb{R})$. We introduce some notations as follows.

- For any $N>0$, we define

$$\mathcal{H}_T^N = \left\{ h \in L^2([0,T]\times K); \int_0^T \int_K h(s,y)dyds \le N \right\},$$
$$\mathcal{P}_2^N = \{u \in \mathcal{P}_2; u \in \mathcal{H}_T^N \quad \text{a.s.}\}$$

and let $\mathcal{H}_T^N$ endow with the weak topology.

- For $\varepsilon>0$, we let $\eta^\varepsilon: \mathcal{C}([0,T]\times K;\mathbb{R}) \to \mathcal{E}$ be a family of measurable maps.
- For every $u \in L^2([0,T]\times K)$, we define

$$\mathrm{Int}(u) := \int_{[0,t]\times(K\cap(-\infty,x))} u(s,y)dsdy.$$

We suppose that there exists a measurable map $\eta^0: \mathcal{C}([0,T]\times K;\mathbb{R}) \to \mathcal{E}$ such that

LP$_1$: for every $M<\infty$, the set $\left\{\eta^0(\mathrm{Int}(v)),\ v \in \mathcal{H}_T^M\right\}$ is a compact subset of $\mathcal{E}$;

LP$_2$: for every $M<\infty$, if the family $\{v^\varepsilon\} \subset \mathcal{P}_2^M$ converges to $v \in \mathcal{P}_2^M$ in distribution. Then

$$\eta^\varepsilon\left(\sqrt{\varepsilon}W + \mathrm{Int}(v^\varepsilon)\right) \to \eta^0\left(\mathrm{Int}(v)\right)$$

in distribution, as $\varepsilon \to 0$.

Theorem 5.1 (Budhiraja etc [9]). *Under* **LP**$_1$ *and* **LP**$_2$*, the family*

$$X^\varepsilon := \eta^\varepsilon(\sqrt{\varepsilon}W), \quad \varepsilon>0$$

satisfies the Laplace principle with rate function $I(f)$ given as follows

$$I(f) := \inf_{\{h\in L^2([0,T]\times K): f=\eta^0(\mathrm{Int}(u))\}} \left\{ \frac{1}{2}\int_0^T \int_K u^2(s,x)dsdx \right\} \tag{5.2}$$

with $f \in \mathcal{E}$.

Consider the Banach space $C^\theta([0,T]\times K;\mathbb{R})$ of $\theta=(\theta_1,\theta_2)$-Hölder continuous functions equipped with the norm defined by

$$\|f\|_\theta := \sup_{(t,x)\in[0,T]\times K} |f(t,x)| + \sup_{s\neq t\in[0,T]} \sup_{x\neq y\in K} \frac{|f(t,x)-f(s,y)|}{|t-s|^{\theta_1}+|x-y|^{\theta_2}},$$

where $0<\theta_1<\vartheta_1,\ 0<\theta_2<\vartheta_2$. Then the solution $\{u^\varepsilon(t,x);(t,x)\in[0,T]\times K\}$ lives in the Hölder space $C^\theta([0,T]\times K;\mathbb{R})$ for every compact subset $K\subset\mathbb{R}$. Since $C^\theta([0,T]\times K;\mathbb{R})$ is not separable, we consider the space $C^{\theta',0}([0,T]\times K;\mathbb{R})$ of Hölder continuous functions f with $\theta'=(\theta_1',\theta_2')$ and $\theta_1'<\theta_1, \theta_2'<\theta_2$ such that

$$\lim_{\delta\to 0}\left(\sup_{|t-s|+|x-y|<\delta} \frac{|f(t,x)-f(s,y)|}{|t-s|^{\theta_1'}+|x-y|^{\theta_2'}} \right) = 0.$$

Clearly, the space $C^{\theta',0}([0,T]\times K;\mathbb{R})$ is a Polish space containing $C^\theta([0,T]\times K;\mathbb{R})$.

Let $\mathcal{E}^\nu := C^{\nu,0}([0,T]\times K;\mathbb{R})$ denote the space of ν-Hölder continuous functions equipped with the Hölder norm of order ν denoted by $\|\cdot\|_\nu$. Consider the deterministic equation:

$$\begin{aligned} Z^h(t,x) = &\int_0^t \int_K \frac{\partial}{\partial y} G_{\alpha,\beta}(s,y;t,x) f(s,y,Z^h(s,y))dyds \\ &+ \int_0^t \int_K (K^*_{H_1,H_2} G_{\alpha,\beta}(*,\star;t,x))(s,r)h(s,r)drds \end{aligned} \tag{5.3}$$

for $h \in L^2([0,T]\times K)$.

Lemma 5.1. *For any $h \in L^2([0,T]\times K)$, the equation (5.3) admits a unique solution $\{Z^h(t,x); (t,x)\in[0,T]\times K\}$ and $Z^h(t,x)\in\mathcal{E}^\theta$.*

The above Lemma is an immediate conclusion of Lemma 5.2 and Proposition 5.1 with $\varepsilon = 0$ given later. In this section, our object is to expound and prove the following theorem.

Theorem 5.2. *Let for all $1 < \frac{1+\alpha^2}{2\alpha} < \beta < \alpha < 2$ and let the assumptions 1 and 2 hold. If $\{u^\varepsilon(t,x),(t,x)\in[0,T]\times\mathbb{R}\}$ is the solution of (5.1) for every $0<\varepsilon\le 1$, then the family of laws of $\{u^\varepsilon(t,x),(t,x)\in[0,T]\times K\}$ satisfies, on $\mathcal{E}^\theta$, a large deviation principle with rate function*

$$I(f) = \inf_{\{h\in L^2([0,T]\times K): Z^h = f\}} \left\{ \frac{1}{2}\int_0^T\int_K h^2(s,x)dxds \right\} \tag{5.4}$$

for every compact subset $K\subset\mathbb{R}$.

To prove the theorem we need some preliminaries. For $v\in\mathcal{P}_2^N$, $\varepsilon\in(0,1]$, $t\in[0,T]$ and compact subset $K\subset\mathbb{R}$, we define the controlled equation as follows

$$\begin{aligned} u^{\varepsilon,v}(t,x) = &\int_0^t\int_K \frac{\partial G_{\alpha,\beta}(s,y;t,x)}{\partial y} f(s,y,u^{\varepsilon,v}(s,y))dyds \\ &+ \sqrt{\varepsilon}\int_0^t\int_K (K^*_{H_1,H_2}G_{\alpha,\beta}(*,\star;x,t))(s,y)W(ds,dy) \\ &+ \int_0^t\int_K (K^*_{H_1,H_2}G_{\alpha,\beta}(*,\star;t,x))(s,y)v(s,y)dyds. \end{aligned} \tag{5.5}$$

Lemma 5.2. *Let for all $1<\frac{1+\alpha^2}{2\alpha}<\beta<\alpha\le 2$ and the assumptions 1 and 2 hold, then the equation (5.5) admits a unique solution $\{u^{\varepsilon,v}(t,x);(t,x)\in[0,T]\times K\}$ such that*

$$\sup_{0<\varepsilon\le 1}\sup_{v\in\mathcal{P}_2^N}\sup_{t\in[0,T],x\in K}\mathbb{E}|u^{\varepsilon,v}(t,x)|^p<\infty$$

for any $v\in\mathcal{P}_2^N$, $\varepsilon\in(0,1]$ and $p\ge 2$.

Proof. The proof of the lemma is similar to Theorem 3.1, and we omit it. □

In view of Theorem 5.1, to prove Theorem 5.2 it suffices to verify the above conditions $\mathbf{LP}_1$ and $\mathbf{LP}_2$.

The condition $\mathbf{LP}_1$ can be obtained by proving that the mapping $\mathcal{P}_2^N \ni h\mapsto Z^h\in\mathcal{E}^\theta$ is continuous, i.e.,

$$\lim_{n\to\infty}\|Z^{h_n}-Z^h\|_\theta = 0 \tag{5.6}$$

provided $\lim_{\varepsilon\to 0}\langle h_n - h, g\rangle_{L^2([0,T]\times K)} = 0$ for $\{h, h_n; n \geq 1\} \subset \mathcal{H}_T^N$ and $g \in L^2([0,T]\times K)$.

For the condition $\mathbf{LP}_2$, by Skorohod representation theorem, there exist a probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$ carrying a sequence $(\overline{v}^\varepsilon, v, \overline{W}^\varepsilon, \varepsilon > 0)$ of random fields such that the joint law of $(\overline{v}^\varepsilon, \overline{W}^\varepsilon)$ and (v^ε, W) coincides and

$$\lim_{\varepsilon\to 0}\langle \overline{v}^\varepsilon - \overline{v}, g\rangle_{\mathcal{H}_T} = 0, \quad \overline{P} - a.s.$$

for all $g \in L^2([0,T]\times K)$. Let $\overline{u}^{\varepsilon,\overline{v}^\varepsilon}(t,x)$ be the solution to a similar equation as (5.5) obtained by changing v into $\overline{v}^\varepsilon$ and W by $\overline{W}$. Thus, the condition $\mathbf{LP}_2$ will be valid if we can prove that

$$\lim_{\varepsilon\to 0}\overline{\mathbb{E}}\left(\|\overline{u}^{\varepsilon,\overline{v}^\varepsilon} - \overline{Z}^{\overline{v}}\|_\theta^q\right) = 0 \tag{5.7}$$

for any $q \geq 1$. Clearly, the convergence (5.6) can be obtained as a particular case of (5.7) by taking $\varepsilon = 0$.

Now, to check $\mathbf{LP}_1$ and $\mathbf{LP}_2$ we prove that the convergence (5.7) holds. According to Bally etc [5], the proof of (5.7) can be carried out into the next statements:

(1) *Estimates on increments*

$$\begin{aligned}&\sup_{0<\varepsilon\leq 1}\overline{\mathbb{E}}\left(\left|(\overline{u}^{\varepsilon,\overline{v}^\varepsilon}(t,x) - Z^{\overline{v}}(t,x)) - (\overline{u}^{\varepsilon,\overline{v}^\varepsilon}(s,y) - Z^{\overline{v}}(s,y))\right|^q\right)\\ &\leq C_q\left[|t-s|^{\theta_1} + |x-y|^{\theta_2}\right]^q\end{aligned} \tag{5.8}$$

for all $q \geq 2$ and $(t,x),(s,y) \in [0,T]\times K$.

(2) *Point-wise convergence*

$$\lim_{\varepsilon\to 0}\overline{\mathbb{E}}\left(|\overline{u}^{\varepsilon,\overline{v}^\varepsilon}(t,x) - Z^{\overline{v}}(t,x)|^q\right) = 0 \tag{5.9}$$

for all $q \geq 2$ and $(t,x),(s,y) \in [0,T]\times K$.

The two statements above follow from the next propositions.

Proposition 5.1. *Let* $1 < \frac{1+\alpha^2}{2\alpha} < \beta < \alpha \leq 2$ *and the assumptions 1 and 2 hold. Let* $\{\overline{u}^{\varepsilon,\overline{v}^\varepsilon}\}$ *be the solution to equation* (5.5)*. Then, we have*

$$\overline{\mathbb{E}}|\overline{u}^{\varepsilon,\overline{v}^\varepsilon}(t,x) - \overline{u}^{\varepsilon,\overline{v}^\varepsilon}(t',x')|^q \leq C_q\left(|t-t'|^{q\theta_1} + |x-x'|^{q\theta_2}\right) \tag{5.10}$$

for any $(t,x),(t',x') \in [0,T]\times K$, $\varepsilon \in (0,1]$ *and* $q \geq 2$*, and*

$$\sup_{0<\varepsilon\leq 1}\sup_{\overline{v}^\varepsilon\in\mathcal{P}_2^N}\overline{\mathbb{E}}\|\overline{u}^{\varepsilon,\overline{v}^\varepsilon}\|_\theta^q < \infty. \tag{5.11}$$

Proof. For any $(t,x),(t',x') \in [0,T]\times K$ and $q \geq 2$, we have

$$\begin{aligned}&\overline{\mathbb{E}}|\overline{u}^{\varepsilon,\overline{v}^\varepsilon}(t,x) - \overline{u}^{\varepsilon,\overline{v}^\varepsilon}(t',x')|^q\\ &\leq C_q\overline{\mathbb{E}}\left|\sqrt{\varepsilon}\int_0^t\int_K (K^*_{H_1,H_2}G_{\alpha,\beta}(*,\star;t,x))(s,y)W(ds,dy)\right.\end{aligned}$$

$$
\begin{aligned}
& - \sqrt{\varepsilon}\int_0^{t'}\int_K (K^*_{H_1,H_2}G_{\alpha,\beta}(*,\star;t,x))(s,y)W(ds,dy)\Bigg|^q \\
+ & C_q\overline{\mathbb{E}}\Bigg|\int_0^t\int_K (K^*_{H_1,H_2}G_{\alpha,\beta}(*,\star;t,x))(s,y)\overline{v}^{\varepsilon}(s,y)dyds \\
& - \int_0^{t'}\int_K (K^*_{H_1,H_2}G_{\alpha,\beta}(*,\star;t',x'))(s,y)\overline{v}^{\varepsilon}(s,y)dyds\Bigg|^q \\
+ & C_q\overline{\mathbb{E}}\Bigg|\int_0^t\int_K G_{\alpha,\beta}(s,y;t,x)f(s,y,\overline{u}^{\varepsilon,\overline{v}^{\varepsilon}}(s,y))dyds \\
& - \int_0^{t'}\int_K G_{\alpha,\beta}(s,y;t',x')f(s,y,\overline{u}^{\varepsilon,\overline{v}^{\varepsilon}}(s,y))dyds\Bigg|^q \\
\equiv & C_q(\Phi_1+\Phi_2+\Phi_3).
\end{aligned}
$$

In a similar argument as the proof of Theorem 4.1, we can prove

$$\Phi_1+\Phi_2 \le C_q\left(|t-t'|^{q\theta_1}+|x-x'|^{q\theta_2}\right)$$

for $0<\theta_1<\vartheta_1$, $0<\theta_2<\vartheta_2$. Moreover, Hölder inequality and the assumptions 1 and 2 imply that

$$
\begin{aligned}
\Phi_3 = \overline{\mathbb{E}}\Bigg| & \int_0^{t'}\int_K \frac{\partial G_{\alpha,\beta}}{\partial y}(s,y;t',x') \\
& \times\Big(f\Big(s+t-t',y+x-x',\overline{u}^{\varepsilon,\overline{v}^{\varepsilon}}(s+t-t',y+x-x')\Big)-f\left(s,y,\overline{u}^{\varepsilon,\overline{v}^{\varepsilon}}(s,y)\right)\Big)dyds \\
& + \int_0^{t-t'}\int_K \frac{\partial G_{\alpha,\beta}}{\partial y}(s,y;t,x)f(s,y,\overline{u}^{\varepsilon,\overline{v}^{\varepsilon}}(s,y))dyds\Bigg|^q \\
\le C_q & \Big(|t-t'|^{q\left(1-\frac{2}{\beta}+\frac{1+\beta}{\beta(1+\alpha)}\right)}+|x-x'|^q \\
& + \int_0^t \sup_{y\in K}\overline{\mathbb{E}}\left|\overline{u}^{\varepsilon,\overline{v}^{\varepsilon}}(s+t-t',y+x-x')-\overline{u}^{\varepsilon,\overline{v}^{\varepsilon}}(s,y)\right|^q ds\Big)
\end{aligned}
$$

for any $(t,x),(t',x')\in[0,T]\times K$ and $q\ge 2$. Thus, we have introduced the estimate

$$
\begin{aligned}
& \overline{\mathbb{E}}|\overline{u}^{\varepsilon,\overline{v}^{\varepsilon}}(t,x)-\overline{u}^{\varepsilon,\overline{v}^{\varepsilon}}(t',x')|^q \\
\le & C_q\left(|t-t'|^{q\theta_1}+|x-x'|^{q\theta_2}\right) \\
& + C_q\int_0^t \sup_{y\in K}\overline{\mathbb{E}}\left|\overline{u}^{\varepsilon,\overline{v}^{\varepsilon}}(s+t-t',y+x-x')-\overline{u}^{\varepsilon,\overline{v}^{\varepsilon}}(s,y)\right|^q ds
\end{aligned}
$$

for all $(t,x),(t',x')\in[0,T]\times K$ and $q\ge 2$, and the proposition follows from Gronwall's lemma. □

Proposition 5.2. *Let $1<\frac{1+\alpha^2}{2\alpha}<\beta<\alpha\le 2$ and the assumptions 1 and 2 hold. If*

$$\lim_{\varepsilon\to 0}\langle\overline{v}^{\varepsilon}-\overline{v},g\rangle_{L^2([0,T]\times K)}=0,\quad \overline{\mathbb{P}}-a.s.$$

for $\{\overline{v},\overline{v}^{\varepsilon};\varepsilon>0\}\subset\mathcal{P}_2^N$ and $g\in L^2([0,T]\times K)$, we then have

$$\lim_{\varepsilon\to 0}\overline{\mathbb{E}}\left(\left|\overline{u}^{\varepsilon,\overline{v}^{\varepsilon}}(t,x)-Z^{\overline{v}}(t,x)\right|^q\right)=0$$

for all $q \geq 2$, $(t,x) \in [0,T] \times K$.

Proof. Let $q \geq 2$ be given. We then have

$$\begin{aligned}
&\overline{\mathbb{E}}\left|\overline{u}^{\varepsilon,\overline{v}^{\varepsilon}}(t,x) - Z^{\overline{v}}(t,x)\right|^q \\
\leq& C_q\left(\overline{\mathbb{E}}\left|\sqrt{\varepsilon}\int_0^t\int_K (K^*_{H_1,H_2}G_{\alpha,\beta}(*,\star;t,x))(s,y)B(ds,dy)\right|^q\right. \\
&+\overline{\mathbb{E}}\left|\int_0^t\int_K (K^*_{H_1,H_2}G_{\alpha,\beta}(*,\star;t,x))(s,y)(\overline{v}^{\varepsilon}(s,y)-\overline{v}(s,y))dyds\right|^q \\
&\left.+\overline{\mathbb{E}}\left|\int_0^t\int_K G_{\alpha,\beta}(s,y;t,x)(f(s,y,\overline{u}^{\varepsilon,\overline{v}^{\varepsilon}}(s,y)-f(s,y,Z^{\overline{v}}(s,y)))dyds\right|^q\right) \\
\equiv& C_q(\Psi_1+\Psi_2+\Psi_3)
\end{aligned}$$

for all $(t,x) \in [0,T] \times K$ and $\varepsilon \in (0,1]$. Clearly, Cauchy-Schwartz's inequality and the proof of Theorem 4.1 imply that

$$\Psi_1 \leq C_{\alpha,q,H,T}\varepsilon^{\frac{q}{2}}$$

and

$$\begin{aligned}
\Psi_2 \leq& \overline{\mathbb{E}}\left(\int_0^t\int_K (K^*_{H_1,H_2}G_{\alpha,\beta}(*,\star;t,x))^2(s,y)dyds\right. \\
&\left.\times\int_0^t\int_K |\overline{v}^{\varepsilon}(s,y)-\overline{v}(s,y)|^2 dyds\right)^{\frac{q}{2}} \\
\leq& C_{\alpha,q,H,T}\|\overline{v}^{\varepsilon}-\overline{v}\|^q_{L^2([0,T]\times K)}
\end{aligned}$$

for all $(t,x) \in [0,T] \times K$ and $\varepsilon \in (0,1]$. Finally, applying Hölder inequality and the assumptions 1 and 2 to lead

$$\begin{aligned}
\Psi_3 \leq& C_q\int_0^t\int_K \frac{\partial G_{\alpha,\beta}}{\partial y}(s,y;t,x)\overline{\mathbb{E}}\left|f(s,y,\overline{u}^{\varepsilon,\overline{v}^{\varepsilon}}(s,y)) - f(s,y,Z^{\overline{v}}(s,y))\right|^q dyds \\
\leq& C_q\int_0^t \sup_{(r,y)\in[0,s]\times K}\mathbb{E}\left|\overline{u}^{\varepsilon,\overline{v}^{\varepsilon}}(s,y) - Z^{\overline{v}}(s,y)\right|^q ds.
\end{aligned}$$

Thus, we have obtained the desired estimate

$$\begin{aligned}
\Psi_{\varepsilon}(t) :=& \sup_{(t,x)\in[0,T]\times K}\overline{\mathbb{E}}\left|\overline{u}^{\varepsilon,\overline{v}^{\varepsilon}}(t,x) - Z^{\overline{v}}(t,x)\right|^q \\
\leq& C_{\alpha,q,H,T}\left(\varepsilon^{\frac{q}{2}} + \|\overline{v}^{\varepsilon}-\overline{v}\|^q_{L^2([0,T]\times K)} + \int_0^t \Psi_{\varepsilon}(s)ds\right)
\end{aligned}$$

for all $\varepsilon \in (0,1]$ and $(t,x) \in [0,T] \times K$, which proves

$$\lim_{\varepsilon\to 0}\sup_{(t,x)\in[0,T]\times K}\overline{\mathbb{E}}\left(|\overline{u}^{\varepsilon,\overline{v}^{\varepsilon}}(t,x) - Z^{\overline{v}}(t,x)|^q\right) = 0$$

by the Gronwall's lemma, and the proposition follows. □

Now, we can easily prove the Theorem 5.2.

Proof of Theorem 5.2. Clearly, the estimate (5.8) and convergence (5.9) are two immediate consequence of Proposition 5.1 and Proposition 5.2, and Theorem 5.2 follows. □

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